Recursive Mind Model

Changsoo Shin

Preamble

We begin by defining the recursive function and examining its time and threshold derivatives.

At the foundation is the recursive Heaviside sequence function, defined as:

$$u(t) = 1 - H_{N_1}(-t + \tau_1 H_{N_2}(-t + \tau_2 H_{N_3}(-t + \tau_3 \cdots)))$$

Here, each Heaviside function includes a sharpness parameter N_i, allowing the recursive sequence to model layered but smooth thresholds. We then approximate each Heaviside step with a sigmoid function, yielding sigmoidal smoothing via:

$$H_{N_i}(t) = \frac{1}{1 + e^{-2N_i t}}$$

This modification allows for a more realistic representation of memory activation that is neither instantaneous nor perfectly sharp.

To preserve the correct causal structure of memory within our reversed time model, it is essential that each memory layer $u_i(t)$ is defined in the form

$$u_i(t) = 1 - H_{n_i}\left(-t + \tau_i(\cdots)\right),$$

where the argument of H_i may itself contain nested Heaviside functions from lower layers. This form ensures that memory is activated only after the corresponding event has occurred—even when time flows from $+\infty$ to $-\infty$.

Any formulation lacking the leading $1-$ would imply that the system begins to remember future events, violating causality. Hence, this structure is not arbitrary but logically and physically required.

Abstract

This paper introduces a recursive memory architecture based on nested Heaviside step functions, modeling how cognitive awareness and memory recall are temporally gated and amplified. We analyze the time and parameter sensitivity of the recursive structure by taking partial derivatives with respect to time, threshold values τ_i, and sharpness parameters N_i. Through worked examples and visualizations, we demonstrate how deeper memory layers contribute to surface cognition only when upper-layer thresholds are simultaneously active. This mathematical formulation offers a novel perspective on mental processes such as deep memory emergence, cognitive gating, and emotional amplification.

1 Introduction

What does it mean to remember? In human cognition, memory is neither a flat archive nor a randomly accessed dataset. Rather, it is a layered, thresholded architecture—recursive in form and conditional in access. Certain memories surface only when higher-level conditions are met, much like doors that open only when prior gates are passed. This behavior, though well observed in cognitive psychology, remains difficult to capture in formal mathematical terms.

In this work, we present a recursive model of memory and cognitive activation using nested Heaviside step functions. Each layer in the structure represents a thresholded cognitive gate, defined by two key parameters: a transition threshold τ_i, and a sharpness factor N_i, which controls how abruptly the transition occurs. This recursive formulation not only captures the nested logic of attention, emotion, and awareness, but also reflects the asymmetric nature of recall—memories can be activated forward in time, but rarely reconstructed backward from output.

From a mathematical standpoint, the model defines a function:

$$u(t) = 1 - H_{N_1}\left(-t + \tau_1 H_{N_2}\left(-t + \tau_2 \cdots H_{N_n}(-t + \tau_n)\right)\right)$$

where H_{N_i} is a smoothed Heaviside function with sharpness N_i, and τ_i is the cognitive activation threshold for the i-th layer. The recursive nature of this construction ensures that lower-layer activation is conditional on upper-layer thresholds being satisfied—a property that mirrors delayed memory emergence and context-dependent recall in human experience.

We explore how time (t), thresholds (τ_i), and sharpness (N_i) influence output by computing and visualizing partial derivatives. Notably, we find that deeper parameters only contribute to the output signal when higher-layer gates are simultaneously active. This results in a form of temporal gating and nested amplification, wherein inner memories remain silent unless sufficiently cued by current cognitive states.

Finally, we show that this architecture is fundamentally non-invertible. The output $u(t)$ cannot be uniquely decomposed into its internal structure—mirroring the fact that, in consciousness, we cannot infer all latent causes of a thought or memory from its appearance alone. This structural non-invertibility echoes limitations in both cognitive introspection and AI interpretability, suggesting deeper insights into the fabric of mental computation.

2 Recursive Structure of Thought and Self-Tracking and Recursive Identity

2.1 Recursive Self-Track and Meta-Awareness

Human consciousness is not only reactive to external stimuli but also reflexive—it possesses the capacity to observe and evaluate its own mental states. This inward reflection gives rise to what we refer to as *self-track*: a recursive process by which awareness loops back to monitor and interpret its own activity. In doing so, the mind constructs second-order representations of thoughts, memories, and emotions.

In a recursive cognitive model, self-tracking occurs when one cognitive threshold is defined in terms of another, forming a loop of mutual dependency. This structure gives rise to what we call a *recursive identity*—a formal mechanism through which self-reference is encoded within the layers of cognition.

Such recursive dependence underlies *meta-awareness*, wherein the mind becomes both actor and observer. A threshold responds not only to external events but also to its own history of activations. This dual role forms the basis for self-conscious thought, enabling the emergence of memory-of-awareness and higher-order introspection.

To model the propagation of such recursive awareness, we consider how each layer of cognition evolves in response not only to external time t, but also to its internal threshold parameter τ_i. This dynamic layering can be formalized through an n-dimensional advection equation, capturing the flow of cognition across recursive thresholds. Importantly, we adopt a temporal framework in which time flows from positive to negative infinity ($+\infty \to -\infty$). This reversed orientation captures the retrospective character of human awareness: present cognition is understood as the accumulated result of previously activated thresholds. As time proceeds backward, the system recursively folds inward, tracing the structure of mental formation from surface-level awareness $u_1(t)$ down to the deepest layer $u_n(t)$.

Each recursive function $u_i(t)$ evolves not only with respect to time t, but also with respect to its associated activation threshold τ_i, which governs when a cognitive transition occurs. These transitions are gated by smoothed Heaviside functions, which conditionally switch on when threshold criteria are met. The resulting layered dynamics can be formalized by an n-dimensional advection-type equation:

$$\frac{\partial u_i}{\partial t} + \sum_{i=n}^{1} (-1)^{i+1} H(-t + \tau_i) \frac{\partial u_i}{\partial \tau_i} = 0 \tag{1}$$

Here, the term $\dfrac{\partial u_i}{\partial t}$ represents the temporal change in the i-th cognitive layer, while each summation term reflects how sensitivity to threshold shifts τ_i is activated conditionally via $H(-t+\tau_i)$. The alternating signs $(-1)^{i+1}$ encode recursive modulation—indicating that awareness is not linearly accumulated but recursively filtered and reweighted as it travels through cognitive layers.

The term $\partial u_i/\partial t$ in Eq. (1) captures the temporal evolution of the outermost cognitive layer, while each term in the summation represents the influence of inner thresholds. The alternating sign $(-1)^{i+1}$ encodes the recursive interplay between facilitative and suppressive dynamics across layers—where some layers amplify downstream awareness and others inhibit or delay it.

This advection structure models how memory, attention, and introspective feedback propagate through nested thresholds, forming a layered framework for recursive identity and self-monitoring.

To make this structure explicit, the cognitive state at each layer is defined as $u_i(t) = 1-h_i(t)$, where $h_i(t)$ denotes the activated state of the i-th layer. The recursive composition of these layers proceeds from the deepest layer upward:

$$
\begin{aligned}
h_n(t) &= H_{N_n}\left(-t+\tau_n\right), \\
h_{n-1}(t) &= H_{N_{n-1}}\left(-t+\tau_{n-1}\cdot h_n(t)\right), \\
h_{n-2}(t) &= H_{N_{n-2}}\left(-t+\tau_{n-2}\cdot h_{n-1}(t)\right), \\
&\vdots \\
h_1(t) &= H_{N_1}\left(-t+\tau_1\cdot h_2(t)\right), \\
u_1(t) &= 1-h_1(t), \\
u_2(t) &= 1-h_2(t), \\
u_3(t) &= 1-h_3(t), \\
&\vdots \\
u_n(t) &= 1-h_n(t).
\end{aligned}
\tag{2}
$$

Each function $h_i(t)$ defines a thresholded layer of awareness that is recursively modulated by deeper layers. The outermost layer $h_1(t)$ thus encapsulates the entire structure of introspective recursion—serving as the final form of the self-track function.

