Systems of Linear Equations

Exercise 7

Solving a Linear System with LU Decomposition

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1 Introduction

This exercise asks to build a tridiagonal matrix using the following rules with the values -1 on the adjacent upper diagonal, the entries $+1$ on the adjacent lower diagonal, and the values b_i, with $i = 1, \dots, n$ given by

$$b_i = \frac{2(i+1)}{3}, \quad i+1 = 3, 6, 9, \dots$$
$$b_i = 1, \quad i+1 = 2, 4, 5, 7, 8, \dots$$

on the main diagonal. This matrix should then be used as the coefficients matrix in the $A\vec{x} = \vec{y}$ linear system. The exercise asks to solve the system using `GEPP` (Gaussian Elimination with Partial Pivoting) and then give x_1, which should be an approximation of the $e - 2$ value.

As we've seen in class, there are multiple ways of solving a linear system $AX = B$. Assume A is a $n \times n$ square matrix, B is a "constant" term matrix $n \times h$, and X is a $n \times h$ unknown matrix. To solve for X, we could compute the inverse of A and find $x = A^{-1}y$. We've seen that this approach, however, requires more computations than necessary and returns a less accurate result.

In this exercise I am going to use solve a linear system using LU decomposition. This technique, used to represent the matrix A in the form of simpler matrices, L and U (lower triangular and upper triangular matrices, respectively), uses forward substitution (solving for Y from $LY = B$) and backward substitution (solving for X from $UX = Y$). As seen in class, this method is numerically stable (as in, there will be no extra truncation errors). I'll also be calculating the condition number and the error.

2 Tools

The following programming language and libraries have been used in this exercise:

- C
- GSL (GNU Scientific Library)

The following double-precision GSL data types have been used in the exercise:

- `gsl_vector`
- `gsl_matrix`
- `gsl_permutation`

The following GSL methods have been used in the exercise:

- `gsl_matrix_alloc(size1, size2)`
- `gsl_matrix_set_zero(matrix)`
- `gsl_matrix_set(matrix, row, column, value)`
- `gsl_matrix_get(matrix, row, column)`
- `gsl_vector_alloc(size)`
- `gsl_vector_set_zero(vector)`
- `gsl_vector_set(vector, index, value)`
- `gsl_vector_get(vector, index)`
- `gsl_matrix_memcpy(matrixToCopyFrom, matrix)`
- `gsl_linalg_SV_decomp(A, V, S, workspaceVector)`
- `gsl_vector_minmax(vector, minInVector, maxInVector)`

In order to factorize a matrix into the LU decomposition, and then solve the square system $Ax = y$ using the decomposition of A, I've used the following methods:

- `gsl_linalg_LU_decomp(A, permutation, signum)`
- `gsl_linalg_LU_solve(LU, permutation, b, x)`
- `gsl_permutation_alloc(size)`

3 Solving the Linear System

By looking closely at the first rule, we see that the $i+1$ are all multiples of 3 ($i+1 = 3*k$, for some k). Hence the i are of the form $i = 3*k-1$, for some k. For $n = 5$, for example, this is what the coefficient matrix looks like:

$$\begin{bmatrix} 1.000000000e+00 & -1.000000000e+00 & 0.000000000e+00 & 0.000000000e+00 & 0.000000000e+00 \\ 1.000000000e+00 & 2.000000000e+00 & -1.000000000e+00 & 0.000000000e+00 & 0.000000000e+00 \\ 0.000000000e+00 & 1.000000000e+00 & 1.000000000e+00 & -1.000000000e+00 & 0.000000000e+00 \\ 0.000000000e+00 & 0.000000000e+00 & 1.000000000e+00 & 1.000000000e+00 & -1.000000000e+00 \\ 0.000000000e+00 & 0.000000000e+00 & 0.000000000e+00 & 1.000000000e+00 & 4.000000000e+00 \end{bmatrix}$$

The coefficients matrix A is first allocated by using the `gsl_matrix_alloc` method, then I set all the elements to zero with `gsl_matrix_set_zero` and finally nested `for` loops fill the diagonal values by checking the indexes. The coefficients reported above on the diagonal have 5 significant digits for improve the readability of this report.

I used the `gsl_vector_alloc` method to create an instance of the vector. All of its elements were set to zero by using `gsl_vector_set_zero(vector)`. The exercise asks us to set the first element of the y vector to one, so I used `gsl_vector_set(vector, 0, 1)` to assign the value 1 to index 0. For $n = 5$, we have:

$$\vec{y} = \begin{bmatrix} 1.000000000e+00 \\ 0.000000000e+00 \\ 0.000000000e+00 \\ 0.000000000e+00 \\ 0.000000000e+00 \end{bmatrix}$$

