Appendix A
Derivation of the Jacobian Vectors and the Hessian Matrices for (54) and (55)

We will first derive the Jacobian vector $\mathbf{g}_1$ and the Hessian matrix $\mathbf{H}_1$ for $\Lambda_1(\boldsymbol{\psi}_1)$ of the single path case. According to (54), the Jacobian vector is expressed as

$$\mathbf{g}_1 = \left[\frac{\partial \Lambda_1(\boldsymbol{\psi}_1)}{\partial \zeta}, \frac{\partial \Lambda_1(\boldsymbol{\psi}_1)}{\partial \xi}\right]^T, \tag{A.1}$$

where

$$\frac{\partial \Lambda_1(\boldsymbol{\psi}_1)}{\partial \zeta} = 2\,\mathrm{Re}\left\{\mathbf{d}(-\zeta)^H \mathbf{r}_\zeta^H \mathbf{r}_\zeta \frac{\partial \mathbf{d}(-\zeta)}{\partial \zeta}\right\}, \tag{A.2}$$

$$\frac{\partial \Lambda_1(\boldsymbol{\psi}_1)}{\partial \xi} = 2\,\mathrm{Re}\left\{\mathbf{p}(-\xi)^H \mathbf{r}_\xi^H \mathbf{r}_\xi \frac{\partial \mathbf{p}(-\xi)}{\partial \xi}\right\}, \tag{A.3}$$

where

$$\mathbf{r}_\zeta = \mathbf{a}(\hat{\theta}, \hat{\varphi})^H \mathcal{W}(\xi)^H \tilde{\mathbf{Y}}, \tag{A.4}$$

$$\mathbf{r}_\xi = \begin{bmatrix} \mathbf{a}(\hat{\theta}, \hat{\varphi})^H \mathbf{W}(\Phi_1)^H \tilde{\mathbf{Y}}_1 \mathbf{d}(-\zeta) \\ \vdots \\ \mathbf{a}(\hat{\theta}, \hat{\varphi})^H \mathbf{W}(\Phi_K)^H \tilde{\mathbf{Y}}_K \mathbf{d}(-\zeta) \end{bmatrix}^T \in \mathbb{C}^{1\times K}, \tag{A.5}$$

and $\mathbf{p}(\cdot)$ is as defined in (17), $\tilde{\mathbf{Y}}_k = \mathbf{Y}_k \,\mathrm{diag}(\mathbf{s}^*)$.

$\frac{\partial \mathbf{d}(-\zeta)}{\partial \zeta}$ and $\frac{\partial \mathbf{p}(-\xi)}{\partial \xi}$ are derived as follows.

$$\frac{\partial \mathbf{d}(-\zeta)}{\partial \zeta} = \mathbf{d}(-\zeta) \odot \mathbf{d}_\zeta, \tag{A.6}$$

where

$$\mathbf{d}_\zeta = -j2\pi\left[-\frac{L}{2}, -\frac{L}{2}+1, \cdots, \frac{L}{2}-1\right]^T, \tag{A.7}$$

and

$$\frac{\partial \mathbf{p}(-\xi)}{\partial \xi} = \mathbf{p}(-\xi) \odot \mathbf{p}_\xi, \tag{A.8}$$

where

$$\mathbf{p}_\xi = -j2\pi N\left[0, 1, \cdots, K-1\right]^T. \tag{A.9}$$

The Hessian matrix can be expressed as

$$\mathbf{H}_1 = \begin{bmatrix} \frac{\partial^2 \Lambda_1(\boldsymbol{\psi}_1)}{\partial \zeta^2} & \frac{\partial^2 \Lambda_1(\boldsymbol{\psi}_1)}{\partial \zeta \partial \xi} \\ \frac{\partial^2 \Lambda_1(\boldsymbol{\psi}_1)}{\partial \xi \partial \zeta} & \frac{\partial^2 \Lambda_1(\boldsymbol{\psi}_1)}{\partial \xi^2} \end{bmatrix}, \tag{A.10}$$

