Introduction - Review

Advanced Algorithms and Data Structures - Lecture 1

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The Maximum Subarray Problem

Maximum Subarray Example

Consider the following list of natural numbers:

$$[4, -2, 3, -7, 5, 2, -6, 8, -4, 3, -2, 1]$$

Problem: Find the sublist with maximum sum

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In this case it is the list with:

$$\text{start index} = 4 \quad \text{end index} = 7 \quad \text{sum} = 9$$

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In this case it is the list with:

start index = 4 end index = 7 sum = 9

Write an algorithm to compute the maximum sublist

(Applications: gene sequence analysis, computer vision, data mining)

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  $[[4], [4, -2], [4, -2, 3], [4, -2, 3, -7], \ldots, [3, -2, 1], [-2], [-2, 1], [1]]$

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  $$[4, 2, 5, -2, \ldots, 2, -2, -1, 1]$$

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$$\text{maximum}\, [4, 2, 5, -2, \ldots, 2, -2, -1, 1] = 9$$

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- Take the maximum

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This is very inefficient (cubic complexity)

Small improvement: reuse the sums already computed

Haskell Code

```
maxSub :: [Int] → (Int,Int,Int)
maxSub [x] = (0,0,x)
maxSub xs = if s0 ≥ s then (0,j0,s0)
                      else (i+1,j+1,s)
  where (j0,s0) = argMax snd (sums xs)
        (i,j,s) = maxSub (tail xs)
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The type of the function `maxSub` says that
`maxSub` is a function that maps a list to a triple (i, j, s)

- i is the index of the first element of the sublist
- j is the index of the last element of the sublist
- s is the sum of the sublist

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If the input is a singleton list `[x]`
then the singleton is the maximum list

- 0: index of `x`, first element of the sublist
- 0: `x` is also the last element of the sublist
- x: the sum of `[x]` is x

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For longer lists we split the computation in two parts

- Sublists that contain the first element of `xs`
  let $(0, i_0, s_0)$ be the maximum of them
- Sublists that do not contain the first element
  Then they are sublists of `(tail xs)`
  let (i, j, s) be the recursive max sublist of `(tail xs)`
- Choose the larger of the two

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Ausiliary functions:

- `sums` computes the progressive sums of lists starting at the beginning
  `sums [1,-2,3,5] = [(0,1),(1,-1),(2,2),(3,7)]` because
  - the sum of `[1]` (indices 0 and 0) is 1 $\Rightarrow (0,1)$
  - the sum of `[1,-2]` (indices 0 and 1) is -1 $\Rightarrow (1,-1)$
  - the sum of `[1,-2,3]` (indices 0 and 2) is 2 $\Rightarrow (2,2)$
  - the sum of `[1,-2,3,5]` (indices 0 and 3) is 7 $\Rightarrow (3,7)$
- `argMax snd` selects the element
  with the maximum second component (the sum)

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  You must be able to understand my code
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- I give examples and solutions in Haskell
  You must be able to understand my code
  I only use basic Haskell (no Advanced Functional Programming)
- You don't need to program in Haskell yourself
  Use your favourite programming language
  The textbook has pseudocode in imperative style

Complexity of the Algorithm

Exercise: What is the complexity of the `maxSub` algorithm?
(How long does it take to to compute on an input of size n?)

PRETTY BAD (we'll see how bad)

Q: Are there more efficient algorithms?
YES: We will see two of them

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But First:
We see the Outline of the course
We review the basics of computational complexity

Introduction and Prerequisites

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- Complexity Analysis
  The Master Method, Amortized Complexity

Extra Subjects: Trendy Structures/Algorithms

- RSA Public-Key Cryptosystem
- Neural Networks (the Gradient-Descent Algorithm)
- Page Rank (the Google Search Algorithm)

Prerequisites

- Discrete Math (MCS)
  IA Ch.3 and Appendices A and B
- Basic Algorithms and Data Structures (ACE)
  stacks, lists, trees (IA Ch.10)
  sorting (IA Ch.2)
  elements of computational complexity
- Programming skills
  In *some* programming language: C/C++, Java, Python, Haskell
  You need to understand Haskell code, but you don't have to write it

Complexity Classes

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We don't mean exact running time
(that depends on the implementation and machine)
but a measure of the number of elementary computation steps

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- input size is often measured by memory locations, for example lists are measured by their lengths, trees by the number of nodes
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Important is not the exact running time, but the Complexity Class: the algorithms runs in linear, or quadratic, ... or exponential time

Big-O notation

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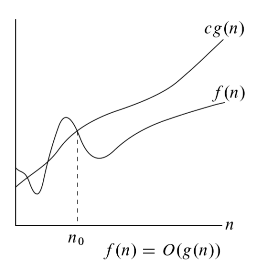

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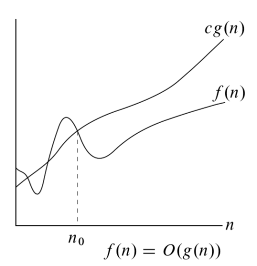

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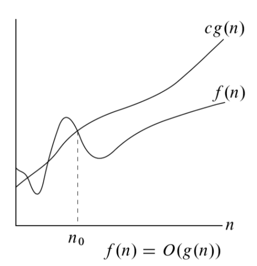

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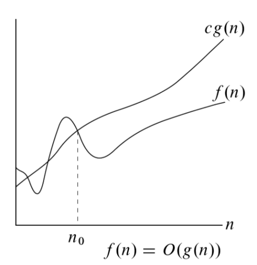

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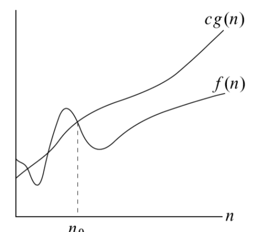

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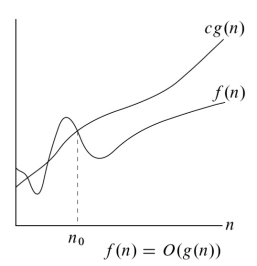

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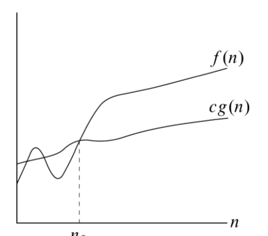

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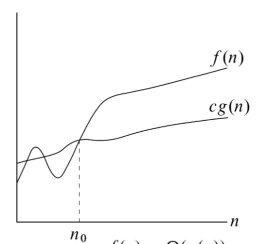

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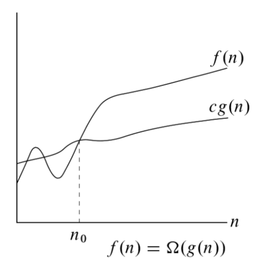

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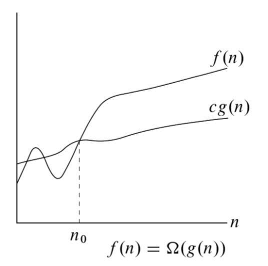

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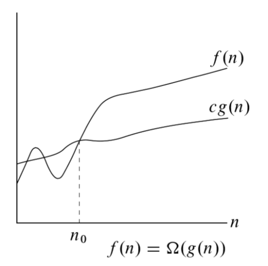

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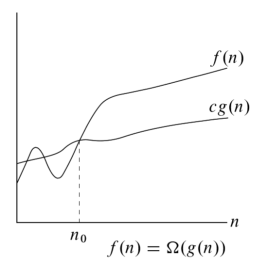

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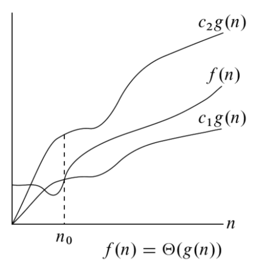

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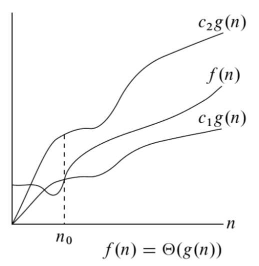

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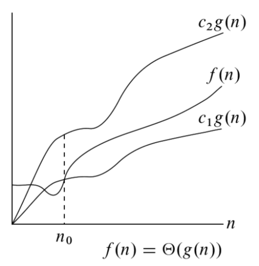

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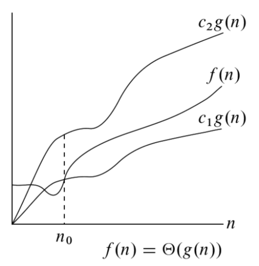

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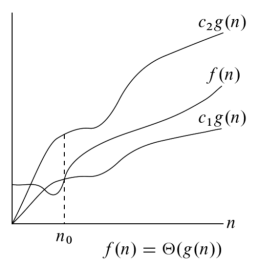

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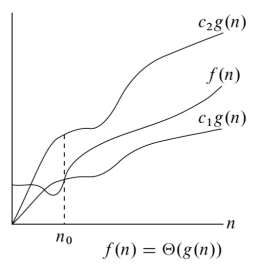

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such that for all $n \geq n_0, 0 \leq f(n) < cg(n)$

$$o(g) = \{f \mid \forall c, \exists n_0, \forall n \geq n_0, 0 \leq f(n) < cg(n)\}$$

$f(n) = \omega(g(n))$ means $f(n)$ grows faster than $g(n)$

o and ω classes

There are also "small" versions of O and Ω
where the inequalities hold strictly

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$f(n) = \omega(g(n))$ means $f(n)$ grows faster than $g(n)$
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$$\omega(g) = \{f \mid \forall c, \exists n_0, \forall n \geq n_0, 0 \leq cg(n) < f(n)\}$$

Complexity of Maximum Subarray

Analysis of the Naive Algorithm

