Elements of Scientific Computing with Julia

March 5, 2015

Fitting a Line to Data

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that is $\mathbf{y}$ can be approximated by a linear combination of n predictor variables.

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In this lecture we will focus our attention on *single* variable regression and leave the multivariable case for the next lecture. In the single variable case we have that the outcome $\mathbf{y}$ can be approximated by a linear relationship involving only one input variable $\mathbf{x}$, or mathematically:

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In this course we will only address supervised learning problems.

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Let us motivate our discussion with an example. Consider the following training data set on the next slide, where we have on the left column the number of years for which a person went to school and on the right column the income they now make, in thousands of dollars per year.

Example: Education v. Income

"Education"	"Income"
10.0	26.6588
10.4013	27.3064
10.8428	22.1324
11.2441	21.1698
11.6455	15.1926
12.087	26.399
12.4883	17.4353
12.8896	25.5079
13.291	36.8846
13.7324	39.6661
14.1338	34.3963
14.5351	41.498
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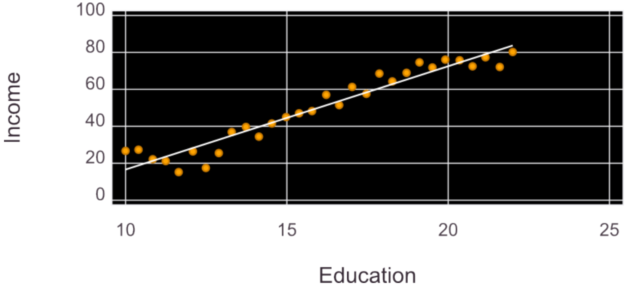

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In the next lecture we will generalize linear regression and gradient descent for multiple variables (that is, multiple input vectors $\mathbf{x}_i$), and also learn the *Normal Method* for solving for the parameters θ_i.

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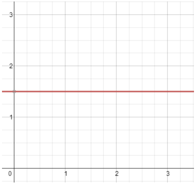

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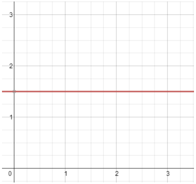

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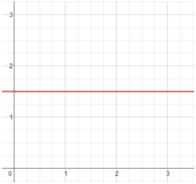

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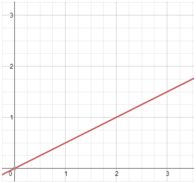

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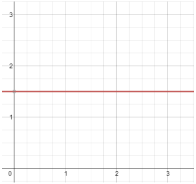

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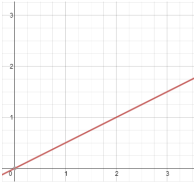

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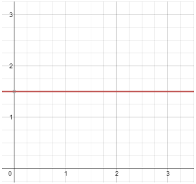

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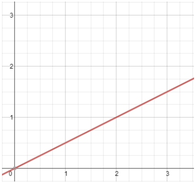

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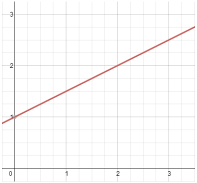

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As an exercise, I encourage you to read different datasets in Julia (with one input variable, where the output is quantitative), scatter plot the data and play around with manually picking the slope and intercept parameters for the line that best fits the data visually.

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$$C(\theta_0, \theta_1) = \frac{1}{2m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2$$

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For our purposes, the explanation is simple - minimization means taking derivatives, and therefore the cost function must be continuous and differentiable.

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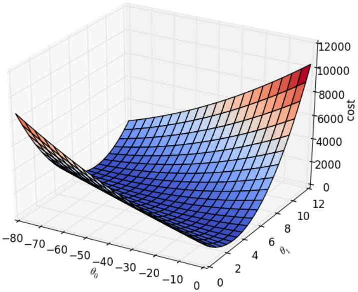
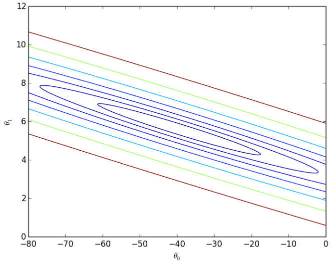
Analogously, in our case C's graph lives in a three dimensional space and it is a "bowl-up" shaped surface, with a global minimum at the base of the bowl.

The Cost Function

The surface plot:



The contour plot:



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The rule for gradient descent is that from our previous guess we should step down (with some step size) toward the direction of steepest descent.

If we iterate this rule in a “bowl-up” shaped surface with a not-to-large step size, we shall converge to a minimum.

The Gradient Descent Algorithm

Gradient Descent Algorithm for Bivariate Functions

repeat until convergence $\{$

for $j = 0$ and $j = 1$, simultaneously update:

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} C(\theta_0, \theta_1)$$

$\}$

The Gradient Descent Algorithm

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The intuition here is that the slope of the tangent line to our current point is the steepest slope we can get on a line that would still include the point.

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The best way to check that you are picking a good α is to graph (or otherwise output) your *cost* at each iteration and verify that it is indeed decreasing and the rate of decrease is not too slow.

Let me take that derivative for you...

Using basic Calculus you can easily verify that:

$$j = 0 \Rightarrow \frac{\partial}{\partial \theta_0} C(\theta_0, \theta_1) = \frac{1}{m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})$$

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Back to our Example

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Let's go over the accompanying IJulia notebook for this lecture which illustrates linear regression applied to the education versus income data.

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1. Using linear regression with least squares and gradient descent, as outlined in lecture 6, determine the line that best fits the relationship between sales and advertising budget for each medium.
2. Plot your solution line against the data for each feature considered (that is, sales versus TV budget, radio budget and newspaper budget).

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Commit and push your .ipynb, .tex, and .pdf files for this homework into your GitHub private repository by 5/19 at 11:59 PM.