# ECE408 / CS483 / CSE408
Summer 2025

Applied Parallel Programming

Lecture 12: Parallel Computation Patterns – Parallel Scan (Prefix Sum)

## What Will You Learn Today?

- parallel scan (prefix sum) algorithms
  based on reductions and reverse reductions
- the concept of double buffering
- tradeoffs between work efficiency and latency
- how to develop hierarchical algorithms
  (across multiple kernels)

## Scan Includes all Partial Results

Reductions are a simplified form of scans.

In scan / parallel prefix,

- we need all of the partial sums
- (or whatever the operator might be).

## (Inclusive) Scan (Prefix-Sum) Definition

**Definition:** *The scan operation takes a binary associative operator* $\oplus$, *and an array of n elements*

$$[x_0, x_1, \ldots, x_{n-1}],$$

*and returns the prefix-sum array*

$$[x_0, (x_0 \oplus x_1), \ldots, (x_0 \oplus x_1 \oplus \ldots \oplus x_{n-1})].$$

**Example:** If $\oplus$ is addition, the scan operation
on the array [3 1 7 0 4 1 6 3],
returns [3 4 11 11 15 16 22 25].

## Example: Sharing a Big Sandwich

You order a 100-inch sandwich to feed 10 people,

and you know how much each person wants in inches:

[3 5 2 7 28 4 3 0 8 1] .

**How do you cut the bread quickly?**

**How much of the sandwich is left over?**

Method 1: sequentially!

Cut 3 inches, then cut 5 inches, then …

Method 2: **calculate cutting offsets with prefix-sum**

[3, 8, 10, 17, 45, 49, 52, 52, 60, 61] (39 inches left)

## Typical Applications of Scan

A simple and useful parallel building block.

Convert sequential recurrences

```
    for (j = 1; j < n; j++)
        out[j] = out[j-1] + f(j);
```

into parallel:

```
    forall (j) { temp[j] = f(j); }
    scan (out, temp);
```

## Typical Applications of Scan

- Useful for many parallel algorithms:
  - radix sort
  - quicksort
  - String comparison
  - Lexical analysis
  - Stream compaction
  - Polynomial evaluation
  - Solving recurrences
  - Tree operations
  - Histograms
  - Etc.

## Other Applications

- Assigning camp slots
- Assigning farmer market space
- Allocating memory to parallel threads
- Allocating memory buffer to communication channels
- …

## An Inclusive Sequential Scan

Given a sequence $[x_0, x_1, x_2, \ldots]$

Calculate output $[y_0, y_1, y_2, \ldots]$

Such that

$$y_0 = x_0$$

$$y_1 = x_0 + x_1$$

$$y_2 = x_0 + x_1 + x_2$$

…

*Using a recursive definition*

$$y_i = y_{i-1} + x_i$$

# A Sequential C Implementation

```
y[0] = x[0];
for (i = 1; i < Max_i; i++)
    y[i] = y[i-1] + x[i];
```

Computationally efficient:

N additions needed for N elements - O(N).

## A Naïve Inclusive Parallel Scan

- Assign one thread to calculate each y element
- Have every thread to add up all x elements needed for the y element

$$y_0 = x_0$$
$$y_1 = x_0 + x_1$$
$$y_2 = x_0 + x_1 + x_2$$

"Parallel programming is easy as long as you do not care about performance."

## Parallel Inclusive Scan using Reduction Trees

Calculate each output element as the reduction of all previous elements

- Some reduction partial sums will be shared among the calculation of output elements
- Based on hardware added design by Peter Kogge and Harold Stone at IBM in the 1970s – Kogge-Stone Trees
- Goal: low latency

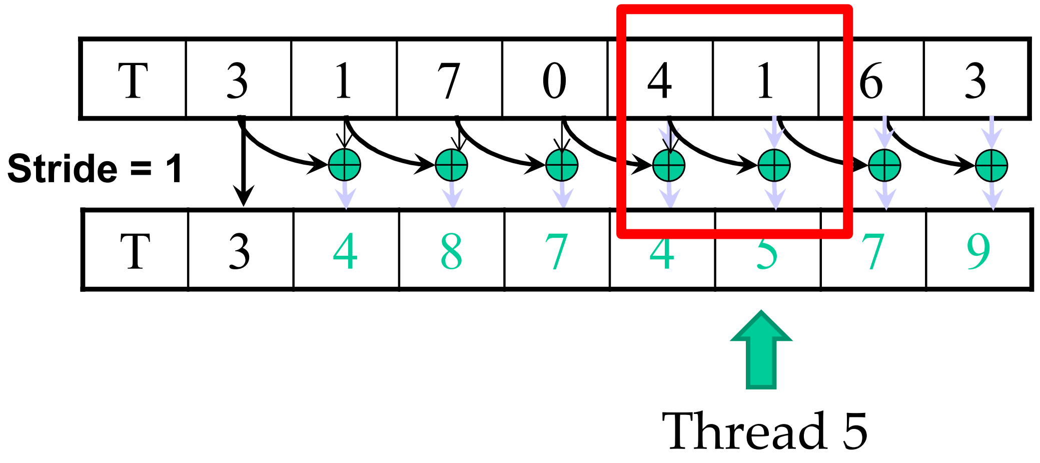
## A Kogge-Stone Parallel Scan Algorithm

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| T | 3 | 1 | 7 | 0 | 4 | 1 | 6 | 3 |

1. Load input from global memory into shared memory array T

Each thread loads one value from the input (global memory) array into shared memory array T.

# A Kogge-Stone Parallel Scan Algorithm



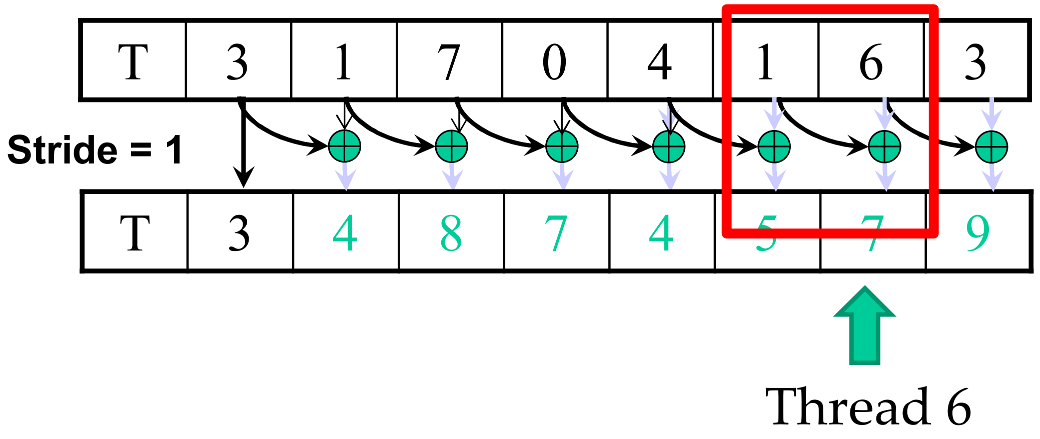
1. (previous slide)

2. Assuming n is a power of 2. Iterate log(n) times, stride from 1 to n/2. Threads *stride* to *n-1 active:* add pairs of elements that are s*tride* elements apart.