where

$$\frac{\partial^2 \Lambda_1(\boldsymbol{\psi}_1)}{\partial \zeta^2} = 2\,\mathrm{Re}\left\{\mathbf{d}(-\zeta)^H \mathbf{r}_\zeta^H \mathbf{r}_\zeta \frac{\partial^2 \mathbf{d}(-\zeta)}{\partial \zeta^2}\right\} + 2\left|\mathbf{r}_\zeta \frac{\partial \mathbf{d}(-\zeta)}{\partial \zeta}\right|^2, \tag{A.11}$$

$$\frac{\partial^2 \Lambda_1(\boldsymbol{\psi}_1)}{\partial \xi^2} = 2\,\mathrm{Re}\left\{\mathbf{p}(-\xi)^H \mathbf{r}_\xi^H \mathbf{r}_\xi \frac{\partial^2 \mathbf{p}(-\xi)}{\partial \xi^2}\right\} + 2\left|\mathbf{r}_\xi \frac{\partial \mathbf{p}(-\xi)}{\partial \xi}\right|^2, \tag{A.12}$$

$$\frac{\partial^2 \Lambda_1(\boldsymbol{\psi}_1)}{\partial \zeta \partial \xi} = 2\,\mathrm{Re}\left\{\frac{\partial \mathbf{d}(-\zeta)^H}{\partial \zeta} \mathbf{r}_\zeta \mathbf{r}_{\zeta\xi} \mathbf{d}(-\zeta)\right\} + 2\,\mathrm{Re}\left\{\mathbf{d}(-\zeta)^H \mathbf{r}_\zeta \mathbf{r}_{\zeta\xi} \frac{\partial \mathbf{d}(-\zeta)}{\partial \zeta}\right\}, \tag{A.13}$$

$$\frac{\partial^2 \Lambda_1(\boldsymbol{\psi}_1)}{\partial \xi \partial \zeta} = \frac{\partial^2 \Lambda_1(\boldsymbol{\psi}_1)}{\partial \zeta \partial \xi} \tag{A.14}$$

where

$$\mathbf{r}_{\zeta\xi} = \left[\mathbf{a}(\hat{\theta}, \hat{\varphi})^H \mathcal{W}(\xi)^H\right] \odot \left(\mathbf{p}_\xi^T \otimes \mathbf{l}_1\right) \tilde{\mathbf{Y}}, \tag{A.15}$$

and all the elements for $\mathbf{l}_1 \in \mathbb{C}^{1\times M_r}$ are 1. Besides, $\frac{\partial^2 \mathbf{d}(-\zeta)}{\partial \zeta^2}$ and $\frac{\partial^2 \mathbf{p}(-\xi)}{\partial \xi^2}$ are derived as follows.

$$\frac{\partial^2 \mathbf{d}(-\zeta)}{\partial \zeta^2} = \frac{\partial \mathbf{d}(-\zeta)}{\partial \zeta} \odot \mathbf{d}_\zeta, \tag{A.16}$$

$$\frac{\partial^2 \mathbf{p}(-\xi)}{\partial \xi^2} = \frac{\partial \mathbf{p}(-\xi)}{\partial \xi} \odot \mathbf{p}_\xi, \tag{A.17}$$

Now we have the Jacobian vector $\mathbf{g}_1$ and the Hessian matrix $\mathbf{H}_1$ for $\Lambda_1(\boldsymbol{\psi}_1)$.

Then we will derive the Jacobian vector $\mathbf{g}_2$ and the Hessian matrix $\mathbf{H}_2$ for $\Lambda_2(\boldsymbol{\psi}_2)$ of the single path case. According to (55), the Jacobian vector is expressed as

$$\mathbf{g}_2 = \left[\frac{\partial \Lambda_2(\boldsymbol{\psi}_2)}{\partial \theta}, \frac{\partial \Lambda_2(\boldsymbol{\psi}_2)}{\partial \varphi}\right]^T, \tag{A.18}$$