```
maxSub :: [Int] → (Int,Int,Int)
maxSub [x] = (0,0,x)
maxSub xs = if s0 ≥ s then (0,j0,s0)
                      else (i+1,j+1,s)
  where (j0,s0) = argMax snd (sums xs)
        (i,j,s) = maxSub (tail xs)
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When the input has length $n = 1$, it is a singleton $[x]$

We immediately return the result $(0, 0, x)$

Constant time: just write the output

$$T(1) = c_0$$

Analysis of the Naive Algorithm

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```

For inputs with length $n > 1$, we must compute

- `(sums xs)` linear time: traverses the input list
- `argMax snd ...` linear time: traverses the sums
- `maxSub (tail xs)` recursive call

$$T(n) = c_1 n + T(n-1)$$

Analysis of the Naive Algorithm

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$$
\begin{aligned}
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$$
\begin{aligned}
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$$

So the complexity is:

$$
\begin{aligned}
T(n) &= c_1 n + T(n-1) = c_1 n + c_1(n-1) + T(n-2) \\
&= c_1 n + c_1(n-1) + \cdots c_1 2 + T(1) \\
&= c_1 n + c_1(n-1) + \cdots c_1 2 + c_0 \\
&= c_1 \sum_{i=2}^{i=n} i + c_0 = c_1(n(n+1)/2 - 1) + c_0 \\
&= \Theta(n^2)
\end{aligned}
$$

More Efficient Algorithms?

Exercise: Write a better algorithm for the Maximum Subarray Problem

Two ideas/strategies:

More Efficient Algorithms?

Exercise: Write a better algorithm for the Maximum Subarray Problem

Two ideas/strategies:

1. A divide-and-conquer algorithm:
   - Split the list in two halves
   - Compute separately the maximum subarray of both halves
   - Compute the maximum *cross-over* subarray

   This has complexity $O(n \log n)$

More Efficient Algorithms?

Exercise: Write a better algorithm for the Maximum Subarray Problem

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1. A divide-and-conquer algorithm:
   - Split the list in two halves
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   This has complexity $O(n \log n)$
2. A linear time algorithm (see Ex. 4.1-5 in IA)
   - Traverse the list from left to right
   - Keep track of the maximum subarray seen so far
   - and the maximum subarray ending at the last seen element

   This has complexity $O(n)$