Given the $Ax = y$ system, my goal is now to find the vector of the unknowns x. To do so, I first factorize A into its LU decomposition by allocating a new matrix (so that the matrix which represents A doesn't get overridden) using `gsl_matrix_memcpy` and then by calling `gsl_linalg_LU_decomp`. This method utilizes Gaussian Elimination with partial pivoting to compute the decomposition. The following is the LU matrix for $n = 5$:

$$\begin{bmatrix} 1.000000000e+00 & -1.000000000e+00 & 0.000000000e+00 & 0.000000000e+00 & 0.000000000e+00 \\ 1.000000000e+00 & 3.000000000e+00 & -1.000000000e+00 & 0.000000000e+00 & 0.000000000e+00 \\ 0.000000000e+00 & 3.333333333e-01 & 1.333333333e+00 & -1.000000000e+00 & 0.000000000e+00 \\ 0.000000000e+00 & 0.000000000e+00 & 7.500000000e-01 & 1.750000000e+00 & -1.000000000e+00 \\ 0.000000000e+00 & 0.000000000e+00 & 0.000000000e+00 & 5.714285714e-01 & 4.571428571e+00 \end{bmatrix}$$

I can now use the LU matrix to solve the system by passing LU, x, a permutation structure `gsl_permutation` (it contains the order of the indexes of the equations in the system to keep track of swapping) and y to `gsl_linalg_LU_solve`. This method uses forward and back-substitution to modify the contents of the x vector given in input, which now looks like this (for $n = 5$):

$$\vec{x} = \begin{bmatrix} 7.187500000e-01 \\ -2.812500000e-01 \\ 1.562500000e-01 \\ -1.250000000e-01 \\ 3.125000000e-02 \end{bmatrix}$$

Then, I calculate the condition number of the matrix A of order n which will give me a better idea if this is a well-conditioned or an ill-conditioned linear system. In GSL there is no direct function that calculates the condition number, but it's possible to use the ratio of the largest singular value of matrix A, $\sigma_n(A)$, to the smallest $\sigma_1(A)$:

$$\kappa(A) := \frac{\sigma_n(A)}{\sigma_1(A)} = \frac{\|A\|}{\|A^{-1}\|^{-1}}$$

I proceed to factorize A into its singular value decomposition SVD using the `gsl_linalg_SV_decomp` method, and then use `gsl_vector_minmax` to extract the minimum and maximum singular values out of the vector S that contains the diagonal elements of the singular value matrix.

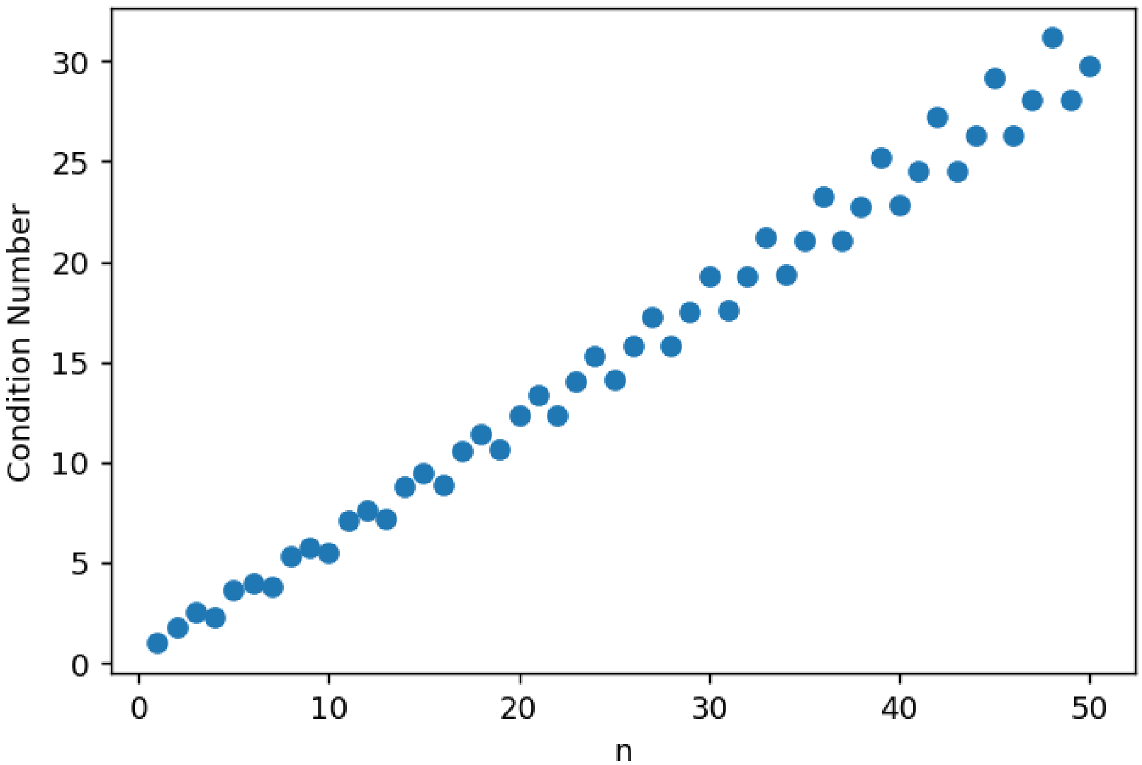
For $n = 5$, the condition number is

$$\kappa(A) = \frac{\sigma_n(A)}{\sigma_1(A)} = \frac{4.205100611e+00}{1.142643287e+00} = 3.680151678e+00$$

