Section 6.4 - Reflections

Mittwoch, 21. Februar 2024 15:00

- black text: this stems from the book
- blue text: deviates from book bc of typos etc

Def: $\underline{B}$ τ-algebra, A set, $h: B \to A$ and $g: A \to B$. The reflection of $\underline{B}$ w.r.t. g and h is τ-algebra $\underline{A}$ where for all $f \in \tau$ of arity n and all $x_1, \dots, x_n \in A$,

$$f^{\underline{A}}(x_1, \dots, x_n) := h(f^{\underline{B}}(g(x_1), \dots, g(x_n)))$$

For a class $\mathcal{C}$ of τ-algebras, the class of their reflections is denoted by $\mathrm{Refl}(\mathcal{C})$.

wonderland: for every operation f of $\underline{B}$ then: apply Refl to operation clones. Need this at one point in proof of the theorem.

Analogous to the HSP-Lemma, the following holds:

Lemma $\mathcal{C}$ class of τ-algebras.

- The smallest class of τ-algebras containing $\mathcal{C}$ and closed under Refl, H, S and P is $\mathrm{Refl}\,P(\mathcal{C})$.
- The smallest class of τ-algebras containing $\mathcal{C}$ and closed under Refl, H, S and P^{fin} is $\mathrm{Refl}\,P^{fin}(\mathcal{C})$.

Proof: We show first statement. Suffices to show closedness of $\mathrm{Refl}\,P(\mathcal{C})$ under Refl, H, S, P.

Refl: By the definition of Reflection, the following diagram commutes.

$$\begin{tikzcd} B \arrow[r, shift left, "h_1"] \arrow[rr, bend right, shift right, "h_2 \circ h_1"'] & A \arrow[l, shift left, "g_1"] \arrow[r, shift left, "h_2"] & C \arrow[l, shift left, "g_2"] \arrow[ll, bend left, "g_1 \circ g_2"] \end{tikzcd}$$

Refl Refl Refl

$\sim h_2(h_1(f^A(g_1(g_2(x)))))$

$\sim (h_2 \circ h_1)(f^A((g_1 \circ g_2)(x)))$

Thus, $\mathrm{Refl}(\mathrm{Refl}(\mathcal{K})) = \mathrm{Refl}(\mathcal{K})$ for any class $\mathcal{K}$.

H: Show that $H(\mathcal{K}) \subseteq \mathrm{Refl}(\mathcal{K})$ for any class $\mathcal{K}$.

Given $\underline{A} \in H(\mathcal{K})$, there is a surjective homomorphism h from some $\underline{B} \in \mathcal{K}$ to $\underline{A}$. Let $g: A \to B$ s.t. $h \circ g = id_A$. It holds for $f \in \tau$ of arity n and $x_1, \dots, x_n \in A$ that

$$h(f^{\underline{B}}(g(x_1), \dots, g(x_n))) = f^{\underline{A}}(h \circ g(x_1), \dots, h \circ g(x_n)) = f^{\underline{A}}(x_1, \dots, x_n).$$

Thus, $\underline{A}$ is the reflection of $\underline{B}$ w.r.t. h and g, so $\underline{A} \in \mathrm{Refl}(\mathcal{K})$.

S: Show that $H(\mathcal{K}) \subseteq \mathrm{Refl}(\mathcal{K})$ for any class $\mathcal{K}$.

Given $A \leq B \in \mathcal{C}$, set $g: A \to B$ to be the identity and $h: B \to A$ any extension of g to B. Then $\underline{A}$ is the reflection of $\underline{B}$ w.r.t. h and g, analogous to H.

P: Show that $P(\mathrm{Refl}\,P(\mathcal{K})) \subseteq \mathrm{Refl}(P(\mathcal{K}))$.

Given a set I, algebras $(\underline{B}_i)_{i \in I} \in (P(\mathcal{K}))^I$ and τ-algebras $(A_i)_{i \in I}$ s.t.

