Title: MVSK Portfolio Optimisation with the PGP Algorithm for Carbon-Efficient Assets

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Abstract:

This research explores the Mean-Variance-Skewness-Kurtosis (MVSK) framework of portfolio optimisation, which is an extension to the more traditional mean-variance model to include the third and fourth moments (Skewness and Kurtosis) of returns. To deal with these types of portfolios, we invoke the Polynomial-Goal-Programming (PGP) algorithm, which allows the incorporation of multiple conflicting objectives and investor preferences into one objective function. The paper also introduces two risk-adjusted performance metrics, the Omega Ratio, and the Stutzer Index to enhance portfolio evaluation tailored to the MVSK framework.

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1- Introduction

In this paper, we will focus on the Mean-Variance-Skewness-Kurtosis (MVSK) framework of portfolio optimisation. This approach goes beyond the conventional mean-variance model developed by Markowitz (1952), and it is part of the more recent field of modern portfolio theory. While the standard model uses only the first two moments of returns, this innovative approach considers the third and fourth moments, skewness and kurtosis. In doing so, it aims to provide investors with a better understanding of risk, return and its trade-offs, especially in cases where returns are not normally distributed. We start by navigating the initial work by Markowitz and then inspect the complexities of this new approach which is characterised by non-convex and non-linear challenges. To address these complexities, we introduce the Polynomial Goal Programming (PGP) technique. This enables us to incorporate multiple objectives while considering what investors prefer in terms of maximising returns or accommodating risk aversion, and whether they wish to avoid negative skewness and the likelihood of extreme events. We introduce two appropriate risk-adjusted performance measures to enhance portfolio evaluation tailored for the MVSK portfolio, the Omega Ratio, and the Stutzer Index. The empirical analysis reveals optimal asset allocations based on different investor preferences and risk tolerance.

The aim of the paper would be to provide the investor or portfolio manager with an effective way to gain insights into MVSK portfolio optimisation. The rest of this paper is structured as follows: Section 2 looks at the literature review since the work of Markowitz. Sections 3 and 4 describe what methodology and data are used to create this analysis. Section 5 displays the empirical analysis. A summary of the results is given in Section 6.

2- Literature Review

An extensive amount of literature is available on portfolio construction methods. These studies rely on the radical work of Markowitz (1952), who developed the mean-variance efficiency framework by optimally allocating capital across risky assets. The mean-variance frontier can be described as a way to characterise the relationship between the expected returns (minus the risk-free rate) over a specific amount of risk (variance). The portfolio with the maximum reward-to-risk is equivalent to the 'tangency portfolio' on the capital allocation line (CAL). The slope of the CAL can also be addressed as the Sharpe ratio, as established by Sharpe (1967). As a result, the larger the ratio is, the greater the reward-to-risk is for the selected portfolio. However, one of the major drawbacks of the conventional mean-variance approach is that it relies on the assumption that asset returns are normally distributed. This is not so accurate in practice (Mandelbrot, 1963), (Fama, 1965). Asymmetry and heavy tails do occur frequently in returns distribution and should be addressed by studying higher moments such as skewness and kurtosis (Keel et al, 2006). Furthermore, Samuelson (1970) proved that the mean-variance efficient frontier becomes insufficient when the investment decision is based on a finite time period. A newer approach by Scott and Horvath (1980) examined the application of higher-order moments in portfolios and found that if an investor has a preference for positive skewness and negative kurtosis, its utility function marginally increases. Similarly, Lai (1991) had similar findings, considering the importance of including skewness in portfolio selection, proving that investors might have preferences for positive skewness. In line with this analysis, Maroua & Prigent (2010) promote the use of the Mean-Variance-Skewness-Kurtosis (MVSK) and conclude that the incorporation of higher moments causes a major positive change in the resulting portfolio.

When dealing with skewness and kurtosis, the portfolio optimisation process becomes more complex and challenging. When only the returns and variance are included, the problem is