Define f_{num} and f_{den} as the numerator and denominator of (55) respectively, then

$$\frac{\partial \Lambda_2(\boldsymbol{\psi}_2)}{\partial \theta} = \frac{\frac{\partial f_{num}}{\partial \theta} f_{den} - f_{num}\frac{\partial f_{den}}{\partial \theta}}{f_{den}^2}, \tag{A.19}$$

$$\frac{\partial \Lambda_2(\boldsymbol{\psi}_2)}{\partial \varphi} = \frac{\frac{\partial f_{num}}{\partial \varphi} f_{den} - f_{num}\frac{\partial f_{den}}{\partial \varphi}}{f_{den}^2}, \tag{A.20}$$

where

$$\frac{\partial f_{den}}{\partial \theta} = 2\,\mathrm{Re}\left\{\mathbf{a}(\theta,\varphi)^H \mathbf{W}^H \mathbf{W} \frac{\partial \mathbf{a}(\theta,\varphi)}{\partial \theta}\right\}, \tag{A.21}$$

$$\frac{\partial f_{den}}{\partial \varphi} = 2\,\mathrm{Re}\left\{\mathbf{a}(\theta,\varphi)^H \mathbf{W}^H \mathbf{W} \frac{\partial \mathbf{a}(\theta,\varphi)}{\partial \varphi}\right\}, \tag{A.22}$$

$$\frac{\partial f_{num}}{\partial \theta} = 2\,\mathrm{Re}\left\{\mathbf{a}(\theta,\varphi)^H \mathbf{r}\mathbf{r}^H \frac{\partial \mathbf{a}(\theta,\varphi)}{\partial \theta}\right\}, \tag{A.23}$$

$$\frac{\partial f_{num}}{\partial \varphi} = 2\,\mathrm{Re}\left\{\mathbf{a}(\theta,\varphi)^H \mathbf{r}\mathbf{r}^H \frac{\partial \mathbf{a}(\theta,\varphi)}{\partial \varphi}\right\}, \tag{A.24}$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}(\Phi_1) \\ \mathbf{W}(\Phi_2) \\ \vdots \\ \mathbf{W}(\Phi_K) \end{bmatrix} \tag{A.25}$$

where $\mathbf{r}$ is defined as (53). $\frac{\partial \mathbf{a}(\theta,\varphi)}{\partial \theta}$ and $\frac{\partial \mathbf{a}(\theta,\varphi)}{\partial \varphi}$ are derived as follows. According to (1), $\mathbf{a}(\theta,\varphi)$ can be rewritten as

$$\mathbf{a}(\theta,\varphi) = \mathbf{a}_x(\theta,\varphi) \otimes \mathbf{a}_z(\varphi) \tag{A.26}$$

where

$$\mathbf{a}_x(\theta,\varphi)=\left[1,e^{-j\frac{2\pi}{\lambda}d_x\sin\varphi\cos\theta},\cdots,e^{-j\frac{2\pi}{\lambda}(Q-1)d_x\sin\varphi\cos\theta}\right]^T, \tag{A.27}$$

$$\mathbf{a}_z(\varphi)=\left[1,e^{j\frac{2\pi}{\lambda}d_z\cos\varphi},\cdots,e^{j\frac{2\pi}{\lambda}(P-1)d_z\cos\varphi}\right]^T, \tag{A.28}$$

then we have

$$\frac{\partial\mathbf{a}(\theta,\varphi)}{\partial\theta}=[\mathbf{a}_x(\theta,\varphi)\odot\boldsymbol{\theta}_1]\otimes\mathbf{a}_z(\varphi), \tag{A.29}$$

$$\frac{\partial\mathbf{a}(\theta,\varphi)}{\partial\varphi}=[\mathbf{a}_x(\theta,\varphi)\odot\boldsymbol{\varphi}_1]\otimes\mathbf{a}_z(\varphi)+\mathbf{a}_x(\theta,\varphi)\otimes[\mathbf{a}_z(\varphi)\odot\boldsymbol{\varphi}_2], \tag{A.30}$$

where

$$\boldsymbol{\theta}_1=j\frac{2\pi}{\lambda}d_x\sin\varphi\sin\theta[0,1,\cdots,Q-1]^T, \tag{A.31}$$

$$\boldsymbol{\varphi}_1=-j\frac{2\pi}{\lambda}d_x\cos\varphi\cos\theta[0,1,\cdots,Q-1]^T, \tag{A.32}$$

$$\boldsymbol{\varphi}_2=-j\frac{2\pi}{\lambda}d_z\sin\varphi[0,1,\cdots,P-1]^T. \tag{A.33}$$

