Measured MRI signal (single voxel)

$\mathrm{x_i}$

Parameters

y

Acquisition settings

z

Predicted MRI signal

$\widehat{\mathrm{x}_\mathrm{i}} = \mathrm{M}(\mathrm{z}|\widehat{\mathrm{y}_\mathrm{i}})$

Gaussian diffusion model

$\widehat{\mathrm{x}_\mathrm{i}} = \mathrm{S_0} \exp(-\mathrm{bD})$

IVIM model

$\widehat{\mathrm{x}_\mathrm{i}} = \mathrm{S_0}(\mathrm{f} \cdot \exp\left(-\mathrm{b}(\mathrm{D_{slow}} + \mathrm{D_{fast}})\right) + (1-\mathrm{f}) \cdot \exp(-\mathrm{bD_{slow}}))$

Kurtosis model

$$\widehat{\mathrm{x}_\mathrm{i}} = \mathrm{S_0} \exp\left(-\mathrm{bD} + \frac{1}{6}\mathrm{b^2D^2K}\right)$$

NODDI model

$\widehat{\mathrm{x}_\mathrm{i}} = \mathrm{S_0}(f_{ISO} \exp(-bD_{iso}) + (1-f_{iso})(f_{in} \int f(n) \exp\left(-bd_{||}(\boldsymbol{q} \cdot \boldsymbol{n})^2\right) d\boldsymbol{n} + (1-f_{in}) \exp(-b\boldsymbol{q}^T \left(\int f(\boldsymbol{n}) D(\boldsymbol{n}) d\boldsymbol{n}\right) \boldsymbol{q})))$

Ball and sticks model (one fiber population)

$\widehat{\mathrm{x}_\mathrm{i}} = \mathrm{S_0}(1-f) \exp(-bD) + f exp(-bD\boldsymbol{q}^T \boldsymbol{RAR}^T \boldsymbol{q})$

Least squares fitting (single voxel)

$$\widehat{y_i} = \underset{y}{argmin}(\|M(z|y) - x_i\|)$$

Maximum likelihood fitting (single voxel)

$$\widehat{y_i} = \underset{y}{argmax}\, P(x_i|M(z|y); \sigma)$$

ML - Least squares loss function (batch)

$$L_{LS} = \sum_{i=1}^{n_{train}} (M(z|y = F(x_i|p)) - x_i)$$

Likelihood loss function (batch)

$$L_{LK} = \sum_{i=1}^{n_{train}} P(x_i|M(z|F(x_i|p)); \sigma)$$

ML – network weights

$$\hat{p} = \underset{p}{argmin}\, L$$

ML – single voxel parameter prediction
$\hat{y}_i = F(x_i|\hat{p})$

Rician distribution
$$p\left(x_{i,j}\middle|\hat{x}_{i,j},\sigma\right) = \frac{x_{i,j}}{\sigma^2}\exp\left(\frac{-\left({x_{i,j}}^2 + {\hat{x}_{i,j}}^2\right)}{2\sigma^2}\right) I_0\left(\frac{x_{i,j}\hat{x}_{i,j}}{\sigma^2}\right)$$

Likelihood with Rician distribution
$$P(x_i|\hat{x}_i,\sigma) = \sum_j p\left(x_{i,j}\middle|\hat{x}_{i,j},\sigma\right)$$

Identities

$$k! = \Gamma(k+1)$$

$$I_0(z) = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}(z)^2\right)^k}{(k!)^2} \approx \sum_{k=0}^{N_k}\frac{\left(\frac{1}{4}(z)^2\right)^k}{(k!)^2} = \sum_{k=0}^{N_k}\frac{\left(\frac{1}{4}(z)^2\right)^k}{\Gamma(k+1)^2}$$

$$I_0\left(\frac{x_{i,j}\hat{x}_{i,j}}{\sigma^2}\right) = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}\left(\frac{x_{i,j}\hat{x}_{i,j}}{\sigma^2}\right)^2\right)^k}{\Gamma(k+1)^2} \approx \sum_{k=0}^{N_k}\frac{\left(\frac{1}{4}\left(\frac{x_{i,j}\hat{x}_{i,j}}{\sigma^2}\right)^2\right)^k}{\Gamma(k+1)^2}$$

$$\mathrm{logsumexp}(\mathbf{z}) = \log\left(\sum\nolimits_i \exp(z_i)\right)$$

$$\log(x\cdot y\cdot z) = \log(x) + \log(y) + \log(z)$$

$$\sum\nolimits_i x_i = \sum\nolimits_i \exp(\log(x_i))$$

$$x = \exp(\log(x))$$

$$\log_e(x^a) = b\ (what\ should\ you\ raise\ e\ by\ to\ get\ x^a?)$$

$e^b = x^a$, so $e^{\log(x^a)} = x^a$

$$log\left(\frac{\left(\frac{1}{4}(z)^2\right)^k}{\Gamma(k+1)^2}\right) = \log\left(\left(\frac{1}{4}(z)^2\right)^k\right) - \log(\Gamma(k+1)^2) = klog\left(\frac{1}{4}(z)^2\right) - 2\log\left(\Gamma(k+1)\right)$$

