Basic orthogonal and periodic functions

Exercise 3

Least-squares Approximation of Functions Using Orthogonal Polynomials

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1 Introduction

This exercise asks to compute the Chebyshev approximation

$$t(x) = \sum_{j=0}^{n} a_j T_j(x)$$

for the functions $f(x) = \cos(2x)$ and $f(x) = cos(4x)$ over the interval $[-\pi, \pi]$ for $n = 6$ and then plot the original functions with the approximations and draw conclusions on the results.

2 Tools

To solve this problem, I've used MATLAB as requested by the instructions. To make the computation more efficient, I wrote my own function which calculates the Chebyshev polynomial instead of using `chebyshevT` which was slower.

3 Computation

Given $n = 6$ and $N = n + 1$, I first calculate the roots of the Chebyshev polynomial by using the formula:

$$x_j = \cos\left(\frac{2j-1}{2n}\pi\right), j = 1, \ldots, N$$

The resulting zeros are:

$$\begin{bmatrix} 9.749279121818236e-01 \\ 7.818314824680298e-01 \\ 4.338837391175582e-01 \\ 6.123233995736766e-17 \\ -4.338837391175581e-01 \\ -7.818314824680295e-01 \\ -9.749279121818237e-01 \end{bmatrix}$$

Let's first analyze the function $\cos(2x)$. We've seen in class we can calculate the coefficients with the formula given by:

$$c_j = \frac{2}{N}\sum_{k=0}^{n} f(x_j)T_j(x_k)$$

To better exploit the properties of Chebyshev polynomials, I rescale the interval from $[-\pi, \pi]$ to $[-1, 1]$. This is done in the function calculation $f(x_j)$ by multiplying the argument by π. I find the following coefficients:

$$\begin{bmatrix} 4.407675118300712e-01 \\ 3.489272363107635e-16 \\ 5.739892333362830e-01 \\ -2.537652627714643e-16 \\ 6.516380108719009e-01 \\ -1.554312234475219e-15 \\ -7.019674665493458e-01 \end{bmatrix}$$

To plot these coefficients in the graph, I need to calculate their associated y values:

$$\begin{bmatrix} 9.876173836808355e-01 \\ 1.986723718905843e-01 \\ -9.149466098687946e-01 \\ 9.999999999999993e-01 \\ -9.149466098687926e-01 \\ 1.986723718905768e-01 \\ 9.876173836808371e-01 \end{bmatrix}$$

Calculating the coefficients for the $\cos(4x)$ function now only requires minor changes in the code. The following are the coefficients of the polynomial:

$$\begin{bmatrix} 6.879841220704104e-01 \\ 2.537652627714643e-16 \\ -1.535576743556636e-01 \\ -1.459150260935920e-15 \\ 1.012919200612446e+00 \\ -1.094362695701940e-15 \\ 5.104689360033142e-01 \end{bmatrix}$$

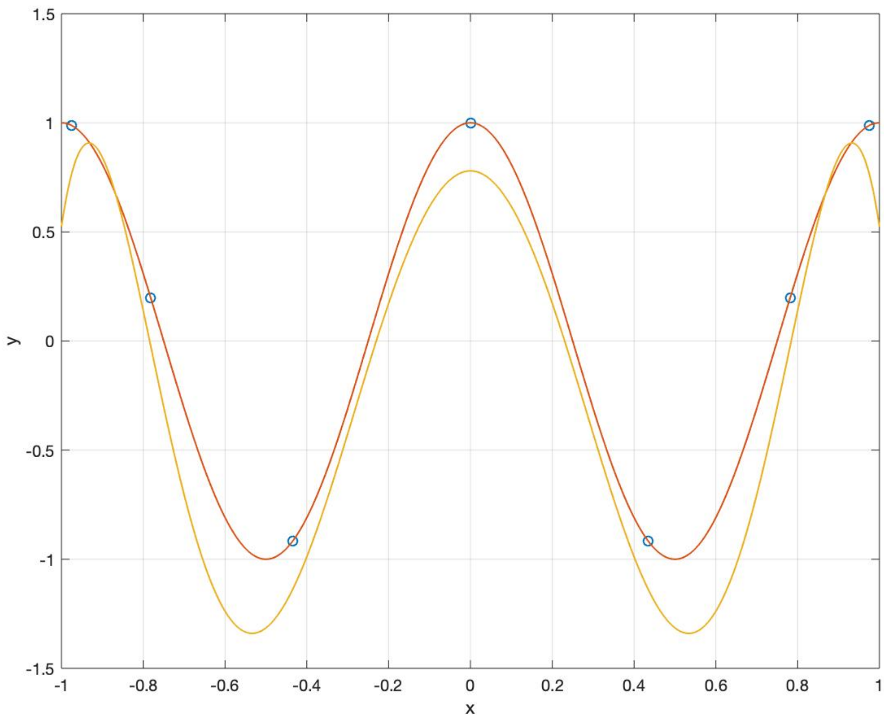
The associated y values for the given roots are the following:

