Red-Black Trees

Advanced Algorithms and Data Structures - Lecture 3

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BST: Keeping the Balance

Binary Search Trees implement the basic operations of Dynamic Sets in $O(h)$ time, where h is the height of the tree

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Red-Black Trees:

- Not perfect balance
- Some paths may be twice as long as others
- Still guarantees that the height is $O(\log n)$

Red-Black Trees Definition

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- Color the nodes of a BST either **Red** or **Black**
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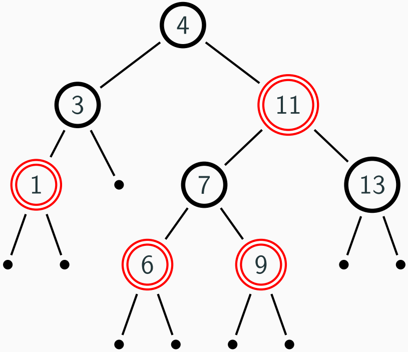
Black-height of a node:
The number of black nodes in any path from the node to any leaf

Example of Red-Black Tree

Example (red nodes have double circles)



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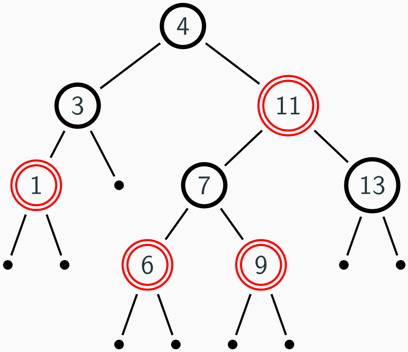

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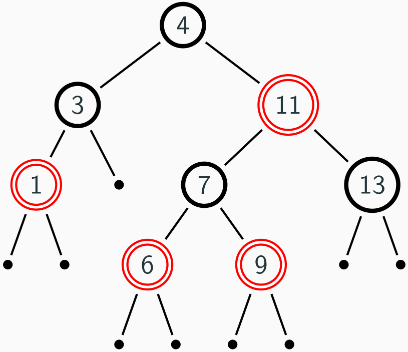

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Longest paths at most twice as long as shortest

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Definition of the type of Red-Black trees in Haskell

Similar to Binary Search Trees, with extra field for color

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data Color = Red | Black
data RBTree = Leaf | Node Color RBTree Key RBTree
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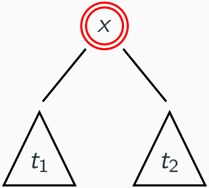

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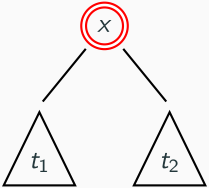

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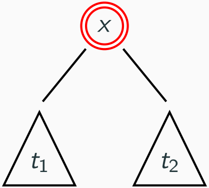

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Ensure that the properties are satisfied when you create and modify trees:

The element must be a correct Binary Search Tree and

It must satisfy the extra Red-Black properties

Height of Trees

If a R-B tree contains n elements, then
the maximum length of a path is $2\log(n+1)$

Even if the tree is not perfectly balanced, its height is $O(\log n)$

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We define an auxiliary function `balance` that rotates a tree when there are two consecutive red nodes in one of its children

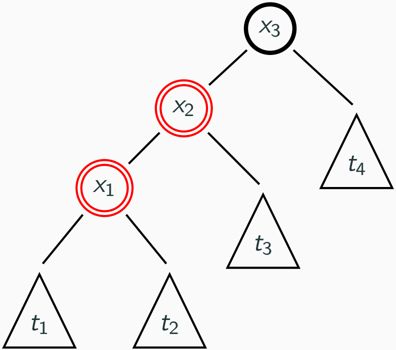
Balance Rotation I

Assume that the top node is **black**,

but there are two consecutive red nodes under it

There are four cases, according to the position of the red nodes

First Case:



BST property: $t_1 < x_1 < t_2 < x_2 < t_3 < x_3 < t_4$

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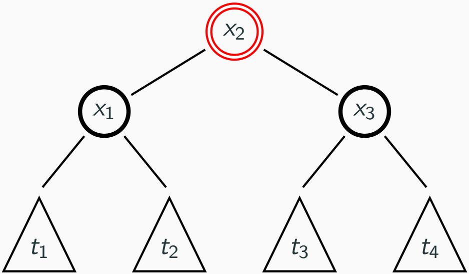

BST property: $t_1 < x_1 < t_2 < x_2 < t_3 < x_3 < t_4$

The black-height of every node remains the same

No consecutive red nodes any more

(but there may be above if the parent is red)

Balance Rotation II

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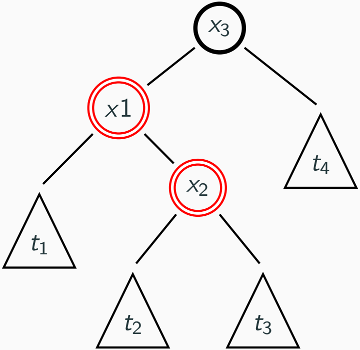

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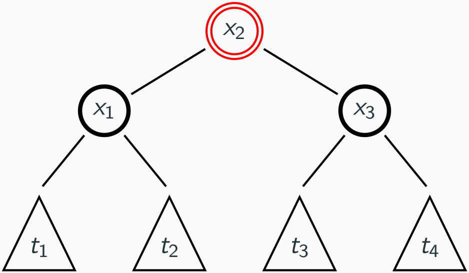

BST property: $t_1 < x_1 < t_2 < x_2 < t_3 < x_3 < t_4$

If the consecutive red nodes are in the right child

rotate symmetrically in the other direction

Balance Rotation III

Haskell program that fixes one double occurrence of red nodes:
It receives the input tree already divided into
color, left-child, key, right-child

```
balance :: Color → RBTree → Key → RBTree → RBTree
balance Black (Node Red (Node Red t1 x1 t2) x2 t3) x3 t4
  = NodeRB Red (Node Black t1 x1 t2) x2 (Node Black t3 x3 t4)

...

balance Black t1 x1 (Node Red t2 x2 (Node Red t3 x3 t4))
  = Node Red (Node Black t1 x1 t2) x2 (Node Black t3 x3 t4)
```

Insertion

Insert a new element into a R-B tree by:

