Extension of Drift Wave Eigenmode Solver to Include Electromagnetic Effects

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1 Introduction

Understanding turbulence in a magnetized plasma is a difficult task. Through the use of numerical simulations and models we can gain insight into the behavior of waves inside plasmas. This paper presents my work in modifying a linear eigenmode solver to incorporate electromagnetic effects. Previously, this eigenmode solver used an electrostatic approximation in which the magnetic field was assumed to be fixed and uniform. This program uses a simplified version of the Braginskii two-fluid equations solved using finite differencing methods to determine the frequency of wave instabilites as eigenvalues of the system. The eigenmode solver program uses numerical methods from the BOUT code developed in the 1990's for plasma modeling. The new eigenmode solver gives us a more accurate picture of instabilities in plasma, particularly with high β.

This paper is organized as follows. The first section goes through the derivation of the equation set from MHD theory and is simplified using approximations appropriate for the LAPD machine. The next section discusses the electromagnetic corrections to the program, which was the main goal of the project. Next is a section discussing the technique of eigenmode solving in general and how the corrections I implemented changed the process as well as noting the issues I ran into. Finally we see results from the code, first showcasing the different instabilities that can be seen from the code as well as the difference the electromagnetic terms made as a function of β.

2 Fluid Equation Derivation

The equations used in the program are a reduced version of the Braginskii Two-Fluid Equations. This section begins with a derivation from first principles of MHD theory.

2.1 Initial Equation Set

We start with Ideal MHD Equations that include resistive terms: the Continuity Equation, Momentum Conservation, and Conservation of Charge.

$$\frac{\partial n_j}{\partial t} = -\nabla \cdot (n_j \mathbf{v}_j) \tag{1}$$

$$n_j m_j \frac{d\mathbf{v}_j}{dt} = -\nabla p_j - \nabla \cdot \mathbf{\Pi}_j + n_j e_j \left(\mathbf{E} + \mathbf{v_j} \times \mathbf{B}\right) + \mathbf{R}_j \tag{2}$$

$$\nabla \cdot \mathbf{j} = 0 \tag{3}$$

The MHD equation for energy conservation equation below is not used here as it it not needed when modifying the conservation of charge equation to the vorticity equation to match the BOUT++ code.

$$\frac{3}{2} n_j \left(\frac{dT_j}{dt}\right) = -p_j \nabla \cdot \mathbf{v}_j - \nabla \cdot (\mathbf{v}_j \cdot \mathbf{P}_j) - \nabla \cdot \mathbf{q}_j + Q_j \tag{4}$$

2.2 Vorticity

In the LAPD the parallel current is dominated by electron flow and the perpendicular current by ion flow. We project the momentum equation above in the parallel direction for electrons, and we use perpendicular ion flow and parallel electron velocity to derive the vorticity equation.

$$\nabla \cdot \mathbf{j} = \nabla \cdot \left(en\left(v_{\|i} - v_{\|e}\right) + en\left(\mathbf{v}_{\perp i} - \mathbf{v}_{\perp e}\right)\right) = 0 \rightarrow \nabla \cdot \left(n\left(\mathbf{v}_{\perp i} - v_{\|e}\right)\right) = 0 \tag{5}$$

Noticing that

$$\mathbf{v}_{\perp i} \simeq \mathbf{v}_E + \mathbf{v}_{pi} + \mathbf{v}_{\nu i} \tag{6}$$

Which is

$$\mathbf{v}_E = \mathbf{E} \times \mathbf{B}/B^2 \qquad E \times B \text{ Drift} \tag{7}$$

$$\mathbf{v}_{pi} = \left(1/\omega_{ci}\right) \mathbf{b} \times \left(\partial_t + \mathbf{v}_i \cdot \nabla\right) \mathbf{v}_i \qquad \text{Polarization Drift} \tag{8}$$

$$\mathbf{v}_{\nu i} = \left(\nu_{in}/\omega_{ci}\right) \mathbf{b} \times \mathbf{v}_i \qquad \text{Pedersen Velocity} \tag{9}$$

We are ignoring the ExB velocity for now since ions will produce equal and opposite current.