Given a set I, algebras $(\underline{B}_i)_{i\in I} \in (P(K))^I$ and τ-algebras $(A_i)_{i\in I}$ s.t. for all i, A_i is the reflection of B_i w.r.t. h_i and g_i.
Then $h: (b_i)_{i\in I} \mapsto (h_i(b_i))_{i\in I}$ and $g: (a_i)_{i\in I} \mapsto (g_i(a_i))$ witness that $\prod_{i\in I} A_i$ is a reflection of $\prod_{i\in I} B_i$, which proves

$P(\mathrm{Refl}\, P(K)) \subseteq \mathrm{Refl}\, P(K)$. $\square$

Notation for a class $\mathcal{C}$ of relational structures, write

$H(\mathcal{C})$ for the class of all homomorphically equivalent structures.

$\mathrm{Red}(\mathcal{C})$ for the class of all pp-reducts $\mathbb{A}$ of structures $\mathbb{B}$ in $\mathcal{C}$, i.e. same domain and all relations pp-definable.

$I(\mathcal{C})$ for the class of structures with a pp-interpretation in a structure from $\mathcal{C}$.

Remark Chapter 3.6: Given a class $\mathcal{C}$ of structures, $HI(\mathcal{C})$ is the class of structures that can be pp-constructed from $\mathcal{C}$.

Theorem: Let $\mathbb{B}, \mathbb{C}$ be at most countable ω-categorical structures and let $\underline{C}$ be a polymorphism algebra of $\mathbb{C}$. Then:

i) $\mathbb{B} \in H\,\mathrm{Red}(\mathbb{C})$ iff there exists $\underline{B} \in \mathrm{Exp}\,\mathrm{Refl}(\underline{C})$ s.t. $\mathrm{Clo}(\underline{B}) = \mathrm{Pol}(\mathbb{B})$

ii) $\mathbb{B} \in HI(\mathbb{C})$ iff there exists $\underline{B} \in \mathrm{Exp}\,\mathrm{Refl}\, P^{fin}(\underline{C})$ s.t. $\mathrm{Clo}(\underline{B}) = \mathrm{Pol}(\mathbb{B})$

Proof: i) "$\Rightarrow$" Let $\mathbb{C}' \in \mathrm{Red}(\mathbb{C})$ s.t. $\mathbb{B} \in H(\mathbb{C}')$ witnessed by $h: \mathbb{C}' \to \mathbb{B}$ as well as $g: \mathbb{B} \to \mathbb{C}'$. Set $\underline{C}'$ to be the expansion of $\underline{C}$ obtained by adding $\mathrm{Pol}(\mathbb{C}') \setminus \mathrm{Clo}(\underline{C})$ to the signature, making $\underline{C}'$ a polymorphism algebra of $\mathbb{C}'$.

$$\begin{tikzcd} B \arrow[r, bend left, "h"] & A \arrow[l, bend left, "g"] \end{tikzcd}$$

Let $\underline{B}'$ be the reflection of $\underline{C}'$ w.r.t. h and g.
All operations on $\underline{B}'$ are obtained as compositions of operations of the form $f^{\underline{B}'}(x_1,\dots,x_n) := h(f^{\underline{C}'}(g(x_1),\dots,g(x_n)))$. As h, g are

All operations on $\underline{B}$ are obtained as compositions of operations of the form $f^{\underline{B}'}(x_1,\dots,x_n) := h(f^{\underline{C}'}(g(x_1),\dots,g(x_n))$. As h, g are homomorphisms and $f^{\underline{C}'} \in Pol(\mathbb{C}')$, they preserve all relations of $\mathbb{B}$, which gives $Clo(\underline{B}') \subseteq Pol(\mathbb{B})$.
Extending $\underline{B}'$ by adding $Pol(\mathbb{B}) \setminus Clo(\underline{B}')$ to it's signature gives us

$$\begin{aligned}\underline{B} \in Exp(\underline{B}') &\subseteq Exp\, Refl(\underline{C}') \\ &\subseteq Exp\, Refl\, Exp(\underline{C}) \\ &= Exp\, Refl(\underline{C})\end{aligned}$$

"$\mathbb{B} \in HRed(\mathbb{C}) \Leftarrow$ there exists $\underline{B} \in Exp\, Refl(\underline{C})$ s.t. $Clo(\underline{B}) = Pol(\mathbb{B})$"