convex and linear. However, in the MVSK approach, the portfolio optimisation turns into non-convex and non-linear structure. The way this optimisation problem is dealt with is by incorporating the Polynomial Goal Programming (PGP) technique. The PGP allows the incorporation of multiple conflicting and competing objective functions together, while employing investors' preferences into the equation. This clever algorithm was first introduced by Tayi & Leonard (1988) to help facilitate banks' balance sheet management with multiple competing objectives. Thereafter, it was implemented by Lai (1991), Chunhachinda et al. (1997), Leung et al. (2001), Sun and Yan (2003), Prakash et al. (2003) and others to solve portfolio selection with mean, variance, and skewness only. Lai (2006) has then extended the model from mean-variance-skewness to mean-variance-skewness-kurtosis. The usefulness of the MVSK portfolio has proven effective since its inception. More recently, Davies et al. (2009) made use of the MVSK procedure to develop a technique for hedge funds to allocate capital across different hedge fund strategies and traditional asset classes. Boudt et al. (2019) have built a tilting algorithm which converts a standard mean-variance portfolio into an MVSK efficient portfolio by increasing the mean and skewness while decreasing the variance and kurtosis, with the only disadvantage of deviating from the initial optimal criteria. Although the MVSK has many benefits and provides a comprehensive understanding of asset returns and risk, it presents some challenges in terms of computational complexity. With the inclusion of third and fourth moments, the calculations of the gradients for these moments increase considerably as more portfolios are taken into account (Abid et al., 2023). To overcome this problem, several optimisation approaches have been analysed, ranging from stochastic algorithms such as Differential Evolution (Maringer & Parpas, 2009) to genetic algorithms (Kshatrya & Prasanna, 2018). More recently, a different method developed by Zhou et al. (2021) is based on the successive convex approximation (SCA) algorithm and appears to be much more efficient in solving higher-order portfolios.
Moreover, while the MVSK has been proven effective in many circumstances and applications, previous research primarily concentrated on the formulation and optimisation of the model, often neglecting a comparative analysis through the use of appropriate risk-adjusted metrics. This research paper addresses this lack of comparative analysis by introducing the Omega Ratio and Stutzer Index to the portfolios to facilitate a comprehensive analysis of the optimal MVSK portfolio suggested by the PGP algorithm. Traditional performance benchmarks such as the Sharpe Ratio approximate the returns distribution with the mean and the standard deviation. As aforementioned, this assumes that returns are normally distributed, thereby discarding the effects of skewness and kurtosis. Measures like the Omega Ratio, however, are able to capture all information in the return's distribution as it incorporates higher moment effects in its calculation (Keating & Shadwick 2002). Likewise, the Stutzer Index, introduced by Stutzer (2000), attempts to estimate the rate at which the probability of the portfolio underperforming a benchmark will decay to zero. In addition, it does not assume that returns are normally distributed and penalises negative skewness and high kurtosis. These features make those metrics a good candidate to be used in our comparative analysis of optimal MVSK portfolios. The next section of this paper will examine in detail how the MVSK portfolio optimisation model was constructed, and the methodology applied to evaluate the optimal portfolios.

3 - Methodology

In this section, we discuss the PGP algorithm for computing the MVSK framework. If we consider $R = (R_1, R_2, R_3, \dots, R_n)$ being the distribution of the returns in our portfolio and $\overline{R}$ the mean of those returns, then $X^T = (x_1, x_2, x_3, \dots, x_n)$ is the weight vector where x_i is the proportion of capital allocated to the ith asset. Furthermore, we computed the Ledoit-Wolf shrinkage for the variance-covariance matrix estimation as it effectively improves covariance estimation by incorporating prior information, which results in more stable and accurate estimates (Ledoit and Wolf, 2001). Lastly, S is the skewness-coskewness and K is the kurtosis-cokurtosis matrices of R. The first four moments of R are computed as follows:

$$Mean = R(x) = X^T\overline{R} = \sum_{i=1}^{n} x_i R_i \tag{1}$$

$$Variance = V(x) = X^T X V = \sum_{Shrink} = \widehat{\delta^*} * F + (1 - \widehat{\delta^*})S \tag{2}$$

$$Skewness = S(x) = E(X^T(R - \overline{R}))^3 = \sum\nolimits_{i=1}^{n} x_i^3 s_i^3 + 3\sum_{i=1}^{n}\left(\sum_{j=1}^{n} x_i^2 x_j s_{iij} + \sum_{j=1}^{n} x_i x_j^2 s_{ijj}\right)(i \neq j) \tag{3}$$

$$Kurtosis = K(x) = E(X^T(R - \overline{R}))^4 = \sum\nolimits_{i=1}^{n} x_i^4 k_i^4 + 4\sum_{i=1}^{n}(\sum_{j=1}^{n} x_i^3 x_j k_{iiij} + \sum_{j=1}^{n} x_i x_j^3 s_{ijjj}) + 6\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^2 x_j^2 k_{iijj}(i \neq j) \tag{4}$$