But also if we notice that $\mathbf{v}_i \simeq v_E$, we can then replace that to get an equation fully in terms of $v_{\parallel e}$

$$\nabla_\parallel \left(n v_{\parallel e}\right) = -\frac{m_i}{eB^2} \nabla_\perp \cdot \left[n\mathbf{b} \times \left(\partial_t + \mathbf{v}_E \cdot \nabla + \nu_{in}\right) \nabla_\perp \phi\right] \tag{10}$$

Next, if we define the quantity called the vorticity, $\varpi \equiv \nabla_\perp \cdot (n \nabla_\perp \phi)$, we can use it to modify the above equation into the form we want below.

$$\boxed{\frac{\partial \varpi}{\partial t} = -\mathbf{v}_E \cdot \nabla_\perp \varpi + \frac{1}{2} \left(\mathbf{b} \times \nabla_\perp n\right) \cdot \nabla_\perp \mathbf{v}_E^2 - \frac{eB^2}{m_i} \nabla_\parallel \left(n v_{\parallel e}\right) - \nu_{in} \varpi} \tag{11}$$

2.3 Simplification using LAPD Assumptions

2.3.1 Continuity Equation

Quasi-neutrality lets us only need to consider the continuity equation for one species. Let $n = n_e$ going forward. In the LAPD, the following is a suitable approximation.

$$\nabla \cdot (n \mathbf{v}_{\perp e}) = \mathbf{v}_E \cdot \nabla n$$

Therefore the continuity equation used in the eigsolver code is the following:

$$\boxed{\frac{\partial n}{\partial t} = -\mathbf{v}_E \cdot \nabla n - \nabla_\parallel \left(n v_{\parallel e}\right)}$$

2.3.2 Momentum balance

The vorticity equation was derived using the perpendicular momentum equations. We can now use the parallel component. We know that $v_{\parallel e} \gg v_{\parallel i}$ so we consider the parallel electron equation.

Using $\mathbf{R}_e = -m_e n_e \nu_e \left(0.51 u_{\parallel e} + \mathbf{u}_{\perp e}\right) - 0.71 n_e \nabla_\parallel T_e - \frac{3}{2} \frac{n_e \nu_e}{\omega_{ce}} \mathbf{b} \times \nabla T_e$. In equation (2), we get the following momentum equation. Several terms of this equation are dropped for final use in the eigsolver code.

$$\boxed{n m_e \frac{\partial v_{\parallel e}}{\partial t} = -n m_e \mathbf{v}_E \cdot \nabla v_{\parallel e} - \nabla_\parallel p_e - e n E_\parallel - 0.71 n \nabla_\parallel T_e - 0.51 m_e n \nu_e v_{\parallel e}}$$

2.4 Final Equations

These are the equations as they appear in the code, where $N, v_\parallel, \varpi$ are total quantities, i.e. the equilibrium profile and the perturbation.

$$\frac{\partial N}{\partial t} = -\mathbf{v}_E \cdot \nabla N - \nabla_\parallel (N v_\parallel) \tag{12}$$

$$\frac{\partial v_\parallel}{\partial t} = -\mathbf{v}_E \cdot \nabla v_\parallel - \frac{\mu}{N_0} \nabla_\parallel (N T_{e0}) + \mu \nabla_\parallel \phi - \nu_e v_\parallel \tag{13}$$

$$\frac{\partial \varpi}{\partial t} = -\nabla_\parallel (N v_\parallel) - \mathbf{v}_E \cdot \nabla \varpi + \frac{1}{2} (\mathbf{b} \times \nabla N) \cdot \nabla (\mathbf{v}_E \cdot \mathbf{v}_E) - \nu_{in} \varpi + \mu_{ii} \nabla_\perp^2 \varpi \tag{14}$$