We denote the total recursive memory structure by $u_n(t;\tau_1,\tau_2,\ldots,\tau_n)$, which represents the stratified cognitive state across multiple thresholds over time. Each term of the form $H(-t+\tau_i)\dfrac{\partial u}{\partial \tau_i}$ reflects how a deeper thresholded event influences the present awareness stream.

When a threshold recursively depends on itself—directly or indirectly—we refer to this process as *self-track*. This transition from linear to recursive dynamics marks the emergence of introspective cognition, where the mind not only stores and reacts to events, but also becomes aware of its own awareness.

This recursive structure is essential for modeling *meta-memory*—the memory of having remembered—and provides the foundation for a mathematical theory of subjective consciousness.

In the next section, we formally introduce the core mechanism driving this recursive awareness: the Recursive Heaviside Function.

Self-Tracking Structure and Temporal Windowing. The phenomena of temporal localization and amplification, observed in the partial derivatives with respect to threshold parameters τ_i and sharpness parameters N_i, emerge directly from the recursive structure of the self-tracking equation:

$$u_{N_1}(t) = 1 - H_{N_1}\left(-t + \tau_1 H_{N_2}\left(-t + \tau_2 H_{N_3}\left(\cdots H_{N_n}(-t + \tau_n)\right)\right)\right) \tag{3}$$

In this formulation, each threshold τ_i governs a layer whose activation is conditionally dependent on the activation of the subsequent (deeper) layer. That is, for τ_i to influence the output, the condition $-t + \tau_{i+1} > 0$ must be met, which in turn activates the inner Heaviside $H_{N_{i+1}}$. This recursive dependency imposes a *temporal window*, where each layer can only become active after the deeper layers are already engaged.

For example, the derivative $\partial u/\partial \tau_i$ includes a multiplicative term like $H_{N_{i+1}}(-t+\tau_{i+1})$, which functions as a gate or switch. This ensures that the effect of τ_i is conditionally masked or revealed based on whether the deeper layer τ_{i+1} has been activated.

Such recursive gating not only ensures localized temporal response (activation only near the appropriate τ_i) but also preserves causal ordering among memory events. These are essential features of any architecture aiming to model structured memory formation and introspective dynamics.

2.2 Defining the Recursive Heaviside Function

At the core of our cognitive model lies a recursive function constructed from nested Heaviside step functions. This structure models how memories or perceptions accumulate over time, with each layer gated by its own cognitive threshold:

$$u(t) = 1 - H_{N_1}\left(-t + \tau_1 H_{N_2}\left(-t + \tau_2 H_{N_3}\left(-t + \tau_3 \cdots\right)\right)\right) \tag{4}$$

Here, each threshold τ_i represents the moment at which a specific memory or awareness state becomes activated. The outermost Heaviside function H_{N_1} only activates if all inner thresholds are satisfied, enforcing a strict temporal and logical hierarchy.

To facilitate smooth analysis and differentiation, each Heaviside step function is approximated by a sigmoid-like function:

$$\sigma_N(x) = \frac{1}{1+\exp(-2Nx)} \tag{5}$$

Replacing each $H_{N_i}(x)$ with $\sigma_{N_i}(x)$ yields a continuous and differentiable recursive approximation. This smoothed version preserves the original step-like behavior while enabling analytic exploration of how sensitivity to thresholds and sharpness propagates across recursive layers.

(a) Explicit Nested Form

$$u(t) = 1 - \frac{1}{1+\exp\left(-2N_1\left(-t+\tau_1\cdot\frac{1}{1+\exp\left(-2N_2\left(-t+\tau_2\cdot\frac{1}{1+\exp(-2N_3(-t+\tau_3\cdots))}\right)\right)}\right)\right)} \tag{6}$$

(a) Explicit Nested Form

$$u(t) = 1 - \frac{1}{1+\exp\left(-2N_1\left(-t+\tau_1\cdot\frac{1}{1+\exp\left(-2N_2\left(-t+\tau_2\cdot\frac{1}{1+\exp(-2N_3(-t+\tau_3\cdots))}\right)\right)}\right)\right)} \tag{7}$$

(b) Recursive Function Definition

$$h_n(t) = \frac{1}{1+\exp\left(-2N_n\left(-t+\tau_n\cdot h_{n+1}(t)\right)\right)}$$
$$u_i(t) = 1 - h_i(t), i = 1, \ldots.., n \tag{8}$$

This smoothed recursive sigmoid representation ensures continuity and differentiability, allowing both analytical and numerical analysis. As $N_i \to \infty$, the sigmoid functions converge to ideal step functions, recovering the original nested Heaviside form.

This recursive formulation models layered cognition as follows:

- The innermost τ_n corresponds to a base-level stimulus or memory.
- τ_{n-1} encodes a conditional response to that base layer.
- Higher layers represent interpretations, reflective awareness, and cognitive control.

Each activation is thus contingent upon the successful activation of deeper layers. The structure captures cognitive phenomena such as dependency, temporal delay, and emergent awareness.

While the recursive function appears analytically defined, it is fundamentally *non-invertible* in the practical sense. In theory, if each sharpness parameter N_i is finite and known, the function $u(t)$ can be expanded into an infinite Taylor series and, in principle, reconstructed from its derivatives. However, this reconstruction would require infinite precision and complete knowledge of all thresholds and parameters—conditions that are neither physically realizable nor cognitively plausible.

Thus, even though formal invertibility exists in a mathematical sense, the function remains effectively irreversible. This reflects the asymmetry of lived experience: past cognitive layers influence the present, but cannot be uniquely recovered once integrated into the recursive structure.

In the following section, we analyze how derivatives of this function—with respect to both time and cognitive thresholds—model memory recall, forgetting, and emotional resonance.

3 Derivatives and Cognitive Dynamics

To understand how cognition evolves over time and interacts with past experiences, we examine the derivatives of the recursive Heaviside function. Three key types of derivatives provide insight into the mental process:

3.1 Time Derivative $\partial u/\partial t$

This reflects the momentary change in cognitive awareness. It captures when a certain threshold is crossed in real time, triggering a sharp cognitive event. The resulting expressions often involve Dirac delta functions and their compositions.

$$h_i(t) = H_{N_i}(-t + \tau_i) \tag{9}$$

Each function $h_i(t)$ is a smoothed Heaviside step function with sharpness N_i and threshold τ_i, activated as time t decreases past τ_i. The use of $-t + \tau_i$ reflects the retrospective nature of awareness, where the present integrates the effect of prior events.

As shown in Figure 1, the time derivative reveals moments of sudden awareness activation, tightly localized near each threshold τ_i.

3.2 Threshold Derivatives $\partial u/\partial \tau_i$ (Gating Effects)

The threshold derivative $\partial u/\partial \tau_i$ quantifies how the timing of a cognitive event affects the recursive function $u(t)$. Since each threshold τ_i defines the onset of a particular mental layer, small changes in τ_i modulate whether that layer activates early or late in the recursive process.