Suppose that the reflection $\underline{B}$ of $\underline{C}$ w.r.t. $h: C \to B$ and $g: B \to C$ is such that $Clo(\underline{B}) \subseteq Pol(\mathbb{B})$.
Set $\mathbb{C}'$ to be the structure with domain C, the same signature as $\mathbb{B}$ and for all k-ary rel. symb. R of $\mathbb{B}$ the relation

$$R^{\mathbb{C}'} := \{ (f(g(b_1^1),\dots,g(b_1^\ell)), \dots, f(g(b_k^1),\dots,g(b_k^\ell))) \mid \ell \in \mathbb{N}, f \in Pol(\mathbb{C})^{(\ell)}, \overline{b^1},\dots,\overline{b^\ell} \in R^{\mathbb{B}} \}$$

These relations are preserved by $Pol(\mathbb{C})$ and therefore pp-def. in $\mathbb{C} \Rightarrow \mathbb{C}' \in Red(\mathbb{C})$.
As $Pol(\mathbb{C})$ contains the projections, g is a hom from $\mathbb{B}$ to $\mathbb{C}'$.

h is a homomorphism from $\mathbb{C}'$ to $\mathbb{B}$:
Let $f(g(b^1),\dots,g(b^\ell)) \in R^{\mathbb{C}'}$ (non-unary case works analogously)
then $\varphi: (x^1,\dots,x^\ell) \mapsto h(f(g(x^1),\dots,g(x^\ell)))$ is an operation on $\underline{B} \in Refl(\underline{C})$. Since $Clo(\underline{B}) \subseteq Pol(\mathbb{B})$, φ is a polymorphism and thus $h(f(g(b^1),\dots,g(b^\ell))) \in R^{\mathbb{B}}$.
Thus, we have $\mathbb{B} \in H(\mathbb{C}') \subseteq H(Red(\mathbb{C}))$.

ii) "$\mathbb{B} \in HI(\mathbb{C}) \Rightarrow$ there exists $\underline{B} \in Exp\, Refl^{fin}(\underline{C})$ s.t. $Clo(\underline{B}) = Pol(\mathbb{B})$"

Let $\mathbb{B} \in HI(\mathbb{C})$.
$\Rightarrow \exists\ \mathbb{D} \in I(\mathbb{C})$ s.t. $\mathbb{B} \in H(\mathbb{D})$

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$\Rightarrow \exists\ \underline{D} \in \mathrm{Exp}\, HSP^{fin}(\underline{C})$ s.t. $Clo(\underline{D}) = Pol(\mathbb{D})$

$\overset{i)}{\Rightarrow} \exists\ \underline{B} \in \mathrm{Exp}\, \mathrm{Refl}(\underline{D})$ s.t. $Clo(\mathbb{B}) = Pol(\underline{B})$

$$\underline{B} \in \mathrm{Exp}\,\mathrm{Refl}(\underline{D}) \subseteq \mathrm{Exp}\,\mathrm{Refl}(\mathrm{Exp}\, HSP^{fin}(\underline{C})) = \mathrm{Exp}\,\mathrm{Refl}\, P^{fin}(\underline{C}).$$

Below is a proof of part ii) of the theorem (both implications), as the "$\Leftarrow$" seems to be missing in the current version of the book. A different proof, using techniques from [29] and results of the third chapter is given at the end of the document.

Recall theorem 3.6.2, which states that, given a class $\mathfrak{C}$ of structures, $HI(\mathfrak{C}) = H\,\mathrm{Red}\, P^{fin}_{full}(\mathfrak{C})$.

Lemma: Let $\mathbb{A}$, $\mathbb{B}$ be at most countable ω-cat. structures, and let $\underline{B}$ be a polymorphism algebra of $\mathbb{B}$. Then, the following holds:

i) $\mathbb{A} \in P^{fin}_{full}(\mathbb{B}) \Rightarrow \exists\, \underline{A} \in \mathrm{Exp}\, P^{fin}(\underline{B}) : Clo(\underline{A}) = Pol(\mathbb{A})$

ii) $\mathbb{A} \in P^{fin}_{full}(\mathbb{B}) \Leftarrow \exists\, \underline{A} \in P^{fin}(\underline{B}) : Clo(\underline{A}) = Pol(\mathbb{A})$

Proof Lemma:

"$\Leftarrow$" Suppose that $\underline{A} = \underline{B}^n$. Then, the domain A of $\mathbb{A}$ is B^n.

