Basic orthogonal and periodic functions

Exercise 3

Least-squares Approximation of Functions Using Orthogonal Polynomials

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1 Introduction

This exercise asks to compute the Chebyshev approximation

$$t(x) = \sum_{j=0}^{n} a_j T_j(x)$$

for the functions $f(x) = \cos(2x)$ and $f(x) = cos(4x)$ over the interval $[-\pi, \pi]$ for $n = 6$ and then plot the original functions with the approximations and draw conclusions on the results.

As we've seen in class, Chebyshev polynomials are a set of orthogonal polynomials that can be used to approximate to a least squares fit. For this exercise, I'll first compute the roots of the polynomial (also called Chebyshev nodes), then calculate its coefficients for both the functions. Finally, I'll calculate the y values of the approximation to see how well it approximates the original function. During this process, I'll make sure to choose the correct data points and the correct number of data points to exploit the properties of these basis functions.

2 Tools

To solve this problem, I've used MATLAB as requested by the exercise. To make the computation more efficient, I wrote my own function which calculates the Chebyshev polynomial instead of using the built-in MATLAB function `chebyshevT` which was noticeably slower. Writing your own function is just a matter of using the compact closed-form expression for the Chebyshev polynomials:

$$T_i(x) = \cos\left(i \arccos(x)\right)$$

3 Computation

Given $n = 6$ and $N = n + 1$, I first calculate the roots of the Chebyshev polynomial by using the formula:

$$x_j = \cos\left(\frac{2j-1}{2n}\pi\right), j = 1, \ldots, N$$

The resulting zeros are:

$$\begin{bmatrix} 9.749279121818236e-01 \\ 7.818314824680298e-01 \\ 4.338837391175582e-01 \\ 6.123233995736766e-17 \\ -4.338837391175581e-01 \\ -7.818314824680295e-01 \\ -9.749279121818237e-01 \end{bmatrix}$$

Let's first analyze the function $\cos(2x)$. We've seen in class we can calculate the coefficients with the formula given by:

$$c_j = \frac{2}{N} \sum_{k=0}^{n} f(x_j) T_j(x_k)$$

To better exploit the properties of Chebyshev polynomials, I rescale the interval from $[-\pi, \pi]$ to $[-1, 1]$. This is done in the function calculation $f(x_j)$ by multiplying the argument by π. I find the following coefficients:

$$\begin{bmatrix} 4.407675118300712e-01 \\ 3.489272363107635e-16 \\ 5.739892333362830e-01 \\ -2.537652627714643e-16 \\ 6.516380108719009e-01 \\ -1.554312234475219e-15 \\ -7.019674665493458e-01 \end{bmatrix}$$

To plot these coefficients in the graph, I need to calculate their associated y values:

$$\begin{bmatrix} 9.876173836808355e-01 \\ 1.986723718905843e-01 \\ -9.149466098687946e-01 \\ 9.999999999999993e-01 \\ -9.149466098687926e-01 \\ 1.986723718905768e-01 \\ 9.876173836808371e-01 \end{bmatrix}$$

Calculating the coefficients for the $\cos(4x)$ function now only requires minor changes in the code. The following are the coefficients of the polynomial:

$$\begin{bmatrix} 6.879841220704104e-01 \\ 2.537652627714643e-16 \\ -1.535576743556636e-01 \\ -1.459150260935920e-15 \\ 1.012919200612446e+00 \\ -1.094362695701940e-15 \\ 5.104689360033142e-01 \end{bmatrix}$$