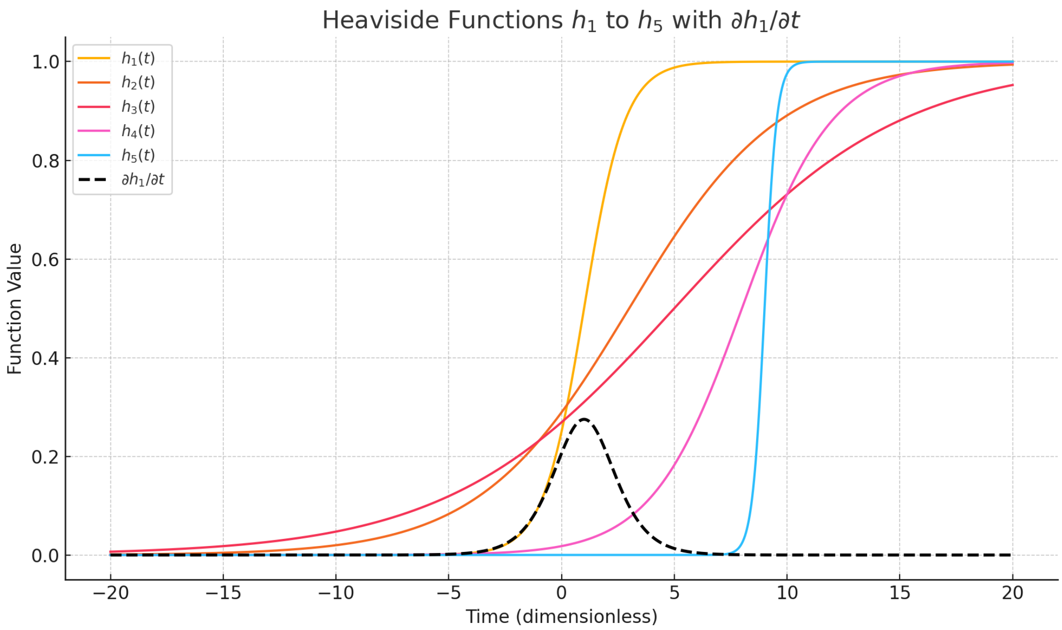

Figure 1: Partial derivatives of the recursive Heaviside memory model. The five solid curves represent the smoothed Heaviside components $h_1(t)$ to $h_5(t)$, each defined as $h_i(t) = H_{N_i}(-t + \tau_i)$. The dashed black curve represents the total time derivative $\partial u/\partial t$, showing sharp peaks at each τ_i and illustrating the temporal localization of memory events.

In a 3-layer recursive Heaviside function:

$$u(t) = 1 - H_{N_1}\left(-t + \tau_1 H_{N_2}\left(-t + \tau_2 H_{N_3}(-t + \tau_3)\right)\right)$$

The derivative with respect to τ_3—the innermost threshold—is given by:

$$\frac{\partial u}{\partial \tau_3}(t) = -\delta(A_1) \cdot \tau_1 \cdot \delta(A_2) \cdot \tau_2 \cdot \delta(A_3) \tag{10}$$

where:

$$A_3 = -t + \tau_3, \quad A_2 = -t + \tau_2 H_{N_3}(A_3), \quad A_1 = -t + \tau_1 H_{N_2}(A_2)$$

This expression shows that the effect of τ_3 is gated by the higher-level thresholds τ_1 and τ_2. The delta functions are only active when all inner layers are aligned—i.e., when the nested Heaviside conditions are simultaneously satisfied. This multiplicative structure models **temporal gating**: sensitivity to deep events is revealed only through the recursive opening of upper layers.

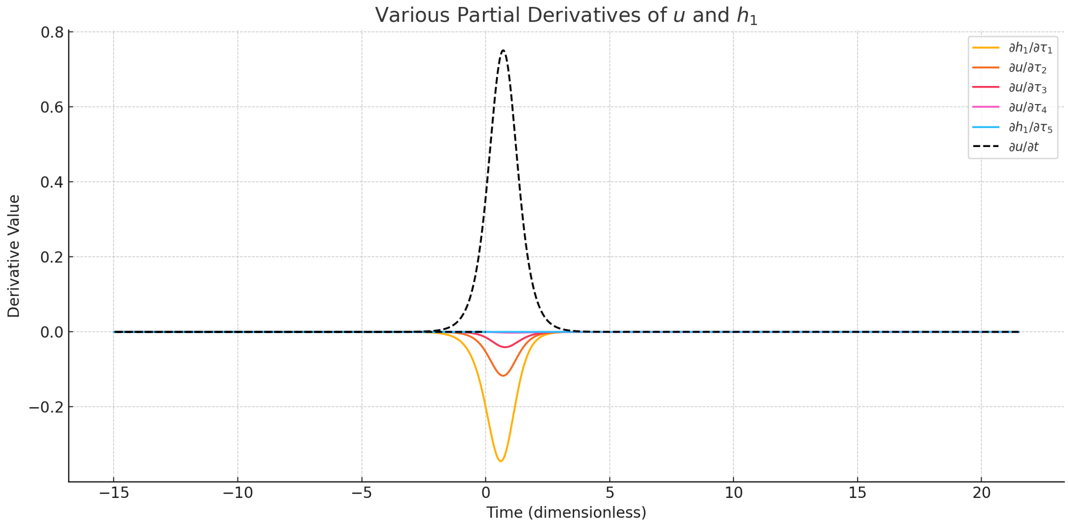

Figure 2: Partial derivatives of the recursive Heaviside components. The plot shows $\partial h_1/\partial\tau_1$, $\partial u/\partial\tau_2$, $\partial u/\partial\tau_3$, $\partial u/\partial\tau_4$, and $\partial h_1/\partial\tau_5$ as well as the time derivative $\partial u/\partial t$ (dashed line). Each peak corresponds to a transition point τ_i, and the magnitude reflects the sharpness N_i of the smoothed Heaviside step function.

3.3 Threshold Derivative** $\partial u/\partial\tau_i$

In order to analyze the sensitivity of the recursive Heaviside-based memory components to their respective threshold parameters τ_i, we computed several partial derivatives. Figure 2 presents the results, where the curve labeled $\partial h_1/\partial\tau_1$ represents the derivative of the first Heaviside component $h_1(t) = H_{N_1}(t-\tau_1)$ with respect to its own threshold. This derivative exhibits a smooth peak centered around $t = \tau_1$, and its shape reflects the sharpness parameter N_1.

Additionally, the figure shows the partial derivatives $\partial u/\partial\tau_2$, $\partial u/\partial\tau_3$, and $\partial u/\partial\tau_4$, which measure how the combined function $u(t) = \sum_i h_i(t)$ changes in response to variations in the respective thresholds. Although the function $h_1(t)$ does not directly depend on τ_5, we include $\partial h_1/\partial\tau_5$ to illustrate the lack of influence from unrelated thresholds.

For comparison, the sixth curve (dashed) in the figure corresponds to the time derivative $\partial u/\partial t$, which features multiple sharp peaks at each τ_i. This derivative captures the instantaneous rate of change of the composite function $u(t)$ and reflects the temporal localization induced by the smoothed step transitions.

* This describes how sensitive the entire mental structure is to the timing of a particular memory or stimulus. * It represents the influence of modifying a single cognitive threshold.

3.4 Sharpness Derivative $\partial u/\partial N_i$

In addition to time and threshold derivatives, we examine how the cognitive structure responds to variations in the sharpness parameter N_i, which controls the steepness of each smoothed Heaviside transition. To illustrate this effect, we focus on the partial derivative of the recursive function with respect to N_3 in a three-layer structure:

$$u(t) = 1 - H_{N_1}\left(-t + \tau_1 H_{N_2}\left(-t + \tau_2 H_{N_3}(-t + \tau_3)\right)\right). \tag{11}$$

The innermost component $H_{N_3}(-t + \tau_3)$ is defined as:

$$H_{N_3}(-t + \tau_3) = \frac{1}{1 + \exp\left(-2N_3(-t + \tau_3)\right)}, \tag{12}$$

whose derivative with respect to N_3 becomes:

$$\frac{\partial}{\partial N_3} H_{N_3}(-t + \tau_3) = \frac{(t - \tau_3) \cdot \exp\left(2N_3(t - \tau_3)\right)}{\left(1 + \exp\left(2N_3(t - \tau_3)\right)\right)^2}. \tag{13}$$

This function peaks sharply near $t = \tau_3$, and vanishes rapidly as time moves away from the threshold. Under our temporal model in which time flows from future to past ($t \to -\infty$), this derivative grows increasingly influential as the system approaches τ_3 from the future side. Thus, even modest changes in sharpness can generate amplified responses when the recursive structure is near cognitive activation.

When this sharpness derivative is embedded into the recursive hierarchy, the full derivative of $u(t)$ with respect to N_3 is given by:

$$\frac{\partial u}{\partial N_3} = \delta_{N_1}(A) \cdot \tau_1 \delta_{N_2}(B) \cdot \tau_2 \cdot \frac{2(t - \tau_3)}{\left(e^{N_3(t-\tau_3)} + e^{-N_3(t-\tau_3)}\right)^2}, \tag{14}$$

where the intermediate arguments are defined as:

$$A = -t + \tau_1 H_{N_2}(B), \quad B = -t + \tau_2 H_{N_3}(-t + \tau_3). \tag{15}$$

This result illustrates that deeply nested sharpness parameters can still influence the output through recursive activation pathways. The localized response around τ_3 reflects the structure's sensitivity to precision modulation, especially in the context of emotion, trauma, or delayed cognitive recall.