- Let R be any k-ary, pp-def. relation on $\mathbb{A}$.

  Then, R can be viewed as an nk-ary relation on B via

  $R^{\mathbb{B}} = \{ (b^1_1, \dots, b^1_n, \dots, b^k_1, \dots, b^k_n) \mid (b^1, \dots, b^k) \in R^{\mathbb{A}} \}$

  This relation is pp-def. from $\mathbb{B}$:

  Let $f^{\mathbb{B}} \in Pol(\mathbb{B})^{(e)}$ and $({}_1b^j_i, \dots, {}_e b^j_i) \in B^e$, $i = 1, \dots, n$, $j = 1, \dots, k$

  s.t. $R^{\mathbb{B}}({}_1b^1_1, \dots, {}_1b^k_n), \dots, R^{\mathbb{B}}({}_e b^1_1, \dots, {}_e b^k_n)$

  By our assumption, $R^{\mathbb{A}}$ is pp-def, and thus $f^{\mathbb{A}}$, which acts componentwise like $f^{\mathbb{B}}$, preserves it:

  $$R^{\mathbb{A}}(\underbrace{({}_1b^1_1, \dots, {}_1b^1_n)}_{=:\,{}_1b^1}, {}_1b^2, \dots, {}_1b^k), \dots, R^{\mathbb{A}}({}_e b^1, \dots, {}_e b^k)$$

  $$\Rightarrow R^{\mathbb{A}}(\underbrace{f^{\mathbb{A}}({}_1b^1, {}_2b^1, \dots, {}_e b^1)}, \dots, f^{\mathbb{A}}({}_1b^k, \dots, {}_e b^k))$$

  $$= (f^{\mathbb{B}}({}_1b^1_1, \dots, {}_e b^1_1), \dots, f^{\mathbb{B}}({}_1b^1_n, \dots, {}_e b^1_n))$$

$$= (f^{\mathbb{B}}({}_1b_1^1, \ldots, {}_eb_1^1), \ldots, f^{\mathbb{B}}({}_1b_n^1, \ldots, {}_eb_n^1))$$

$$\Leftrightarrow R^{\mathbb{B}}(f^{\mathbb{B}}({}_1b_1^1, \ldots, {}_eb_1^1), \ldots, f^{\mathbb{B}}({}_1b_n^k, \ldots, {}_eb_n^k))$$

$\Rightarrow R^{\mathbb{B}}$ is pp-def from $\mathbb{B}$.

- with a similar proof we also see that a relation is pp-def from $\mathbb{A}$ if it (as nn-ary rel.) is from $\mathbb{A}$.

"$\Rightarrow$"

Let $\mathbb{A} \in P_{full}^{fin}(\mathbb{B})$ be an n-th full power. Set $\underline{A} := \underline{B}^n$.
Clearly, $Clo(\underline{A}) \subseteq Pol(\mathbb{A})$, as elements of $Clo(\underline{A})$ act componentwise $\square$

We get the following corollary from the proof of ii):

Corollary: Let $\mathbb{B}$ be an at most countable, ω-categorical structure, and let $\underline{B}$ be a polymorphism algebra of $\mathbb{B}$. Let $\underline{A} \in P^{fin}(\underline{B})$. Then $\underline{A}$ is a polymorphism algebra of a structure $\mathbb{A} \in P_{full}^{fin}(\mathbb{B})$.

Proof theorem ii)

"$\Rightarrow$" $\mathbb{B} \in HI(\mathbb{C}) \overset{3.6.2}{\Leftrightarrow} \mathbb{B} \in HRed\, P_{full}^{fin}(\mathbb{C})$

$$\overset{i)+\text{Lemma}}{\Rightarrow} \exists \underline{B} \in Exp\, Refl\, Exp\, P^{fin}(\underline{C}) : Clo(\underline{B}) = Pol(\mathbb{B}).$$

$$\| \\ Exp\, Refl\, P^{fin}(\underline{C})$$

"$\Leftarrow$" $\exists \underline{B} \in Exp\, Refl\, P^{fin}(\underline{C}) : Clo(\underline{B}) = Pol(\mathbb{B})$.

