

# CS 260: Assignment #1

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*Prof. Wim van Dam*

Chad Spensky

## 1 Question 1

The problem asks to prove that:

$$H(X) \leq 2\log\left(\sum_{x \in \mathcal{X}} \sqrt{p(x)}\right)$$

This can be done in the following way:

$$H(X) = \sum_{x \in \mathcal{X}} -p(x)\log(p(x)) \quad (1)$$

$$= \sum_{x \in \mathcal{X}} -2\frac{1}{2}p(x)\log(p(x)) \quad (2)$$

$$= \sum_{x \in \mathcal{X}} 2p(x)\log\left(\frac{1}{\sqrt{p(x)}}\right) \quad (3)$$

$$= E_p\left[2\log\left(\frac{1}{\sqrt{p(x)}}\right)\right] \quad (4)$$

$$\leq 2\log\left[E_p\left(\frac{1}{\sqrt{p(x)}}\right)\right] \quad (5)$$

$$2\log\left[E_p\left(\frac{1}{\sqrt{p(x)}}\right)\right] = 2\log\left[\sum_{x \in \mathcal{X}} \frac{p(x)}{\sqrt{p(x)}}\right] \quad (6)$$

$$= 2\log\left[\sum_{x \in \mathcal{X}} \sqrt{p(x)}\right] \quad (7)$$

where 5 is an applications of Jensen's inequality.  $\square$

## 2 Question 2

In this question we are asked for the various entropies give all of the joint probabilities.

$$\begin{aligned} Pr[\mathcal{X} = 0, \mathcal{Y} = a] &= 0.15 & Pr[\mathcal{X} = 0, \mathcal{Y} = b] &= 0.3 & Pr[\mathcal{X} = 0, \mathcal{Y} = c] &= 0.05 \\ Pr[\mathcal{X} = 1, \mathcal{Y} = a] &= 0.25 & Pr[\mathcal{X} = 1, \mathcal{Y} = b] &= 0.15 & Pr[\mathcal{X} = 1, \mathcal{Y} = c] &= 0.1 \end{aligned}$$

First, we'll define the individual probability distributions:

$$\begin{aligned} Pr[\mathcal{X} = 0] &= Pr[\mathcal{X} = 0, \mathcal{Y} = a] + Pr[\mathcal{X} = 0, \mathcal{Y} = b] + Pr[\mathcal{X} = 0, \mathcal{Y} = c] &= 0.5 \\ Pr[\mathcal{X} = 1] &= Pr[\mathcal{X} = 1, \mathcal{Y} = a] + Pr[\mathcal{X} = 1, \mathcal{Y} = b] + Pr[\mathcal{X} = 1, \mathcal{Y} = c] &= 0.5 \end{aligned}$$

$$\begin{aligned} Pr[\mathcal{Y} = a] &= Pr[\mathcal{X} = 0, \mathcal{Y} = a] + Pr[\mathcal{X} = 1, \mathcal{Y} = a] &= 0.4 \\ Pr[\mathcal{Y} = b] &= Pr[\mathcal{X} = 0, \mathcal{Y} = b] + Pr[\mathcal{X} = 1, \mathcal{Y} = b] &= 0.45 \\ Pr[\mathcal{Y} = c] &= Pr[\mathcal{X} = 0, \mathcal{Y} = c] + Pr[\mathcal{X} = 1, \mathcal{Y} = c] &= 0.15 \end{aligned}$$

First, we will calculate the basic entropies:

$$\begin{aligned} H(\mathcal{X}) &= \sum_{x \in \mathcal{X}} -p(x) \log(p(x)) \\ &= -0.5 * \log(0.5) - 0.5 * \log(0.5) &= 0.301 \\ H(\mathcal{Y}) &= \sum_{y \in \mathcal{Y}} -p(y) \log(p(y)) \\ &= -0.4 * \log(0.4) - 0.45 * \log(0.45) - 0.15 * \log(0.15) &= 0.439 \end{aligned}$$

Recall:

$$\begin{aligned} Pr[X, Y] &= Pr[X] * Pr[Y|X] \\ Pr[Y|X] &= \frac{Pr[X, Y]}{Pr[X]} \end{aligned}$$

Next, we will compute the conditional entropies:

$$\begin{aligned}
H(X|Y = a) &= \sum_{x \in \mathcal{X}} -p(x|y = a) \log(p(x|y = a)) \\
&= -\frac{0.15}{0.5} \log\left(\frac{0.15}{0.5}\right) - \frac{0.25}{0.5} \log\left(\frac{0.25}{0.5}\right) \\
&= 0.307
\end{aligned}$$

$$\begin{aligned}
H(X|Y = b) &= \sum_{x \in \mathcal{X}} -p(x|y = b) \log(p(x|y = b)) \\
&= -\frac{0.3}{0.5} \log\left(\frac{0.3}{0.5}\right) - \frac{0.15}{0.5} \log\left(\frac{0.15}{0.5}\right) \\
&= 0.290
\end{aligned}$$

$$\begin{aligned}
H(X|Y = c) &= \sum_{x \in \mathcal{X}} -p(x|y = c) \log(p(x|y = c)) \\
&= -\frac{0.05}{0.5} \log\left(\frac{0.05}{0.5}\right) - \frac{0.1}{0.5} \log\left(\frac{0.1}{0.5}\right) \\
&= 0.240
\end{aligned}$$

$$\begin{aligned}
H(X|Y) &= \sum_{y \in \mathcal{Y}} p(y) H(X|Y = y) \\
&= 0.4 * 0.307 + 0.45 * 0.29 + 0.15 * 0.240 \\
&= .289
\end{aligned}$$