Worked Example: Amplification and Temporal Gating in a 3-Layer Structure

To illustrate the role of nested sharpness and threshold parameters in recursive memory models, we consider a 3-layer function:

$$u(t) = 1 - H_{N_1}\left(-t + \tau_1 H_{N_2}\left(-t + \tau_2 H_{N_3}(-t + \tau_3)\right)\right) \tag{16}$$

We set:

$$\tau_1 = 1.0, \quad \tau_2 = 1.5, \quad \tau_3 = 2.0, \quad N_1 = 1.0, \quad N_2 = 0.3, \quad N_3 = 2.0 \tag{17}$$

Let the recursive activation arguments be defined as:

$$A_3 = -t + \tau_3, \quad A_2 = -t + \tau_2 H_{N_3}(A_3), \quad A_1 = -t + \tau_1 H_{N_2}(A_2) \tag{18}$$

Derivative with respect to the sharpness parameter N_3:

$$\frac{\partial u}{\partial N_3}(t) = -\delta(A_1) \cdot \tau_1 \cdot \delta(A_2) \cdot \tau_2 \cdot \frac{\partial H_{N_3}(A_3)}{\partial N_3} \cdot \delta(A_3) \tag{19}$$

Derivative with respect to the threshold parameter τ_3:

$$\frac{\partial u}{\partial \tau_3}(t) = -\delta(A_1) \cdot \tau_1 \cdot \delta(A_2) \cdot \tau_2 \cdot \delta(A_3) \tag{20}$$

These two expressions demonstrate distinct behaviors: - The term $\partial u / \partial N_3$ includes both sensitivity and amplification effects due to nesting. - The term $\partial u / \partial \tau_3$ reflects sharp localization near the innermost threshold.

Worked Example: Amplification and Temporal Gating in a 3-Layer Structure

To illustrate the role of nested sharpness and threshold parameters in recursive memory models, we consider a 3-layer function:

$$u(t) = 1 - H_{N_1}\left(-t + \tau_1 H_{N_2}\left(-t + \tau_2 H_{N_3}(-t + \tau_3)\right)\right) \tag{21}$$

We set:

$$\tau_1 = 1.0, \quad \tau_2 = 1.5, \quad \tau_3 = 2.0, \quad N_1 = 1.0, \quad N_2 = 0.3, \quad N_3 = 2.0 \tag{22}$$

Let the recursive activation arguments be defined as:

$$A_3 = -t + \tau_3, \quad A_2 = -t + \tau_2 H_{N_3}(A_3), \quad A_1 = -t + \tau_1 H_{N_2}(A_2) \tag{23}$$

Derivative with respect to the sharpness parameter N_3:

$$\frac{\partial u}{\partial N_3}(t) = -\delta(A_1) \cdot \tau_1 \cdot \delta(A_2) \cdot \tau_2 \cdot \frac{\partial H_{N_3}(A_3)}{\partial N_3} \cdot \delta(A_3) \tag{24}$$

Derivative with respect to the threshold parameter τ_3:

$$\frac{\partial u}{\partial \tau_3}(t) = -\delta(A_1) \cdot \tau_1 \cdot \delta(A_2) \cdot \tau_2 \cdot \delta(A_3) \tag{25}$$

These two expressions demonstrate distinct behaviors: - The term $\partial u/\partial N_3$ includes both sensitivity and amplification effects due to nesting. - The term $\partial u/\partial \tau_3$ reflects sharp localization near the innermost threshold.

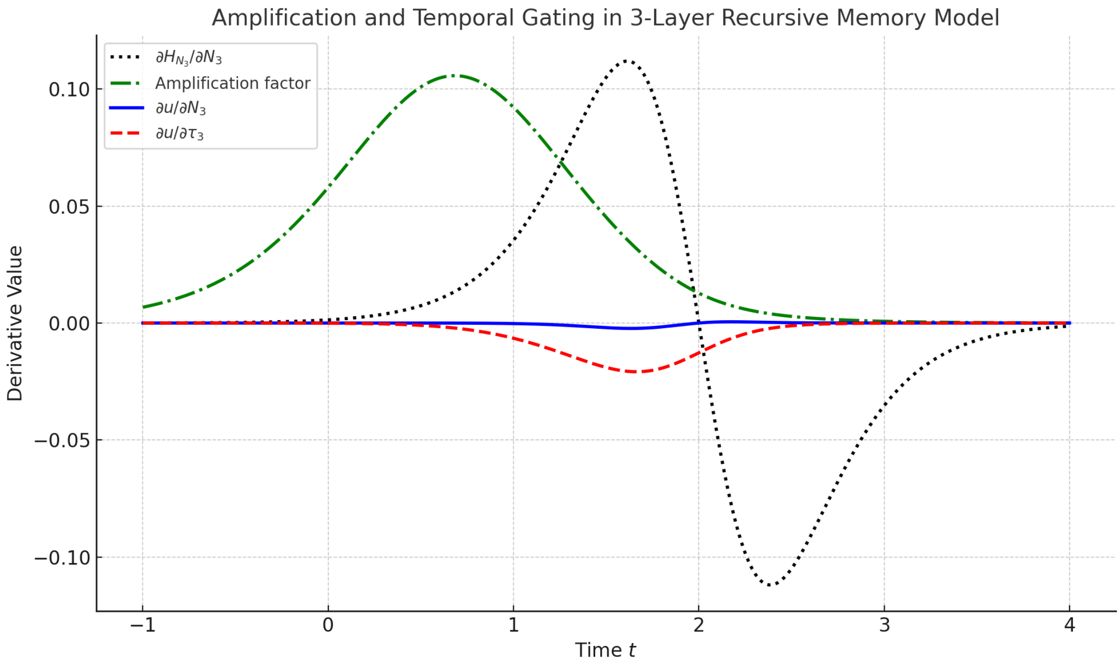

Figure 3: Amplification effects in a 3-layer recursive memory model with parameters $\tau_1 = 1.0$, $\tau_2 = 1.5$, $\tau_3 = 2.0$, and $N_1 = 1.0$, $N_2 = 0.3$, $N_3 = 2.0$. The dotted black curve shows $\partial H_{N_3}/\partial N_3$, the dash-dotted green curve shows the amplification factor, the solid blue curve is $\partial u/\partial N_3$, and the dashed red curve is $\partial u/\partial \tau_3$.

As shown in Figure 3, the recursive architecture amplifies the influence of deep-layer parameters such as N_3 only when upper-layer thresholds τ_1 and τ_2 are also activated. In contrast, the threshold sensitivity $\partial u/\partial \tau_3$ remains temporally localized, demonstrating strong gating characteristics. The partial derivative of the recursive function with respect to N_i in a n layered structure is given C.

3.5 Cross-Threshold Derivatives $\partial^2 u/\partial \tau_i \partial \tau_j$

While mixed derivatives with respect to time and threshold describe how real-time awareness responds to latent memories, the second-order derivatives with respect to two threshold parameters—$\partial^2 u/\partial \tau_i \partial \tau_j$—capture interactions between distinct cognitive layers. These terms quantify how a small shift in one memory threshold modulates the sensitivity of another, and are essential in modeling emotionally or contextually coupled memories.

Such cross-threshold derivatives become particularly meaningful when the recursive structure includes repeated or closely spaced thresholds:

$$u(t) = 1 - H_{N_1}(-t + \tau_1 H_{N_2}(-t + \tau_2 H_{N_3}(-t + \tau_3))). \tag{26}$$

In this case, the derivative $\partial^2 u/\partial\tau_2\partial\tau_3$ includes:

$$\frac{\partial^2 u}{\partial\tau_2\partial\tau_3} = \delta_{welN_1}(A) \cdot \tau_1 \frac{\partial}{\partial\tau_2}\left[\delta_{N_2}(B) \cdot \tau_2 \frac{\partial H_{N_3}(-t+\tau_3)}{\partial\tau_3}\right], \tag{27}$$

with:

$$A = -t + \tau_1 H_{N_2}(B), \quad B = -t + \tau_2 H_{N_3}(-t + \tau_3). \tag{28}$$

The resulting expressions involve products of delta functions and their derivatives. These terms are sharply localized in time, and they express how the shift of one memory layer (e.g., τ_3) influences another (e.g., τ_2). When both thresholds are close or repeated, the interaction is amplified—this corresponds to linked or compounded memory effects, such as emotionally charged associative recall or trauma chaining.