The PGP algorithm allows us to optimise multiple objectives at the same time. The goal is to maximise returns and skewness while minimising variance and kurtosis. For the model to run this optimisation, we must provide the aspired levels of each moment, which define the ideal scenario for setting the optimal portfolio. This model can be represented as follows:

$$(A1)\begin{cases} \text{Maximize } R^*(X) = X^T R \\ \text{Minimize } V^*(X) = X^T V X \\ \text{Maximize } S^*(X) = E(X^T(R - \overline{R}))^3 \\ \text{Minimize } K^*(X) = E(X^T(R - \overline{R}))^4 \\ \text{subject to:} \\ \qquad X^T I = 1 \\ \qquad X \geq 0 \end{cases} \tag{5}$$

An efficient way to solve A1 is by consolidating each function into a single combined objective function. This allows us to utilise the PGP to solve these objectives simultaneously. To achieve this, the final objective can be defined as minimising the deviations from the ideal scenario, where the ideal scenario is set by the aspired levels. To measure the deviations from each moment from their respective aspired levels, we make use of goal variables. By denoting these deviations as d_1, d_2 , d_3 and d_4 and calling the aspired levels R^*, V^*, S^* and K^* we can solve four sub-problems (A2, A3, A4, and A5) as shown below. These sub-problems are optimised in a way that each aspired level represents the ideal scenario independently of each other.

$$(A2)\begin{cases}\text{Maximize } R^*(X) = X^T R \\ \text{subject to:} \\ \qquad X^T I = 1 \\ \qquad X \geq 0 \end{cases} \tag{6}$$

$$(A3)\begin{cases}\text{Minimize } V^*(X) = X^T V X \\ \text{subject to:} \\ \qquad X^T I = 1 \\ \qquad X \geq 0 \end{cases} \tag{7}$$

$$(A4)\begin{cases}\text{Maximize } S^*(X) = E(X^T(R-\overline{R}))^3 \\ \text{subject to:} \\ \qquad X^T I = 1 \\ \qquad X \geq 0 \end{cases} \tag{8}$$

$$(A5)\begin{cases}\text{Minimize } K^*(X) = E(X^T(R-\overline{R}))^4 \\ \text{subject to:} \\ \qquad X^T I = 1 \\ \qquad X \geq 0 \end{cases} \tag{9}$$

Those sub-problems will be solved by using both linear and non-linear programming techniques. To deal with the non-linearity of subproblems A4 and A5 we utilise the Sequential Least Squares Quadratic Programming (SLSQP) (Kraft et al., 1988). Once the aspired levels are computed the final problem will be to specify the objective function in the PGP. Spronk (1981) suggests that we invoke the Minkovski distance, which is described below.

$$Z = \left(\sum_{k=1}^{m} \left| \frac{d_k}{Z_k} \right|^p \right)^{\frac{1}{p}} \tag{10}$$

If we modify the Minkovski distance to address our particular situation, we normalise the deviations by dividing them by the respective aspired levels. This normalisation enables us to compare and weigh deviations relative to their best-case scenario, independently for each moment. We can now bring into consideration the investors' preferences $\lambda_1, \lambda_2, \lambda_3$ and λ_4. Those parameters express the investors' relative significance for different portfolio moments in the optimisation process. Therefore, if we take each normalisation within the Minkovski distance and raise it to the power of λ_i where each i corresponds to specific moments, we can effectively incorporate the investors' preferences in the optimisation outcome. The final PGP model (A6) will be equal to:

$$(A6)\begin{cases}\text{Minimize } Z = \left|\frac{d_1}{R^*}\right|^{\lambda_1} + \left|\frac{d_2}{V^*}\right|^{\lambda_2} + \left|\frac{d_3}{S^*}\right|^{\lambda_3} + \left|\frac{d_4}{K^*}\right|^{\lambda_4} \\ \text{subject to:} \\ \qquad X^T\overline{R} + d_1 = R^* \\ \qquad X^T V X - d_2 = V^* \\ \qquad E(X^T(R-\overline{R}))^3 + d_3 = S^* \\ \qquad E(X^T(R-\overline{R}))^4 - d_4 = K^* \\ \qquad X^T I = 1 \\ \qquad 5 \leq X \leq 75 \\ \qquad d_i \geq 0, \; i = 1,2,3,4 \end{cases} \tag{11}$$

