§6.1.6 ARITY REDUCTION

We often want to bound arity of polys with some property.

Lemma (Reducing arity of f not preserving R)
Let $\mathbb{B}$ be relational, R a k-ary rel contained in m many orbits of k-tuples under $\mathrm{Aut}(\mathbb{B})$. If $\mathbb{B}$ has a polymorphism f that does not preserve R, $\mathbb{B}$ also has a poly g of arity m that does not preserve R.

<u>proof</u>: Let f' be of minimal arity ℓ s.t. f' does not preserve R.

So $\exists\ t_1 \dots t_\ell \in R$ s.t. $f'(t_1, \dots, t_\ell) \notin R$.

<u>For $\ell > m$:</u> some t_i and t_j must lie in the same orbit (so $t_i = \alpha t_j$ for $\alpha \in \mathrm{Aut}(\mathbb{B})$) (Say WLOG $i=1$, $j=2$). Define $(\ell-1)$-ary g by

$$g(x_2 \dots x_\ell) := f'(\alpha(x_2), x_2, \dots, x_\ell)$$

g is also a poly. And it does <u>NOT</u> preserve R by construction.

※ we contradict minimality.

So $\ell \le m$ and we can build an m-ary poly from f' adding fictitious variables ☒

COROLLARY Let $\mathrm{Aut}(\mathbb{B})$ have r many orbitals & s many orbits on $\mathbb{B}$. If $\mathbb{B}$ has an essential polymorphism, then $\mathbb{B}$ also has an essential poly of arity $2r-s$

Proof:

$\mathbb{B}$ has an essential poly iff some poly does not preserve $P^3_{\mathbb{B}} = \{(a,b,c) \mid a=b \vee b=c\}$

$P^3_{\mathbb{B}}$ consists of at most $2r-s$-many orbits of triplets:

$\leq r(\mathrm{Aut}(\mathbb{B}))$ orbits of triplets (a,b,c) with $a=b$

// // with $b=c$

the orbits of $t_1=t_2=t_3$ are counted twice.

So by previous lemma, we have an essential poly of arity $\leq 2r-s$

COROLLARY² Say $\mathrm{Aut}(\mathbb{B})$ is 2-transitive on $\mathbb{B}$. Then $\mathbb{B}$ has an essential poly $\Rightarrow$ $\mathbb{B}$ has a 3-ary essential poly

§6.1.7 KARA's METHOD

improves this to "binary" under weaker assumptions

B has the ORBITAL EXTENSION PROPERTY (OEP) if there is an orbital O s.t. $\forall b_1, b_2 \in B \ \exists c$ s.t. $(b_1, c), (b_2, c) \in O$.

OLIGOMORPHIC EXAMPLES

- B with $Aut(B)$ 2-transitive
- Random graph
- $(\mathbb{Q}, <)$
- Any ω-categorical B with $Aut(B)$ transitive and an invariant type
- If A and B have OEP then $A * B$ has OEP.

OLIGOMORPHIC NON-EXAMPLE

- $K_{\omega,\omega}$

KARA'S METHOD Let $\mathcal{C}$ be a clone with an essential operation that contains a permutation group $\mathcal{G}$ with the OEP. Then, $\mathcal{C}$ must also contain a binary essential operation.

Proof (of KARA'S METHOD):

Let f be essential depending on all arguments and with arity $k \geq 3$.

So $\exists\, a_1 \dots a_k, a_1'$ s.t. $f(a_1 \dots a_k) \neq f(a_1' a_2 \dots a_k)$.

Let O be the orbital from OEP (s.t. $\forall\, b_1, b_2\ \exists\, c$ $(b_1, c), (b_2, c) \in O$)

CASE 1: $\exists\, b_1 \dots b_k$ s.t. $(b_i, a_i) \in O$ for $2 \leq i \leq k$ and

$f(b_1, a_2, \dots, a_k) \neq f(b_1 \dots b_k)$

So $\exists\, \alpha_3 \dots \alpha_k$ s.t. $\alpha_i(a_2) = a_i$ $\alpha_i(b_2) = b_i$

$$g(x, y) = f(x, y, \alpha_3(y), \dots, \alpha_k(y))$$

so g is binary and essential:

$$\left.\begin{aligned} g(a_1 a_2) &= f(a_1, a_2, \dots, a_k) \neq f(a_1', a_2, \dots, a_k) = g(a_1' a_2) \\ g(b_1, a_2) &= f(b_1, a_2, \dots, a_k) \neq f(b_1, b_2 \dots b_k) = g(b_1, b_2) \end{aligned}\right]$$

So it depends on both arguments

CASE 2: $\forall\, b_1 \dots b_k$ if $(a_i, b_i) \in O$ for $2 \le i \le k$

$f(b_1, a_2, \dots, a_k) = f(b_1, b_2, \dots, b_k)$ $\circledast$

f depends on 2nd coordinate $\Rightarrow \exists\, c_1 \dots c_k$ and c_2' s.t. $f(c_1, c_2, \dots c_k) \neq f(c_1, c_2', \dots c_k)$

So, $f(c_1 a_2 \dots a_k) \neq f(c_1 c_2 \dots c_k)$ WLOG.