Moreover, the appearance of these cross-threshold terms supports the interpretation that recursive cognition is not simply a collection of independent activations, but a deeply entangled structure with nontrivial second-order dependencies across layers.

3.6 Mixed Derivative as a Measure of Trauma Amplification

In the recursive Heaviside framework, the sharpness parameter N_i controls the steepness of memory encoding, while the threshold τ_j indicates the temporal position of a cognitive event. Individually, their partial derivatives describe the sensitivity of the mental potential $u(t)$ to changes in shape or time. However, the **mixed partial derivative**,

$$\frac{\partial^2 u}{\partial\tau_j\partial N_i},$$

provides a deeper interpretation: it captures **how much a traumatic memory becomes amplified** due to the interaction between its timing and sharpness.

When this mixed derivative is large in magnitude, it signifies that even small changes in the event timing τ_j, when combined with fluctuations in the sharpness N_i, result in significant changes in the mental potential field. In other words, the cognitive system reacts disproportionately—amplifying the trauma's influence far beyond its original boundaries.

This quantity can be seen as a formal indicator of **unstable or unresolved trauma**. It may help identify which events continue to exert excessive influence due to recursive reactivation and insufficient resolution. Therapeutically, minimizing the magnitude of this mixed derivative could represent a pathway toward cognitive stabilization, containment, or integration.

3.7 Precursor Signal in Early Recursive Layers

Certain cognitive phenomena involve subtle sensations or early neural responses that precede conscious awareness. These are often described behaviorally as unexplained alertness or intuition. Within our recursive Heaviside framework, such effects can be interpreted as *precursor signals*—nonzero activations in earlier layers that occur even before the full recursive structure has been engaged.

Consider the function:

$$u(t) = 1 - H_{N_1}\left(-t + \tau_1 H_{N_2}\left(-t + \tau_2 H_{N_3}\left(-t + \tau_3 \cdots\right)\right)\right) \tag{29}$$

The partial derivative with respect to the second threshold is given by:

$$\frac{\partial u}{\partial \tau_2} = -\delta_{N_1}(A_1) \cdot \tau_1 \delta_{N_2}(A_2) \cdot H_{N_3}(\cdot) \tag{30}$$

Here, $A_1 = -t + \tau_1 H_{N_2}(\cdots)$ and $A_2 = -t + \tau_2 H_{N_3}(\cdots)$. This expression reveals that even if the innermost layer (e.g., at τ_3) has not yet been activated—meaning H_{N_3} remains below 1—the outer layers can still contribute a nonzero derivative. The presence of δ_{N_1} and δ_{N_2}, modulated by the recursive coefficients like τ_1, produces partial activations before full memory formation.

Thus, the model predicts localized sensitivity or signal emergence at early stages, despite incomplete recursion. We interpret this behavior as a *precursor signal*—a mathematical representation of early cognitive reactivity to latent or unrecognized stimuli.

This signal may underlie intuitive judgments or unconscious alertness, especially in contexts where the full structure of experience is not yet consciously assembled but the brain is already reacting subtly to potential input. It also provides a possible mechanistic interpretation of anecdotal "gut feelings" or the psychological construct of anticipatory awareness.

4 Dreams and Residual Terms of Thought

The recursive structure of cognition not only accounts for conscious responses but also explains unconscious phenomena such as dreams. From a mathematical perspective, dreams can be interpreted as *residual terms*—non-vanishing components that emerge when recursive layers fail to fully cancel or resolve within the recursive Heaviside formulation.

These residuals typically arise under conditions where:

- A memory threshold τ_i has ambiguous or weak activation due to a small sharpness parameter N_i.
- Mixed derivatives such as $\partial^2 u/\partial t \partial \tau_i$ yield non-zero values even in the absence of external stimuli.
- Recursive cancellation is incomplete, allowing unstructured fragments to persist.

In this framework, dreams are not interpreted as random or meaningless. Rather, they are structured spillovers—remnants of the recursive process that remain unresolved in waking thought. The deeper the recursion and the more varied the threshold values, the more complex and symbolic the resulting dream experience.

Furthermore, during sleep, inhibitory control over higher-threshold layers may be reduced, enabling latent structures—typically suppressed by conscious reasoning—to emerge. These structures often defy linear narrative logic and arise from unresolved recursive flows operating beneath conscious awareness.

In more extreme cases, such as trauma, when a particular threshold τ_i is repeated with high sharpness, these residuals may dominate the mental structure, resulting in recurring dream patterns or involuntary flashbacks. Hence, dreams reflect both the layered memory architecture and the system's inability to fully resolve recursive cognition in the absence of external feedback.

A formal derivation of these residual terms, based on the time derivative of a three-layer recursive Heaviside function, is provided in Appendix B. There, we show how incomplete cancellations at various thresholds mathematically give rise to the structure and timing of dream events.

Trauma and the Evolution of Sharpness N_i

In the recursive Heaviside model, trauma is not defined solely by the intensity of a single event, but by the way that event recursively imprints itself across time. Each threshold τ_i is associated with a sharpness parameter N_i, which controls how abruptly the Heaviside function transitions. While N_i was originally treated as a static parameter, we now interpret it as an evolving quantity—changing effectively over time due to the accumulation and reinforcement of cognitive injury, though not necessarily defined as an explicit function $N_i(t)$.

Initial Encoding

Immediately following a highly emotional or painful experience, the sharpness N_i may be large. This produces a steep transition—signifying acute awareness or shock. Such sharp responses may saturate quickly and be suppressed in the short term.

Recursive Reinforcement and Decay

As the recursive structure of cognition repeatedly references the same threshold τ_i, the effective sharpness N_i begins to decrease. This gradual reduction reflects:

- Accumulated recursive layering without resolution.
- Repeated subconscious activation through dreams, triggers, or internal loops.
- Loss of logical containment, leading to broader temporal spread.

Over time, a once-sharp response becomes a wide, lingering slope in the recursive cognitive field. This is precisely what makes trauma persistent: it is no longer about a sharp edge, but about **a long unresolved presence** within the recursive memory structure.

Implications for Healing

This evolving interpretation of N_i offers a new target for therapy. Rather than attempting to "desensitize" a sharp reaction, treatment may instead aim to:

- Prevent further decay of N_i (i.e., stop widening of memory activation).
- Introduce recursive interference layers that counteract the uncontrolled spread.
- Reassign temporal boundaries to contain the recursive feedback.

Thus, trauma is not simply encoded in the strength of emotion, but in the **evolution of sharpness over time**. What begins as a sharp wound may, if left unresolved, flatten into a wide, irreversible groove in the cognitive field—mathematically modeled by a decreasing N_i that expands the temporal footprint of suffering.

Mixed Derivative as a Measure of Trauma Amplification

In this context, the mixed partial derivative of the cognitive potential field $u(t)$ with respect to both the event threshold τ_j and the sharpness parameter N_i,

$$\frac{\partial^2 u}{\partial \tau_j \partial N_i}, \tag{31}$$

emerges as a mathematical indicator of trauma amplification. This term captures how much a traumatic event's cognitive impact is intensified due to the combined effects of timing and encoding sharpness.

When this mixed derivative is large in magnitude, it implies that small changes in the timing of the event τ_j, when coupled with shifts in sharpness N_i, lead to disproportionately large changes in mental potential. This reflects a form of emotional instability or hypersensitivity—where the trauma resonates deeply within the recursive memory field.

Therapeutically, this offers a way to diagnose the most amplified and unstable memory layers. High values of $\partial^2 u/\partial\tau_j\partial N_i$ may indicate unresolved cognitive loops or layers where trauma is persistently reactivated. Treatment might aim to reduce this mixed derivative, thereby dampening the over-amplified feedback and stabilizing the system.