- Insert in place of a leaf as in BSTs
- Initially color the new node red
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ins :: Key → RBTree → RBTree
ins a Leaf = Node Red Leaf a Leaf
ins a t@(Node color t1 x t2)
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Except: Its root could be red - just paint it black:

```
insert :: Key → RBTree → RBTree
insert a tree = blackRoot (ins a tree)
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Insert Observations

Let's say that a tree is weakly R-B if it satisfies all the R-B properties except that its root may be red and one of its children may also be red (so there could be two consecutive red nodes at the top.

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Observation:

- If `t` is a weakly R-B tree, then also `(ins a t)` is a weakly R-B tree
- If `t` is a weakly R-B tree, then we can turn it into a fully R-B tree by painting its root black

This will increase the black-height by one,
but since we do it at the root,
all paths will increase their black-lenghts equally.

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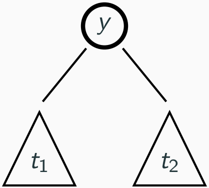

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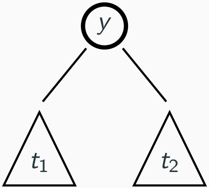

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Define rebalancing functions for
when one child has a black-height larger by one than the other

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- `fuse :: RBTree -> RBTree -> RBTree`
  merges two trees t_1 and t_2 when all elements of t_1 are smaller than all elements of t_2

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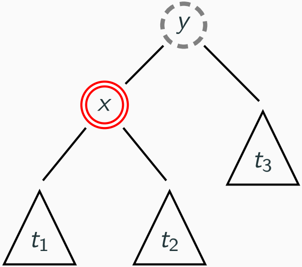

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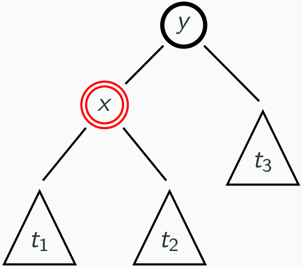

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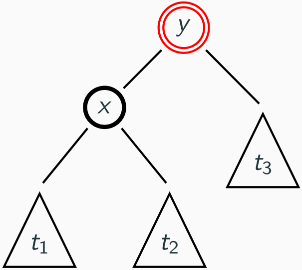

y must be black We swap the colors of x and y
The black-height of the right child decreases by 1,
the black-height of the left child is unchanged
(There could now be two red nodes at the top)

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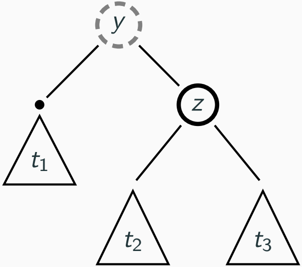

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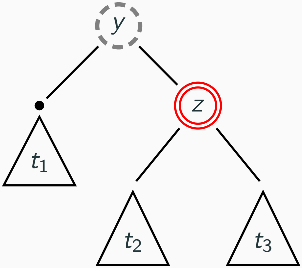

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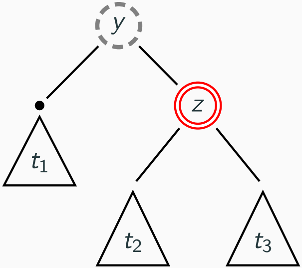

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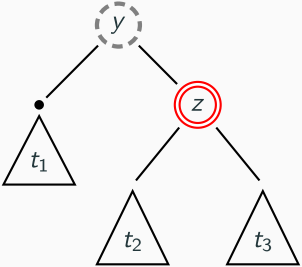

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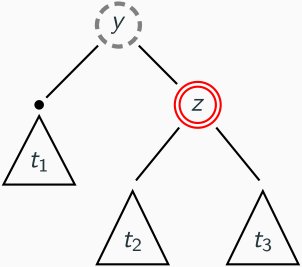

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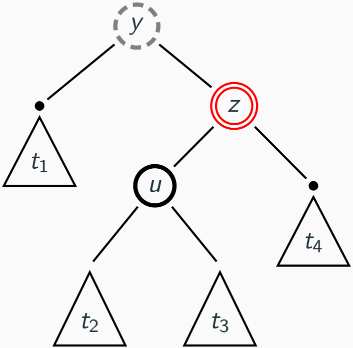
Apply `balance` to fix this problem

Balancing Left III

Third Case (left child black or leaf, right red):

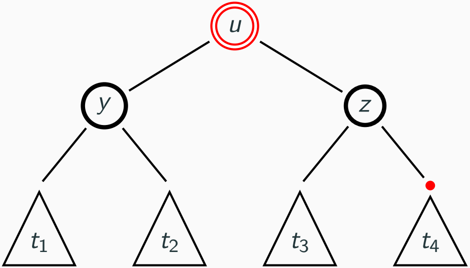


There must be at least a black node under z for the right child to have higher black-height

t_4 must have a black top node

Balancing Left III

Third Case (left child black or leaf, right red):



There must be at least a black node under z for the right child to have higher black-height

t_4 must have a black top node

We might have created consecutive red nodes in the right child

Apply `balance` to the right child

Balance Left in Haskell

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```
balL :: RBTree → Key → RBTree → RBTree
balL (Node Red t1 x t2) y t3
   = Node Red (Node Black t1 x t2) y t3
balL t1 y (Node Black t2 z t3)
   = balance Black t1 y (Node Red t2 z t3)
balL t1 y (Node Red (Node Black t2 u t3) z t4)
   = Node Red (Node Black t1 y t2)
              u
              (balance Black t3 z (redRoot t4))
```