Iteration #1
Stride = 1

- Active threads: *stride* to *n*-1 (*n* - *stride* active threads)
- Thread *j* adds elements T[*j*] and T[*j-stride*] and writes result into element T[*j*]
- Each iteration requires two syncthreads
  - make sure that input is in place
  - make sure that all input elements have been used

# A Kogge-Stone Parallel Scan Algorithm

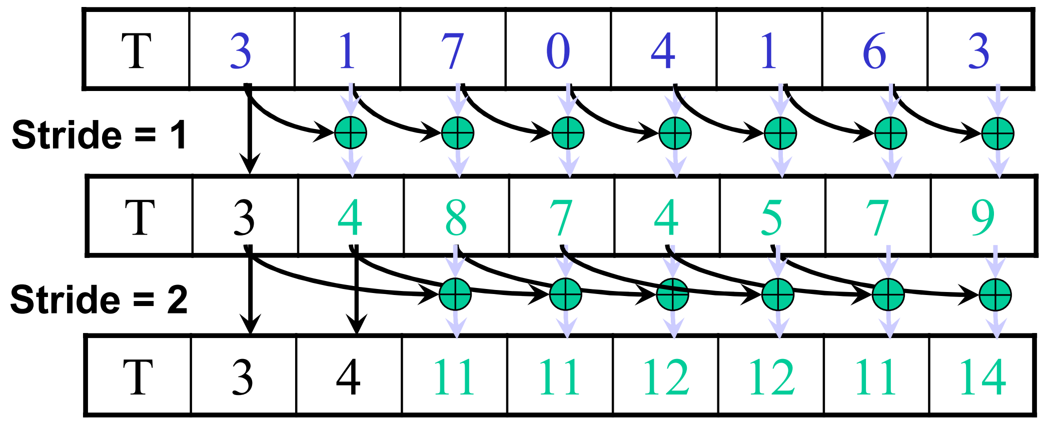


1. (previous slide)
2. Assuming n is a power of 2. Iterate log(n) times, stride from 1 to n/2. Threads *stride* to *n-1* *active:* add pairs of elements that are *stride* elements apart.

Iteration #1
Stride = 1

- Active threads: *stride* to *n*-1 (*n* - *stride* active threads)
- Thread *j* adds elements T[*j*] and T[*j-stride*] and writes result into element T[*j*]
- Each iteration requires two syncthreads
  - syncthreads(); // make sure that input is in place
  - float temp = T[*j*] + T[*j-stride*];
  - syncthreads(); // make sure that previous output has been consumed
  - T[*j*] = temp;

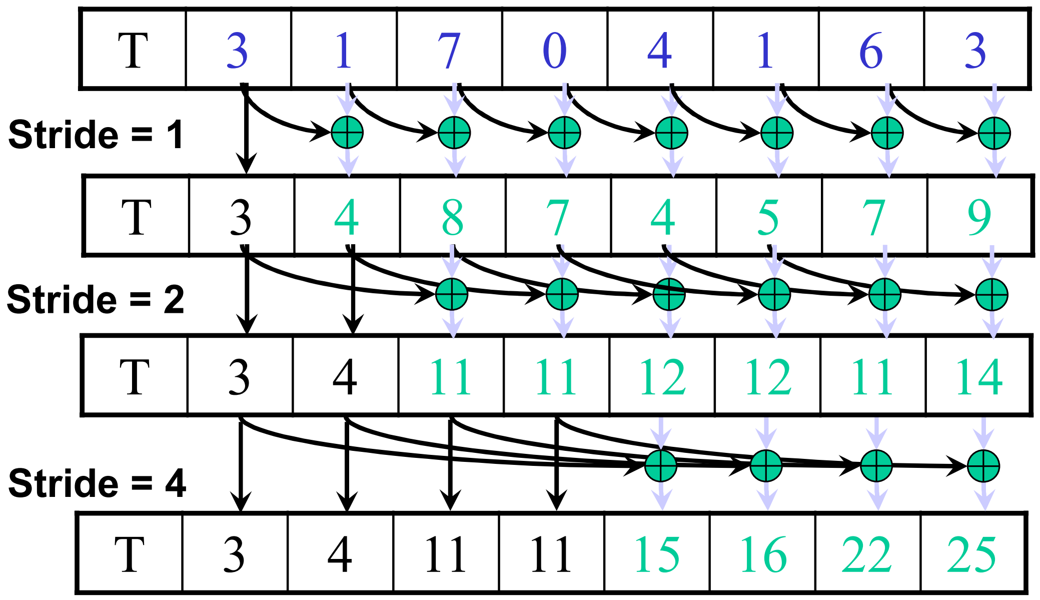
# A Kogge-Stone Parallel Scan Algorithm



1. …
2. Assuming n is a power of 2. Iterate log(n) times, stride from 1 to n/2. Threads *stride* to *n-1 active:* add pairs of elements that are *stride* elements apart.