3 Electromagnetic Corrections

To include full electromagnetic effects in this simulation we need to drop the assumption that the B-field is constant, and we will work with the vector potential to provide a more accurate version of the electric field. The two main corrections to the equations used in the eigsolver are as follows

$$E = -\nabla \phi \quad \rightarrow \quad E = -\nabla \phi - \frac{\partial A_\parallel}{\partial t} \tag{15}$$

$$\nabla_\parallel = \hat{z} \cdot \nabla \quad \rightarrow \quad \mathbf{b} \cdot \nabla \tag{16}$$

Where $\mathbf{b} = \nabla \times A$. In particular, only the parallel component of the vector potential is considered in this paper as it is significantly easier to implement its effects, and it gives us the fluctuations in the magnetic field perpendicular to the fixed main magnetic field.

In the existing momentum equation, $E_\parallel$ is simplified to $-\nabla_\parallel \phi$, but it is more precise to use Equation (15). Additionally, all of the parallel derivatives in the equations are taken in the direction of the main magnetic field $\mathbf{b_0}$ which is pointed in the $\hat{z}$ direction. To achieve this, we want to calculate $A_\parallel$, take its time derivative to find the corrected electric field, take its "curl" to find the perturbations in the magnetic field, and define the parallel derivative operator in terms of that direction.

3.1 Implementing in Practice

To find $A_\parallel$ for use in the above equations, we need to add a fourth equation to the numerical solving in the code.

$$\nabla_\perp^2 A_\parallel = -\mu_0 j_\parallel = \mu_0 n e v_{\parallel e} \tag{17}$$

This gives us an easily solved differential equation for $A_\parallel$ since we are already solving for the density, n, and parallel electron velocity, $v_{\parallel e}$. To implement this in practice, we simply add this equation to finite difference matrix.

4 Eigenmode Solving

When using the eigenmode method, we write all the unknonwn quantities in terms of a complex exponential to transform differential equations to algebraic equations. The quantity that we are after is the complex frequency ω, which we get when taking time derivatives of the waves. Once we transform this equation into a matrix equation, it becomes an eigenvalue problem that is solved quickly by the computer for the eigenvalue ω. This section describes my understanding of the theory behind eigenmode solving in general and how it is implemented in this instance.

4.1 Linearization of the Equations

To solve the set of equations we first want to linearize the equations by writing the independent variables, $n, v_{\parallel e}, \phi$ as a perturbation about an equilibrium value.

$$\begin{aligned} n &\to n_0 + n_1 \\ \phi &\to \phi_0 + \phi_1 \\ v &\to v_1 \end{aligned}$$

Where the equilibrium velocity is zero as it represents a plasma completely at rest. We now want to convert a set of differential equations into a set of linear equations by making the ansatz that the perturbations will look like complex exponentials, which is justified as they will always be a linear combination of them. For each of the independent variables we write the equilibrium value and the perturbation in the following form.

$$f(\vec{x}) = f^0(r) + f^1(r) e^{i(m_\theta \theta + k_\parallel z - \omega t)}$$

This way, parallel (z), azimuthal (θ), and time derivatives simply pull out a factor of $k_\parallel, m_\theta$, or ω respectively.

Next we put the system in the following form, as all of the three original equations have a time derivative on the left hand side:

$$\frac{\partial}{\partial t} \begin{pmatrix} N \\ v_\parallel \\ \varpi \end{pmatrix} = \mathbf{M} \begin{pmatrix} N \\ v_\parallel \\ \varpi \end{pmatrix}$$

Where M is a matrix full of coefficients and differential operators. The exponential in the solution will convert all the differential operators except ∂_r into algebraic factors. This system is then solved on a spatial mesh across the radial interval, producing a square matrix that has dimensions of the number of spatial intervals times the number of equations to solve. Then this becomes an eigenvalue problem to solve for complex eigenvalue ω, which pops out of the left side of the equation and is the temporal frequency/growth rate of the perturbations.

This method is particularly useful as we can use fourier filtering to pick out specific mode numbers to analyze their properties specifically. In general however, we are only after the fastest growing wave, regardless of its mode number.