The Hessian matrix can be expressed as

$$\mathbf{H}_2=\begin{bmatrix}\frac{\partial^2\Lambda_2(\boldsymbol{\psi}_2)}{\partial\theta^2} & \frac{\partial^2\Lambda_2(\boldsymbol{\psi}_2)}{\partial\theta\partial\varphi}\\ \frac{\partial^2\Lambda_2(\boldsymbol{\psi}_2)}{\partial\varphi\partial\theta} & \frac{\partial^2\Lambda_2(\boldsymbol{\psi}_2)}{\partial\varphi^2}\end{bmatrix}, \tag{A.34}$$

where

$$\frac{\partial^2\Lambda_2(\boldsymbol{\psi}_2)}{\partial\theta^2}=\frac{1}{f_{den}^2}\left[\frac{\partial^2 f_{num}}{\partial\theta^2}f_{den}-f_{num}\frac{\partial^2 f_{den}}{\partial\theta^2}\right.\\\left.-2f_{den}\frac{\partial\Lambda_2(\boldsymbol{\psi}_2)}{\partial\theta}\frac{\partial f_{den}}{\partial\theta}\right], \tag{A.35}$$

$$\frac{\partial^2\Lambda_2(\boldsymbol{\psi}_2)}{\partial\varphi^2}=\frac{1}{f_{den}^2}\left[\frac{\partial^2 f_{num}}{\partial\varphi^2}f_{den}-f_{num}\frac{\partial^2 f_{den}}{\partial\varphi^2}\right.\\\left.-2f_{den}\frac{\partial\Lambda_2(\boldsymbol{\psi}_2)}{\partial\varphi}\frac{\partial f_{den}}{\partial\varphi}\right], \tag{A.36}$$

$$\frac{\partial^2\Lambda_2(\boldsymbol{\psi}_2)}{\partial\theta\partial\varphi}=\frac{1}{f_{den}^2}\left[\frac{\partial^2 f_{num}}{\partial\theta\partial\varphi}f_{den}+\frac{\partial f_{num}}{\partial\theta}\frac{\partial f_{den}}{\partial\varphi}\right.\\\left.-\frac{\partial f_{num}}{\partial\varphi}\frac{\partial f_{den}}{\partial\theta}-f_{num}\frac{\partial^2 f_{den}}{\partial\theta\partial\varphi}-2f_{den}\frac{\partial\Lambda_2(\boldsymbol{\psi}_2)}{\partial\theta}\frac{\partial f_{den}}{\partial\varphi}\right], \tag{A.37}$$

$$\frac{\partial^2\Lambda_2(\boldsymbol{\psi}_2)}{\partial\varphi\partial\theta}=\frac{\partial^2\Lambda_2(\boldsymbol{\psi}_2)}{\partial\theta\partial\varphi} \tag{A.38}$$

where

$$\frac{\partial^2 f_{den}}{\partial\theta^2}=2\,\mathrm{Re}\left\{\mathbf{a}(\theta,\varphi)^H\mathbf{W}^H\mathbf{W}\frac{\partial^2\mathbf{a}(\theta,\varphi)}{\partial\theta^2}\right\}\\+2\left\|\mathbf{W}\frac{\partial\mathbf{a}(\theta,\varphi)}{\partial\theta}\right\|^2, \tag{A.39}$$

$$\frac{\partial^2 f_{den}}{\partial\varphi^2}=2\,\mathrm{Re}\left\{\mathbf{a}(\theta,\varphi)^H\mathbf{W}^H\mathbf{W}\frac{\partial^2\mathbf{a}(\theta,\varphi)}{\partial\varphi^2}\right\}\\+2\left\|\mathbf{W}\frac{\partial\mathbf{a}(\theta,\varphi)}{\partial\varphi}\right\|^2, \tag{A.40}$$