$$= klog\left(\frac{1}{4}\right) + klog(z^2) - 2log\big(\Gamma(k+1)\big) = klog\left(\frac{1}{4}\right) + 2klog(z) - 2log\big(\Gamma(k+1)\big)$$

$$= k(\log(0.25) + 2\cdot\log(z)) - 2\log\left(\Gamma(k+1)\right)$$

$$\frac{\left(\frac{1}{4}(z)^2\right)^k}{\Gamma(k+1)^2} = exp\left(log\left(\frac{\left(\frac{1}{4}(z)^2\right)^k}{\Gamma(k+1)^2}\right)\right)$$

$$log(I_0(z)) \approx \log\left(\sum_{k=0}^{N_k}\frac{\left(\frac{1}{4}(z)^2\right)^k}{\Gamma(k+1)^2}\right) = \log\left(\sum_{k=0}^{N_k} exp\left(log\left(\frac{\left(\frac{1}{4}(z)^2\right)^k}{\Gamma(k+1)^2}\right)\right)\right) =$$

$$= \log\left(\sum_{k=0}^{N_k} exp(k(\log(0.25) + 2\cdot\log(z)) - 2\log\left(\Gamma(k+1)\right))\right) = logsumexp(\boldsymbol{l})$$

$\boldsymbol{l}$ is vector where $\boldsymbol{l}_i = \boldsymbol{k}_i(\log(0.25) + 2\cdot\log(z)) - 2\log\big(\Gamma(\boldsymbol{k}_i + 1)\big)$, for $k = \{0, \ldots, N_k\}$

Differentiable Rician distribution

$$p\left(x_{i,j}\middle|\hat{x}_{i,j},\sigma\right) = \frac{x_{i,j}}{\sigma^2}\exp\left(\frac{-\left({x_{i,j}}^2 + {\hat{x}_{i,j}}^2\right)}{2\sigma^2}\right)\sum_{k=0}^{N_k}\frac{\left(\frac{1}{4}\left(\frac{x_{i,j}\hat{x}_{i,j}}{\sigma^2}\right)^2\right)^k}{\Gamma(k+1)^2}$$

Log of differentiable rician distribution

$$\log\left(p\left(x_{i,j}\middle|\hat{x}_{i,j},\sigma\right)\right) = log\left(\frac{x_{i,j}}{\sigma^2}\right) + log\left(\exp\left(\frac{-\left({x_{i,j}}^2 + {\hat{x}_{i,j}}^2\right)}{2\sigma^2}\right)\right) + log\left(\sum_{k=0}^{N_k}\frac{\left(\frac{1}{4}\left(\frac{x_{i,j}\hat{x}_{i,j}}{\sigma^2}\right)^2\right)^k}{\Gamma(k+1)^2}\right)$$

Likelihood with differentiable Rician distribution

$$P(x_i|\hat{x}_i,\sigma)=\prod_{j=0}^{n_z}p\left(x_{i,j}\middle|\hat{x}_{i,j},\sigma\right)\approx\prod_{j=0}^{n_z}\left[\frac{x_{i,j}}{\sigma^2}\exp\left(\frac{-\left({x_{i,j}}^2+{\hat{x}_{i,j}}^2\right)}{2\sigma^2}\right)\sum_{k=0}^{N_k}\frac{\left(\frac{1}{4}\left(\frac{x_{i,j}\hat{x}_{i,j}}{\sigma^2}\right)^2\right)^k}{\Gamma(k+1)^2}\right]$$

Likelihood loss function (batch, differentiable Rician)

$$L_{LK}=\sum_{i=1}^{n_{train}}P\left(x_i\middle|M\left(z\middle|F(x_i|p)\right);\sigma\right)$$

$$=\sum_{i=1}^{n_{train}}\prod_{j=0}^{n_z}\left[\frac{x_{i,j}}{\sigma^2}\exp\left(\frac{-\left({x_{i,j}}^2+{\hat{x}_{i,j}}^2\right)}{2\sigma^2}\right)\sum_{k=0}^{N_k}\frac{\left(\frac{1}{4}\left(\frac{x_{i,j}\hat{x}_{i,j}}{\sigma^2}\right)^2\right)^k}{\Gamma(k+1)^2}\right]$$

Log-Likelihood loss function (batch, differentiable Rician)

$$L_{LLK}=\sum_{i=1}^{n_{train}}\log\left(P\left(x_i\middle|M\left(z\middle|F(x_i|p)\right);\sigma\right)\right)$$

$$=\sum_{i=1}^{n_{train}}\sum_{j=1}^{n_z}\left[log\left(\frac{x_{i,j}}{\sigma^2}\right)+log\left(\exp\left(\frac{-\left({x_{i,j}}^2+{\hat{x}_{i,j}}^2\right)}{2\sigma^2}\right)\right)+logsumexp(\boldsymbol{l})\right]$$

NOTES

Rician distribution

$$p(x|v,\sigma) = \frac{x}{\sigma^2} \exp\left(\frac{-(x^2 + v^2)}{2\sigma^2}\right) I_0\left(\frac{xv}{\sigma^2}\right)$$