Systems of Linear Equations

Exercise 7

Solving a Linear System with LU Decomposition

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October 31, 2019

1 Introduction

In this exercise I am going to use solve a linear system $A\vec{x} = \vec{y}$ using LU decomposition. The goal is to verify that first element of the unknowns vector, x_1, will contain an approximation of $e - 2$.

As we've seen in class, there are multiple ways of solving a linear system $AX = B$. Assume A is a $n \times n$ square matrix, B is a "constant" term matrix $n \times h$, and X is a $n \times h$ unknown matrix. To solve for X, we could compute the inverse of A and find $x = A^{-1}y$. We've seen that this approach, however, requires more computations than necessary and returns a less accurate result.

On the other hand, the LU decomposition technique is a way to represent the matrix A in the form of simpler matrices, L and U (lower triangular and upper triangular matrices, respectively):

$$PA = LU$$

This method uses forward substitution (solving for Y from $LY = B$) and backward substitution (solving for X from $UX = Y$). I'll be specifically solving the system by using Gaussian Elimination with partial pivoting, which reduces round-offs errors compared to its naive implementation. I'll also be calculating the error and the condition number as the variable n increases, and plot the results.

2 Tools

The following programming language and libraries have been used in this exercise:

- C
- GSL (GNU Scientific Library)

The following double-precision GSL data types have been used in the exercise:

- `gsl_vector`
- `gsl_matrix`
- `gsl_permutation`

The following GSL methods have been used in the exercise:

- `gsl_matrix_alloc(size1, size2)`
- `gsl_matrix_set_zero(matrix)`
- `gsl_matrix_set(matrix, row, column, value)`
- `gsl_matrix_get(matrix, row, column)`
- `gsl_vector_alloc(size)`
- `gsl_vector_set_zero(vector)`
- `gsl_vector_set(vector, index, value)`
- `gsl_vector_get(vector, index)`
- `gsl_matrix_memcpy(matrixToCopyFrom, matrix)`
- `gsl_linalg_SV_decomp(A, V, S, workspaceVector)`
- `gsl_vector_minmax(vector, minInVector, maxInVector)`

In order to factorize a matrix into the LU decomposition, and then solve the square system $Ax = y$ using the decomposition of A, I've used the following methods:

- `gsl_linalg_LU_decomp(A, permutation, signum)`
- `gsl_linalg_LU_solve(LU, permutation, b, x)`
- `gsl_permutation_alloc(size)`

3 Solving the linear system

In order to solve the system $Ax = y$, I first need to build the matrix A by understanding how it's build. The requirements are to build a tridiagonal matrix with the values -1 on the adjacent upper diagonal, the entries $+1$ on the adjacent lower diagonal, and on the main diagonal the values b_i, with $i = 1, \dots, n$ given by

$$b_i = \frac{2(i+1)}{3}, \quad i+1 = 3, 6, 9, \dots$$
$$b_i = 1, \quad i+1 = 2, 4, 5, 7, 8, \dots$$

By looking closely at the first rule, we see that the $i+1$ are all multiples of 3 ($i+1 = 3*k$, for some k). Hence the i are of the form $i = 3*k-1$, for some k. For $n = 5$, for example, this is what the matrix looks like:

$$\begin{bmatrix} 1.0000000000 & -1.0000000000 & 0.0000000000 & 0.0000000000 & 0.0000000000 \\ 1.0000000000 & 2.0000000000 & -1.0000000000 & 0.0000000000 & 0.0000000000 \\ 0.0000000000 & 1.0000000000 & 1.0000000000 & -1.0000000000 & 0.0000000000 \\ 0.0000000000 & 0.0000000000 & 1.0000000000 & 1.0000000000 & -1.0000000000 \\ 0.0000000000 & 0.0000000000 & 0.0000000000 & 1.0000000000 & 4.0000000000 \end{bmatrix}$$

The coefficients matrix A is first allocated by using the `gsl_matrix_alloc` method, then I set all the elements to zero with `gsl_matrix_set_zero` and finally nested `for` loops fill the diagonal values by checking the indexes. The coefficients reported above on the diagonal have 5 significant digits for improve the readability of this report.

I used the `gsl_vector_alloc` method to create an instance of the vector. All of its elements were set to zero by using `gsl_vector_set_zero(vector)`. The exercise asks us to set the first element of the y vector to one, so I used `gsl_vector_set(vector, 0, 1)` to assign the value 1 to index 0. For $n = 5$, we have:

$$\vec{y} = \begin{bmatrix} 1.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \\ 0.0000000000 \end{bmatrix}$$

Given the $Ax = y$ system, my goal is now to find the vector of the unknowns x. To do so, I first factorize A into its LU decomposition by allocating a new matrix (so that the matrix which represents A doesn't get overridden) using `gsl_matrix_memcpy` and then by calling `gsl_linalg_LU_decomp`. This method utilizes Gaussian Elimination with partial pivoting to compute the decomposition. The following is the LU matrix for $n = 5$:

$$LU = \begin{bmatrix} 1.0000000000 & -1.0000000000 & 0.0000000000 & 0.0000000000 & 0.0000000000 \\ 1.0000000000 & 3.0000000000 & -1.0000000000 & 0.0000000000 & 0.0000000000 \\ 0.0000000000 & 0.3333333333 & 1.3333333333 & -1.0000000000 & 0.0000000000 \\ 0.0000000000 & 0.0000000000 & 0.7500000000 & 1.7500000000 & -1.0000000000 \\ 0.0000000000 & 0.0000000000 & 0.0000000000 & 0.5714285714 & 4.5714285714 \end{bmatrix}$$