$$\frac{\partial^2 f_{den}}{\partial\theta\partial\varphi}=2\,\mathrm{Re}\left\{\mathbf{a}(\theta,\varphi)^H\mathbf{W}^H\mathbf{W}\frac{\partial^2\mathbf{a}(\theta,\varphi)}{\partial\theta\partial\varphi}\right\}\\+2\,\mathrm{Re}\left\{\frac{\partial\mathbf{a}(\theta,\varphi)^H}{\partial\theta}\mathbf{W}^H\mathbf{W}\frac{\partial\mathbf{a}(\theta,\varphi)}{\partial\varphi}\right\}, \tag{A.41}$$

$$\frac{\partial^2 f_{den}}{\partial\theta^2}=2\,\mathrm{Re}\left\{\mathbf{a}(\theta,\varphi)^H\mathbf{r}\mathbf{r}^H\frac{\partial^2\mathbf{a}(\theta,\varphi)}{\partial\theta^2}\right\}\\+2\left\|\mathbf{r}^H\frac{\partial\mathbf{a}(\theta,\varphi)}{\partial\theta}\right\|^2, \tag{A.42}$$

$$\frac{\partial^2 f_{den}}{\partial\varphi^2}=2\,\mathrm{Re}\left\{\mathbf{a}(\theta,\varphi)^H\mathbf{r}\mathbf{r}^H\frac{\partial^2\mathbf{a}(\theta,\varphi)}{\partial\varphi^2}\right\}\\+2\left\|\mathbf{r}^H\frac{\partial\mathbf{a}(\theta,\varphi)}{\partial\varphi}\right\|^2, \tag{A.43}$$

$$\frac{\partial^2 f_{den}}{\partial\theta\partial\varphi}=2\,\mathrm{Re}\left\{\mathbf{a}(\theta,\varphi)^H\mathbf{r}\mathbf{r}^H\frac{\partial^2\mathbf{a}(\theta,\varphi)}{\partial\theta\partial\varphi}\right\}\\+2\,\mathrm{Re}\left\{\frac{\partial\mathbf{a}(\theta,\varphi)^H}{\partial\theta}\mathbf{r}\mathbf{r}^H\frac{\partial\mathbf{a}(\theta,\varphi)}{\partial\varphi}\right\}. \tag{A.44}$$

Besides, $\frac{\partial^2\mathbf{a}(\theta,\varphi)}{\partial\theta^2}$, $\frac{\partial^2\mathbf{a}(\theta,\varphi)}{\partial\varphi^2}$ and $\frac{\partial^2\mathbf{a}(\theta,\varphi)}{\partial\theta\partial\varphi}$ are derived as follows.

$$\frac{\partial^2\mathbf{a}(\theta,\varphi)}{\partial\theta^2}=[\mathbf{a}_x(\theta,\varphi)\odot(\boldsymbol{\theta}_2+\boldsymbol{\theta}_1\odot\boldsymbol{\theta}_1)]\otimes\mathbf{a}_z(\varphi), \tag{A.45}$$

$$\frac{\partial^2\mathbf{a}(\theta,\varphi)}{\partial\varphi^2}=[\mathbf{a}_x(\theta,\varphi)\odot(\boldsymbol{\theta}_2+\boldsymbol{\varphi}_1\odot\boldsymbol{\varphi}_1)]\otimes\mathbf{a}_z(\varphi)\\+2\,[\mathbf{a}_x(\theta,\varphi)\odot\boldsymbol{\varphi}_1]\otimes[\mathbf{a}_z(\varphi)\odot\boldsymbol{\varphi}_2]\\+\mathbf{a}_x(\theta,\varphi)\otimes[\mathbf{a}_z(\varphi)\odot(\boldsymbol{\varphi}_3+\boldsymbol{\varphi}_2\odot\boldsymbol{\varphi}_2)], \tag{A.46}$$