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delL :: Key → RBTree → Key → RBTree → RBTree
delL x t1 y t2 =
  if (color t1) == Black
  then balL (del x t1) y t2
  else NodeRB Red (del x t1) y t2
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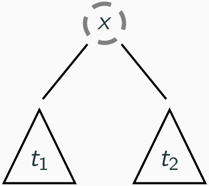
Define similar functions `balR` and `delR`
to rebalance and delete on the right

Fuse

In the case when $x = y$, we must delete the root of the tree

If we delete x from



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We must put them back together into a single tree

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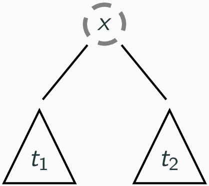

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of replacing the deleted node with the minimum of the right child
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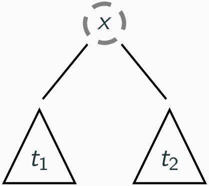

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doesn't work any more, because it may disrupt the R-B properties

We must come up with a cleverer way of fusing t_1 and t_2

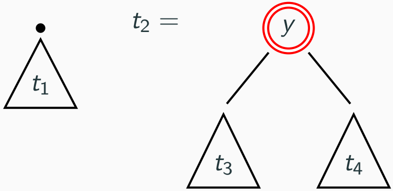
```
fuse ::  RBTree -> RBTree -> RBTree
```

We know that all elements of t_1 are smaller than all elements of t_2

Fuse: Different Color

If the two trees have top nodes of different color

$t_1 =$ $t_2 =$



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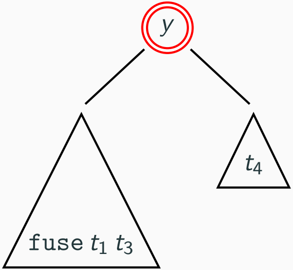

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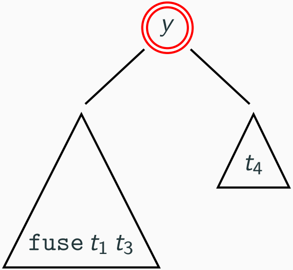

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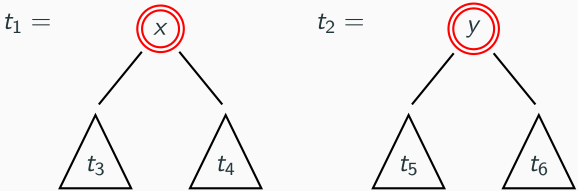
Similarly when the first is red and the second is black

If both trees have a red top node



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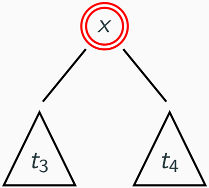
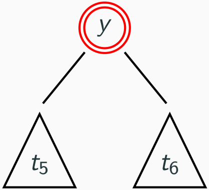
$t_1 =$ 

$t_2 =$ 

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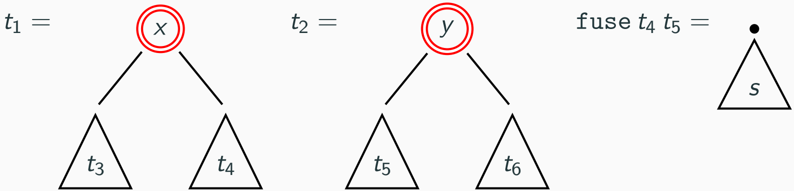

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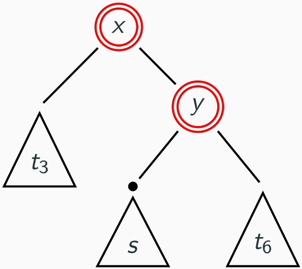

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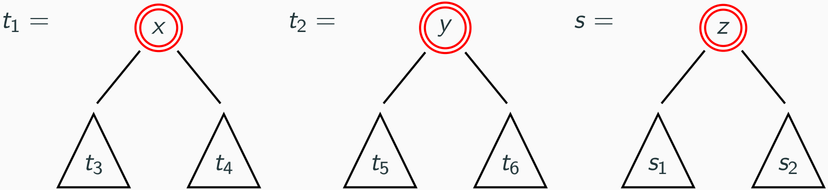

