On Julia and Mandelbrot Sets

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We are interested in polynomials of the form

$$f(z) = z^2 + c \tag{1}$$

where z and c are complex numbers. We can consider the sequence of iterates

$$z_{n+1} = f(z_n) = z_n^2 + c \tag{2}$$

We want to investigate the converge of this sequence. It will depend on the choice of initial value z_0. It will also depend on the parameter c.

There are three possible situations. One, the sequence converges. This means that the sequence becomes or approaches a **fixed point** where it is no longer changing. Two, the sequence diverges, meaning that it gets farther and father away from any fixed value (its absolute value approaches infinity). Three, it does not converge, yet it remains within a bounded region. For instance, if we have $c = 0$ and start with

$$z_0 = \cos(2\pi/3) + i\sin(2\pi/3) \quad \text{or} \quad z_0 = \cos(4\pi/3) + i\sin(4\pi/3) \tag{3}$$

the sequence will neither converge nor diverge, but rather alternate between those two values (complex non-unit cube roots of 1). Those values are called **periodic points**.

If we are performing iteration on a computer for some purpose, arguably the worst behavior is for a sequence to diverge to infinity. The sequence would escape the limits for which our numbers are defined. Thus we wish to investigate the set of initial values z_0 for which the sequence *does not diverge*. This is called the **filled Julia set**.

Linear functions often come up, and it is fairly straightforward to analyze the Julia sets of those. Thus we jump to the next-simplest kind of function: the quadratic. We widen our scope to include complex numbers, not just real numbers, because these come up in the quadratic formula. This justifies the need to consider the function $z^2 + c$. Even such a simple function already exhibits fractal behavior.

The most obvious candidate to serve as an initial value is $z_0 = 0$. Define the **Mandelbrot Set** as the set of complex numbers c for which 0 is in the filled Julia set of $z^2 + c$. It turns out that this set has astoundingly complex fractal behavior, as we will describe.

Consider the behavior of the point 0 under iteration by $z^2 + c$. It can be a fixed point in the exceptional case $c = 0$. It can be a periodic point (such will be the case, for example, if c satisfies

$c^3 + 2c^2 + c + 1 = 0$). Another possible behavior is that 0 is not a periodic point itself, but that it becomes a periodic point after some finite number of steps. If 0 is such a point for a given c, then c has a special name. It is called a **Misiurewicz point** of the Mandelbrot set.

It is these Misiurewicz points that lead to fractal (self-similar) behavior in the Mandelbrot set. Strictly speaking, the behavior is not fractal, but it is almost fractal. As you zoom in upon a neighborhood of a Misiurewicz point, you see a little almost-copy of the Mandelbrot set. As you zoom in more and more, this continues ad infinitum.