The Master Method

Advanced Algorithms and Data Structures - Lecture 2A

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Maximum Array - Divide and Conquer

Back to the Maximum Array Problem.

We solve it in a recursive way, similar to Merge Sort:

$$l = [4, -2, 3, -7, 5, 2, -3, 4, -8, 6, -2, 1]$$

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- Split the input array in two halves
- Compute the maximum subarray of each half
- Compute the maximum *cross-over* subarray

The result is the maximum of the three partial subproblems

Maximum Array DC in Haskell

```
maxSub :: [Int] → (Int,Int,Int)
maxSub [x] = (0,0,x)
maxSub xs = let mid = length xs `div` 2
                (xs1,xs2) = splitAt mid xs
                (i1,j1,max1) = maxSub xs1
                (i2,j2,max2) = maxSub xs2
                (i3,j3,max3) = maxCross xs1 xs2
            in if max1 ≥ max2 && max1 ≥ max3
                then (i1,j1,max1)
                else if max2 ≥ max3
                     then (i2+mid,j2+mid,max2)
                     else (i3,j3+mid,max3)
```

`maxCross` is an auxiliary functions that finds the maximum *crossover* sublist, with `i3` the start index in `xs1` and `j3` the end index in `xs2`

It has linear complexity in the sum of the lengths of `xs1` and `xs2`

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Putting all the components together we get (with $c = c_1 + c_2$):

$$T(n) = 2T(n/2) + c_1 n + c_2 n + d = 2T(n/2) + cn + d$$

Simplifying the Equations

Strictly speaking, if the length n of the list is not even, the splitting is not exact: we get a sublist of length $\lfloor n/2 \rfloor$ and one of length $\lceil n/2 \rceil$
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We can rewrite the equation using complexity classes for the terms:

$$\begin{aligned} T(1) &= \Theta(1) \\ T(n) &= 2T(n/2) + \Theta(n) \end{aligned}$$

Solving Recursive Equations

Three methods to solve a recursive equation:

- Substitution Method: make a guess on the complexity class, verify and derive the parameters by recursion
- Recursion Tree Method: Draw a tree with all the recursive calls of the function and add up all the steps in each node
- Master Method: A general theorem that gives you the complexity class depending on the form of the equation

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Let's apply all three to the simplify system of equations

$$T(1) = 1$$
$$T(n) = 2T(n/2) + n$$

The solution will be the same as for the equations for the Maximum Subarray algorithm (and merge sort)

Substitution Method

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Let's check that this works for the inductive step:

Assume that it is true for values smaller than n

Prove that it also must hold for n:

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &\leq 2c\tfrac{n}{2} \log \tfrac{n}{2} + n \qquad \text{by Induction Hypothesis} \\ &= cn(\log n - \log 2) + n = cn(\log n - 1) + n \\ &= cn \log n - cn + n \leq cn \log n \qquad \text{if } c \geq 1 \end{aligned}$$

Substitution Method - Base Case

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So everything works if we choose $n_0 = 2$ and $c = 2$

We proved that $T(n) = O(n \log n)$

(We've been a bit simplistic: $n/2$ is not guaranteed to be an integer.
Either assume that n is a power of two, or replace $n/2$ with $\lfloor n/2 \rfloor$)

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Root: $T(n)$ Children: two calls $T(n/2)$ And so on

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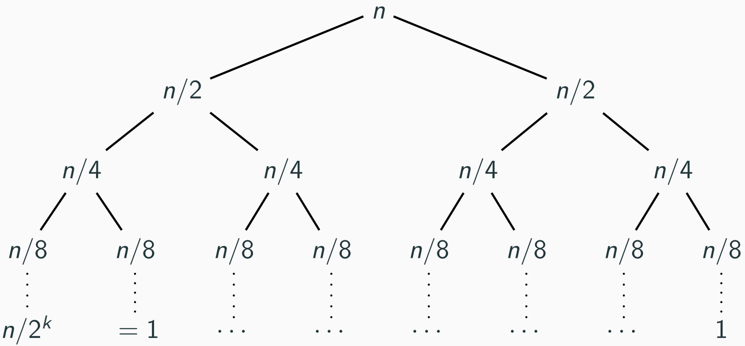

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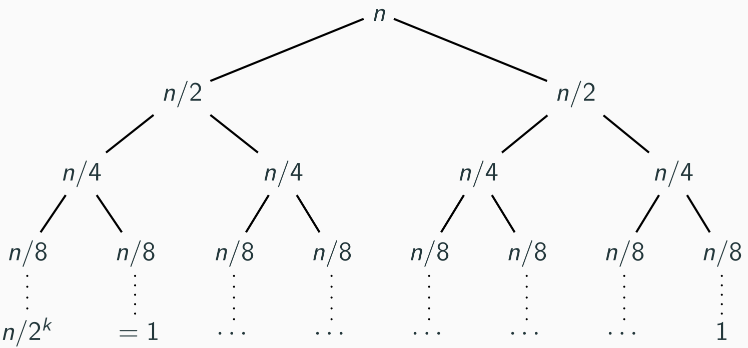