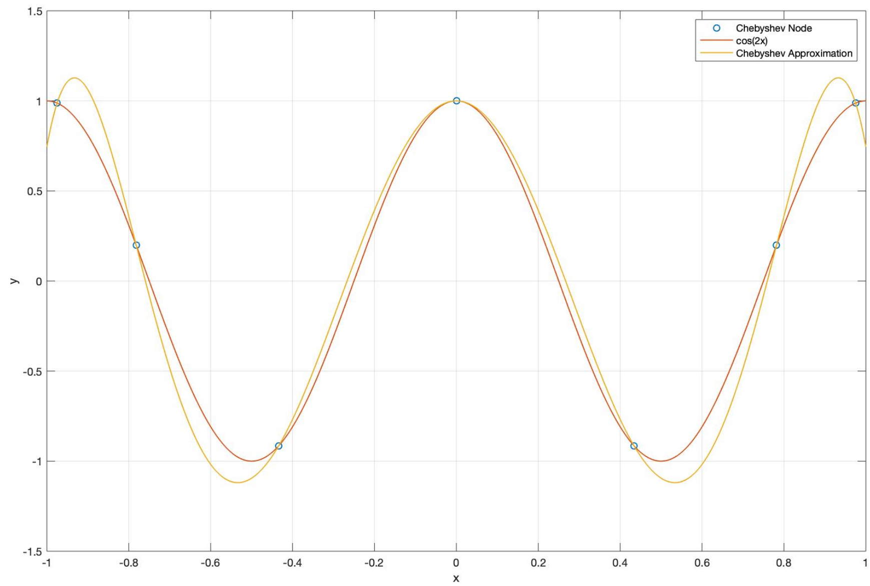
The associated y values for the given roots are the following:

$$\begin{bmatrix} 9.507761930971577e-01 \\ -9.210585772947382e-01 \\ 6.742545978208001e-01 \\ 1.000000000000000e+00 \\ 6.742545978207990e-01 \\ -9.210585772947411e-01 \\ 9.507761930971639e-01 \end{bmatrix}$$

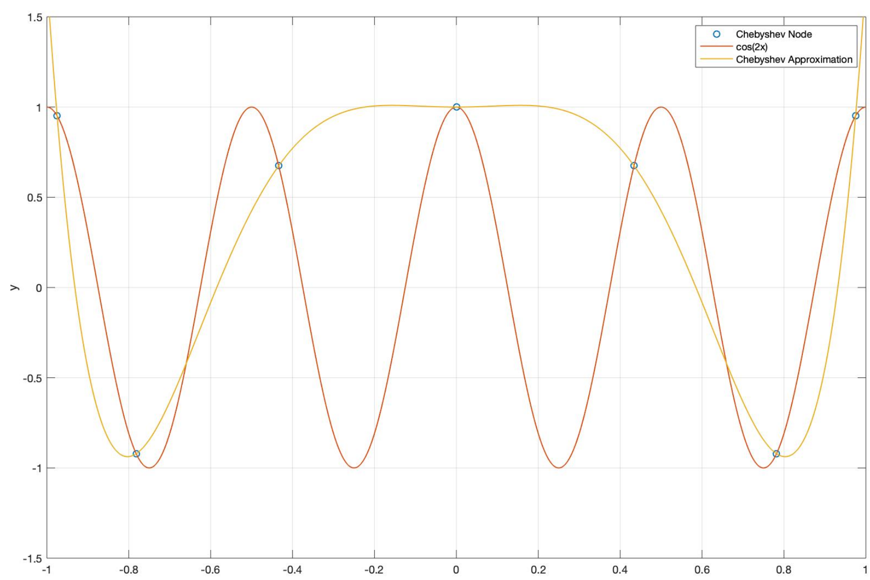
In order to properly calculate the associated y values, I had to subtract half of the first coefficient from the calculation. This is done because the formula I used to calculate the coefficients isn't correct for the first coefficient. The plots for both functions have been added to the next section.

4 Plots

Function $cos(2x)$:



Function $cos(4x)$:



5 Observations

I first calculated the zeros of the polynomial so I could use them as nodes in polynomial interpolation because the resulting interpolation polynomial minimizes the effect of Runge's phenomenon. The functions still display a little bit of Runge's phenomenon.

The Chebyshev approximation for the function $cos(2x)$ is pretty accurate, whereas the approximation for $cos(4x)$ not so much as it doesn't take into account some of the curves of the original function.