$$\frac{\partial^2\mathbf{a}(\theta,\varphi)}{\partial\theta\partial\varphi}=[\mathbf{a}_x(\theta,\varphi)\odot(\boldsymbol{\varphi}_4+\boldsymbol{\theta}_1\odot\boldsymbol{\varphi}_1)]\otimes\mathbf{a}_z(\varphi)\\+[\mathbf{a}_x(\theta,\varphi)\odot\boldsymbol{\theta}_1]\otimes[\mathbf{a}_z(\varphi)\odot\boldsymbol{\varphi}_2], \tag{A.47}$$

where

$$\boldsymbol{\theta}_2=j\frac{2\pi}{\lambda}d_x\sin\varphi\cos\theta[0,1,\cdots,Q-1]^T, \tag{A.48}$$

$$\boldsymbol{\varphi}_3=-j\frac{2\pi}{\lambda}d_z\cos\varphi[0,1,\cdots,P-1]^T, \tag{A.49}$$

$$\boldsymbol{\varphi}_4=j\frac{2\pi}{\lambda}d_x\cos\varphi\sin\theta[0,1,\cdots,Q-1]^T. \tag{A.50}$$

Now we have the Jacobian vector $\mathbf{g}_2$ and the Hessian matrix $\mathbf{H}_2$ for $\Lambda_2(\boldsymbol{\psi}_2)$.

Appendix B
Derivation of the Jacobian Vector and the Hessian Matrix for (92)

According to (92), the objective function $\Lambda_3(\boldsymbol{\psi}_3)$ of $\boldsymbol{\psi}_3\triangleq[\bar{\zeta},\bar{\xi},\bar{\theta}]^T$ can be rewritten as

$$\Lambda_3(\boldsymbol{\psi}_3)=\left|\tilde{\mathbf{y}}^H\left\{\mathbf{d}(\bar{\zeta})\otimes\left[\overline{\mathcal{B}}(\bar{\xi})\mathbf{c}(\bar{\theta})\right]\right\}\right|^2=\left|\tilde{\mathbf{y}}^H\check{\mathbf{x}}\right|^2, \tag{B.1}$$

where

$$\tilde{\mathbf{y}}\triangleq\mathrm{vec}(\tilde{\mathbf{Y}}), \tag{B.2}$$

$$\breve{\mathbf{x}} = \mathbf{d}(\bar{\zeta}) \otimes \left[\overline{\mathcal{B}}(\bar{\xi})\mathbf{c}(\bar{\theta})\right], \tag{B.3}$$

Then we will derive the Jacobian vector $\mathbf{g}_3$ and Hessian matrix $\mathbf{H}_3$ of the single path case. The Jacobian vector is expressed as

$$\mathbf{g}_3 = \left[\frac{\partial \Lambda(\boldsymbol{\psi}_3)}{\partial \bar{\zeta}}, \frac{\partial \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\xi}}, \frac{\partial \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\theta}}\right]^T, \tag{B.4}$$

where

$$\frac{\partial \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\zeta}} = 2\,\mathrm{Re}\left\{\breve{\mathbf{x}}^H \tilde{\mathbf{y}}\tilde{\mathbf{y}}^H \frac{\partial \breve{\mathbf{x}}}{\partial \bar{\zeta}}\right\}, \tag{B.5}$$

$$\frac{\partial \Lambda_1(\boldsymbol{\psi}_1)}{\partial \bar{\xi}} = 2\,\mathrm{Re}\left\{\breve{\mathbf{x}}^H \tilde{\mathbf{y}}\tilde{\mathbf{y}}^H \frac{\partial \breve{\mathbf{x}}}{\partial \bar{\xi}}\right\}, \tag{B.6}$$

$$\frac{\partial \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\theta}} = 2\,\mathrm{Re}\left\{\breve{\mathbf{x}}^H \tilde{\mathbf{y}}\tilde{\mathbf{y}}^H \frac{\partial \breve{\mathbf{x}}}{\partial \bar{\theta}}\right\}, \tag{B.7}$$

where $\frac{\partial \breve{\mathbf{x}}}{\partial \bar{\zeta}}$, $\frac{\partial \breve{\mathbf{x}}}{\partial \bar{\xi}}$ and $\frac{\partial \breve{\mathbf{x}}}{\partial \bar{\theta}}$ are derived as follows.