I calculate the error by subtracting the computed solution x_1^* from the exact mathematical solution $\widetilde{x}$ (which can be obtained by using the `M_E` GSL constant minus 2).

n	$\widetilde{x}_1$	$x_1^* - \widetilde{x}_1$	$\kappa(A_n)$
1	1.000000000e+00	-2.817181715e-01	1.000000000e+00
2	6.666666667e-01	5.161516179e-02	1.767591879e+00
3	7.500000000e-01	-3.171817154e-02	2.561552813e+00
4	7.142857143e-01	3.996114173e-03	2.258696038e+00
5	7.187500000e-01	-4.681715410e-04	3.680151678e+00
6	7.179487179e-01	3.331105103e-04	3.953864002e+00
7	7.183098592e-01	-2.803069588e-05	3.847674609e+00
8	7.182795699e-01	2.258566572e-06	5.377037588e+00
9	7.182835821e-01	-1.753630507e-06	5.727581839e+00
10	7.182817183e-01	1.101773268e-07	5.498872833e+00
11	7.182818352e-01	-6.746947445e-09	7.100335770e+00
12	7.182818229e-01	5.515095380e-09	7.582164638e+00
13	7.182818287e-01	-2.766507023e-10	7.195531702e+00
14	7.182818284e-01	1.364375279e-11	8.833149892e+00
15	7.182818285e-01	-1.153854789e-11	9.488074730e+00
16	7.182818285e-01	4.816147481e-13	8.911558696e+00
17	7.182818285e-01	-1.998401444e-14	1.057152285e+01
18	7.182818285e-01	1.709743458e-14	1.142018246e+01
19	7.182818285e-01	-6.661338148e-16	1.063813407e+01
20	7.182818285e-01	-1.110223025e-16	1.231319966e+01
21	7.182818285e-01	-2.220446049e-16	1.336883104e+01
22	7.182818285e-01	-2.220446049e-16	1.237107821e+01
23	7.182818285e-01	-2.220446049e-16	1.405700479e+01
24	7.182818285e-01	-2.220446049e-16	1.532862983e+01
25	7.182818285e-01	-2.220446049e-16	1.410816377e+01
26	7.182818285e-01	-2.220446049e-16	1.580226249e+01
27	7.182818285e-01	-2.220446049e-16	1.729630706e+01
28	7.182818285e-01	-2.220446049e-16	1.584809348e+01
29	7.182818285e-01	-2.220446049e-16	1.754855617e+01
30	7.182818285e-01	-2.220446049e-16	1.926975724e+01
31	7.182818285e-01	-2.220446049e-16	1.759006043e+01
32	7.182818285e-01	-2.220446049e-16	1.929561485e+01
33	7.182818285e-01	-2.220446049e-16	2.124756325e+01
34	7.182818285e-01	-2.220446049e-16	1.933353645e+01
35	7.182818285e-01	-2.220446049e-16	2.104325456e+01
36	7.182818285e-01	-2.220446049e-16	2.322873622e+01
37	7.182818285e-01	-2.220446049e-16	2.107816128e+01
38	7.182818285e-01	-2.220446049e-16	2.279134599e+01
39	7.182818285e-01	-2.220446049e-16	2.521256520e+01
40	7.182818285e-01	-2.220446049e-16	2.282368084e+01
41	7.182818285e-01	-2.220446049e-16	2.453979556e+01
42	7.182818285e-01	-2.220446049e-16	2.719852600e+01
43	7.182818285e-01	-2.220446049e-16	2.456991077e+01
44	7.182818285e-01	-2.220446049e-16	2.628853390e+01
45	7.182818285e-01	-2.220446049e-16	2.918622370e+01
46	7.182818285e-01	-2.220446049e-16	2.631671410e+01
47	7.182818285e-01	-2.220446049e-16	2.803750846e+01
48	7.182818285e-01	-2.220446049e-16	3.117535515e+01
49	7.182818285e-01	-2.220446049e-16	2.806398692e+01
50	7.182818285e-01	-2.220446049e-16	2.978667872e+01

4 Plot



5 Observations

The linear system presented in this exercise gets increasingly ill-conditioned as n grows (since $\kappa(A_n) > 1$ for most n). From the plot, it can be observed that the condition number grows linearly. It can be noticed, however, that a large condition number doesn't necessarily mean that the error will be large in all cases, just that it is possible to have a large error. However, it can be observed that as n increases, the error gets incrementally smaller.

The error that I have calculated represents how well the computed solution $\widetilde{x}_1$ approximates the true solution x_1^*. It can be noted that the Gaussian elimination with partial pivoting doesn't introduce any additional truncation errors and therefore it is numerically stable.