Iteration #2
Stride = 2

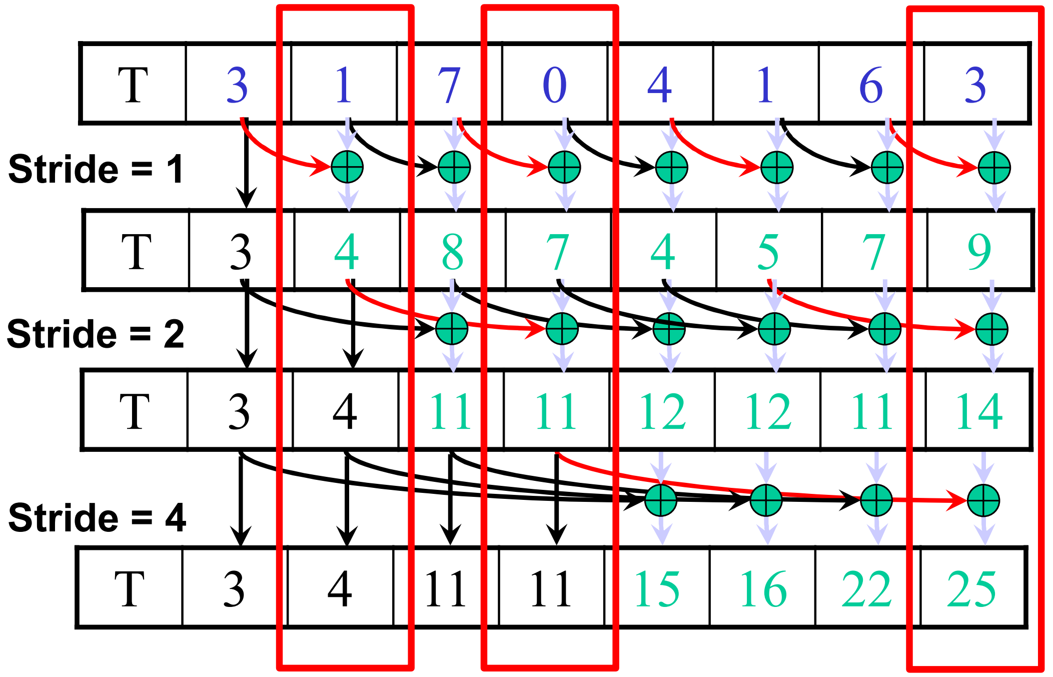
# A Kogge-Stone Parallel Scan Algorithm



1. ...
2. …
3. Write output from shared memory to device memory

Iteration #3
Stride = 4

# Sharing Computation in Kogge-Stone



Iteration #3
Stride = 4

## (*Incorrect*) Implementation with Single Barrier

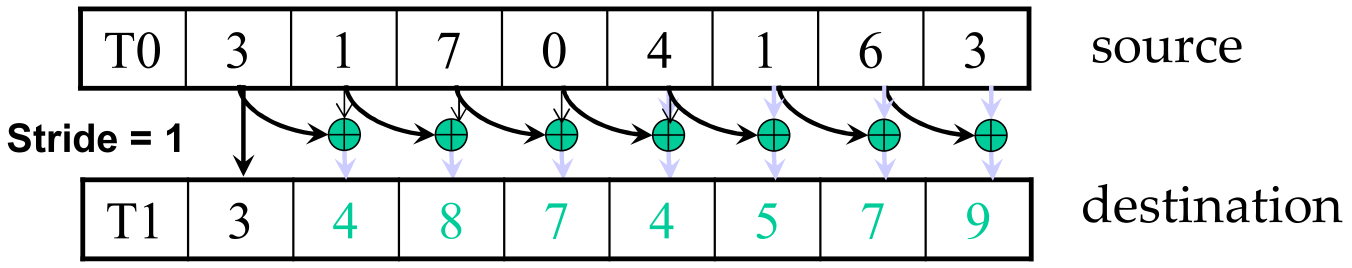
```
__global__
void Kogge_Stone_scan_kernel(float *X, float *Y, int InputSize)
{
 __shared__ float T[SECTION_SIZE];
 int i = blockIdx.x*blockDim.x + threadIdx.x;
 if (i < InputSize) T[threadIdx.x] = X[i];

 for (unsigned int stride = 1; stride < blockDim.x; stride *= 2) {
   __syncthreads();
   if (threadIdx.x >= stride) // This code has a data race condition
      T[threadIdx.x] += T[threadIdx.x-stride];
 }
 Y[i] = T[threadIdx.x];
}
```

## Double Buffering

- Use two copies of data T0 and T1
- Start by using T0 as input and T1 as output
- Switch input/output roles after each iteration
  - Iteration 0: T0 as input and T1 as output
  - Iteration 1: T1 as input and T0 and output
  - Iteration 2: T0 as input and T1 as output
- This is typically implemented with two pointers, *source* and *destination* that swap their contents from one iteration to the next
- This eliminates the need for the second __syncthreads() call

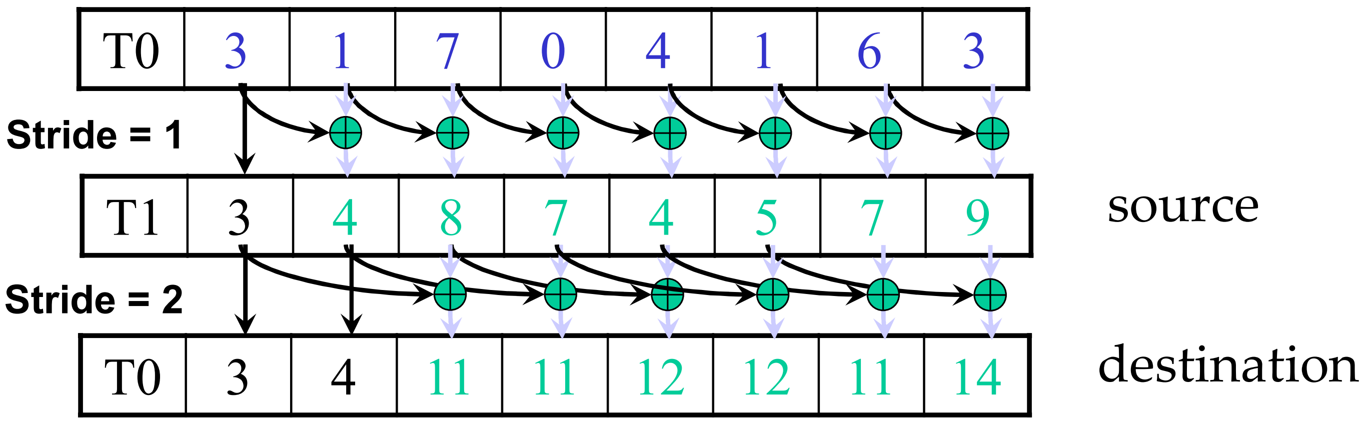
## A Double-Buffered Kogge-Stone Parallel Scan Algorithm



Iteration #1
Stride = 1

- source = &T0[0]; destination = &T1[0];
- Each iteration requires only one syncthreads()
  - syncthreads(); // make sure that input is in place
  - float destination[*j*] = source[*j*] + source[*j-stride*];
  - temp = destination; destination = source; source = temp;
- After the loop, write source (swapped) contents to global memory

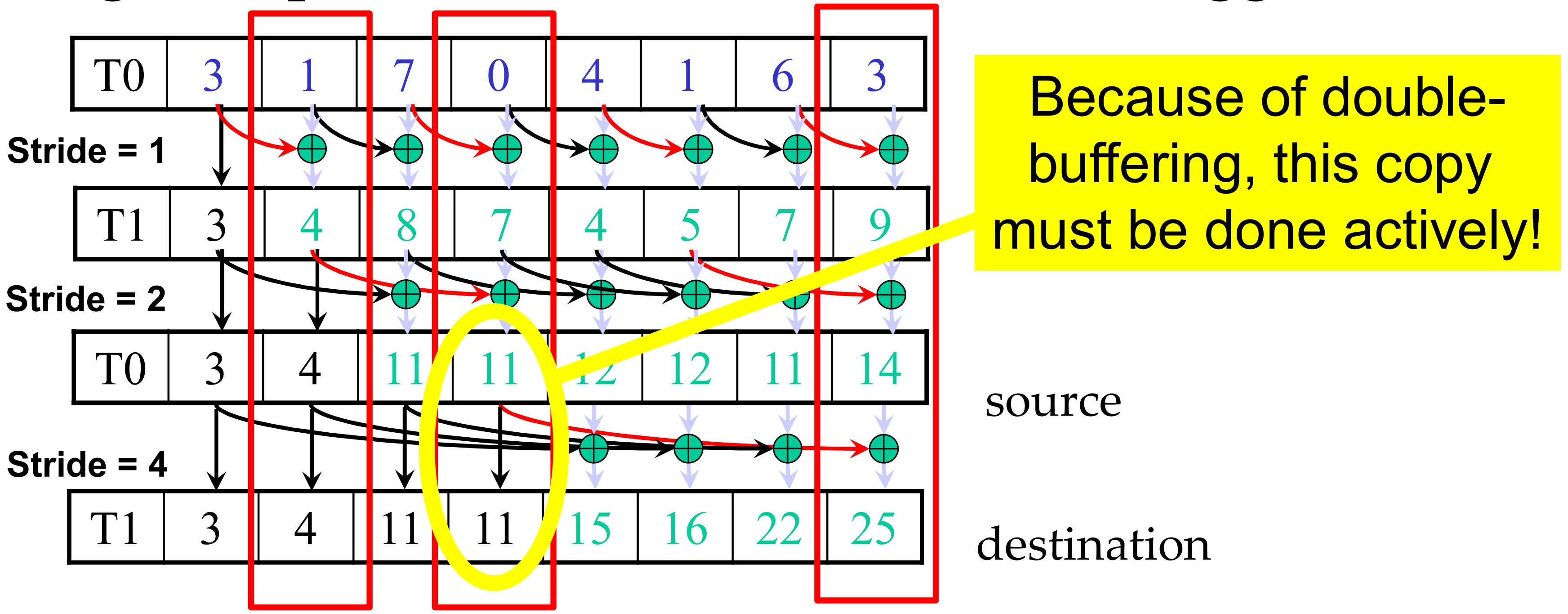
# A Double-Buffered Kogge-Stone Parallel Scan Algorithm