By OEP let $d_2, \dots, d_k$ s.t. $\forall\, 2 \le i \le k$ (d_i, a_i) and $(d_i, c_i) \in O$

So $\exists\, \alpha_3 \dots \alpha_k \in \mathcal{G}$ s.t. $\alpha_i(c_2) = c_i$ and $\alpha_i(d_2) = d_i$. Let

$g(x, y) := f(x, y, \alpha_3(y), \dots, \alpha_k(y))$

g depends on both arguments.

$g(a_1, d_2) = f(a_1 d_2 \dots d_k) \overset{\circledast}{=} f(a_1 \dots a_k) \neq f(a_1' \dots a_k) \overset{\circledast}{=} f(a_1' d_2 \dots d_k) = g(a_1', d_2)$

$g(c_1, d_2) = f(c_1, d_2 \dots d_k) \overset{\circledast}{=} f(c_1, a_2 \dots a_k) \neq f(c_1 \dots c_k) = f(c_1, c_2)$

So g is essentially binary. $\square$

§ 6.1.8 MINIMAL CLONES

Let $\mathcal{C}$ and $\mathcal{D}$ be closed subclones of $\mathcal{O}$. We say $\mathcal{D}$ is minimal above $\mathcal{C}$ if for all closed subclone $\mathcal{E}$ of $\mathcal{O}$

$$\mathcal{C} \subsetneq \mathcal{E} \subseteq \mathcal{D} \Rightarrow \mathcal{E} = \mathcal{D}$$

$f \in \mathcal{O} \setminus \mathcal{C}$ is minimal above $\mathcal{C}$ if

① $\forall g \in \mathcal{O} \setminus \mathcal{C}$ $\quad f \in \overline{\langle \mathcal{C} \cup \{g\} \rangle} \Rightarrow g \in \overline{\langle \mathcal{C} \cup \{f\} \rangle}$

② If f' satisfies ①, the arity of f' is $\geq$ the arity of f

EQUIVALENCE OF DEFINITIONS Let $\mathcal{C}$ be a closed subclone of $\mathcal{O}$

ⓐ f is minimal above $\mathcal{C}$ iff $\overline{\langle \mathcal{C} \cup \{f\} \rangle}$ is minimal above $\mathcal{C}$

ⓑ $\mathcal{D}$ (closed subclone of $\mathcal{O}$) is minimal above $\mathcal{C}$ iff $\mathcal{D} = \overline{\langle \mathcal{C} \cup \{f\} \rangle}$ for f minimal above $\mathcal{C}$.

Proof: ⓐ by ①.

ⓑ Just take f of minimal arity among $f \in \mathcal{D} \setminus \mathcal{C}$ and use minimality. ∎

RECALL, the Inv-Pol connection

THEOREM Let B be countable ω-categorical.

① For f.o. reducts C of B, the sets $\langle C\rangle_{pp}$ ordered by inclusion form a lattice

② The closed subclones of $\mathcal{O}_B$ containing $\mathrm{Aut}(B)$ ordered by inclusion form a lattice

③ Inv is an anti-isomorphism between the two lattices and Pol is its inverse.

PROPOSITION Let B be ω-categorical and let C be pp-definable in B. Then, $\mathrm{Pol}(C)$ is minimal above $\mathrm{Pol}(B)$ if and only if for every $R \in \langle B\rangle_{pp} \setminus \langle C\rangle_{pp}$ B has a pp-definition in (C, R).

Proof: ($\Rightarrow$) Since $R \in \langle B\rangle_{pp} \setminus \langle C\rangle_{pp}$ $\langle B\rangle_{pp} \supseteq \langle (C,R)\rangle_{pp} \supsetneq \langle C\rangle_{pp}$.

Applying Pol, $\mathrm{Pol}(B) \subseteq \mathrm{Pol}(C,R) \subsetneq \mathrm{Pol}(C) \xRightarrow{\text{minimality}} \mathrm{Pol}(B) = \mathrm{Pol}(C,R)$

$\xRightarrow{\text{Applying Inv}} \langle B\rangle_{pp} = \langle (C,R)\rangle_{pp}$.