4.1.1 Including Electromagnetic Terms

To include the new equation in this system we use equation (17). The perpendicular laplacian in cylindrical coordinates takes the form

$$\nabla_\perp^2 A_\parallel = \frac{1}{r}\partial_r(r\partial_r A_\parallel) + \frac{1}{r^2}\partial_\theta^2 A_\parallel$$

Therefore we want to write $A_\parallel$ in the same way as the other perturbed quantities, where the equilibrium $A_{\parallel 0} = 0$, as the equilibrium vector potential is in the θ direction for the field $\mathbf{b_0} = b_0\hat{z}$. So we have

$$A_\parallel(r,\theta,z,t) = A_\parallel(r)e^{i(m_\theta\theta + k_\parallel z - \omega t)}$$

To match with the perturbed velocity and to easily solve the differential equation as a part of the system.

To implement this in practice, I have created symbolic quantities for $A_\parallel(r)$ as well as $A_\parallel(r)e^{i(m_\theta\theta+kz-\omega t)}$ in the `SymbolicEq` class, as well as equation (17) in symbolic form. I have modified the symbolic equation for the momentum equation in the following way, to include this $A_\parallel$ term.

$$\partial_t v_{\parallel e} = -\mathbf{v}_E \cdot \nabla v_{\parallel e} - \mu\frac{T_{e0}}{N_0}\nabla_\parallel N + \mu\nabla_\parallel\phi - \nu_e v_{\parallel e}$$

$$\downarrow$$

$$\partial_t v_{\parallel e} = -\mathbf{v}_E \cdot \nabla v_{\parallel e} - \mu\frac{T_{e0}}{N_0}\nabla_\parallel N + \mu\left(\nabla_\parallel\phi - \frac{\partial A_\parallel}{\partial t}\right) - \nu_e v_{\parallel e}$$

As in the original equation, the term $\nabla_\parallel\phi$ is really just the electric field, but was written in this way due to the electrostatic approximation. To solve the equation for $A_\parallel$ I expanded the finite difference matrix to make room for the new equation (17) in the `EigSolve` class.

5 Results

5.1 Results from the Unmodified Code

Below in Figures (1-3) are some figures generated using the old version of the code which plot the frequency and growth rate of the largest growing mode as a function of the azimuthal wave number m_θ for the given profiles on the right. The profiles used for this data come from an old 2010 BaPSF paper about BOUT. The potential profiles, as well as the parallel wave number were chosen differently in each run to highlight three different common instabilities in the LAPD, resistive drift wave, Kelvin-Helmholtz instability, and rotational interchange, shown in Figures 1-3.