$\Rightarrow \exists \underline{A} \in P^{fin}(\underline{C}) : \underline{B} \in Exp\, Refl(\underline{A})$. By the previous corollary, $\underline{A}$ is a polymorphism algebra of $\mathbb{A} \in P_{full}^{fin}(\mathbb{C})$.

Thus: $\mathbb{B} \in HRed\, P_{full}^{fin}(\mathbb{C}) = HI(\mathbb{C})$. $\square$

Corollary: Let $\mathbb{B}$ be an at most countable, ω-categorical structure and let $\underline{B}$ be a polymorphism algebra of $\mathbb{B}$. TFAE:

i) $HI(\mathbb{B})$ contains $\mathbb{K}_3$;

ii) $HI(\mathbb{B})$ contains all finite structures;

i) $HI(\mathbb{B})$ contains $\mathbb{K}_3$;

ii) $HI(\mathbb{B})$ contains all finite structures;

iii) $HI(\mathbb{B})$ contains $(\{0,1\}; 1IN3)$;

iv) $\mathrm{Refl}\,P^{fin}(\underline{B})$ contains an algebra of size ≥ 2 all of whose operations are projections;

v) $\mathrm{Refl}\,P^{fin}(\underline{B})$ contains for every finite set A an algebra on A all of whose operations are projections.

If these conditions apply, $\mathbb{B}$ has a finite-signature reduct with an NP hard CSP.

Proof: **i) ⇒ ii)** $I(\mathbb{K}_3)$ contains all finite structures (Cor. 3.2.1) and $IHI(\mathbb{B}) \subseteq HI(\mathbb{B})$ (Th 3.6.2)

ii) ⇒ iii) trivial

iii) ⇒ iv) The polymorphisms of $1IN3$ only contain projections.

Theorem $\Rightarrow \exists\ \underline{A} \in \cancel{Exp}\,\mathrm{Refl}\,P^{fin}(\underline{B}): Clo(\underline{A}) \subseteq Pol(1IN3)$

iv) ⇒ v) Let $\underline{A} \in \mathrm{Refl}\,P^{fin}(\underline{B})$ s.t. $|A| \geq 2$ and $Clo(\underline{A}) \subseteq Proj_A$

Th. 6.3.10 $\Rightarrow$ $HSP^{fin}(\underline{A})$ contains for every finite set S an alg. $\underline{S}$ on S s.t. $Clo(\underline{S}) \subseteq Proj_S$.

Now $\underline{S} \in HSP^{fin}(\underline{A}) \subseteq HSP^{fin}(\mathrm{Refl}\ P^{fin}(\underline{B})) \overset{\text{Lemma}}{\subseteq} \mathrm{Refl}\,P^{fin}(\underline{B})$

v) ⇒ i) Let $\underline{A} \in \mathrm{Refl}\,P^{fin}(\underline{B})$ s.t. $A = \{0,1,2\}$ and $Clo(\underline{A}) \subseteq Proj_A$. Then $Clo(\underline{A}) \subseteq Pol(\mathbb{K}_3)$

$\Rightarrow \exists\, \underline{C} \in Exp\,\mathrm{Refl}\,P^{fin}(\underline{B})$ s.t. $Clo(\underline{C}) = Pol(\mathbb{K}_3)$

$\overset{\text{Theorem}}{\Rightarrow}$ $\mathbb{K}_3 \in HI(\mathbb{B})$

The hardness follows from Cor. 3.7.1 ($\mathbb{K}_3 \in HI(\mathbb{B}) \Rightarrow \exists$ finite sign. reduct whose CSP is NP hard). □

This concludes the section. A different proof of Theorem ii) is given below.

