G53KRR: Knowledge Representation & Reasoning

Lecture 5: Resolution

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Outline of this lecture

- recap: propositional calculus
- soundness & completeness
- proof systems
- resolution

Recap: propositional calculus

- *atomic propositions* are simple declarative sentences about the world that can be *true* or *false*
- a *model* is an assignment of *true* or *false* to each atomic proposition *p*
- we can construct more complex sentences using the *logical connectives*: $\neg$ “not, $\wedge$ “and, $\vee$ “or and $\rightarrow$ “implies
- the meaning of the logical connectives is given in terms of *truth tables* which specify how the truth of complex sentences is built up from the truth of their components

Recap: truth tables

- the truth tables for the four basic connectives are:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$
$true$	$true$	$false$	$true$	$true$	$true$
$true$	$false$	$false$	$false$	$true$	$false$
$false$	$true$	$true$	$false$	$true$	$true$
$false$	$false$	$true$	$false$	$false$	$true$

- using truth tables we can determine the truth or falsity of *any* complex sentence in a given model

Example: truth tables

- by enumerating *all* possible models (all possible truth assignments) we can determine the truth value of, e.g.,

$$\neg p \vee q \rightarrow r$$

in all models

p	q	r	$\neg p$	$\neg p \vee q$	$\neg p \vee q \rightarrow r$
$true$	$true$	$true$	$false$	$true$	$true$
$true$	$true$	$false$	$false$	$true$	$false$
$true$	$false$	$true$	$false$	$false$	$true$
$true$	$false$	$false$	$false$	$false$	$true$
$false$	$true$	$true$	$true$	$true$	$true$
$false$	$true$	$false$	$true$	$true$	$false$
$false$	$false$	$true$	$true$	$true$	$true$
$false$	$false$	$false$	$true$	$true$	$false$

Recap: entailment

- given a notion of truth, we can say what it means for the truth of one statement to follow necessarily from the truth (or falsity) of other statements

- a set of sentences $\{\phi_1, \phi_2, \ldots, \phi_n\}$ *entails* a sentence ψ
  $\{\phi_1, \phi_2, \ldots, \phi_n\} \models \psi$ if in all models where $\{\phi_1, \phi_2, \ldots, \phi_n\}$ are *true*, ψ is also *true*

- e.g., $p \vee q, \ \neg p \models q$ since in all models where $p \vee q$ and $\neg p$ are *true* q is also *true*

- note that $p \vee q, \ p \not\models q$ since there is a model where $p \vee q$ and p are *true* but q is *false*

Inference

- entailment can be used to derive conclusions—i.e., to carry out *logical inference*

- by enumerating all possible models we can determine if a sentence ψ follows logically from sentences $\{\phi_1, \phi_2, \ldots, \phi_n\}$

- gives us a reasoning process whose conclusions are guaranteed to be *true* if the premises are *true*

  – if $\{\phi_1, \phi_2, \ldots, \phi_n\}$ are *true* in the world, then ψ is necessarily *true* in the world

- pattern of inference relies only on the truth values of propositions and the meaning of the logical connectives, not on a detailed knowledge of a particular problem domain

Example: valid inference

- given the atomic propositions

$$\begin{aligned} p &: \textit{the train is late} \\ q &: \textit{there are taxis at the station} \\ r &: \textit{John is late} \end{aligned}$$

express in propositional calculus

  – if the train is late and there are no taxis at the station, John is late for the meeting

  – the train is late

  – John is not late for the meeting

and show using truth tables that there must have been taxis at the station

Example: valid inference

• we need to show that in all models where $p \wedge \neg q \rightarrow r,\ p,\ \neg r$ are *true* q is also *true*

p	q	r	$\neg q$	$p \wedge \neg q$	$p \wedge \neg q \rightarrow r$	$\neg r$
$true$	$true$	$true$	$false$	$false$	$true$	$false$
$\boxed{true}$	$\boxed{true}$	$false$	$false$	$false$	$\boxed{true}$	$\boxed{true}$
$true$	$false$	$true$	$true$	$true$	$true$	$false$
$true$	$false$	$false$	$true$	$true$	$false$	$true$
$false$	$true$	$true$	$false$	$false$	$true$	$false$
$false$	$true$	$false$	$false$	$false$	$true$	$true$
$false$	$false$	$true$	$true$	$false$	$true$	$false$
$false$	$false$	$false$	$true$	$false$	$true$	$true$

Computational complexity

- with n propositions, i.e., $n = |P|$, the number of possible models (rows in the truth table) is 2^n
- computing entailment by exhaustive enumeration of models is therefore exponential in the number of propositional variables
- time complexity is $O(2^n)$, space complexity is $O(n)$