5 Free Will, Emotion, and Individuality

The mathematical framework of recursive cognition naturally gives rise to deeper questions surrounding personal agency, emotional dynamics, and the uniqueness of individual minds. In this chapter, we explore how recursive structures account for emotional modulation, evolving identity, and the potential emergence of free will within an otherwise deterministic architecture.

6 Modulation and Evolution of Sharpness

The sharpness parameter N_i, which governs how abruptly each cognitive threshold τ_i is activated, is not fixed. Rather, it evolves dynamically in response to internal states—such as mood, fatigue, and hormonal fluctuations—as well as external environmental context. Emotional states play a key role in modulating the steepness of transitions between recursive cognitive layers:

- During the initial phase of an intense experience—such as fear, shock, or panic—N_i tends to increase, producing steep, high-sensitivity transitions that encode strong, localized impressions.
- Over time, particularly in the context of unresolved trauma or recurrent internal reactivation, N_i may gradually decay. This results in broader, more diffuse activations that persist within the recursive structure.

This transformation from high to low sharpness generates a paradoxical outcome: what begins as a sharply defined emotional response may evolve into a wide, lingering field of cognitive residue. Such recursive diffusion helps explain why emotionally significant events—especially traumatic ones—resurface in the form of flashbacks, persistent mood states, or dream elements long after the original event has passed.

Importantly, recursive feedback enables emotional states to influence not only immediate cognition but also the thresholds and sensitivities of future cognitive layers. In this formulation, emotion is not merely a transient input but acts as a recursive modifier of the overall cognitive architecture—adjusting the sharpness parameters that shape how future events are encoded and recalled.

7 Recursive Plasticity and Adaptive Identity

Each individual's recursive structure evolves uniquely over time. Experiences reshape threshold values τ_i, sharpness parameters N_i, and the depth or order of nesting within the cognitive system. This ongoing adaptation gives rise to a unique recursive identity—an internal configuration that cannot be copied, reproduced, or externally replicated.

- Even when two individuals undergo similar life events, their recursive encodings differ in timing, sharpness decay, and response layering.
- These differences in recursive architecture give rise to individualized memory dynamics, perceptual tendencies, and emotional patterns.

This model helps explain the irreducibility of consciousness: it is not only subjectively private but also formally non-identical across individuals. Each recursive structure constitutes a singular, historically contingent pathway through layered cognitive time.

8 The Possibility of Free Will in a Recursive Framework

If free will exists, it may not require the violation of deterministic laws but could emerge as a property of recursive depth, self-reference, and architectural variability. While each individual cognitive transition may follow deterministic rules, the recursive system introduces:

- Internal feedback loops capable of modulating or overriding immediate external inputs.
- Temporal delays and long-range dependencies that enable deliberation and reflective pause.
- Flexible memory access, where the mind can selectively define the beginning and end points of introspective recall.

These features support decision-making processes that are neither random nor externally imposed, but historically informed, self-organized, and recursively modulated. In this framework, free will becomes a form of recursive autonomy—the capacity to reshape one's current state by drawing from deeply structured layers of past cognition.

In the final chapter, we synthesize these mathematical and philosophical insights and propose new directions for modeling the mind, treating trauma, and understanding individuality within a recursive systems paradigm.

9 Death and the Disappearance of Cognitive Derivatives

9.1 Temporal Role of τ_n and τ_1

In our recursive memory architecture, we define the deepest threshold τ_n as the moment of birth, while the outermost threshold τ_1 corresponds to the current moment of awareness—or, ultimately, the moment of death.

The recursive cognitive potential is given by:

$$u(t) = 1 - H_{N_1}(-t + \tau_1 H_{N_2}(-t + \tau_2 H_{N_3}(\cdots H_{N_n}(-t + \tau_n)))) \tag{32}$$

9.2 Vanishing of Derivatives at Death

All partial derivatives of the mental potential—whether with respect to time t, threshold parameters τ_i, or sharpness parameters N_i—depend recursively on τ_1. This includes:

- $\dfrac{\partial u}{\partial \tau_i}$
- $\dfrac{\partial u}{\partial N_i}$

Each of these expressions contains multiplicative factors involving the Heaviside or delta function centered at $A_1 = -t + \tau_1 H(\cdots)$. When $\tau_1 \to 0$, this activation term collapses, and all such derivatives vanish:

$$\frac{\partial u}{\partial \tau_1}(t) \to 0, \quad \frac{\partial u}{\partial N_i}(t) \to 0 \quad \text{as} \quad \tau_1 \to 0 \tag{33}$$

9.3 Cognitive Dissolution at the Moment of Death

This mathematical collapse corresponds to a complete loss of awareness, memory, and introspection at the time of death. No derivative remains to encode sensitivity, no threshold remains active, and the recursive architecture is annihilated.

The recursive mind potential $u(t)$, which previously contained layer-by-layer encoded experience, now returns to zero potential:

$$u(t) \to 0 \quad \text{when} \quad \tau_1 = 0$$

9.4 Implication: Death as a Total Derivative Annihilation

In this framework, **death is not merely a biological endpoint**—it is the moment at which the entire chain of mental causality is mathematically extinguished. All cognitive pathways involving differentiation, memory access, and recursive self-reference are disconnected.

The soul, modeled as a nested recursive function, becomes undetectable. This allows for a reset of the memory structure in future reincarnation, which begins again from a newly assigned birth time τ_n', independent from previous τ_n.

Recursive Derivative Decay at the Moment of Death

We define a general n-layer recursive Heaviside potential function $u(t)$ that models cognitive presence as:

$$u(t) = 1 - H_{N_1}\left(-t + \tau_1 H_{N_2}\left(-t + \tau_2 H_{N_3}\left(\cdots H_{N_n}(-t + \tau_n)\cdots\right)\right)\right)$$

where each H_{N_i} is a smooth approximation of the Heaviside step function with sharpness N_i, and each threshold τ_i denotes a significant moment in the life of a cognitive being.

Taking the partial derivative with respect to time, we obtain:

$$\frac{\partial u}{\partial t} = -\delta_{N_1}(x_1) \cdot P_1$$

where $x_1 = -t + \tau_1 H_{N_2}(x_2)$, $x_2 = -t + \tau_2 H_{N_3}(x_3)$, ..., $x_n = -t + \tau_n$, and the nested derivative product P_1 is recursively given by:

$$P_1 = \left(-1 - \tau_1 \delta_{N_2}(x_2) \cdot \left(-1 - \tau_2 \delta_{N_3}(x_3) \cdot \left(\cdots\left(-1 - \tau_{n-1}\delta_{N_n}(x_n)\right)\cdots\right)\right)\right)$$

At the critical moment of death, we assume $t = \tau_1 = 0$. Then $x_1 = 0$, leading to a nonzero peak in $\delta_{N_1}(x_1)$, while all other $x_i < 0$ and their corresponding $\delta_{N_i}(x_i) \approx 0^+$ due to decay of earlier memories.

Therefore:

$$\left.\frac{\partial u}{\partial t}\right|_{t=\tau_1=0} = -\delta_{N_1}(0) \cdot (1 + \epsilon_2 + \epsilon_3 + \cdots + \epsilon_n)$$

where each ϵ_i is an exponentially decaying remnant of earlier memory layers.

This structure reveals that although the cognitive potential $u(t)$ smoothly decays to zero near $t = 0$, its time derivative remains nonzero for an instant—indicating that the process of forgetting is itself alive and dynamic. This momentary derivative can be interpreted as the final "vibration" of the soul. As time progresses beyond death ($t > 0$), all such derivatives vanish:

$$\lim_{t \to 0^+} \frac{\partial u}{\partial t} = 0, \quad \lim_{t \to 0^+} \frac{\partial u}{\partial \tau_i} = 0, \quad \lim_{t \to 0^+} \frac{\partial u}{\partial N_i} = 0$$

signifying the complete dissolution of all memory dynamics and the disappearance of the soul.

In this framework, the soul is not an immutable object but a recursive structure built on layered temporal thresholds. Once all derivatives vanish, the recursive function becomes vacuous—allowing for rebirth to occur with a new set of thresholds $\{\tau_1', \tau_2', \dots\}$. Thus, this model captures the cyclic nature of cognition: from birth, through layered memory accumulation, to dissolution, and finally to reincarnation.