The following definition stems from "The wonderland of reflections" [29]. The lemma thereafter is a shortened version of Lemma 3.8,

The following definition stems from "The wonderland of reflections" [29].
The lemma thereafter is a shortened version of Lemma 3.8, Lemma 3.9 and Corollary 3.10 in that same paper, using our knowledge from section 3.6

Def: Let $\mathbb{A}$, $\mathbb{B}$ be relational structures. We say that $\mathbb{B}$ is a pp-power of $\mathbb{A}$ if $\mathbb{B}$ is isomorphic to a structure $\mathbb{C}$ with domain A^n for some $n \geq 1$ all of whose relations are pp-definable from $\mathbb{A}$ (viewing k-ary relations on A^n as nk-ary relations on $\mathbb{A}$.)

Lemma 1 Let $\mathcal{C}$ be a class of relational structures. Then $HI(\mathcal{C}) = HP_{pp}(\mathcal{C})$

Proof: The inclusion "$\supseteq$" is trivial.
To proof the other inclusion first note that $HH(\mathbb{C}) = H(\mathbb{C})$ for any rel str. $\mathbb{C}$. Therefore, it suffices to show that $I(\mathcal{C}) \subseteq HP_{pp}(\mathcal{C})$. Let $\mathbb{B} \in I(\mathcal{C})$ as well as $\mathbb{A} \in \mathcal{C}$ s.t. $\mathbb{B} \in I(\mathbb{A})$. Let $J: A^n \rightharpoonup B$ be the coordinate map of the pp-interpretation. Now set $\mathbb{C}$ to be the rel. str. with domain A^n, the same signature as $\mathbb{B}$ and relations $R^{\mathbb{C}} := J^{-1}(R^{\mathbb{B}})$ in the sense of Def. 2.4.1., which are pp-definable from $\mathbb{A}$.

Set h to be any extension of J to the entirety of A^n and g to be any right-inverse of J. Now $\mathbb{C} \in P_{pp}(\mathbb{A})$ and $h: \mathbb{C} \to \mathbb{B}$ as well as $g: \mathbb{B} \to \mathbb{C}$ are homomorphisms, which implies $\mathbb{B} \in HP_{pp}(\mathbb{A})$. $\square$

The following fact is part of Prop. 4.6 in [29], and an easy observation when keeping in mind that, given at most countable ω-cat structures $\mathbb{A}$ and $\mathbb{B}$ as well as a polymorphism algebra $\underline{B}$ of $\mathbb{B}$,

$$\mathbb{A} \in Red(\mathbb{B}) \Leftrightarrow \exists \underline{A} \in Exp(\underline{B}): Clo(\underline{A}) = Pol(\mathbb{A}).$$

Lemma 2 Let $\mathbb{A}$, $\mathbb{B}$ be at most countable ω-cat. structures, and let $\underline{B}$ be a polymorphism algebra of $\mathbb{B}$. Then, the following holds:

$$\mathbb{A} \in P_{pp}(\mathbb{B}) \Leftrightarrow \exists \underline{A} \in Exp\, P^{fin}(\underline{B}): Clo(\underline{A}) = Pol(\mathbb{A}).$$

"$\mathbb{B} \in HI(\mathbb{C}) \iff$ there exists $\underline{\mathbb{B}} \in \mathrm{Exp\,Refl}\, P^{fin}(\underline{\mathbb{C}})$ s.t. $\mathrm{Clo}(\underline{\mathbb{B}}) = \mathrm{Pol}(\mathbb{B})$"

$$\mathbb{B} \in HI(\mathbb{C}) \overset{\text{Lemma } 1}{\iff} \mathbb{B} \in HP_{pp}(\mathbb{C})$$

[Keeping in mind that a reduct of a pp-power is a reduct, Part I) of the theorem + Lemma 2 show us that]

$$\iff \exists\, \underline{\mathbb{B}} \in \underbrace{\mathrm{Exp\,Refl\,Exp}\, P^{fin}(\underline{\mathbb{C}})}_{= \mathrm{Exp\,Refl}\, P^{fin}(\underline{\mathbb{C}})} : \mathrm{Clo}(\underline{\mathbb{B}}) = \mathrm{Pol}(\mathbb{B})$$

$\square$