A minor amendment from Lai et al. (2006) was to set the final weights of the PGP, X, between 5% and 75% to ensure that the optimisation process maintains a balance diversification among of all the assets at our disposal. Moreover, to facilitate the comparison between the different outcomes of the PGP we introduce the two key risk-adjusted performance metrics, the Omega Ratio, and the Stutzer Index. The Omega Ratio can be defined as:

$$\Omega = \frac{\int_r^b (1 - F(X))\, dX}{\int_a^r F(X)\, dX} \tag{12}$$

where F is the cumulative probability distribution of the returns, and r is the target return threshold rate. The ratio can be expressed as the probability-weighted ratio of gains versus losses for a specific threshold rate. On the other hand, the Stutzer Index expressed as:

$$I_p = max_\theta \left[-log \left(\frac{1}{T} \sum_{t=1}^{T} e^{\theta r_t} \right) \right] \tag{13}$$
$$Stutzer = \frac{|\overline{R}|}{\overline{R}} \sqrt{2 I_p}$$

here the r_t is the excess returns, I_p is the information statistic, θ is the parameter that maximises I_p and $\overline{R}$ is the average excess return. This measure estimates the speed at which the probability of underperformance decays, with higher values indicating a better performance.
Having defined the methodology of this research we now turn to the next section where we describe what type of data has been used and what are its characteristics.

4 – Data Description

In this section, we describe the data collection procedure and its sources. The research data in this thesis is composed of five indices, namely:

1- S&P 500 Carbon Efficient Index,
2- S&P Europe 350 Carbon Efficient Index,
3- S&P Asia Pacific LargeMidCap Carbon Efficient Index,
4- S&P China A Carbon Efficient Index and
5- S&P Latin America LargeMidCap Carbon Efficient Index.

The S&P Carbon Efficient Index series aims to integrate low carbon consideration into easily accessible market indices. The methodology of each index is to over-weight companies with lower levels of carbon emissions and under-weight companies with higher levels per unit of revenue. The constituents of the indices are assigned an annual carbon to revenue footprint, which is defined by the company's annual greenhouse gas (GHG) emission, divided by annual revenues[1].
Each index is constructed based on the constituents' country of domicile while its benchmark is the respective standard index. For example, the benchmark for the S&P 500 Carbon Efficient

[1] For more information on the S&P Carbon Efficient Index family refer to the document below: spglobal.com/spdji/en/documents/methodologies/methodology-sp-global-carbon-efficient-index-series.pdf

Index is the standard S&P 500 (SPX) and then the relative under weighting and over weighting is applied. The returns from each index are taken from the Bloomberg terminal. However, the data is also accessible from the S&P website at: spglobal.com/spdji/en/index-family/esg/esg-climate/carbon-efficient. For simplicity, each index name has been shortened by its location. The datasets span from the 30th Aug 2012 to the 30th Dec 2022, just slightly over 10 years' worth of data. We compute the weekly log returns, as shown in Equation (14) below, and we arrive at a total of 475 observations.

$$R_t = ln\,(\frac{p_t}{p_{t-1}}) \tag{14}$$

The choice of this dataset is driven by the ongoing interest from both individual and institutional investors in green investments. The selected indices are created from a reliable source, and they can be conveniently utilised by investors who desire to construct a diversified portfolio of carbon-efficient equities. Table 1 below outlines some important characteristics of indices' returns.

Table 1 Returns statistics from the five funds of the portfolio

Index	Mean (%)	StDev	Skewness	Kurtosis	Jarque-Bera Value	p-value
United States	0.0901	0.0222	-1.2045	5.6331	724.7011	0
Europe	0.0041	0.0261	-0.8277	10.0074	1985.3799	0
Asia Pacific	0.0056	0.0223	-0.9995	5.3591	631.1391	0
China	0.0151	0.0344	-1.0533	5.8305	741.5605	0
Latin America	-0.0807	0.0390	-0.5267	4.2599	370.5708	0

The table shows several descriptive statistics calculated from the returns of the five assets used in the research. To address whether the distribution of returns is normally distributed or not, the Jarque-Bera test is performed. The respective p-values are calculated at the 5% significance levels and the critical value is equal to 5.99.

All funds except the Latin America index have positive average returns. The United States has the largest average rate of return (9.01%), followed by China (1.51%). Both Asia Pacific (0.56%) and Europe (0.41%) have negligible positive returns. Skewness is negative for all the five indices while kurtosis is relatively high, especially for the European Index. In terms of the Jarque-Bera test, the results reject the null hypothesis, in each case, that the sample data have skewness and kurtosis matching a normal distribution. Therefore, it is statistically fair to say that the returns are not normally distributed. The next section of the paper will outline the empirical results obtained from running the multi-objective optimisation through the PGP algorithm.

5 - Empirical Analysis

The main objective of this study is to calculate how much capital to allocate to each index in our portfolio, based on a set of investors' preferences. The first set of computations in the model is to arrive at the aspired levels of return, variance, skewness, and kurtosis. Table 2 below reveals these metrics.

Table 2 The aspired moments for returns, variance, skewness, and kurtosis

Sub-Problems	R^*	V^*	S^*	K^*
Apired Moments	0.0901	0.0867	-0.0413	0.1651

Each value corresponds to the results derived from the independent optimisations. Maximising R^*, minimising V^*, maximising S^* and minimising K^*.