Proof systems

- instead of trying to show semantically that $\phi \models \psi$ a proof system uses *rules of inference* to derive valid formulas from other formulas *syntactically*

- a *proof* consists of a sequence of inference steps—applications of inference rules which lead from the initial formulas to the formula to be derived

- proofs may still be exponential (i.e., contain an exponential number of steps) in the worst case, but they can be much shorter

Modus ponens

- one example of an inference rule is the rule of *modus ponens*

$$\frac{\phi,\ \phi \to \psi}{\psi}$$

- from the last lecture we know that $\phi,\ \phi \to \psi\ \models\ \psi$

ϕ	ψ	$\phi \to \psi$
true	**true**	**true**
true	*false*	*false*
false	*true*	*true*
false	*false*	*true*

- ϕ and ψ can be *arbitrarily complex formulas*—we can replace many rows in the truth table with the application of a single inference rule

Sound and complete inference

- rules of inference are chosen to give a *sound* and *complete* inference procedure

- a *sound* inference procedure is one which derives only entailed sentences, i.e., derives true conclusions given true premises

- a *complete* inference procedure is one that can derive any sentence that is entailed, i.e., derives all true conclusions from a set of premises

Resolution

- many proof systems for propositional calculus, e.g,. natural deduction, tableaux methods, etc.—we shall focus on *resolution*
- resolution has a single rule of inference—the *resolution rule*
- sound and (refutation) complete
- widely used in AI theorem proving and problem solving systems
- requires that the logical description of the problem is formulated in terms of *clauses*

Clausal form

- a *literal* is an atomic proposition or its negation, e.g., $p, \neg p$ are literals

- a *clause* is a disjunction of literals, e.g., $p \vee q$, $\neg p \vee r$, and $p \vee q \vee r$ are all clauses

- a sentence expressed as a conjunction of disjunctions of literals (i.e,. clauses) is said to be in *conjunctive normal form* (CNF)

- any complex sentence in propositional calculus can be re-expressed in conjunctive normal form

Converting to clausal form

1. eliminate $\rightarrow$ using

$$(\phi \rightarrow \psi) \equiv (\neg\phi \vee \psi)$$

2. move $\neg$ inward so that it appears only in front of a propositonal variable, using

$$\neg\neg\phi \equiv \phi$$

$$\neg(\phi \wedge \psi) \equiv (\neg\phi \vee \neg\psi)$$

$$\neg(\phi \vee \psi) \equiv (\neg\phi \wedge \neg\psi)$$

3. collect terms:

$$(\phi \vee \phi) \equiv \phi$$

$$(\phi \wedge \phi) \equiv \phi$$

The resolution rule

$$\frac{l_1 \vee \ldots \vee \boxed{l_i} \vee \ldots \vee l_k, \qquad m_1 \vee \ldots \vee \boxed{m_j} \vee \ldots \vee m_n}{l_1 \vee \ldots \vee l_{i-1} \vee l_{i+1} \vee \ldots \vee l_k \vee m_1 \vee \ldots \vee m_{j-1} \vee m_{j+1} \vee \ldots \vee m_n}$$

- where l_i and m_j are *complementary literals*, i.e., one is the negation of the other
- resolution takes two clauses and produces a new clause containing all the literals of the two original clauses *except* the two complementary literals
- derived clause is called the *resolvent*

Example: resolution

- from the clauses $\neg p \vee q$ and $p \vee q$ we can derive q by resolution

$$\frac{\neg p \vee q, \quad p \vee q}{q}$$

- this is clearly a valid inference as $\neg p \vee q,\ p \vee q \models q$

p	q	$\neg p$	$\neg p \vee q$	$p \vee q$
$true$	$\boxed{true}$	$false$	$\boxed{true}$	$\boxed{true}$
$true$	$false$	$false$	$false$	$true$
$false$	$\boxed{true}$	$true$	$\boxed{true}$	$\boxed{true}$
$false$	$false$	$true$	$true$	$false$

Soundness of resolution

- resolution is *sound*
- we can see this by considering the literal l_i
  - if l_i is *true* then m_j is *false* and hence

    $$m_1 \vee \ldots \vee m_{j-1} \vee m_{j+1} \vee \ldots \vee m_n$$

    must be *true*, since $m_1 \vee \ldots \vee m_j \vee \ldots \vee m_n$ is *true*
  - conversely, if l_i is *false*, then $l_1 \vee \ldots \vee l_{i-1} \vee l_{i+1} \vee \ldots \vee l_k$

    must be true since $l_1 \vee \ldots \vee l_i \vee \ldots \vee l_k$ is *true*
- so $l_1 \vee \ldots \vee l_{i-1} \vee l_{i+1} \vee \ldots \vee l_k \vee m_1 \vee \ldots \vee m_{j-1} \vee m_{j+1} \vee \ldots \vee m_n$ is *true*