T0 | 3 | 1 | 7 | 0 | 4 | 1 | 6 | 3
Stride = 1
T1 | 3 | 4 | 8 | 7 | 4 | 5 | 7 | 9
source
Stride = 2
T0 | 3 | 4 | 11 | 11 | 12 | 12 | 11 | 14
destination

Iteration #2
Stride = 2

# Sharing Computation in Double-Buffered Kogge-Stone



T0 | 3 | 1 | 7 | 0 | 4 | 1 | 6 | 3
Stride = 1
T1 | 3 | 4 | 8 | 7 | 4 | 5 | 7 | 9
Stride = 2
T0 | 3 | 4 | 11 | 11 | 12 | 12 | 11 | 14
Stride = 4
T1 | 3 | 4 | 11 | 11 | 15 | 16 | 22 | 25
Because of double-buffering, this copy must be done actively!
source
destination

Iteration #3
Stride = 4

- Each iteration requires only one syncthreads()
  - syncthreads(); // make sure that input is in place
  - float destination[*j*] = source[*j*] + source[*j-stride*];
  - temp = destination; destination = source; source = temp;
- After the loop, write source (swapped) contents to global memory

## Work Efficiency Analysis

- A Kogge-Stone scan kernel executes log(n) parallel iterations
  - The steps do (n-1), (n-2), (n-4),..(n- n/2) add operations each
  - Total # of add operations: n * log(n) - (n-1) → O(n*log(n)) work
- This scan algorithm is not very work efficient
  - Sequential scan algorithm does n adds
  - A factor of log(n) hurts: 20x for 1,000,000 elements!
  - Typically used within each block, where n ≤ 1,024
- A parallel algorithm can be slow when execution resources are saturated due to low work efficiency.

## Improving Efficiency

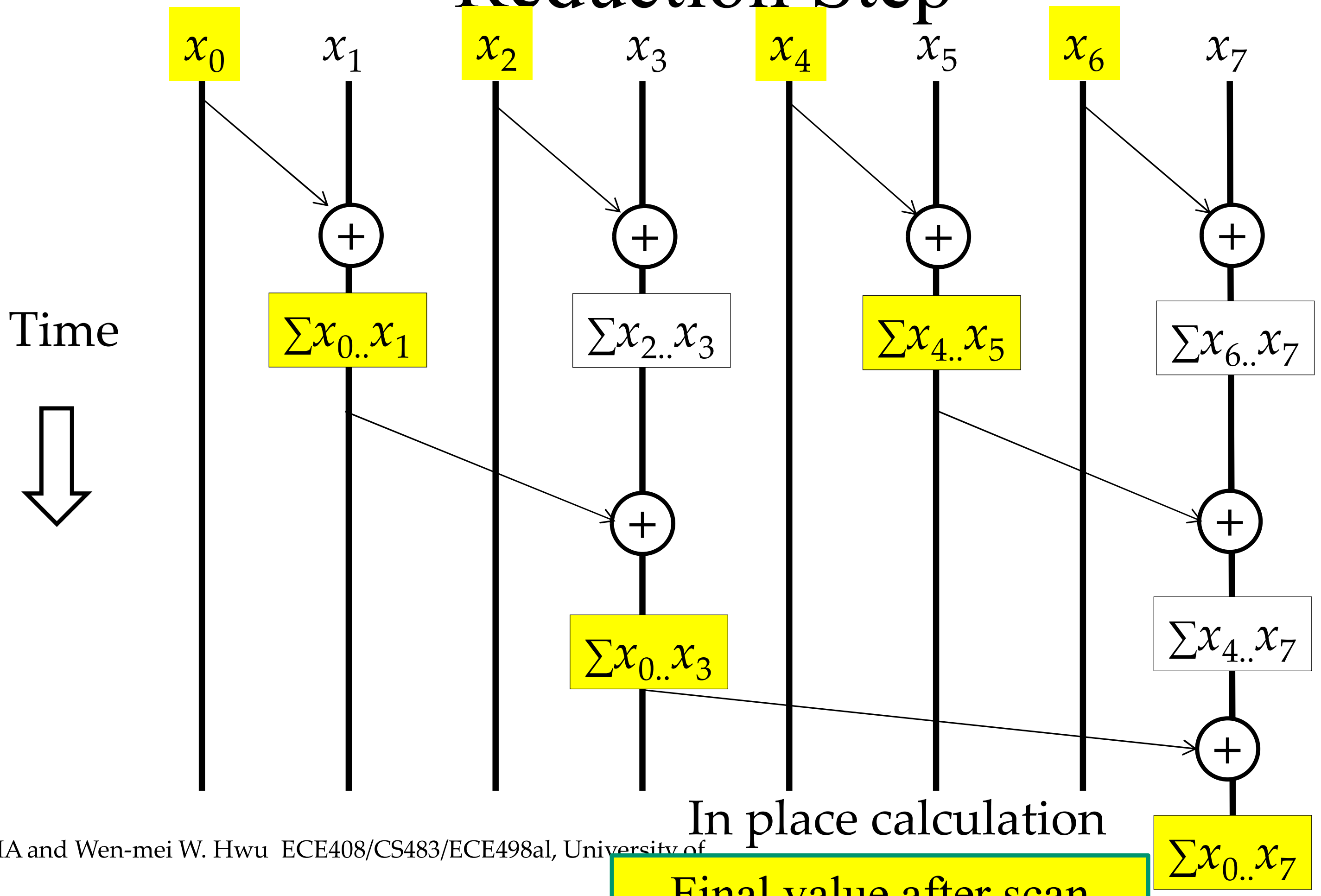
A common parallel algorithm pattern: *Balanced Trees*

- Build balanced binary tree on input data
- Sweep it to and from the root
- Tree is not an actual data structure, but a concept to determine what each thread does at each step

For scan:

- Traverse down from leaves to root building partial sums at internal nodes in the tree
- Root holds sum of all leaves
- Traverse back up the tree building the scan from the partial sums