The table above illustrates the optimal portfolio moments independently of each other. As expected, the aspired level obtained when maximising the returns is equal to the average rate

of return from the best-performing index, the United States. As shown in Fig 1, this is equivalent to allocating 100% to this only index. For the portfolio minimising the variance, the weighting is more balanced, 28% in the United States, 27.6% in Asia Pacific, 22% in Europe and the rest between China and Latin America. As for the portfolio maximising skewness, Fig. 1 shows that 30% is allocated to Latina America, 23.8% to the United States, and no more than 20% to other assets, and following Table 2 this allocation corresponds to a maximum level of skewness of -0.0413. For the final portfolio, by minimising the kurtosis, the allocation appears to be somewhat equally weighted, slightly overweighting the Asia Pacific and the United States, two of the indices with a low kurtosis value.
The next step in the process is to feed the calculated aspired levels into the PGP model and let the optimisation run on a set of investors' preferences. The output will be equal to a desired level of asset allocation, and given a set of preferences, which minimises the deviations from the aspired portfolio moment.

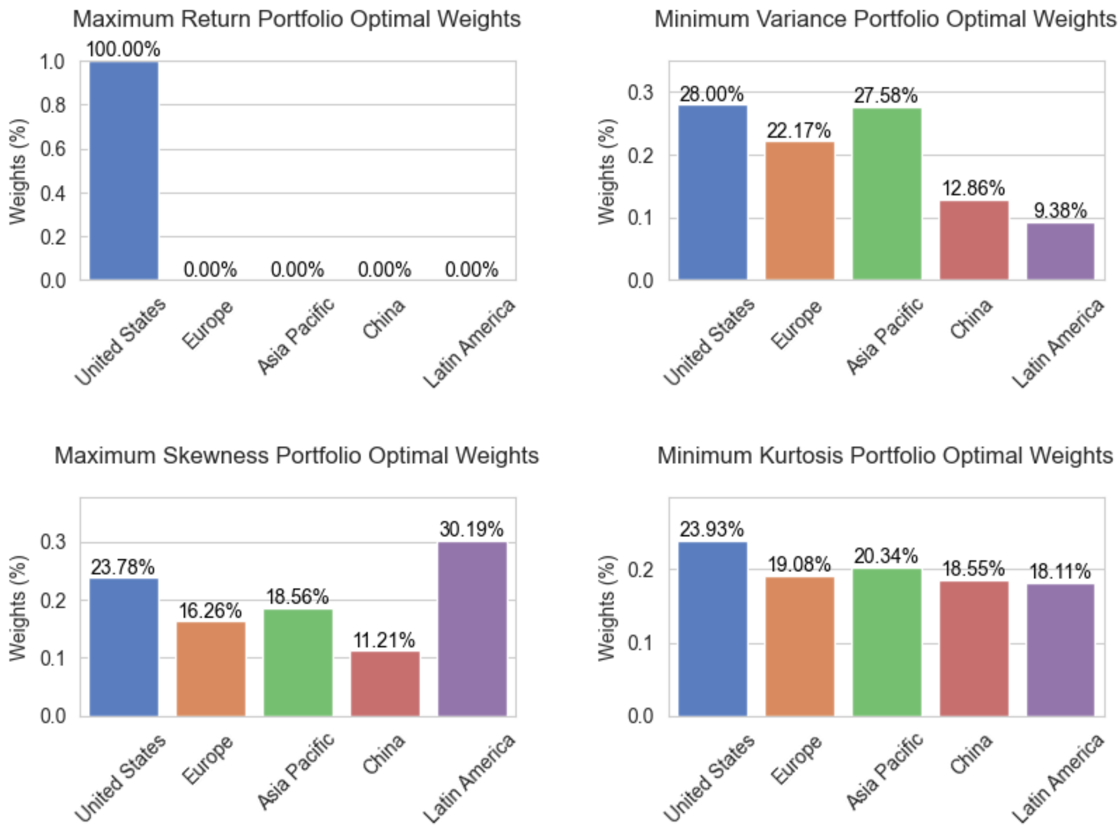

Fig. 1 Optimal allocations for the four portfolio moments

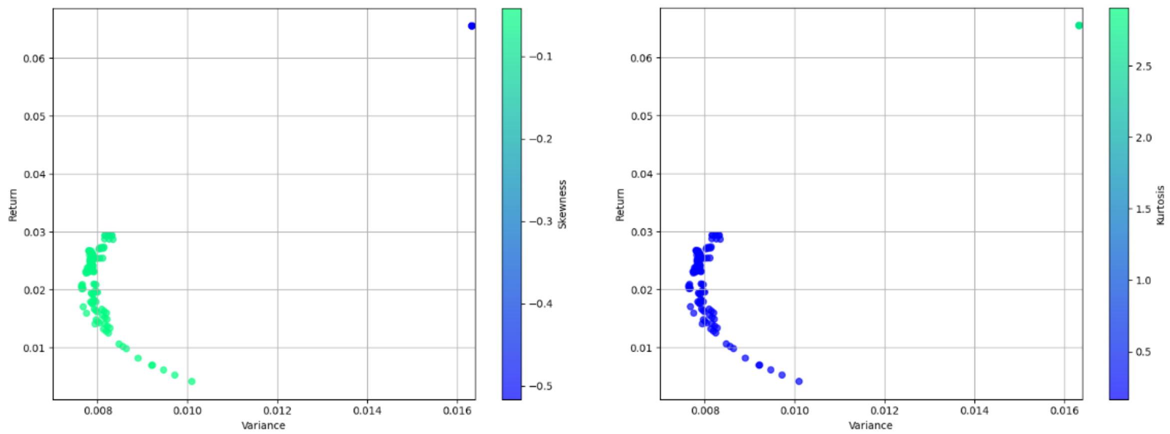

Fig. 2 Efficient Frontier of the MVSK portfolio calculated from the PGP algorithm
The frontier on the left has the skewness as a colour map. The frontier on the right instead has the kurtosis as a colour map.

To analyse the effect of preference on portfolio selection, different combinations were analysed. Specifically, λ_1 and λ_2 could take values [1, 2, 3] while λ_3 and λ_4 could take values [0, 1, 2, 3]. This is equivalent to 144 portfolios with different combinations of preferences.