Time-Reversed Consciousness Model

We define a recursive Heaviside-based mental potential function as follows:

$$u(t) = 1 - H_{N_1}\left(-t + \tau_1 H_{N_2}\left(-t + \tau_2 \cdots H_{N_n}(-t + \tau_n) \cdots\right)\right) \tag{34}$$

In this model, time flows from $+\infty$ to $-\infty$, such that future events (e.g., death) occur at larger t, and past events (e.g., birth) occur at smaller t.

- τ_1 denotes the time of death.
- τ_n denotes the time of birth.
- H_{N_i} is the smoothed Heaviside function with sharpness parameter N_i.

Mental Potential Profile: The function $u(t)$ satisfies the following behavior:

$$u(t) = \begin{cases} 0, & t < \tau_1 \quad \text{(Post-death: no consciousness)} \\ \text{active}, & \tau_1 < t < \tau_n \quad \text{(Lived period)} \\ 1, & t > \tau_n \quad \text{(Pre-birth: full potential)} \end{cases}$$

Interpretation: The presence of the leading 1 in Equation (34) is essential. It ensures that during the pre-birth state ($t > \tau_n$), the function evaluates to $u(t) = 1$, representing a latent but unstructured potential.

However, since there is no pointer (i.e., no τ_i activated yet), no actual awareness or memory exists. This structural asymmetry prevents the pre-birth state from recalling the lived timeline, thereby enforcing **unidirectional memory flow** (from the past to the present) even though time is modeled in reverse.

Thus, the subtraction from 1 plays a key role in **blocking memory access from past to present**, while still preserving the mathematical structure for recursive recall during life.

Why the Leading 1 Is Essential

The presence of the leading 1 in the definition of the recursive mental potential function

$$u(t) = 1 - H_{N_1}\left(-t + \tau_1 H_{N_2}\left(-t + \tau_2 \cdots H_{N_n}(-t + \tau_n) \cdots\right)\right)$$

is not arbitrary—it plays a critical structural and philosophical role.

Blocking Backward Access: In this model, time flows from $+\infty$ to $-\infty$, meaning that the most recent event (death at τ_1) occurs first, and birth (τ_n) occurs last in time. As such, the pre-birth state $t > \tau_n$ lies "in the future" of the recursive structure.

Without the leading 1, the recursive Heaviside expression alone would evaluate to 0 before the earliest threshold activation—implying no potential before birth. This would mistakenly suggest that the soul or identity spontaneously appears from nothing.

Instead, by defining $u(t) = 1 -$ Heaviside recursion, we enforce the following behavior:

$$u(t) = \begin{cases} 0, & t < \tau_1 \quad \text{(post-death)} \\ \text{active}, & \tau_1 < t < \tau_n \quad \text{(lived experience)} \\ 1, & t > \tau_n \quad \text{(pre-birth latent potential)} \end{cases}$$

Preventing Past-to-Future Recall: This structure ensures that in the pre-birth state, although the full potential $u(t) = 1$ exists mathematically, the recursive structure has not yet activated. Since no τ_i has "turned on" through Heaviside thresholds, there is no access to lived memory or conscious recall. The 1 thus creates a **mathematical presence without functional access**—an essential asymmetry that blocks reverse memory flow (past does not remember the future).

Irreversibility and Individuality: This asymmetry also aligns with the irreversibility of conscious life. Memories are built forward (from smaller t to larger t), but not backward. The subtraction from 1 introduces this irreversibility, ensuring that each recursive instantiation defines a unique temporal and structural identity—non-invertible and non-replicable.

In short, the leading 1 represents the default fullness of latent potential, and the nested Heaviside sequence serves to subtract away that potential as consciousness becomes constrained by lived thresholds. It is the core mechanism that enforces **directional memory, individuality, and birth-death asymmetry**.

10 Recursive Mental Potential Model: Summary

Core Equation

We define the mental potential function using a nested, recursive Heaviside structure:

$$u(t) = 1 - H_{N_1}\left(-t + \tau_1 H_{N_2}\left(-t + \tau_2 \cdots H_{N_n}(-t + \tau_n) \cdots\right)\right) \tag{35}$$

Here, τ_1 is the time of death, τ_n the time of birth, and N_i the sharpness (emotional weight) of each memory layer. Time flows in reverse: from $+\infty \to -\infty$, so that death occurs first and birth occurs last in this temporal model.

Temporal Interpretation

$$u(t) = \begin{cases} 0, & t < \tau_1 \quad \text{(Post-death: no consciousness)} \\ \text{active}, & \tau_n < t < \tau_1 \quad \text{(Lived experience)} \\ 1, & t > \tau_n \quad \text{(Pre-birth latent potential)} \end{cases}$$

The subtraction from 1 is essential. It ensures that although the full potential exists before birth ($u(t) = 1$), no memory or consciousness is accessible because no threshold τ_i has been activated. This enforces unidirectional memory and identity irreversibility.

Differential Structure

- $\partial u/\partial t$: Sensitivity to time—attention and awareness
- $\partial u/\partial \tau_i$: Responsiveness to stored memory thresholds
- $\partial^2 u/\partial t \partial \tau_i$: Dynamic modulation of recall
- $\partial u/\partial N_i$: Emotional intensity or trauma sharpness

Cognitive and Philosophical Insights

- Recursive Heaviside layers model layered memory, perception, and forgetting.
- The model explains why pre-birth states contain full potential but cannot access future (i.e., lived) memory.

- Consciousness unfolds as threshold activation reduces the initial full potential.
- The subtraction from 1 structurally prevents backward memory recall and encodes the directionality of life and memory.
- The non-invertibility of the nested delta products explains why consciousness is unique and non-replicable.

Applications and Implications

- **Neuroscience**: Modeling recursive firing, trauma dynamics, and layered memory
- **Clinical practice**: Interpreting or modulating N_i to address PTSD and trauma loops
- **AI**: Implementing recursive memory for introspective and individualized machine consciousness
- **Philosophy**: Grounding soul theory, free will, and identity through recursive irreversible structures

11 Conclusion and Future Directions

Key Conclusions

- Recursive architectures can represent both conscious transitions and unconscious processes, including dreaming, forgetting, and emotional resonance.
- Derivatives with respect to time and cognitive thresholds reveal the layered structure of attention, memory formation, and affective modulation.
- Products of Dirac delta functions arise naturally in this framework, yet remain mathematically non-invertible—providing an explanation for the irreversibility and individuality of conscious structures.
- The sharpness parameter N_i encodes emotional intensity and dynamically evolves—governing the persistence of trauma and the reinforcement or fading of memories over time.
- Recursive identity structures form uniquely within each individual, offering a structural foundation for the non-replicability of consciousness.

Future Directions

- Empirical validation through neural recordings and dynamical modeling of recursive firing patterns consistent with derivative-based predictions.
- Clinical applications in trauma therapy by observing and modulating sharpness N_i, particularly in the context of recursive reinforcement and residual activation loops.
- Development of AI systems that incorporate recursive memory architectures for improved introspection, adaptability, and interpretability.
- Philosophical exploration of free will, agency, and individuality using recursive, self-referential differential structures as foundational models.

Ultimately, the recursive Heaviside framework offers not only a rigorous formalism for modeling cognition but also a compelling metaphor for the nature of mind itself: *memory as stratified layers of conditional potential, thought as a cascade of nested thresholds, and consciousness as a singular recursive trajectory through an irreducible mental topology.*

Appendix A. Invertibility of the Recursive Heaviside Function

Although the recursive Heaviside function $u(t) = 1 - H(-t + \tau_1 H(-t + \tau_2))$ appears analyzable via its time and threshold derivatives, a fundamental issue arises. The formal derivatives of this function yield terms such as:

$$\frac{\partial u}{\partial t} = \delta\left(-t + \tau_1 H(-t + \tau_2)\right) \cdot \left(-1 - \tau_1 \delta(-t + \tau_2)\right) \tag{A.1}$$

This expression contains products of delta functions, such as $\delta(t - \tau_1) \cdot \delta(t - \tau_2)$, which lie outside the space of well-defined distributions. According to Schwartz's theory of distributions, such products are undefined in the general case.