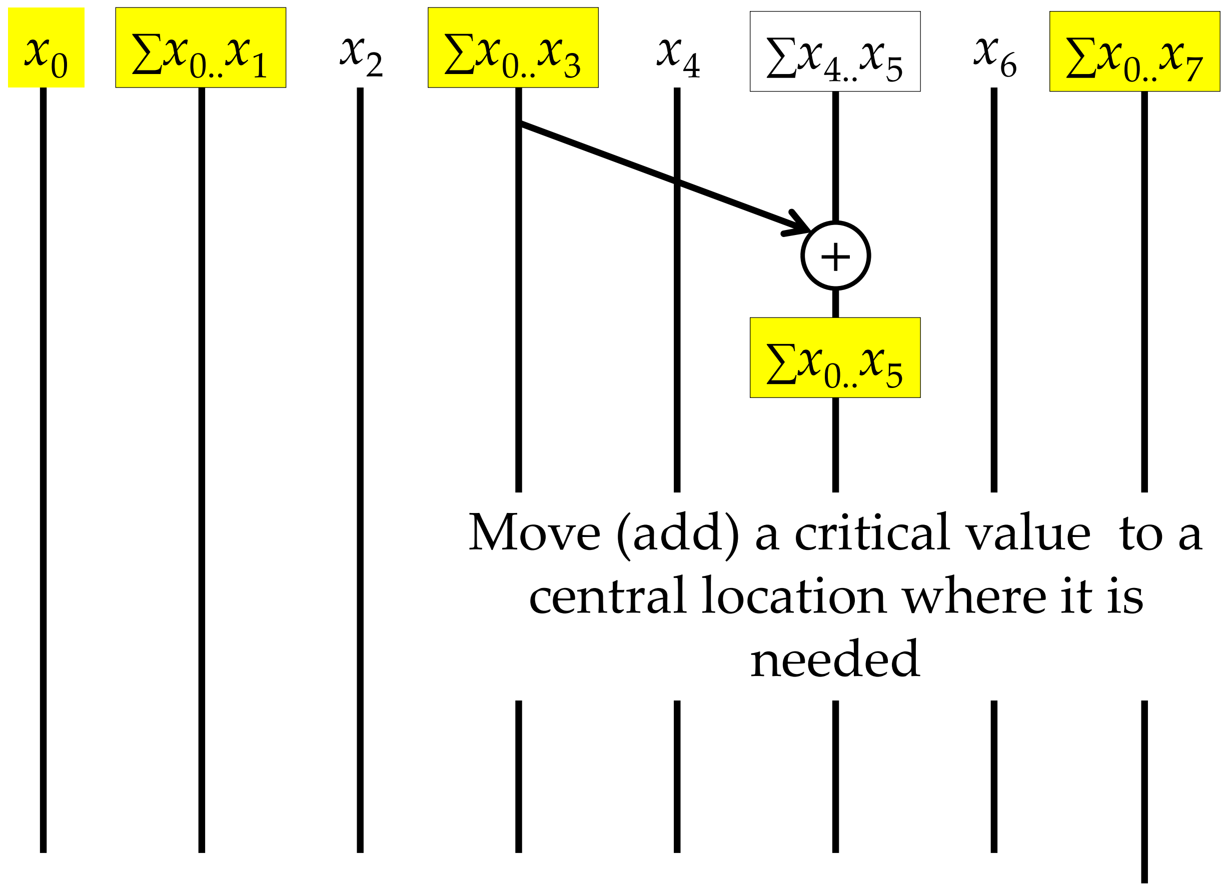
# Brent-Kung Parallel Scan - Reduction Step