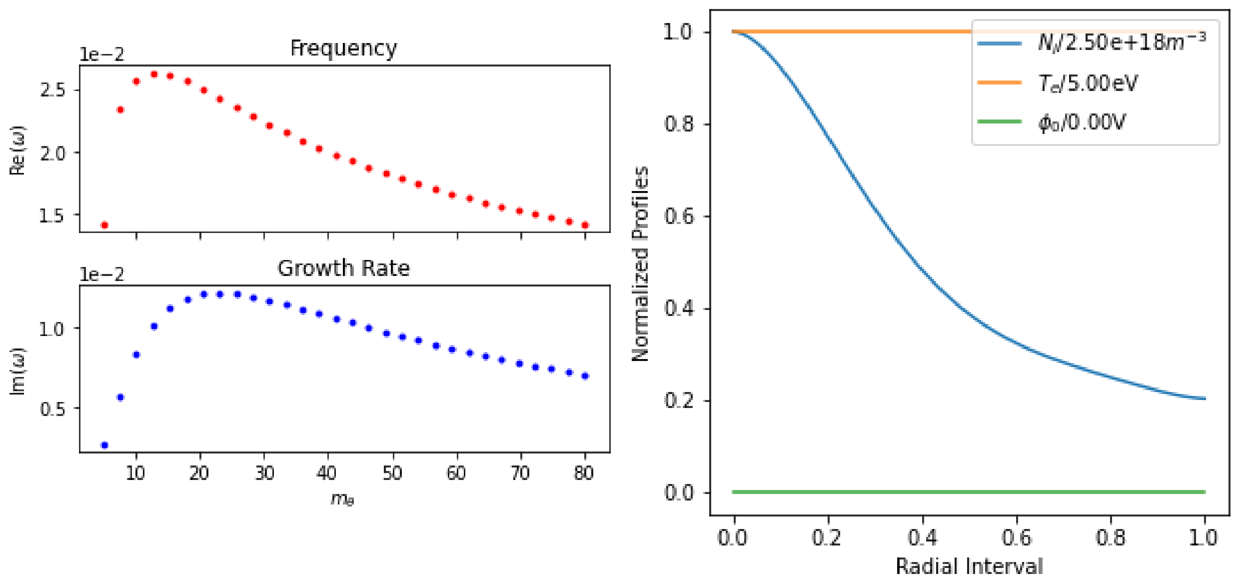

Figure 1: Resistive Drift Wave Instability as a function of the azimuthal mode number as well as the profiles inputted to generate this data. The parallel mode number was $n_z = 0.5$.

Another relevant question to ask would be what the spatial shape of the radial eigenvectors look like for N and φ for each of these simulations. The modifications of the code didn't result in any visible difference from the unmodified code, and the shape of the eigenfunctions are given in Figure 5 for the profiles in the resistive drift wave profiles from Figure 1.

5.2 Modifications

After the code was modified the goal of the project was to determine the difference in growth rates and how they relate to β. On the left in Figure (4) we can see the change in the fastest growing mode as we increase β. This was done by keeping the pressure profile constant and

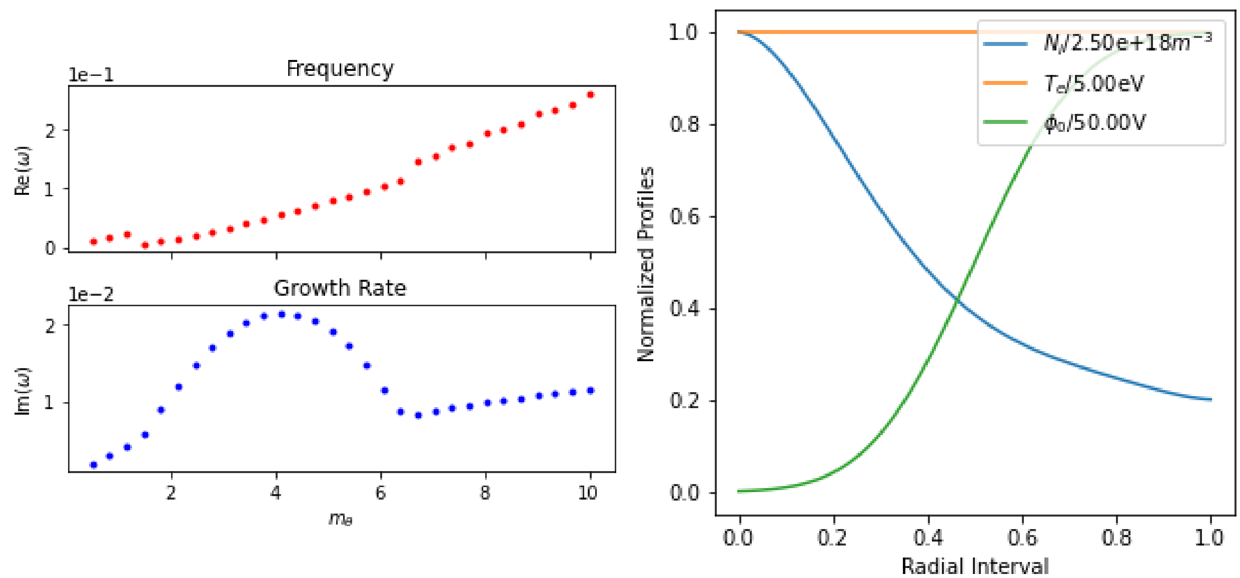

Figure 2: Kelvin-Helmholtz Instability as a function of the azimuthal mode number. The parallel mode number $n_z = 0$ was chosen to eliminate all parallel propagating waves.