From the 144 portfolios, the following set of preferences are analysed in the results below: (1, 1, 0, 0), (1, 1, 0, 3), (1, 1, 3, 0), (1, 2, 3, 2), (1, 3, 0, 1), (1, 3, 1, 3), (2, 3, 3, 1), (3, 1, 1, 0), (3, 1, 2, 3) and (3, 1, 3, 1). The structure of the preferences is as follows: the set (1, 1, 0, 0) is the benchmark case, which represents the Markowitz portfolio, where investors wish to maximise returns and minimise variance. For this reason, λ_1 and λ_2 do not take a value of zero. Moreover, the sets (3, 1, 1, 0), (3, 1, 2, 1) and (3, 1, 3, 1) denote strong preferences to maximise returns without much consideration of the risk level. On the other hand, preferences (1, 3, 1, 1), (1, 1, 1, 3) and (1, 3, 1, 3) tend to give more importance to the risk levels. However, other types of preference sets, such as (1, 2, 3, 2), (3, 1, 2, 3) and (2, 3, 3, 1), seek to optimise multiple objectives at once.
The efficient frontier from the PGP portfolios is depicted in Fig 2. The x-axis represents the risk (i.e., volatility), and the y-axis is the expected return. On the left, the skewness is shown as a colour map, while on the right, the kurtosis is shown as a colour map. One of the first things that stand out from this figure is the portfolio at the top-right corner, the Markowitz portfolio when $\lambda = (1, 1, 0, 0)$. We can already establish that based on its position on the frontier, the investor must give up a large portion of gains when including higher moments in the portfolio, such as skewness and kurtosis. As the inclusion of higher moments in the optimisation problem is considered, the challenge becomes more complex and, as a result, this leads to these portfolios having much lower expected returns compared to the Markowitz portfolio. The trade-off in this case seems to be extensively high when investors seek to enhance their unique risk-return profile by maximising skewness and minimising kurtosis. Nevertheless, this does not mean that the other portfolios should be dismissed. While the Markowitz portfolio may offer higher expected returns compared to other portfolios, it is important to note that investor preferences differ. Some investors may still prefer portfolios on the lower part of the frontier, given their unique risk-level profiles.
To further analyse the results from the PGP algorithm, Table 3 below provides asset allocations and moment statistics from the portfolios mentioned above.