Completeness of resolution

- while resolution is sound, it is not *complete* for arbitrary formulas

- for example, we cannot derive $q \vee \neg q$ from p using resolution, even though $p \models q \vee \neg q$

- however resolution is *refutation complete*—if a set of clauses is inconsistent, it is possible to derive a contradiction

Proof by contradiction

- recall that $\phi \models \psi$ if and only if $\phi \wedge \neg\psi$ is unsatisfiable
- proving ψ from ϕ by checking the unsatisfiability of $\phi \wedge \neg\psi$ is called *proof by refutation* or *proof by contradiction*
- we assume the sentence ψ to be false and show that this leads to a contradiction with the known axioms ϕ
- for resolution, we add the negation of the clause we wish to derive to the premises and show that this leads to an empty clause (i.e., a contradiction)

Example: proof by contradiction

- show that $(p \vee q) \;\wedge\; (p \rightarrow q) \;\wedge (q \rightarrow p) \models p \wedge q$

- convert to clausal form using $p \rightarrow q \equiv \neg p \vee q$
  $q \rightarrow p \equiv p \vee \neg q$

- which gives $(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \models p \wedge q$

- assume $p \wedge q$ to be *false* and convert to clausal form using $\neg(p \wedge q) \equiv \neg p \vee \neg q$

- which gives $(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q) \models \emptyset$

Example: proof by contradiction

- Clauses

$$\begin{array}{ll} (1) & p \vee q \\ (2) & \neg p \vee q \\ (3) & p \vee \neg q \\ (4) & \neg p \vee \neg q \end{array}$$

- Proof

1	$p \vee q$	(1)
2	$\neg p \vee q$	(2)
3	q	1, 2, by resolution
4	$p \vee \neg q$	(3)
5	p	3, 4 by resolution
6	$\neg p \vee \neg q$	(4)
7	$\neg q$	5, 6 by resolution
8	$\emptyset$	3, 7 by resolution

Example: travel advice

- imagine that we want to give advice about travel destinations
- *far* destinations are *Chile* or *Kenya* $Far \rightarrow Chile \vee Kenya$
- *far* destinations are *international* $Far \rightarrow Int$
- *far* destinations are *expensive* $Far \rightarrow Exp$
- in *Kenya*, *yellow fever* vaccination is strongly recommended, and there is a risk of *malaria* when staying in *lodges*

  $Kenya \rightarrow YellowFever \quad Logde \wedge Kenya \rightarrow Malaria$
- accommodation in *Kenya* is in *lodges* and in *Chile* is in *hotels*

  $Kenya \rightarrow Lodge \quad Chile \rightarrow Hotel$
- when there is a risk of *malaria*, mosquito *nets* are recommended

  $Malaria \rightarrow Nets$

Example: travel advice clauses

- Clauses
  (1) $\neg Far \vee Chile \vee Kenya$
  (2) $\neg Far \vee Int$
  (3) $\neg Far \vee Exp$
  (4) $\neg Kenya \vee YellowFever$
  (5) $\neg Lodge \vee \neg Kenya \vee Malaria$
  (6) $\neg Kenya \vee Lodge$
  (7) $\neg Chile \vee Hotel$
  (8) $\neg Malaria \vee Nets$

- Prove that Far and $\neg Hotel$ entails $Kenya$

- and that Far and $\neg Hotel$ entails $Nets$

Example: travel advice clauses

- Clauses (1) $\neg Far \lor Chile \lor Kenya$
  (2) $\neg Far \lor Int$
  (3) $\neg Far \lor Exp$
  (4) $\neg Kenya \lor YellowFever$
  (5) $\neg Lodge \lor \neg Kenya \lor Malaria$
  (6) $\neg Kenya \lor Lodge$
  (7) $\neg Chile \lor Hotel$
  (8) $\neg Malaria \lor Nets$
  (9) Far
  (10) $\neg Hotel$
  (11) $\neg Kenya$

Example: travel advice (proof 1)

- Proof

1	$\neg Far \vee Chile \vee Kenya$	(1)
2	Far	(9)
3	$Chile \vee Kenya$	1, 2 by resolution
4	$\neg Chile \vee Hotel$	(7)
5	$\neg Hotel$	(10)
6	$\neg Chile$	4, 5 by resolution
7	$Kenya$	3, 6 by resolution
8	$\neg Kenya$	(11)
9	$\emptyset$	7, 8 by resolution