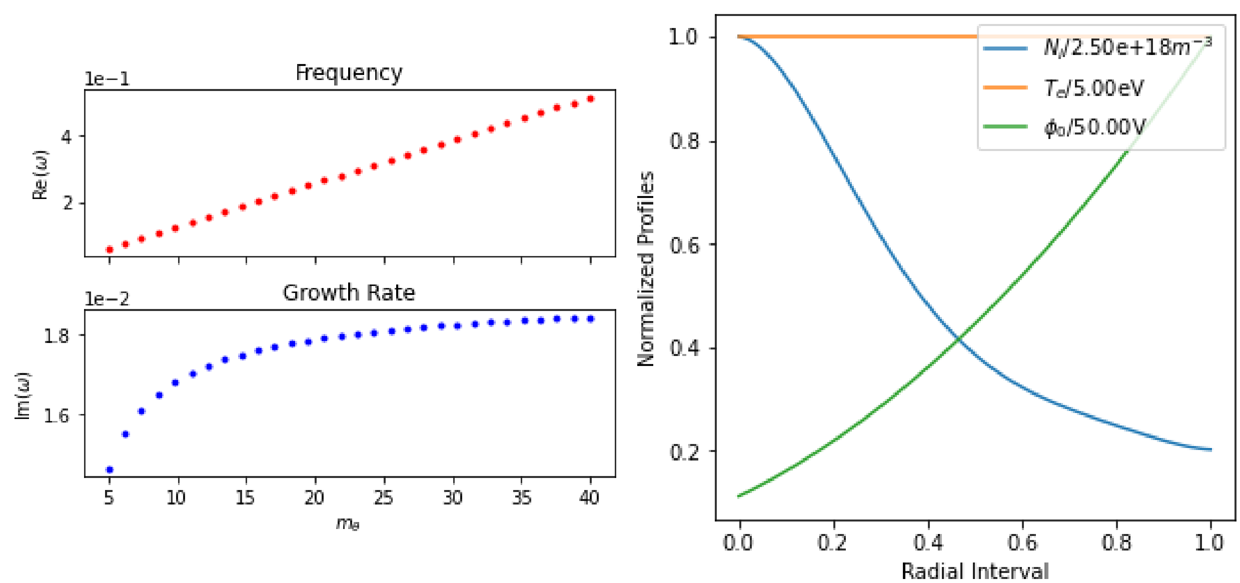

Figure 3: Rotational Interchange Instability as a function of azimuthal mode number. Parallel mode number also $n_z = 0$ to eliminate parallel propagating waves.

decreasing the applied magnetic field, thus increasing β. It appears that the growth rate is significantly reduced as we increase β.

In Figure 5 we plot the radial component of the perturbations for the quantities $N, \varphi, A_\parallel$ that were the eigenfunctions corresponding to the eigenvalue ω with the highest growth rate.

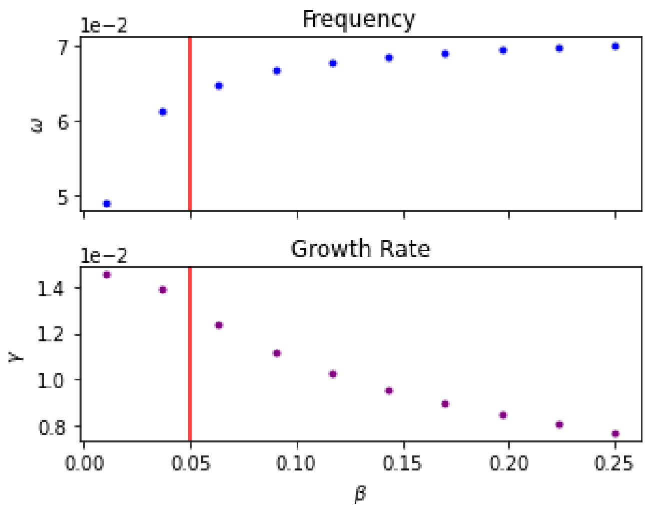

Figure 4: Frequency and growth rate response to changing β. This data uses the same profiles as the resistive drift wave in Figure (1) including the electromagnetic terms. The red line corresponds to a typical LAPD $\beta = 5\%$