In place calculation

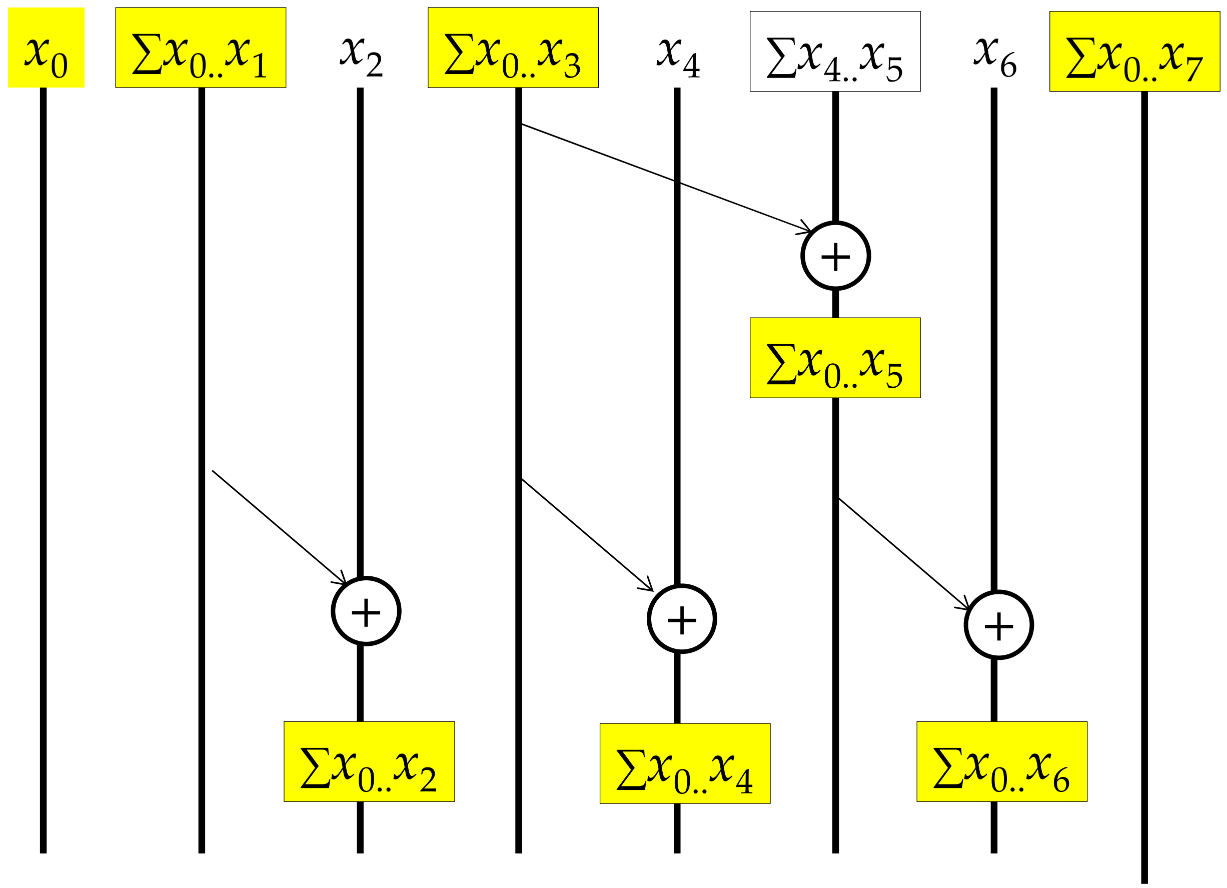
Final value after scan

# Inclusive Post-Scan Step

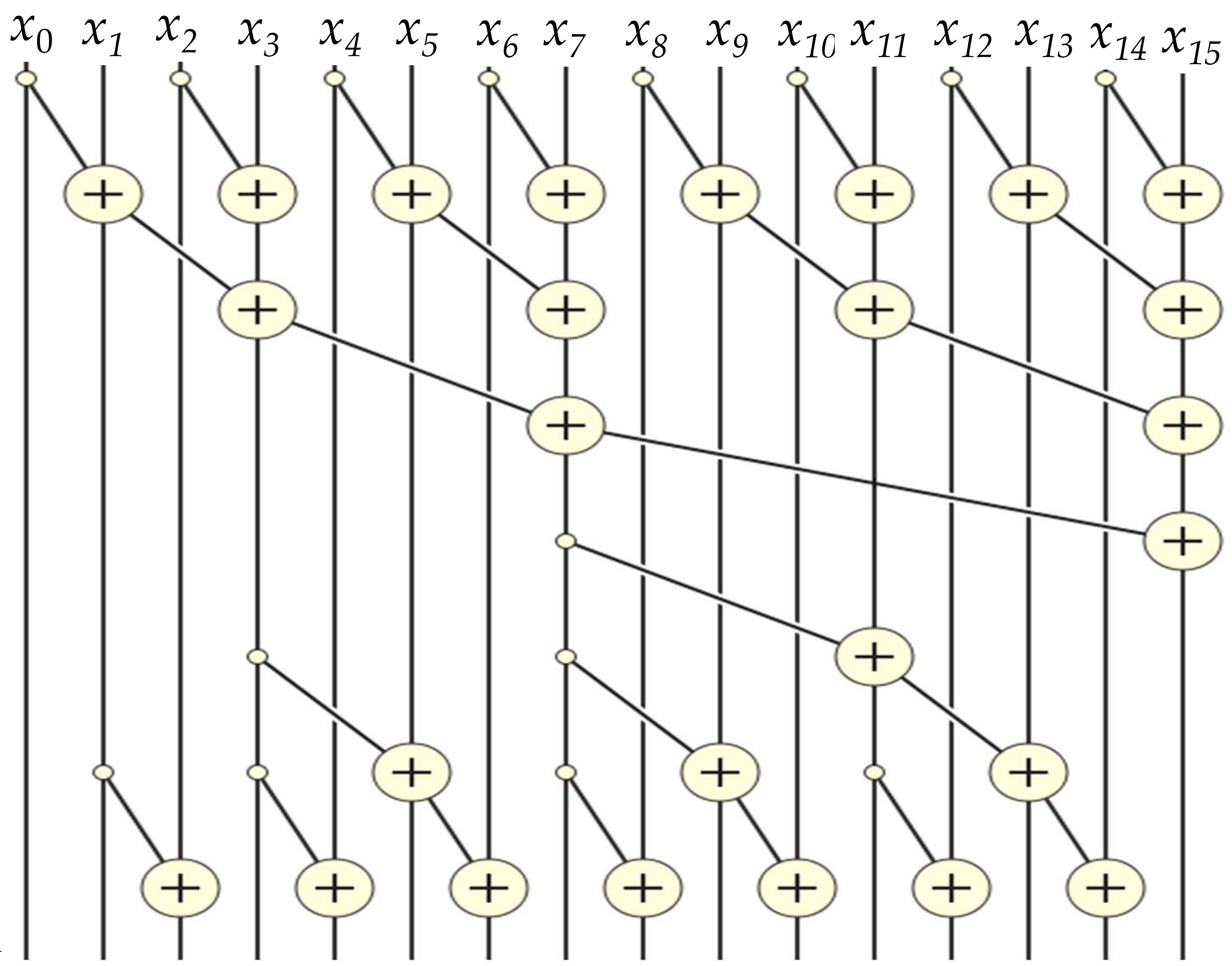


Move (add) a critical value to a central location where it is needed

# Inclusive Post Scan Step



# Putting it Together (Data View)


$x_0$ $x_1$ $x_2$ $x_3$ $x_4$ $x_5$ $x_6$ $x_7$ $x_8$ $x_9$ $x_{10}$ $x_{11}$ $x_{12}$ $x_{13}$ $x_{14}$ $x_{15}$

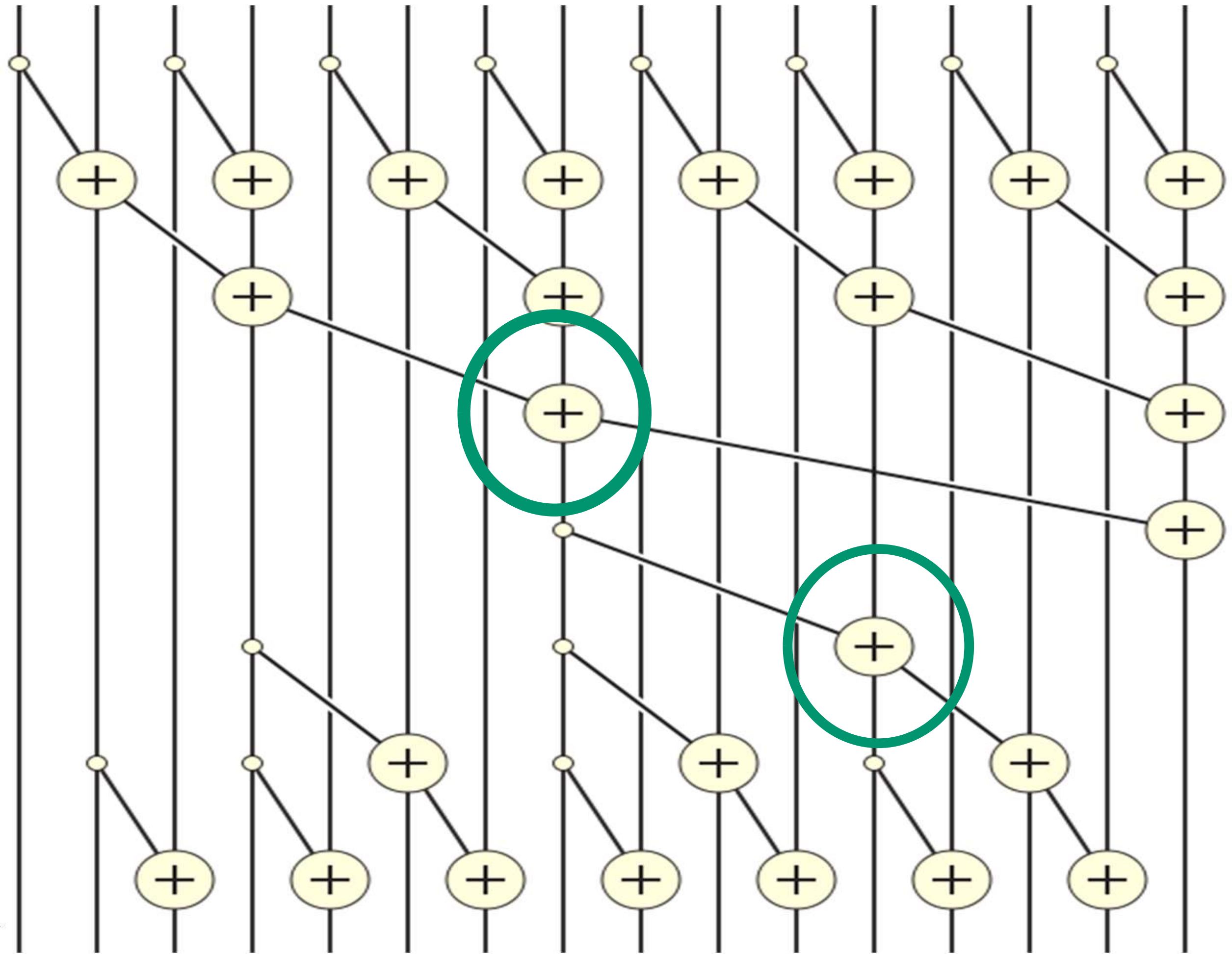
## Reduction Step Kernel Code

```
// float T[2*BLOCK_SIZE] is in shared memory
// for previous slide, BLOCK_SIZE is 8

int stride = 1;
while(stride < 2*BLOCK_SIZE) {
    __syncthreads();
    int index = (threadIdx.x+1)*stride*2 - 1;
    if(index < 2*BLOCK_SIZE && (index-stride) >= 0)
        T[index] += T[index-stride];
    stride = stride*2;
}
```

```
// In our example,
// threadIdx.x+1    = 1, 2, 3, 4, 5, 6, 7, 8
// stride = 1, index = 1, 3, 5, 7, 9, 11, 13, 15
```

## Putting it Together



# Post Scan Step (Distribution Tree)

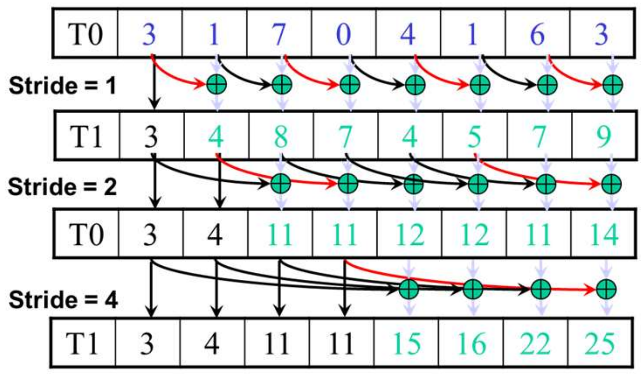
```
int stride = BLOCK_SIZE/2;
while(stride > 0) {
   __syncthreads();
   int index = (threadIdx.x+1)*stride*2 - 1;
   if ((index+stride) < 2*BLOCK_SIZE)
       T[index+stride] += T[index];
   stride = stride / 2;
 }
```