I can now use the LU matrix to solve the system by passing LU, x, a permutation structure `gsl_permutation` (it contains the order of the indexes of the equations in the system to keep track of swapping) and y to `gsl_linalg_LU_solve`. This method uses forward and back-substitution to modify the contents of the x vector given in input, which now looks like this (for $n = 5$):

$$\vec{x} = \begin{bmatrix} 0.7187500000 \\ -0.2812500000 \\ 0.1562500000 \\ -0.1250000000 \\ 0.0312500000 \end{bmatrix}$$

The first element looks contains an approximation of $e-2$. By increasing the size of the matrix n, x_i becomes increasingly more precise. For $n = 10$, for example:

$$\vec{x} = \begin{bmatrix} 0.7182817183 \\ -0.2817182817 \\ 0.1548451548 \\ -0.1268731269 \\ 0.0279720280 \\ -0.0149850150 \\ 0.0129870130 \\ -0.0019980020 \\ 0.0009990010 \\ -0.0009990010 \end{bmatrix}$$

Then, I calculate the condition number of the matrix A of order n which will give me a better idea if this is a well-conditioned or an ill-conditioned linear system. In GSL there is no direct function that calculates the condition number, but it's possible to use the ratio of the largest singular value of matrix A, $\sigma_n(A)$, to the smallest $\sigma_1(A)$:

$$\kappa(A) := \frac{\sigma_n(A)}{\sigma_1(A)} = \frac{\|A\|}{\|A^{-1}\|^{-1}}$$

I proceed to factorize A into its singular value decomposition SVD using the `gsl_linalg_SV_decomp` method, and then use `gsl_vector_minmax` to extract the minimum and maximum singular values out of the vector S that contains the diagonal elements of the singular value matrix.

For $n = 5$, the condition number is

$$\kappa(A) = \frac{\sigma_n(A)}{\sigma_1(A)} = \frac{4.2051006107}{1.1426432872} = 3.6801516779$$

For $n = 10$, the condition number is

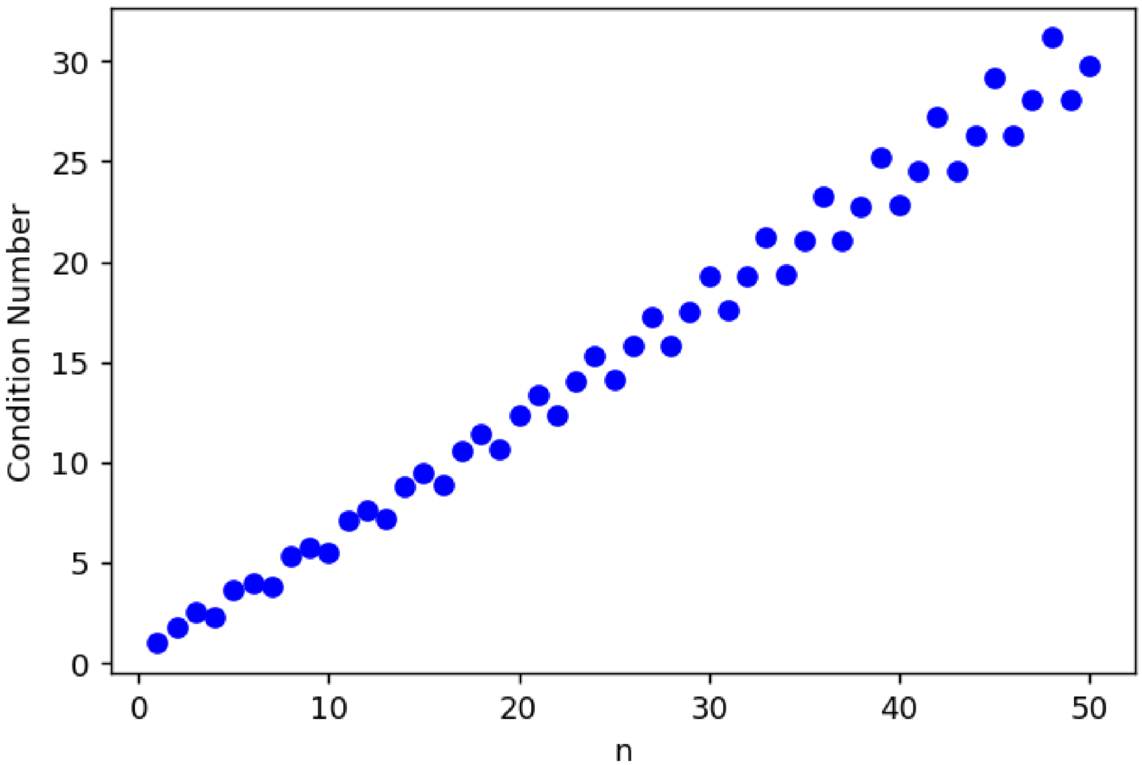
$$\kappa(A) = \frac{\sigma_n(A)}{\sigma_1(A)} = \frac{6.2820508697}{1.1424251953} = 5.4988728328$$

I calculate the error by subtracting the computed solution x_1^* from the exact mathematical solution $\widetilde{x}$ (which can be obtained by using the `M_E` GSL constant minus 2).