In the next figure, we attempted to locate the mode with n_z, m_θ that corresponded to the

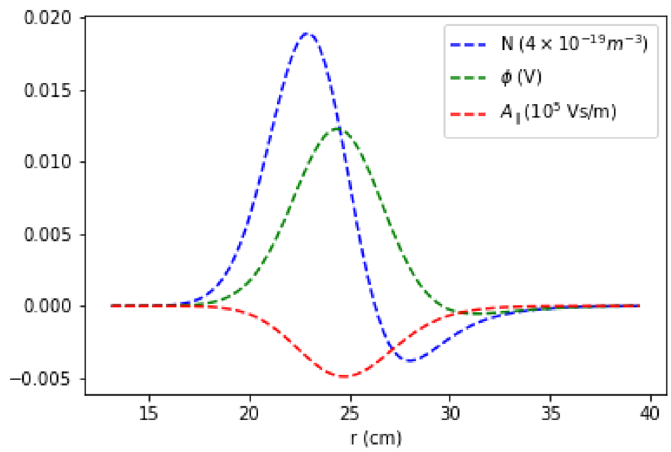

Figure 5: Eigenfunctions of the perturbed density, potential, and parallel electron potential from the modified code using the profiles for the resistive drift wave profile used in Figure 1. The maximal growth rate azimuthal mode number is used at $m_\theta = 27.625$.

highest growth rate eigenvalue and plotted the corresponding $k_\|$ as a function of β, and normalized it with the sound gyroradius, ρ_s. From Figure 6 we can see that the parallel wavenumber increases with β.

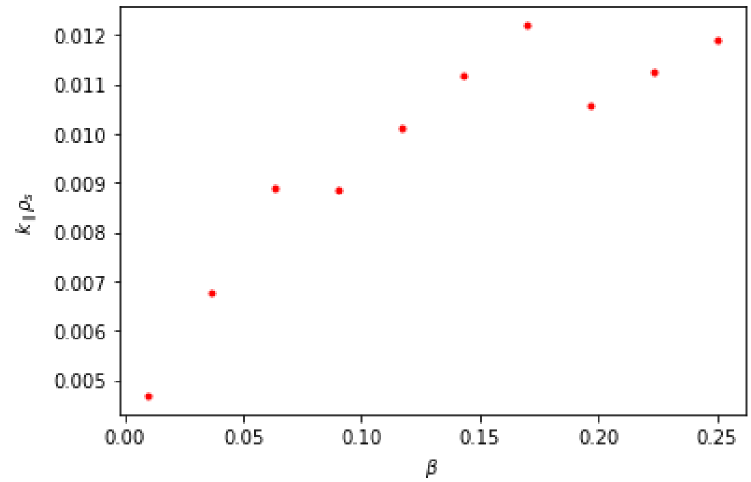

Figure 6: Plotting the parallel wavenumber for the fastest growing mode of the resistive drift wave in Figure 1 as a function of the β.

6 Conclusion

After the inclusion of the single electromagnetic term in the equation for $v_{\parallel e}$ by adding the parallel vector potential, $A_{\parallel}$, as another variable and equation into the finite difference matrix, we can see that it definitely had an effect on the outcome, albeit small. From figure 5 we can see that there were fluctuations in the $A_{\parallel}$ eigenfunction near the center of the radial interval which contributed to the difference in growth rate and frequency of the different modes. Additionally we observed that scaling up the β of the plasma caused a notable increase in the $k_{\parallel}$ of the fastest growing wave.