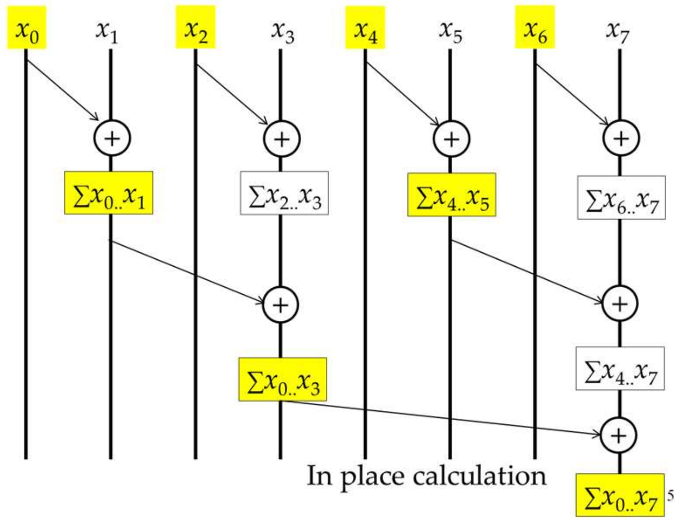
```
// In our example,
// BLOCK_SIZE=8 stride=4, 2, 1
// for first iteration, active thread = 0 index = 7, stride = 11
```

# Work Analysis

- The parallel Scan executes 2*log(n) parallel iterations
  - log(n) in reduction and log(n) in post scan
  - The iterations do n/2, n/4,..1, (2-1), …., (n/4-1), (n/2-1) useful adds
  - In our example, n = 16, the number of useful adds is 16/2 + 16/4 + 16/8 + 16/16 + (16/8-1) + (16/4-1) + (16/2-1)
  - Total adds: (n-1) + (n-2) – (log(n) -1) = 2*(n-1) – log(n)→ O(n) work
- The total number of adds is no more than twice of that done in the efficient sequential algorithm
  - The benefit of parallelism can easily overcome the 2× work when there is sufficient hardware

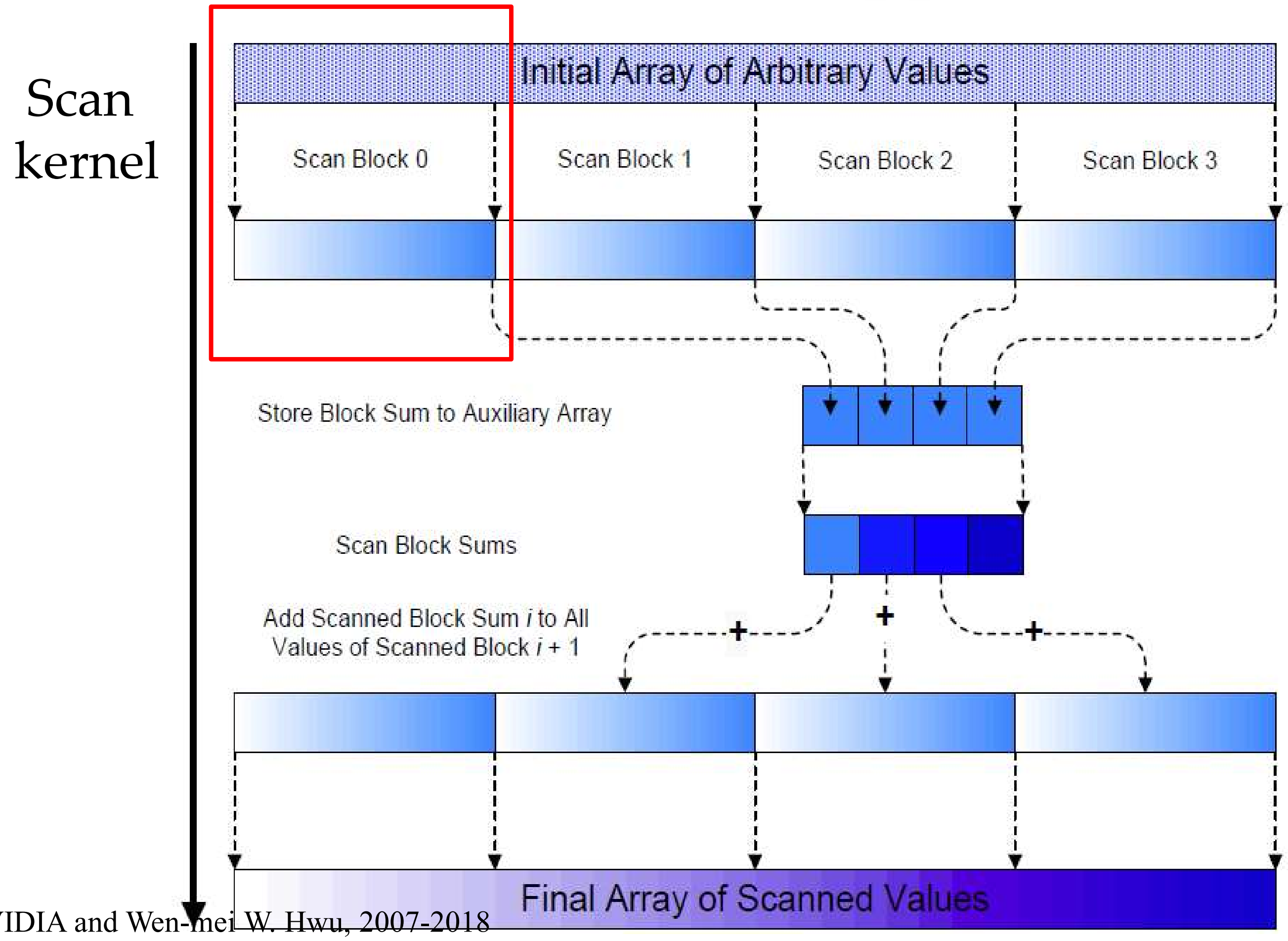
# Kogge-Stone vs. Brent-Kung





- Brent-Kung uses half the number of threads compared to Kogge-Stone
  - Each thread should load two elements into the shared memory
- Brent-Kung takes twice the number of steps compared to Kogge-Stone
  - Kogge-Stone is more popular for parallel scan with blocks in GPUs

## Overall Flow of Complete Scan
## A Hierarchical Approach


Scan kernel
Initial Array of Arbitrary Values
Scan Block 0
Scan Block 1
Scan Block 2
Scan Block 3
Store Block Sum to Auxiliary Array
Scan Block Sums
Add Scanned Block Sum *i* to All Values of Scanned Block *i* + 1
+
+
+
Final Array of Scanned Values