Finally, I am now going to insert the x_1 component, the error, and the condition number for n from 1 to 50 in the next page.

n	$\widetilde{x}_1$	$x_1^* - \widetilde{x}_1$	$\kappa(A_n)$
01	1.000000000000000	-0.281718171540955	1.000000000000000
02	0.666666666666667	0.051615161792378	1.767591879243998
03	0.750000000000000	-0.031718171540955	2.561552812808830
04	0.714285714285714	0.003996114173331	2.258696038055887
05	0.718750000000000	-0.000468171540955	3.680151677879533
06	0.717948717948718	0.000333110510327	3.953864002022550
07	0.718309859154930	-0.000028030695884	3.847674609118915
08	0.718279569892473	0.000002258566572	5.377037588047721
09	0.718283582089552	-0.000001753630507	5.727581839289335
10	0.718281718281718	0.000000110177327	5.498872832802596
11	0.718281835205993	-0.000000006746947	7.100335770367996
12	0.718281822943950	0.000000005515095	7.582164637599711
13	0.718281828735696	-0.000000000276651	7.195531702121659
14	0.718281828445401	0.000000000013644	8.833149892375440
15	0.718281828470584	-0.000000000011539	9.488074730049041
16	0.718281828458563	0.000000000000482	8.911558696408528
17	0.718281828459065	-0.000000000000020	10.571522848352377
18	0.718281828459028	0.000000000000017	11.420182460381190
19	0.718281828459046	-0.000000000000001	10.638134074627382
20	0.718281828459045	-0.000000000000000	12.313199658654300
21	0.718281828459045	-0.000000000000000	13.368831043170752
22	0.718281828459045	-0.000000000000000	12.371078212614004
23	0.718281828459045	-0.000000000000000	14.057004794016557
24	0.718281828459045	-0.000000000000000	15.328629826257806
25	0.718281828459045	-0.000000000000000	14.108163769731830
26	0.718281828459045	-0.000000000000000	15.802262487437682
27	0.718281828459045	-0.000000000000000	17.296307059194255
28	0.718281828459045	-0.000000000000000	15.848093475192302
29	0.718281828459045	-0.000000000000000	17.548556166163607
30	0.718281828459045	-0.000000000000000	19.269757243499750
31	0.718281828459045	-0.000000000000000	17.590060434351265
32	0.718281828459045	-0.000000000000000	19.295614845413013
33	0.718281828459045	-0.000000000000000	21.247563252611098
34	0.718281828459045	-0.000000000000000	19.333536447042945
35	0.718281828459045	-0.000000000000000	21.043254557744980
36	0.718281828459045	-0.000000000000000	23.228736221077760
37	0.718281828459045	-0.000000000000000	21.078161279333553
38	0.718281828459045	-0.000000000000000	22.791345987426897
39	0.718281828459045	-0.000000000000000	25.212565199999691
40	0.718281828459045	-0.000000000000000	22.823680837817037
41	0.718281828459045	-0.000000000000000	24.539795564146115
42	0.718281828459045	-0.000000000000000	27.198526001362339
43	0.718281828459045	-0.000000000000000	24.569910774893234
44	0.718281828459045	-0.000000000000000	26.288533903235187
45	0.718281828459045	-0.000000000000000	29.186223702087091
46	0.718281828459045	-0.000000000000000	26.316714104617667
47	0.718281828459045	-0.000000000000000	28.037508462052152
48	0.718281828459045	-0.000000000000000	31.175355146341044
49	0.718281828459045	-0.000000000000000	28.063986916895736
50	0.718281828459045	-0.000000000000000	29.786678720210102

4 Plot



5 Observations

The linear system presented in this exercise gets increasingly ill-conditioned as n grows (as $\kappa(A_n) \geq$ for most n). From the plot, it can be observed that the condition number grows in a linear fashion. It can be noticed, however, that a large condition number doesn't necessarily mean that the error will be large in all cases, just that it is possible to have a large error. The linear system appears to be ill-conditioned for most n as the condition number gets increasingly bigger = than 1. However, it can be observed that as n increases, the $\widetilde{x}_1$ component gets incrementally closer to the actual $e - 2$ value.

This error I have calculated represents how well the computed solution $\widetilde{x}_1$ approximates the true solution x_1^*, even though I operated in double-precision and there have been truncation errors when calculating $e - 2$ and $\widetilde{x}_1$. It can be noted that the Gaussian elimination with partial pivoting doesn't introduce any additional truncation errors and therefore it is numerically stable.