$$\begin{bmatrix} 9.507761930971577e-01 \\ -9.210585772947382e-01 \\ 6.742545978208001e-01 \\ 1.000000000000000e+00 \\ 6.742545978207990e-01 \\ -9.210585772947411e-01 \\ 9.507761930971639e-01 \end{bmatrix}$$

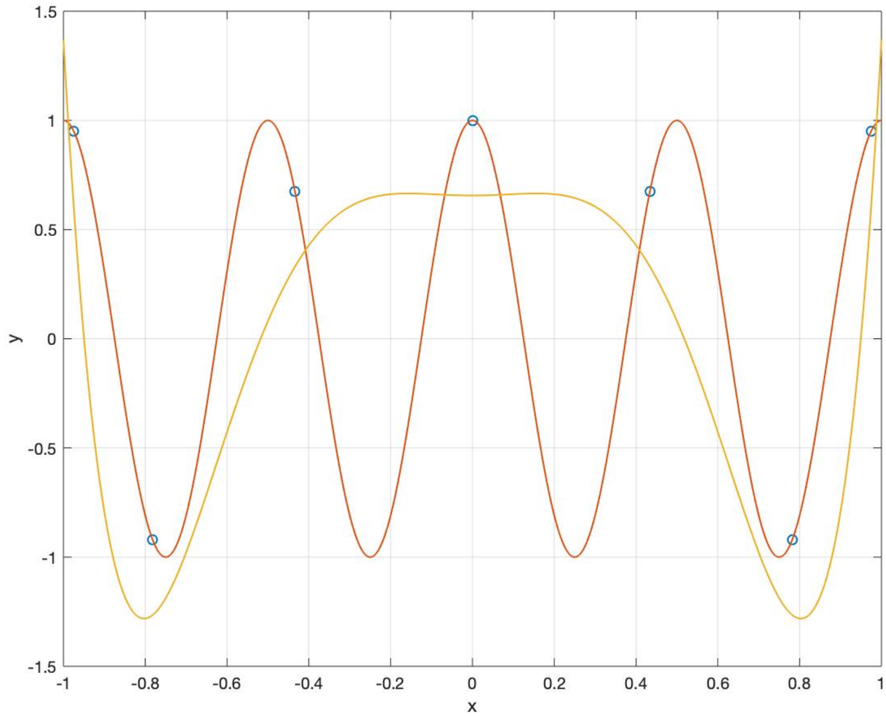
The plots for both functions have been added to the next section.

4 Plots

Function $cos(2x)$:



Function $cos(4x)$:


1.5
1
0.5
0
-0.5
-1
-1.5
y
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
x

5 Observations

I first calculated the zeros of the polynomial so I could use them as nodes in polynomial interpolation because the resulting interpolation polynomial minimizes the effect of Runge's phenomenon.

The Chebyshev approximation for the function $cos(2x)$ is pretty accurate, whereas the approximation for $cos(4x)$ not so much.