Table 3 PGP Optimal portfolios and moments statistics

Portfolio	A	B	C	D	E	F	G	H	I	J
$\lambda 1$	1	1	1	1	1	1	2	3	3	3
$\lambda 2$	1	1	1	2	3	3	3	1	1	1
$\lambda 3$	0	0	3	3	0	1	3	1	2	3
$\lambda 4$	0	3	0	2	1	3	1	0	3	1
United States	75.00%	30.55%	27.97%	26.76%	25.79%	24.90%	26.33%	25.49%	26.69%	26.71%
Europe	5.00%	22.20%	27.70%	24.25%	19.61%	16.51%	20.66%	28.75%	26.99%	21.34%
Asia Pacific	5.00%	21.49%	24.29%	23.26%	20.15%	19.33%	21.10%	25.58%	24.74%	21.69%
China	10.00%	20.76%	15.03%	18.05%	19.41%	12.13%	18.66%	15.18%	16.58%	18.43%
Latin America	5.00%	5.00%	5.00%	7.67%	15.04%	27.13%	13.25%	5.00%	5.00%	11.83%
Mean	0.0655	0.0287	0.0259	0.0230	0.0160	0.0041	0.0179	0.0238	0.0250	0.0194
Variance	0.0163	0.0082	0.0078	0.0078	0.0082	0.0101	0.0080	0.0079	0.0078	0.0079
Skewness	-0.5162	-0.0633	-0.0538	-0.0531	-0.0510	-0.0418	-0.0514	-0.0513	-0.0527	-0.0521
Kurtosis	2.6493	0.2162	0.2152	0.1906	0.1685	0.1926	0.1718	0.2148	0.2082	0.1757

One of the main aspects of the table is that we notice different portfolio moments based on unique investors' preferences. The mean-variance portfolio (A) has the highest expected return. This follows the view that such portfolio must dominate any other portfolio given the same level of risk. On the other hand, portfolio G, which has a strong emphasis on variance and skewness, achieves moderately low variance and a milder negative skewness. Similarly, portfolio F, which prioritises variance and kurtosis, manages to control these moments but to a lesser degree. However, it achieves this by foregoing higher returns to manage the risk and reduce heavy tails. Among the remaining portfolios, interestingly portfolio D and portfolio I deviate from the expected results. Portfolio D, which aims to maximise skewness and have low kurtosis, achieves relatively low skewness and a higher value of kurtosis than other portfolios where the preference for minimising kurtosis is higher. In addition, portfolio I which places a strong preference on kurtosis fails to reach a relatively low value. As a result of these findings, we transition to the next stage of the empirical results section.

5.1 – Omega Ratio Analysis

As aforementioned, when dealing with higher portfolio moments it is not suitable to use common risk-adjusted performance metrics such as the Sharpe Ratio. Instead, we can use the Omega Ratio, which is in this case better suited to our needs. As reported in Table 4, the Omega ratio has been calculated for the 10 portfolios analysed in the previous section. Given that the Omega ratio requires a minimum threshold to be computed, we evaluated the ratio with different threshold levels, ranging from 0% to 3%. Portfolio A is the one with the highest Omega ratio at all thresholds. This is a one-case scenario where investors are neglecting higher moments in their portfolios. Moreover, Portfolio C succeeds in sustaining its omega ratio fairly steadily with the increasing threshold up to 3%. On the other hand, portfolio E struggles to maintain a positive ratio as the threshold increases. Perhaps, this instance underscores the challenges of mitigating risk mainly through variance minimisation. Additionally, portfolio J, which gives a high weight to returns and skewness manages to mitigate the rate of decline in the omega ratio quite well compared to other portfolios. This might be due to the potential usefulness of skewness in stabilising returns during negative periods. As can be seen, while the Omega ratio offers an insight into how the optimal portfolio performs based on gains versus losses for different thresholds, it should not be taken into consideration on its own. Instead, it can be used as an additional metric for the investor to choose their optimal allocation when taking into account skewness and kurtosis. Let us now turn into the next section of the paper where we add an additional measure that can be taken into consideration in the MVSK portfolios optimisation.

Table 4 Omega ratios at different thresholds for the optimal portfolios

Portfolio	λ1	λ2	λ3	λ4	Omega Ratio Ω						
					0%	0.50%	1%	1.50%	2%	2.50%	3%
A	1	1	0	0	78.85	74.91	63.64	53.33	42.17	31.61	23.18
B	1	1	0	3	31.18	25.52	18.76	12.56	7.08	2.60	-0.74
C	1	1	3	0	29.31	27.62	16.60	10.35	4.96	3.26	-2.43
D	1	2	3	2	24.64	21.02	12.79	7.17	2.36	0.02	-4.09
E	1	3	0	1	15.69	10.56	5.38	0.81	-2.99	-5.79	-7.72
F	1	3	1	3	3.80	19.90	-4.88	-8.39	-11.04	-2.11	-13.56
G	2	3	3	1	18.01	14.35	7.28	2.45	-1.59	-3.64	-6.77
H	3	1	1	0	26.90	21.48	14.36	8.33	3.19	-0.57	-3.68
I	3	1	2	3	27.95	24.35	15.45	9.37	4.14	1.54	-2.97
J	3	1	3	1	19.86	16.01	8.83	3.79	-0.46	-2.75	-5.99