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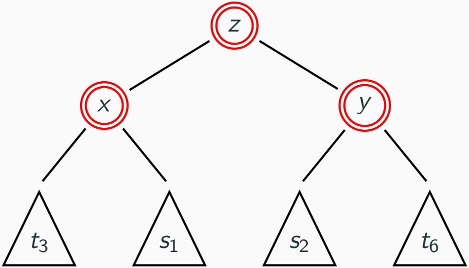

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If s has a red top node, we use its node as new root

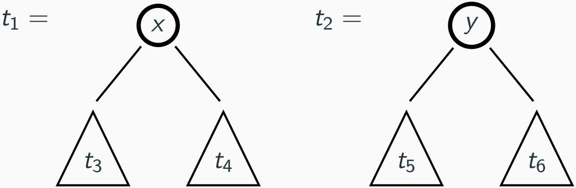
There are double red nodes on both sides, but

the top node will be recolored black either by `balL` or `balR` or `delete`,

according to where we deleted: left, right, or root

Fuse: Both Black

If both trees have a black top node



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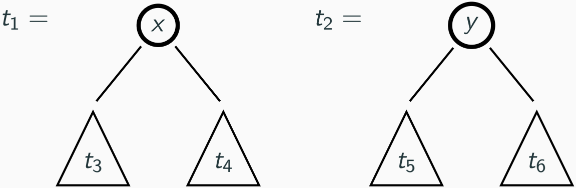

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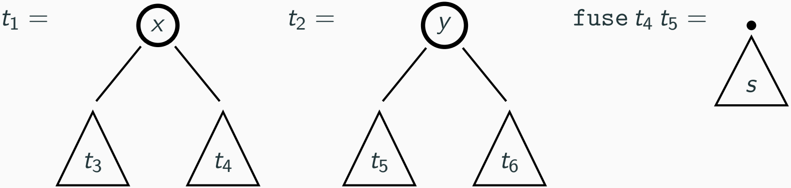

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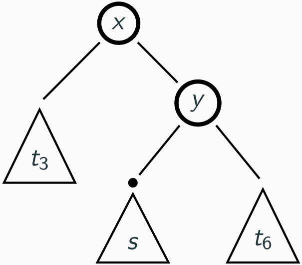

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But this time the right subtree has increased black-height

We must apply `balL`

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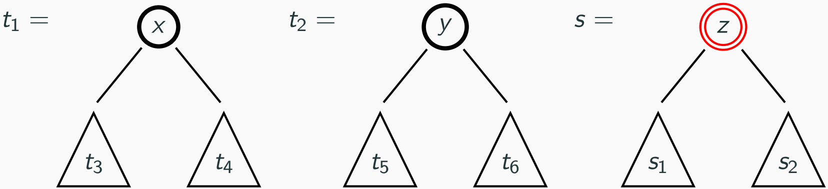

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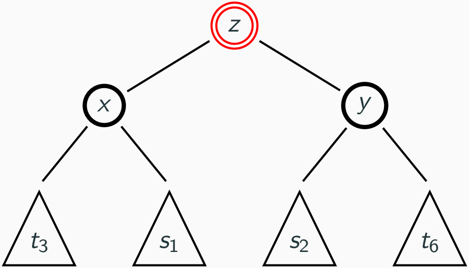

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The main delete function

Having defined all the auxiliary functions, we can now simply implement the main delete function:

```
delete :: Key → RBTree → RBTree
delete x t = blackRoot (del x t)

del :: Key → RBTree → RBTree
del x LeafRB = LeafRB
del x (NodeRB _ t1 y t2)
   | x<y = delL x t1 y t2      -- delete from left child
   | x>y = delR x t1 y t2      -- delete from right child
   | otherwise = fuse t1 t2    -- delete root, fuse children
```