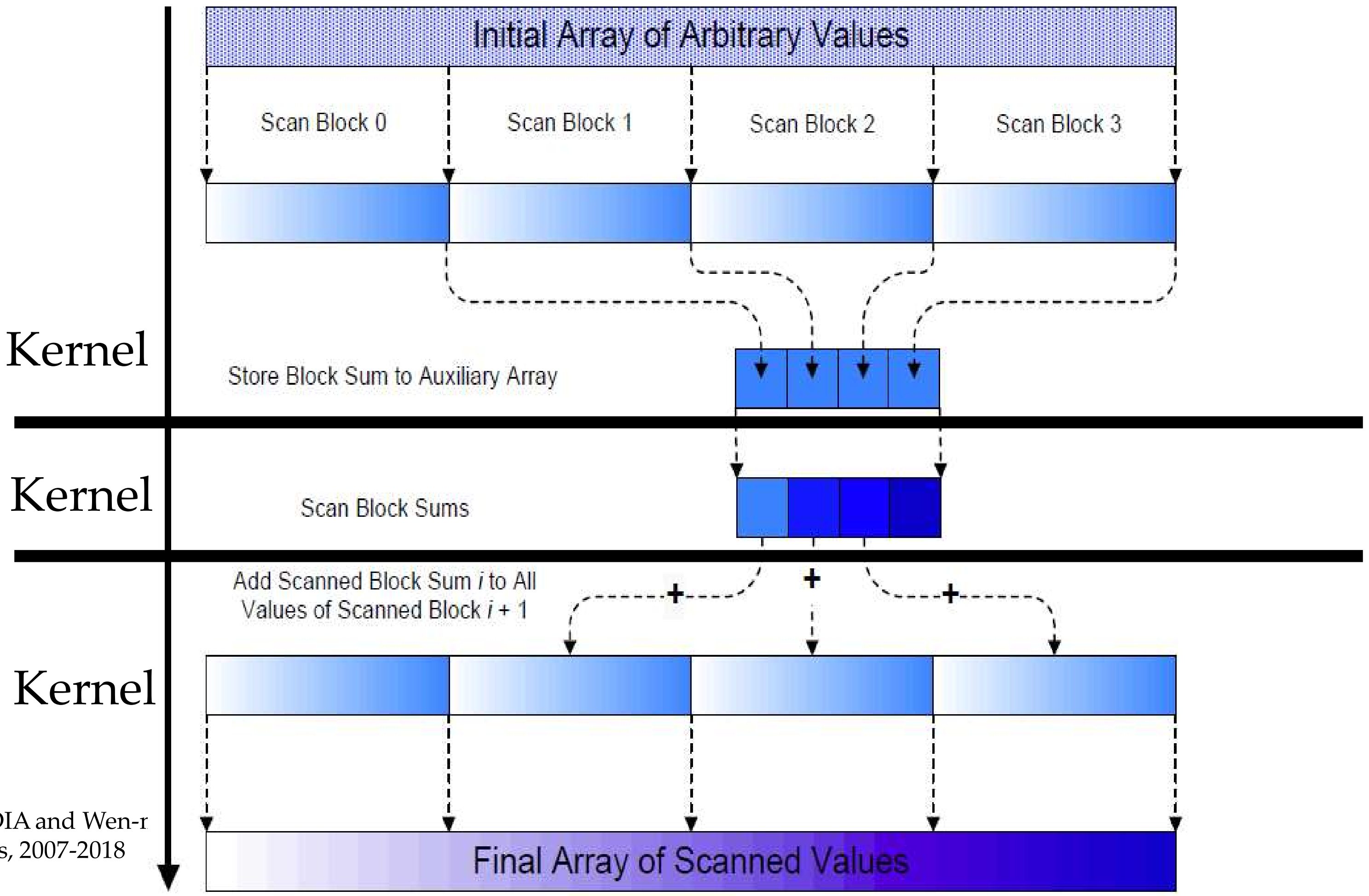
## Using Global Memory Contents in CUDA

- Data in registers and shared memory of one thread block are not visible to other blocks
- To make data visible, the data has to be written into global memory
- However, any data written to the global memory are not visible until a memory fence. This is typically done by terminating the kernel execution
- Launch another kernel to continue the execution. The global memory writes done by the terminated kernels are visible to all tead blocks.

## Scan of Arbitrary Length Input

- Build on the scan kernel that handles up to 2*blockDim.x elements from Brent-Kung
  - For Kogge-Stone, have each section of blockDim.x elements assigned to a block
- Have each block write the sum of its section into a Sum array using its blockIdx.x as index
- Run parallel scan on the Sum array
  - May need to break down Sum into multiple sections if it is too big for a block
- Add the scanned Sum array values to the elements of corresponding sections

# Overall Flow of Complete Scan A Hierarchical Approach


Initial Array of Arbitrary Values
Scan Block 0
Scan Block 1
Scan Block 2
Scan Block 3
Kernel
Store Block Sum to Auxiliary Array
Kernel
Scan Block Sums
Add Scanned Block Sum *i* to All Values of Scanned Block *i* + 1
+
+
+
Kernel
Final Array of Scanned Values

# (Exclusive) Scan Definition

**Definition:** *The exclusive* scan *operation takes a binary associative operator* $\oplus$, *and an array of n elements*

$$[x_0, x_1, \ldots, x_{n-1}]$$

*and returns the array*

$$[0, x_0, (x_0 \oplus x_1), \ldots, (x_0 \oplus x_1 \oplus \ldots \oplus x_{n-2})].$$

**Example:** If $\oplus$ is addition, then the exclusive scan operation on

[3 1 7 0 4 1 6 3],

would return [0 3 4 11 11 15 16 22].

## Why Exclusive Scan

- To find the beginning address of allocated buffers

- Inclusive and Exclusive scans can be easily derived from each other; it is a matter of convenience

[3 1 7 0 4 1 6 3]

Exclusive [0 3 4 11 11 15 16 22]

Inclusive [3 4 11 11 15 16 22 25]

## A simple exclusive scan kernel

- Adapt an inclusive, Kogge-Stone scan kernel
  - Block 0:
    - Thread 0 loads 0 into (shared) T[0]
    - Other threads load (global) X[threadIdx.x-1] into T[threadIdx.x]
  - All other blocks:
    - All thread load X[blockIdx.x*blockDim.x+threadIdx.x-1] into T[threadIdex.x]

- Similar adaption for Brent-Kung kernel but pay attention that each thread loads two elements
  - Only one zero should be loaded
  - All elements should be shifted by only one position

## QUESTIONS?

## READ CHAPTER 8!

## Problem Solving

- Q: Suppose we have a kernel function that performs partial sum reduction. The block dimension is (64,1,1). During the **second** iteration of the for loop, is there any warp that has a control divergence?

```
__shared__ int partialSum[2*BLOCK_SIZE];
// omitted code to fill partialSum from global memory
for (int stride = blockDim.x; stride >= 1; stride /= 2)  {
   __syncthreads();
   if (threadIdx.x < stride) partialSum[t] += partialSum[t + stride];
}
```

- A: No: stride 32 means warp 0 is fully active, while warp 1 is fully inactive

## Problem Solving

- Q: Consider a kernel that performs Brent-Kung scan algorithm and assume that there are 1024 elements in a section, and a warp size is 32. In which iteration will there be at least one warp that has a control divergence? The stride is 1 for the first iteration.

- A: In the 6th iteration: strides are 512, 256, 128, 64, 32, 16, … (and active threads have threadIdx.x < stride)

# Problem Solving

- Q: Suppose we use Brent-Kung in a hierarchical approach to perform parallel scan on a 1D input array of $2^{42}$ elements. We use 1024 threads per block in all our Brent-Kung kernels and our GPU supports at most 2048 blocks per grid. Each block processes 2048 elements. What is the best approximation of the number of floating-point add operations performed per thread block in the reduction and post scan steps in the Brent-Kung kernel (summed together)?

- A: 2048 x 2