($\Leftarrow$) Let $\mathcal{D}$ be a closed subclone s.t. $\mathrm{Pol}(B) \subsetneq \mathcal{D} \subseteq \mathrm{Pol}(C)$. Applying Inv, $\mathrm{Inv}(\mathcal{D}) = \langle D\rangle_{pp}$ for some f.o. reduct of C s.t. $\langle C\rangle_{pp} \subseteq \langle D\rangle_{pp}$. If $\mathcal{D} \subsetneq \mathrm{Pol}(C)$, $\exists R \in \langle B\rangle_{pp} \setminus \langle C\rangle_{pp}$ s.t. $R \in \langle D\rangle_{pp}$. But then, by hypothesis, $\mathcal{D} = \mathrm{Pol}(B)$ contradicting our assumption. So $\mathcal{D} = \mathrm{Pol}(C)$ and $\mathrm{Pol}(C)$ is minimal. $\square$

EXISTENCE OF MINIMAL CLONES Let $\mathbb{B}$ be countable ω-categorical in a finite relational language. Let $\mathcal{D} \supsetneq \mathrm{Pol}(\mathbb{B})$ be a locally closed clone. Then, $\exists\, \mathcal{C} \subseteq \mathcal{D}$ minimal above $\mathrm{Pol}(\mathbb{B})$

Proof:

Consider the POSET $P = \{ \mathcal{S}$ closed clones s.t. $\mathrm{Pol}(\mathbb{B}) \subsetneq \mathcal{S} \subseteq \mathcal{D} \}$ ordered by inclusion.

CLAIM: Every descending chain $\mathcal{S}_1 \supseteq \mathcal{S}_2 \supseteq \dots$ has a lower bound in P (i.e. some $\mathcal{S} \in P$ s.t. $\mathcal{S}_i \supseteq \mathcal{S}$ for each i).

Recall, for each i $\mathrm{Inv}(\mathcal{S}_i) = \langle \mathbb{C}_i \rangle_{pp}$ where $\mathbb{C}_i$ is a f.o. reduct of $\mathbb{B}$.

Now, $\bigcap_{i \geq 1} \mathcal{S}_i$ is also a closed clone containing $\mathrm{Pol}(\mathbb{B})$. WNTS it is in P (i.e. not $= \mathrm{Pol}(\mathbb{B})$)

By the Inv-Pol connection, $\bigcup_{i \geq 1} \mathrm{Inv}(\mathcal{S}_i) = \mathrm{Inv}(\bigcap \mathcal{S}_i) = \langle \mathbb{C} \rangle_{pp}$ for some f.o reduct of $\mathbb{B}$.

Now, $\exists\, R \in \mathbb{B} \setminus \bigcup_{i \geq 1} \mathrm{Inv}(\mathcal{S}_i)$. Otherwise since $\mathbb{B}$ has fin. many relations $\exists\, i$ s.t. all relations of $\mathbb{B}$ are in $\mathrm{Inv}(\mathcal{S}_i)$ and since $\langle \mathbb{B} \rangle_{pp} \subseteq \mathrm{Inv}(\mathcal{S}_i)$, $\mathcal{S}_i \subseteq \mathrm{Pol}(\mathbb{B})$. ※

So $\exists\, f \in \mathrm{Pol}(\mathbb{C}) \setminus \mathrm{Pol}(\mathbb{B})$. So $\mathrm{Pol}(\mathbb{C}) = \bigcap \mathcal{S}_i \in P$. ☒

Now, Theorem follows by ZORN'S LEMMA.

If $\mathcal{D}$ is a clone over a finite domain, there is $\mathcal{C} \subseteq \mathcal{D}$ which is minimal above the clone of projections (unless $\mathcal{D}$ is already Proj)

EXAMPLE (If B is infinite not ω-categorical, there may be no minimal clone above projections)

Let $f: \mathbb{N} \to \mathbb{N}$ $\quad x \mapsto x+1$. Let $\mathcal{D} = \langle f \rangle$

Then, $\mathcal{D}$ does not contain a minimal clone above Proj.

- Every operation in $\mathcal{D}$ is essentially unary
- every unary $g \in \mathcal{D}$ is of the form $x \mapsto x+c$

(i.e. $\underbrace{f(f(\cdots f}_{c \text{ many times}}(x))\cdots)$

Now, g generates $h: x \mapsto x+2c$ $\quad g(g(x))$

But $g \notin \overline{\langle h \rangle}$

MINIMAL CLOSED CLONES ABOVE End(B)

Let $\mathrm{Aut}(B) \curvearrowright B$ have r orbitals and s orbits. Let $\mathcal{B} = \overline{\langle \mathrm{End}(B) \rangle}$.

Then, any minimal closed clone above $\mathcal{B}$ is locally generated by $\mathrm{End}(B) \cup f$ where f has arity $\leq 2r - s$.

<u>Proof</u>: Let $\mathcal{C}$ be minimal above $\mathcal{B}$.

If $\mathcal{C}$ is essentially unary we are fine.

If $\mathcal{C}$ is not essentially unary, then it is locally generated by $\mathrm{End}(B) \cup f$ for f essential.

We know we can find f essential of arity $\leq 2r - s$.

So we are done by minimality of f. $\boxtimes$