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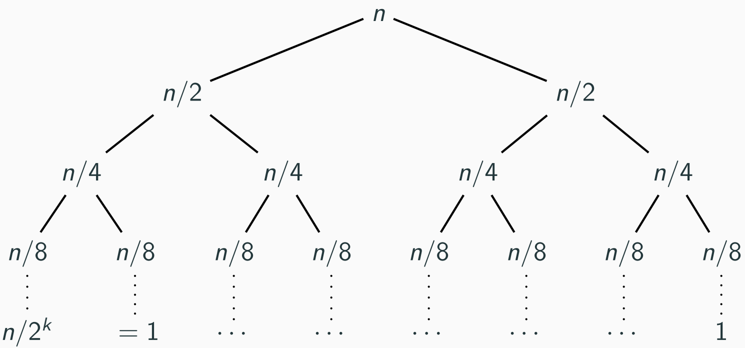

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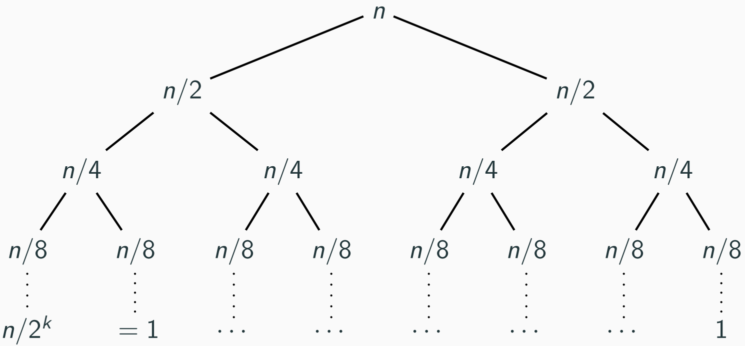

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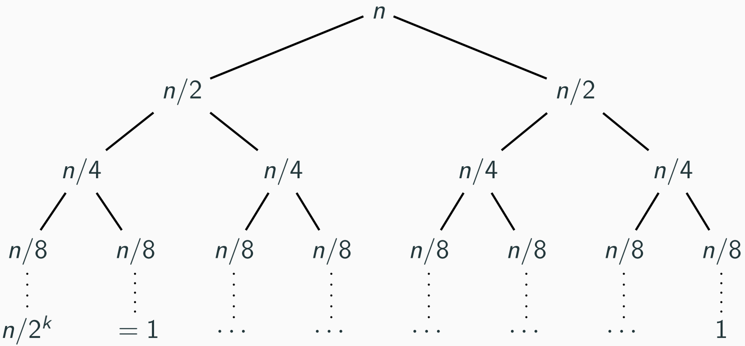

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How many computation steps do we do at each node? At level j, $n/2^j$

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A more general recursive program could have:

- Any number (a) of recursive calls
- Each with an argument of size n/b
- A non-recursive part given by a function $f(n)$

This leads to the equation $T(n) = aT(n/b) + f(n)$

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- a^j nodes at level j

There are $k = \log_b n$ levels, total number of nodes:

$$1 + a + a^2 + a^3 + \cdots a^{\log_b n}$$

This is a geometric series (see IA Appendix A)

Master Method: Computation Steps

Total number of nodes:

$$\sum_{j=0}^{j=k} a^k = \frac{a^{k+1} - 1}{a - 1}$$

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  the recursive part dominates: $T(n) = \Theta(n^{\log_b a})$

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- If they are of the same class: $f(n) = \Theta(n^{\log_b a})$
  each level adds $n^{\log_b a}$ computation steps (check the math)
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  each level adds $n^{\log_b a}$ computation steps (check the math)
  There are $\log_a n$ levels, so: $T(n) = \Theta(n^{\log_b a} \log n)$
- If the non-recursive part grows faster than the number of nodes:

  $$f(n) = \Omega(n^{\log_b a+\epsilon}) \qquad \text{for some } \epsilon > 0$$

  (plus some other condition)
  the non-recursive part dominates: $T(n) = \Theta(f(n))$

Application of the MM to Maximum Array

In the case of the Maximum Array algorithm (and Merge Sort):

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Conclusion $T(n) = \Theta(n^{\log_b a} \log n) = \Theta(n \log n)$