Elements of Scientific Computing with Julia

March 5, 2015

Introduction

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These are systems that allows businesses (or otherwise content providers) to more *intelligently* provide their content based on their customers preferences.

For example: movie recommendations on Netflix or product recommendations on amazon.com.

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1. *Content based systems:* these are systems that base recommendations on the content of the media or information. For example, if you watch actions movies on Netflix and rate those positively, chances are that you will enjoy other action movies.
2. *Collaborative filtering systems:* these are systems that base recommendations on similarities amongst users or customers. For example, if you bought a book on amazon that another user also bought along with other books, then maybe you would like those other books too.

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There are many techniques from scientific computing and machine learning that can be applied to implement these different types of recommender systems - such as singular value decomposition and clustering analysis (which are not covered in this course).

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However, before we "jump in", let us take a detour through the topic of *regularization*.

Regularization

Regularization refers to the technique of adding extra information to our linear (or logistic) regression in other to give preferential treatment to features that seem more relevantly correlated and penalize features whose correlation may not be that relevant to solving the regression.

Regularized Cost Function and its Derivative

$$C(\theta) = \frac{1}{2m}\left[\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2 + \lambda\sum_{j=1}^{n}\theta_j^2\right]$$

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for $j > 0$:

$$\frac{\partial}{\partial \theta_j} C(\theta) = \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_j^{(i)} + \frac{\lambda}{m}\theta_j$$

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Note that we do not regularize θ_0 since it pertains to the "ones" feature and it is not affected by the weight of the features in consideration.

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These features are information like amount of action or romance in a movie, or the different categories in which a particular product may fall.

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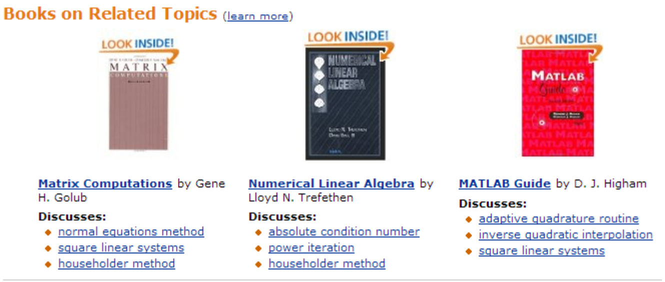
Clicking one of the results and scrolling down a bit, we find yet another example of content based recommendations:



A more simple example...

In order to introduce the topic and formulate the problem, let us work on a simplified example:

Book	Alice	Bob	Charlie	Dave
Advanced Math in Greek	5	0	5	0
The Book of Proofs	?	2	5	0
Down-to-Earth Math	0	5	1	4
Everyday Math	0	?	?	?
Intermediate Crazy Math	5	0	?	?

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An entry $a_{i,j}$ in this matrix is a number ranging from 0 through 5, representing the rating user j gave to book i.

If we do not have a rating from a particular user on a particular book, then the entry is represented by a "?".

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- $\theta^{(j)}$ is the parameter vector for user j;
- $x^{(i)}$ is the feature vector for book i; and
- $(\theta^{(j)})^T(x^{(i)})$ is thus, the predicted rating for user j, for book i.

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That is, someone read all of these books and "graded" them on these features, say on a scale from 0.0 to 1.0.

Problem Formulation

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$$\boxed{\min_{\theta^{(j)}} \left(\frac{1}{2} \sum_{i:r(i,j)=1} ((\theta^{(j)})^T (x^{(i)}) - y^{(i,j)})^2 + \frac{\lambda}{2} \sum_{k=1}^{n} (\theta_k^{(j)})^2 \right)}$$

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Here, the analogous constant factor would be $\frac{1}{m^{(j)}}$ where $m^{(j)}$ is the number of books rated by user j.

Since we will have to optimize for these $\theta^{(j)}$'s across all users j, then the constant factor $\frac{1}{m^{(j)}}$ is no longer convenient.

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For $k = 0$ (since j has a different meaning in our current formulation):

$$\frac{\partial}{\partial \theta_0^{(j)}} C(\theta^{(j)}) = \sum_{i:r(i,j)=1} ((\theta^{(j)})^T (x^{(i)}) - y^{(i,j)}) x_0^{(i)}$$

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Collaborative Filtering Systems

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For example, we know how Sally weighs her genre preferences when it comes to movies - we know that she really likes action movies, then "contains action" is a likely feature of the movies she watches.

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$$\theta^{(1)} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}, \ \theta^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \ \theta^{(3)} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}, \ \theta^{(4)} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

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however it should not look too different from before and working the differentiation is left to you as an exercise.

Going back and forth...

So once we solve the above problem and get a good approximation of $x^{(1)}, x^{(2)}, ..., x^{(n_b)}$, there is no reason why we can't use this new information to estimate $\theta^{(1)}, \theta^{(2)}, ..., \theta^{(n_u)}$

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But there may also be new books over time, and we can go back and forth in this manner estimating features and parameters.

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Combining the Problems

... combined, becomes:

$$\min_{x^{(1)},x^{(2)},\ldots,x^{(n_b)},\theta^{(1)},\theta^{(2)},\ldots,\theta^{(n_u)}} \left(\frac{1}{2} \sum_{(i,j):r(i,j)=1} ((\theta^{(j)})^T(x^{(i)}) - y^{(i,j)})^2 + \frac{\lambda}{2} \sum_{i=1}^{n_b} \sum_{k=1}^{n} (x_k^{(i)})^2 + \frac{\lambda}{2} \sum_{j=1}^{n_u} \sum_{k=1}^{n} (\theta_k^{(j)})^2 \right)$$

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The algorithm just described here is also known as *low-rank matrix factorization*.

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find the ten books l with the smallest distance from book i, that is $||x^{(i)} - x^{(l)}||$.