# CS 260: Assignment #1

Due on Thursday, Jaunary 14, 2016

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# 1 Question 1

The problem asks to prove that:

$$H(X) \leq 2log(\sum_{x \in \mathcal{X}} \sqrt{p(x)}$$

This can be done in the following way:

$$\begin{aligned}
H(X) &= \sum_{x \in \mathcal{X}} -p(x)log(p(x)) && (1)\\
&= \sum_{x \in \mathcal{X}} -2\frac{1}{2}p(x)log(p(x)) && (2)\\
&= \sum_{x \in \mathcal{X}} 2p(x)log(\frac{1}{\sqrt{p(x)}}) && (3)\\
&= E_p[2log(\frac{1}{\sqrt{p(x)}}))] && (4)\\
&\leq 2log[E_p(\frac{1}{\sqrt{p(x)}})] && (5)\\
2log[E_p(\frac{1}{\sqrt{p(x)}})] &= 2log[\sum_{x \in \mathcal{X}} \frac{p(x)}{\sqrt{p(x)}}] && (6)\\
&= 2log[\sum_{x \in \mathcal{X}} \sqrt{p(x)}] && (7)
\end{aligned}$$

where 5 is an applications of Jensen's inequality. □

## 2 Question 2

In this question we are asked for the various entropies give all of the joint probabilities.

$$Pr[\mathcal{X}=0,\mathcal{Y}=a]=0.15 \quad Pr[\mathcal{X}=0,\mathcal{Y}=b]=0.3 \quad Pr[\mathcal{X}=0,\mathcal{Y}=c]=0.05$$
$$Pr[\mathcal{X}=1,\mathcal{Y}=a]=0.25 \quad Pr[\mathcal{X}=1,\mathcal{Y}=b]=0.15 \quad Pr[\mathcal{X}=1,\mathcal{Y}=c]=0.1$$

First, we'll define the individual probability distributions:

$$\begin{aligned}
Pr[\mathcal{X}=0] &= Pr[\mathcal{X}=0,\mathcal{Y}=a]+Pr[\mathcal{X}=0,\mathcal{Y}=b]+Pr[\mathcal{X}=0,\mathcal{Y}=c] && =0.5\\
Pr[\mathcal{X}=1] &= Pr[\mathcal{X}=1,\mathcal{Y}=a]+Pr[\mathcal{X}=1,\mathcal{Y}=b]+Pr[\mathcal{X}=1,\mathcal{Y}=c] && =0.5
\end{aligned}$$

$$\begin{aligned}
Pr[\mathcal{Y}=a] &= Pr[\mathcal{X}=0,\mathcal{Y}=a]+Pr[\mathcal{X}=1,\mathcal{Y}=a] && =0.4\\
Pr[\mathcal{Y}=b] &= Pr[\mathcal{X}=0,\mathcal{Y}=b]+Pr[\mathcal{X}=1,\mathcal{Y}=b] && =0.45\\
Pr[\mathcal{Y}=c] &= Pr[\mathcal{X}=0,\mathcal{Y}=c]+Pr[\mathcal{X}=1,\mathcal{Y}=c] && =0.15
\end{aligned}$$

First, we will calculate the basic entropies:

$$\begin{aligned}
H(X) &= \sum_{x\in\mathcal{X}} -p(x)log(p(x))\\
&= -0.5*log(0.5)-0.5*log(0.5) && =0.301\\
H(Y) &= \sum_{y\in\mathcal{Y}} -p(y)log(p(y))\\
&= -0.4*log(0.4)-0.45*log(0.45)-0.15*log(0.15) && =0.439
\end{aligned}$$

Recall:

$$\begin{aligned}
Pr[X,Y] &= Pr[X]*Pr[Y|X]\\
Pr[Y|X] &= \frac{Pr[X,Y]}{Pr[Y]}
\end{aligned}$$

Next, we will compute the conditional entropies:

$$
\begin{aligned}
H(X|Y=a) &= \sum_{x\in\mathcal{X}} -p(x|y=a)log(p(x|y=a)) \\
&= -\frac{0.15}{0.5}log(\frac{0.15}{0.5}) - \frac{0.25}{0.5}log(\frac{0.25}{0.5}) \\
&= 0.307
\end{aligned}
$$

$$
\begin{aligned}
H(X|Y=b) &= \sum_{x\in\mathcal{X}} -p(x|y=b)log(p(x|y=b)) \\
&= -\frac{0.3}{0.5}log(\frac{0.3}{0.5}) - \frac{0.15}{0.5}log(\frac{0.15}{0.5}) \\
&= 0.290
\end{aligned}
$$

$$
\begin{aligned}
H(X|Y=c) &= \sum_{x\in\mathcal{X}} -p(x|y=c)log(p(x|y=c)) \\
&= -\frac{0.05}{0.5}log(\frac{0.05}{0.5}) - \frac{0.1}{0.5}log(\frac{0.1}{0.5}) \\
&= 0.240
\end{aligned}
$$

$$
\begin{aligned}
H(X|Y) &= \sum_{y\in\mathcal{Y}} p(y)H(X|Y=y) \\
&= 0.4 * 0.307 + 0.45 * 0.29 + 0.15 * 0.240 \\
&= .289
\end{aligned}
$$

or, more concisely (ignoring rounding errors):

$$
\begin{aligned}
H(X|Y) &= \sum_{x\in\mathcal{X}, y\in\mathcal{Y}} p(x,y)log(\frac{p(y)}{p(x,y)}) \\
&= 0.15log(\frac{0.4}{0.15}) + 0.3log(\frac{0.45}{0.3}) + 0.05log(\frac{0.15}{0.05}) \\
&\quad + 0.25log(\frac{0.4}{0.25}) + 0.15log(\frac{0.45}{0.15}) + 0.1log(\frac{0.15}{0.1}) \\
&= 0.281 \\
H(Y|X) &= \sum_{x\in\mathcal{X}, y\in\mathcal{Y}} p(x,y)log(\frac{p(x)}{p(x,y)}) \\
&= 0.15log(\frac{0.5}{0.15}) + 0.3log(\frac{0.5}{0.3}) + 0.05log(\frac{0.5}{0.05}) \\
&\quad + 0.25log(\frac{0.5}{0.25}) + 0.15log(\frac{0.5}{0.15}) + 0.1log(\frac{0.5}{0.1}) \\
&= 0.419
\end{aligned}
$$

Thus, we obtain:

$$\begin{aligned}
H(X,Y) &= H(X) + H(Y|X) \\
&= 0.301 + 0.419 \\
&= 0.720 \\
H(X,Y) &= H(Y) + H(X|Y) \\
&= 0.439 + 0.281 \\
&= 0.720
\end{aligned}$$

Finally, we compute:

$$\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) \\
&= 0.301 - 0.281 \\
&= 0.02 \\
I(Y;X) &= H(Y) - H(Y|X) \\
&= 0.439 - 0.419 \\
&= 0.02
\end{aligned}$$

$\square$

## 3 Question 3

# 4 Question 4