or, more concisely (ignoring rounding errors):

$$\begin{aligned}
H(X|Y) &= \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y) \log\left(\frac{p(y)}{p(x, y)}\right) \\
&= 0.15 \log\left(\frac{0.4}{0.15}\right) + 0.3 \log\left(\frac{0.45}{0.3}\right) + 0.05 \log\left(\frac{0.15}{0.05}\right) \\
&\quad + 0.25 \log\left(\frac{0.4}{0.25}\right) + 0.15 \log\left(\frac{0.45}{0.15}\right) + 0.1 \log\left(\frac{0.15}{0.1}\right) \\
&= 0.281
\end{aligned}$$

$$\begin{aligned}
H(Y|X) &= \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y) \log\left(\frac{p(x)}{p(x, y)}\right) \\
&= 0.15 \log\left(\frac{0.5}{0.15}\right) + 0.3 \log\left(\frac{0.5}{0.3}\right) + 0.05 \log\left(\frac{0.5}{0.05}\right) \\
&\quad + 0.25 \log\left(\frac{0.5}{0.25}\right) + 0.15 \log\left(\frac{0.5}{0.15}\right) + 0.1 \log\left(\frac{0.5}{0.1}\right) \\
&= 0.419
\end{aligned}$$

Thus, we obtain:

$$\begin{aligned}H(X, Y) &= H(X) + H(Y|X) \\&= 0.301 + 0.419 \\&= 0.720 \\H(X, Y) &= H(Y) + H(X|Y) \\&= 0.439 + 0.281 \\&= 0.720\end{aligned}$$

Finally, we compute:

$$\begin{aligned}I(X; Y) &= H(X) - H(X|Y) \\&= 0.301 - 0.281 \\&= 0.02 \\I(Y; X) &= H(Y) - H(Y|X) \\&= 0.439 - 0.419 \\&= 0.02\end{aligned}$$

□

### 3 Question 3

## 4 Question 4

For this question, we were asked to prove or disprove the following relationship:

$$D(p||q) + D(q||r) \geq D(p||r)$$

We can disprove this with a counter example. First, write out our definitions.

$$D(p||q) = \sum_{x \in \mathcal{X}} p(x) \log\left(\frac{p(x)}{q(x)}\right)$$

$$D(q||r) = \sum_{x \in \mathcal{X}} q(x) \log\left(\frac{q(x)}{r(x)}\right)$$

$$D(p||r) = \sum_{x \in \mathcal{X}} p(x) \log\left(\frac{p(x)}{r(x)}\right)$$

Thus, we would like to show the following:

$$\sum_{x \in \mathcal{X}} p(x) \log\left(\frac{p(x)}{q(x)}\right) + \sum_{x \in \mathcal{X}} q(x) \log\left(\frac{q(x)}{r(x)}\right) \geq \sum_{x \in \mathcal{X}} p(x) \log\left(\frac{p(x)}{r(x)}\right) \quad (8)$$

$$\sum_{x \in \mathcal{X}} p(x) \log\left(\frac{p(x)}{q(x)}\right) - \sum_{x \in \mathcal{X}} p(x) \log\left(\frac{p(x)}{r(x)}\right) \geq \sum_{x \in \mathcal{X}} q(x) \log\left(\frac{q(x)}{r(x)}\right) \quad (9)$$

$$\sum_{x \in \mathcal{X}} p(x) [\log\left(\frac{p(x)}{q(x)}\right) - \log\left(\frac{p(x)}{r(x)}\right)] \geq \sum_{x \in \mathcal{X}} q(x) \log\left(\frac{q(x)}{r(x)}\right) \quad (10)$$

$$\sum_{x \in \mathcal{X}} p(x) \log\left(\frac{r(x)}{q(x)}\right) \geq \sum_{x \in \mathcal{X}} q(x) \log\left(\frac{q(x)}{r(x)}\right) \quad (11)$$

$$\sum_{x \in \mathcal{X}} [p(x) \log\left(\frac{r(x)}{q(x)}\right) - q(x) \log\left(\frac{q(x)}{r(x)}\right)] \geq 0 \quad (12)$$

If we assume that  $p(x) = q(x)$ , this reduces to:

$$\sum_{x \in \mathcal{X}} p(x) [\log\left(\frac{r(x)}{p(x)}\right) - \log\left(\frac{p(x)}{r(x)}\right)] \geq 0 \quad (13)$$

$$\sum_{x \in \mathcal{X}} p(x) \log\left(\frac{r(x)^2}{p(x)^2}\right) \geq 0 \quad (14)$$

$$\sum_{x \in \mathcal{X}} p(x) \log\left[\left(\frac{r(x)}{p(x)}\right)^2\right] \geq 0 \quad (15)$$

$$\sum_{x \in \mathcal{X}} 2p(x) \log\left(\frac{r(x)}{p(x)}\right) \geq 0 \quad (16)$$

$$(17)$$

If we let  $\mathcal{X} = \{0, 1\}$  s.t.  $p(0) = p(1) = .5$ , and  $r(0) = .7, r(1) = .3$ , then:

$$\begin{aligned}\sum_{x \in \mathcal{X}} 2p(x) \log\left(\frac{r(x)}{p(x)}\right) &= 2 * 0.5 \log\left(\frac{.7}{.5}\right) + 2 * 0.5 \log\left(\frac{.3}{.5}\right) \\ &= -0.076 < 0\end{aligned}$$

Which violates our assumption, thus this inequality does not hold in the general case.  $\square$