CS 260: Assignment #4
String Prefix

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1 Abstraction Domain Lattice

Let our original lattice $L = (\mathbb{P}(S), \leq)$ and our abstract-domain lattice $L^{\#} = (S \cup \mathbb{P}(S + *) \cup \{\top, \perp\}, \sqsubseteq)$ where S is the set of all strings (including the empty string) where $\top = * = S$ and $\perp$ is *undefined*. Let $\perp \sqsubset s \sqsubseteq s' \sqsubseteq \top, \forall s \in S$ where $s' = x*$ and $\textbf{isPrefix}(x, s)$. For example $\perp \sqsubset abc \sqsubset abc* \sqsubset ab* \sqsubset a* \sqsubset \top$. For connivence let $x*$ denote the set of all stings where x is a prefix, and s denote the singleton set $\{s\}$, thus $\sqsubseteq = \subseteq$ in our lattice.

$$\alpha(x) : L \to L^{\#} = \begin{cases} \perp & \text{if } x \text{ is } \{\} \\ \{s\} & \text{if } x \text{ is } \{\text{s}\} \\ p* & \text{where } p \text{ is } \textbf{commonPrefix}(x) \end{cases}$$

Meet $(x \in L^{\#})$

- $\perp \sqcap x = \perp, \forall x$
- $\top \sqcap x = x, \forall x$
- $x \sqcap y = x, x \cap y$

Join $(x \in L^{\#})$

- $\perp \sqcup x = x, \forall x$
- $\top \sqcup x = \top, \forall x$
- $x \sqcup y = \textbf{commonPrefix}(\{x, y\}), \forall x, y \in S, x \neq y$

The lattice is infinite, but of of finite height since all strings are of finite length, therefore it **is noetherian**.

1.1 Monotone Operators

Concatenation (+) For + to be monotone the following must hold: $x + y \leq x' + y' \Rightarrow \alpha(x) + \alpha(y) \sqsubseteq \alpha(x') + \alpha(y')$, where $x, y, x', y' \in \mathbb{P}(S)$ and $a \leq b \Rightarrow \textbf{isPrefix}(a, b)$ for $a, b \in S$ and $x + y \Rightarrow \{x_i y_i\} \forall x_i \in x, y_i \in y$. Similarly $x \leq y \Rightarrow \textbf{isPrefix}(x_i, y_i) \forall x_i \in x, y_i \in y$. In all cases $|x| = |y|$ and $|x'| = |y'|$.

- For case where $x = y =$ "", this holds trivially.
- For $|x| = |y| = 1$, this holds trivially since the definitions are the same, i.e. straightforward concatenation.
- For $|x| = |y| \geq 1$ and $|x'| = |y'| \geq 1$, $\alpha(x) + \alpha(y) = \alpha(x)$, thus we need only show that $\alpha(x) \sqsubseteq \alpha(x')$. If $\alpha(x') = \top$, this is trivial. Otherwise it must be the case the $\alpha(x)$ and $\alpha(x')$ since $x + y \leq x' + y'$.

$\square$

	$\bot$	y	$y*$	$\top$
$\bot$	$\bot$	y	y	$\top$
x	x	xy	$xy*$	$\top$
$x*$	$x*$	$x*$	$x*$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$

(a) Concatenation (+)

	$\bot$	y	$y*$	$\top$
$\bot$	T	T	T	T
x	F	$\textbf{isPrefix}(x, y)$	$\textbf{isPrefix}(y, x), F$	T, F
$x*$	F	$\textbf{isPrefix}(y, x), F$	$\textbf{isPrefix}(y, x) \vert \textbf{isPrefix}(x, y), F$	T, F
$\top$	F	T, F	T, F	T, F

(b) Comparator ($\leq$)

Figure 1: Arithmetic Tables

Comparison ($\leq$) The same logic follows for $\leq$. For $\leq$ to be monotone the following must hold: $x \leq y \Rightarrow \alpha(x) \sqsubseteq \alpha(y)$, where $x, y \in \mathbb{P}(S)$ and $x \leq y \Rightarrow$ $\textbf{substring}(x_i, y_i) \forall x_i \in x, y_i \in y$.

- For case where $x = y = \{\}$, this holds trivially.
- Similarly for $|x| = |y| = 1$, then the definitions are identical.
- For $|x| = |y| \geq 1$, If $\alpha(y) = \top$, this is trivial. Otherwise it must be the case the $\alpha(x)$ and $\alpha(x')$ since $x \leq y'$.

$\square$

1.2 Galois Connection

$$\gamma(\hat{x}) : L^{\#} \to L = \begin{cases} \{\} & \text{if } \hat{x} \text{ is } \bot \\ s & \text{if } \hat{x} \text{ is s} \\ s* & \text{if } \hat{x} \text{ is s*} \\ S & \text{if } \hat{x} \text{ is } \top \end{cases}$$

We must show that $\alpha(x) \sqsubseteq \hat{x} \iff x \subseteq \gamma(\hat{x})$.

$\alpha(x) \sqsubseteq \hat{x} \Rightarrow x \subseteq \gamma(\hat{x})$:

- If $x = \{\}$, this holds trivially.
- If $|x| = 1$, then either $\hat{x} \in S$, which holds trivially, or $\hat{x} = y*, \textbf{isPrefix}(y, x)$, which by our definition $x \in y* \Rightarrow x \subseteq y*$
- Since $\gamma(\hat{x}) = \hat{x}$ and $x \subseteq \alpha(x)$, this must hold.

$x \subseteq \gamma(\hat{x}) \Rightarrow \alpha(x) \sqsubseteq \hat{x}$:

- If $x = \{\}$, this holds trivially.
- If $|x| = 1$, $\alpha(x) = x, \gamma(\hat{x}) = \hat{x} \Rightarrow \alpha(x) \sqsubseteq \hat{x}$
- For $|x| \geq 1$, $\hat{x} = \top$ must hold. Otherwise x must be of the form $y*$ and $x \subseteq \gamma(y*) \Rightarrow x \subseteq y* \Rightarrow \textbf{commonPrefix}(x) = y \in \hat{x} \Rightarrow \alpha(x) \subseteq \hat{x}$

$\square$

1.3 Soundness

Because a Galois connection exists, our approximation is both sound and precise.