5.2 – Stutzer Index Analysis

In a similar format as the Omega Ratio, Table 5 below illustrates the Stutzer Index values which have been computed for the optimal portfolios in question. For the Stutzer Index to be calculated, it requires a benchmark against which the portfolio returns are compared. The benchmark was calculated by first taking into consideration the returns of individual benchmarks for each asset in the portfolio. Subsequently, an equal-weighted return was computed from the returns of all five benchmarks. The Stutzer index gives an estimate of how quickly the probability of underperforming the benchmark decays to zero. Therefore, a larger number is preferred. For instance, the Stutzer index for Portfolio A is equal to 0.3482 which is a favourable adjusted performance measure compared to the calculated benchmark. Similarly, portfolio B Stutzer Index is significantly positive (0.00609) but lower than Portfolio A. The impact of focusing on maximising returns, minimising variance, and strongly preferring to minimise kurtosis yields a portfolio which effectively manages the risk of underperforming its benchmark. This could be explained by the preference of the Stutzer Index to penalise portfolio with high kurtosis and therefore aligns well with portfolio B. Similarly, portfolio C achieves a positive (0.00059) value, but considerably lower value than portfolio B. The clear focus of portfolio C is to maximise skewness while moderately focusing on returns and variance and no preference to minimise kurtosis. Furthermore, in addition to favouring portfolios with low kurtosis, the Stutzer index effectively captures positive skewness. Consequently, portfolio C, with a strong emphasis on maximising the third moment attains a satisfactory adjusted performance measure against the benchmark. Moreover, portfolio I which leans towards strong preferences for returns, skewness and kurtosis also achieves a moderately positive Stutzer Index (0.00052). These implications follow those from portfolios B and C. The preference in maximising skewness and minimising kurtosis simultaneously implies in this instance a limited impact on decaying the probability to underperform. Finally, portfolios D, E, F, G, H, and J exhibit a Stutzer Index of zero. This result could be attributed to several factors like the inability of these portfolios to effectively decay the probability of underperforming the benchmark. In addition, as the Stutzer index penalises portfolios with negative skewness and high kurtosis, those portfolios might not manage those moments adequately, resulting in a lower or zero value. It is important to note that portfolios D, F, F and G have an elevated preference towards minimising variance. The inclusion of a high value of λ_2 may lead to conflicts with preferences for skewness and kurtosis, as the primary objective is to minimise risk. This, in turn, could potentially compromise the Stutzer Index.

Portfolio	λ1	λ2	λ3	λ4	Stutzer Index
A	1	1	0	0	0.03482
B	1	1	0	3	0.00609
C	1	1	3	0	0.00059
D	1	2	3	2	0
E	1	3	0	1	0
F	1	3	1	3	0
G	2	3	3	1	0
H	3	1	1	0	0
I	3	1	2	3	0.00052
J	3	1	3	1	0

Table 5 Stutzer Index values for the optimal portfolios

The Stutzer Index provides a valuable view of risk-adjusted performance for non-normally distributed returns. Investors can unveil useful insights regarding the risk-return trade-offs and higher moments optimisation. The different combination of investors' preferences underlines the complexity of dealing with higher-order portfolio optimisation. Nonetheless, both the Omega Ratio and the Stutzer Index can help the investor, or the portfolio manager navigate the intricacy of investment management.

Conclusion

In this paper, we have analysed how the PGP model can be used to compute optimal portfolios in the mean-variance-skewness-kurtosis approach based on a set of investors' preferences. The analysis shows that in general, the PGP handles well multiple objectives simultaneously, and the portfolio outcome agrees with the investors' preferences. The technique has proven to be very powerful and efficient in terms of finding the optimal asset allocation and implementing a good risk management process for balancing various risk factors. Intriguingly, in other instances, the optimal portfolios would deviate from the chosen preferences and those outcomes may be unstable and not ideal for the investor. To aid investors in evaluating the PGP outcomes, we have used the Omega ratio and the Stutzer Index, a type of risk-adjusted performance metrics that go beyond the regular Sharpe Ratio by considering higher moments. This comparison analysis serves as an additional step to distinguish the most favourable portfolios available to an investor. However, it is worth noting that these two metrics should not be viewed as a standalone measure for making investment decisions within the MVSK framework. Instead, investors seeking to implement higher moments for portfolio selection should be mindful of the distinctive characteristics represented in each portfolio. The ultimate and optimal choice can be made by understanding how individual preferences influence the portfolios' ability to achieve their most advantageous allocation.

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