$$\frac{\partial \breve{\mathbf{x}}}{\partial \bar{\zeta}} = \frac{\partial \mathbf{d}(\bar{\zeta})}{\partial \bar{\zeta}} \otimes \left[\overline{\mathcal{B}}(\bar{\xi})\mathbf{c}(\bar{\theta})\right], \tag{B.8}$$

where

$$\frac{\partial \mathbf{d}(\bar{\zeta})}{\partial \bar{\zeta}} = \mathbf{d}(\bar{\zeta}) \odot \mathbf{d}_{\bar{\zeta}}, \tag{B.9}$$

$$\mathbf{d}_{\bar{\zeta}} = j2\pi\left[-\frac{L}{2}, -\frac{L}{2}+1, \cdots, \frac{L}{2}-1\right]^T, \tag{B.10}$$

and

$$\frac{\partial \breve{\mathbf{x}}}{\partial \bar{\xi}} = \mathbf{d}(\bar{\zeta}) \otimes \left[\frac{\partial \overline{\mathcal{B}}(\bar{\xi})}{\partial \bar{\xi}}\mathbf{c}(\bar{\theta})\right], \tag{B.11}$$

where

$$\frac{\partial \overline{\mathcal{B}}(\bar{\xi})}{\partial \bar{\xi}} = \left[\overline{\mathbf{p}}(\bar{\xi}) \odot \overline{\mathbf{p}}_{\bar{\xi}}\right] \otimes \mathbf{I}_{M_r}, \tag{B.12}$$

$$\overline{\mathbf{p}}(\bar{\xi}) = \left[1, e^{j2\pi\bar{\xi}N}, \cdots, e^{j2\pi\bar{\xi}(\overline{K}-1)N}\right]^T, \tag{B.13}$$

$$\overline{\mathbf{p}}_{\bar{\xi}} = j2\pi N\left[0, 1, \cdots, \overline{K}-1\right]^T. \tag{B.14}$$

and

$$\frac{\partial \breve{\mathbf{x}}}{\partial \bar{\theta}} = \mathbf{d}(\bar{\zeta}) \otimes \left[\overline{\mathcal{B}}(\bar{\xi})\frac{\partial \mathbf{c}(\bar{\theta})}{\partial \bar{\theta}}\right], \tag{B.15}$$

where

$$\frac{\partial \mathbf{c}(\bar{\theta})}{\partial \bar{\theta}} = \mathbf{c}(\bar{\theta}) \odot \mathbf{c}_{\bar{\theta}}, \tag{B.16}$$

$$\mathbf{c}_{\bar{\theta}} = j\frac{2\pi}{\lambda}\bar{d}_x \sin\bar{\theta}\,[0, 1, \cdots, M-1]^T. \tag{B.17}$$

The Hessian matrix can be expressed as

$$\mathbf{H}_3 = \begin{bmatrix} \frac{\partial^2 \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\zeta}^2} & \frac{\partial^2 \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\zeta}\partial \bar{\xi}} & \frac{\partial^2 \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\zeta}\partial \bar{\theta}} \\ \frac{\partial^2 \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\xi}\partial \bar{\zeta}} & \frac{\partial^2 \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\xi}^2} & \frac{\partial^2 \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\xi}\partial \bar{\theta}} \\ \frac{\partial^2 \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\theta}\partial \bar{\zeta}} & \frac{\partial^2 \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\theta}\partial \bar{\xi}} & \frac{\partial^2 \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\theta}^2} \end{bmatrix}, \tag{B.18}$$

where

$$\frac{\partial^2 \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\zeta}^2} = 2\left|\tilde{\mathbf{y}}^H \frac{\partial \breve{\mathbf{x}}}{\partial \bar{\zeta}}\right|^2 + 2\,\mathrm{Re}\left\{\breve{\mathbf{x}}^H \tilde{\mathbf{y}}\tilde{\mathbf{y}}^H \frac{\partial^2 \breve{\mathbf{x}}}{\partial \bar{\zeta}^2}\right\}, \tag{B.19}$$

$$\frac{\partial^2 \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\xi}^2} = 2\left|\tilde{\mathbf{y}}^H \frac{\partial \breve{\mathbf{x}}}{\partial \bar{\xi}}\right|^2 + 2\,\mathrm{Re}\left\{\breve{\mathbf{x}}^H \tilde{\mathbf{y}}\tilde{\mathbf{y}}^H \frac{\partial^2 \breve{\mathbf{x}}}{\partial \bar{\xi}^2}\right\}, \tag{B.20}$$

$$\frac{\partial^2 \Lambda_3(\boldsymbol{\psi}_3)}{\partial \bar{\theta}^2} = 2\left|\tilde{\mathbf{y}}^H \frac{\partial \breve{\mathbf{x}}}{\partial \bar{\theta}}\right|^2 + 2\,\mathrm{Re}\left\{\breve{\mathbf{x}}^H \tilde{\mathbf{y}}\tilde{\mathbf{y}}^H \frac{\partial^2 \breve{\mathbf{x}}}{\partial \bar{\theta}^2}\right\}, \tag{B.21}$$

and $\frac{\partial^2 \breve{\mathbf{x}}}{\partial \bar{\zeta}^2}$, $\frac{\partial^2 \breve{\mathbf{x}}}{\partial \bar{\xi}^2}$ and $\frac{\partial^2 \breve{\mathbf{x}}}{\partial \bar{\theta}^2}$ are derived as follows.

$$\frac{\partial^2 \breve{\mathbf{x}}}{\partial \bar{\zeta}^2} = \frac{\partial^2 \mathbf{d}(\bar{\zeta})}{\partial \bar{\zeta}^2} \otimes \left[\overline{\mathcal{B}}(\bar{\xi})\mathbf{c}(\bar{\theta})\right], \tag{B.22}$$

where

$$\frac{\partial^2 \mathbf{d}(\bar{\zeta})}{\partial \bar{\zeta}^2} = \frac{\partial \mathbf{d}(\bar{\zeta})}{\partial \bar{\zeta}} \odot \mathbf{d}_{\bar{\zeta}}, \tag{B.23}$$

and

$$\frac{\partial^2 \breve{\mathbf{x}}}{\partial \bar{\xi}^2} = \mathbf{d}(\bar{\zeta}) \otimes \left[\frac{\partial^2 \overline{\mathcal{B}}(\bar{\xi})}{\partial \bar{\xi}^2}\mathbf{c}(\bar{\theta})\right], \tag{B.24}$$

where

$$\frac{\partial^2 \overline{\mathcal{B}}(\bar{\xi})}{\partial \bar{\xi}^2} = \left[\overline{\mathbf{p}}(\bar{\xi}) \odot \overline{\mathbf{p}}_{\bar{\xi}} \odot \overline{\mathbf{p}}_{\bar{\xi}}\right] \otimes \mathbf{I}_{M_r}, \tag{B.25}$$

and

$$\frac{\partial^2 \breve{\mathbf{x}}}{\partial \bar{\theta}^2} = \mathbf{d}(\bar{\zeta}) \otimes \left[\overline{\mathcal{B}}(\bar{\xi})\frac{\partial^2 \mathbf{c}(\bar{\theta})}{\partial \bar{\theta}^2}\right], \tag{B.26}$$

where

$$\frac{\partial^2 \mathbf{c}(\bar{\theta})}{\partial \bar{\theta}^2} = \mathbf{c}(\bar{\theta})\left(\mathbf{c}_{\tilde{\theta}} + \mathbf{c}_{\bar{\theta}} \odot \mathbf{c}_{\bar{\theta}}\right), \tag{B.27}$$

$$\mathbf{c}_{\tilde{\theta}} = j\frac{2\pi}{\lambda}\bar{d}_x \cos\bar{\theta}\,[0, 1, \cdots, M-1]^T. \tag{B.28}$$

Similarly, the rest elements of the Hessian matrix $\mathbf{H}_3$ can be calculated based on the first derivative of $\breve{\mathbf{x}}$.