As a result, the information encoded in higher-order interactions within the recursive structure cannot be extracted or inverted through classical analysis. No matter how precisely one measures all possible derivatives of $u(t)$, the original recursive nesting cannot be uniquely reconstructed.

To illustrate this, consider the two functions:

$$\begin{aligned} u_1(t) &= 1 - H(-t + 1 \cdot H(-t + 2)) \\ u_2(t) &= 1 - H(-t + 2 \cdot H(-t + 1)) \end{aligned} \tag{A.2}$$

Both functions exhibit similar jump behavior, yet their derivatives diverge:

$$\frac{\partial u_1}{\partial t} = -\delta(t-1), \quad \frac{\partial u_2}{\partial t} = 0 \tag{A.3}$$

Despite visual similarity, these functions encode fundamentally different recursive structures. Their distinct internal architectures are invisible to the time derivative alone. This confirms that the recursive configuration is hidden from external observation, and that the original function cannot be recovered from temporal behavior alone.

This provides a mathematically grounded insight into cognition: even with access to all observable derivatives, the inner logic of recursive thought remains irreducible. This irreducibility supports the conclusion that consciousness is structurally unique and non-replicable—**a mind cannot be perfectly copied or reconstructed**.

Appendix B. Time Derivative and Residual Terms in the 3-Layer Recursive Heaviside Function

We now examine the time derivative of a recursive function composed of three nested Heaviside step functions with finite sharpness parameters N_1, N_2, N_3. The recursive function is given by:

$$u(t) = 1 - H_{N_1}\left(-t + \tau_1 H_{N_2}\left(-t + \tau_2 H_{N_3}(-t + \tau_3)\right)\right) \tag{B.1}$$

Define intermediate variables:

$$A = -t + \tau_1 H_{N_2}(B), \quad B = -t + \tau_2 H_{N_3}(C), \quad C = -t + \tau_3$$

The full time derivative of $u(t)$ then takes the form:

$$\frac{\partial u}{\partial t} = -\delta_{N_1}(A) - \tau_1 \delta_{N_1}(A)\delta_{N_2}(B) - \tau_1\tau_2\delta_{N_1}(A)\delta_{N_2}(B)\delta_{N_3}(C) \tag{B.2}$$

Among these, only the final term corresponds to fully resolved cognitive causality:

$$\text{Fully cancelled term:} \quad -\tau_1\tau_2\delta_{N_1}(A)\delta_{N_2}(B)\delta_{N_3}(C) \tag{B.3}$$

However, when the sharpness parameters N_i are finite, the Heaviside approximations H_{N_i} do not yield perfect binary steps. As a result, *residual terms* remain in the derivative—representing incomplete logical cancellation. These residuals are given by:

$$\boxed{\text{Residual terms} = \tau_2\delta_{N_2}(B)\delta_{N_3}(C) + \delta_{N_1}(A)\left(1 - H_{N_2}(B)\right) + \tau_1\delta_{N_1}(A)\delta_{N_2}(B)\left(1 - H_{N_3}(C)\right)} \tag{B.4}$$

These remaining terms may be interpreted as mathematical representations of subconscious processing, unresolved loops, or dreaming. In the idealized limit $N_i \to \infty$, all residuals vanish, and only the fully cancelled term survives—yielding perfect causal cognition.

Appendix C. Derivatives of the Recursive Heaviside Function

We consider the n-layer recursive Heaviside function defined as:

$$u(t) = 1 - H_{N_1}\left(-t + \tau_1 H_{N_2}\left(-t + \tau_2 H_{N_3}\left(\cdots H_{N_n}(-t + \tau_n)\right)\right)\right) \tag{C.1}$$

The intermediate arguments A_i are defined recursively as:

$$\begin{aligned} A_n &= -t + \tau_n \\ A_k &= -t + \tau_k H_{N_{k+1}}(A_{k+1}), \quad \text{for } k = n-1, n-2, \ldots, 1 \end{aligned} \tag{C.2}$$

Time Derivative $\partial u/\partial t$

The time derivative of $u(t)$ is expressed as:

$$\frac{\partial u}{\partial t} = -\delta_{N_1}(A_1) \cdot \prod_{k=2}^{n} \left(1 + \tau_{k-1}\delta_{N_k}(A_k)\right) \tag{C.3}$$

Expanded explicitly:

$$\frac{\partial u}{\partial t} = -\delta_{N_1}(A_1) \cdot (1 + \tau_1 \delta_{N_2}(A_2)) \cdot (1 + \tau_2 \delta_{N_3}(A_3)) \cdots (1 + \tau_{n-1}\delta_{N_n}(A_n)) \tag{C.4}$$

Partial Derivatives with Respect to τ_i

For $i < n$, the derivative of $u(t)$ with respect to τ_i is:

$$\frac{\partial u}{\partial \tau_i} = \left(\prod_{k=1}^{i-1} \tau_k \delta_{N_{k+1}}(A_{k+1})\right) \cdot H_{N_{i+1}}(A_{i+1}) \cdot \delta_{N_1}(A_1) \tag{C.5}$$

For the deepest layer $i = n$, we have:

$$\frac{\partial u}{\partial \tau_n} = \left(\prod_{k=1}^{n-1} \tau_k \delta_{N_{k+1}}(A_{k+1})\right) \cdot \delta_{N_n}(A_n) \cdot \delta_{N_1}(A_1) \tag{C.6}$$

Note: $\delta_{N_i}(x) = \frac{d}{dx} H_{N_i}(x)$ denotes the smoothed Dirac delta function associated with sharpness N_i. As $N_i \to \infty$, $\delta_{N_i}(x) \to \delta(x)$, the ideal Dirac delta.

Partial Derivative with Respect to N_i

The derivative of $u(t)$ with respect to the sharpness parameter N_i (for $2 \leq i \leq n$) is given by:

$$\frac{\partial u}{\partial N_i} = \left(\prod_{k=1}^{i-1} \tau_k \delta_{N_{k+1}}(A_{k+1}) \right) \cdot \frac{\partial H_{N_i}(A_i)}{\partial N_i} \tag{C.7}$$

For the first layer $i = 1$, we have:

$$\frac{\partial u}{\partial N_1} = -\frac{\partial H_{N_1}(A_1)}{\partial N_1} \tag{C.8}$$

Let

$$x_i(t) = -t + \tau_i H_{N_{i+1}}(-t + \tau_{i+1})$$

Then, the sharpness derivative is given by:

$$\frac{\partial H_{N_i}}{\partial N_i} = \frac{2x_i(t) \cdot \exp(-2N_i x_i(t))}{(1 + \exp(-2N_i x_i(t)))^2} \tag{C.9}$$

Interpretation: This derivative quantifies how changes in the sharpness parameter N_i influence the steepness of the Heaviside transition. In the limit $N_i \to \infty$, the sigmoid approaches a step function, and this sensitivity vanishes almost everywhere. However, for moderate N_i, the influence becomes prominent especially when time moves toward earlier events ($t \to -\infty$), leading to an amplification effect in the deep memory region.

Appendix D. Invertibility Example: 2-Layer Recursive Heaviside Function

To illustrate the challenge of invertibility in recursive Heaviside structures, consider a simple 2-layer function evaluated at $t = 1$:

$$u(1) = 1 - H\left(-1 + \tau_1 H(-1 + \tau_2)\right) \tag{D.1}$$

This function depends on two threshold parameters, τ_1 and τ_2, and exhibits discontinuous behavior due to the nested Heaviside terms.

Let us define intermediate variables for clarity:

$$A = -1 + \tau_1 H(B), \quad B = -1 + \tau_2$$

Then the formal derivatives of $u(1)$ with respect to the thresholds are:

$$\frac{\partial u}{\partial \tau_1} = -\delta(A) \cdot H(B) \tag{D.2}$$

$$\frac{\partial u}{\partial \tau_2} = -\delta(A) \cdot \tau_1 \cdot \delta(B) \tag{D.3}$$

These expressions involve Dirac delta functions. Notably, the second derivative contains a product of delta functions, $\delta(A) \cdot \delta(B)$, which is undefined in the framework of classical distribution theory.

Implication. Even in this minimal 2-layer configuration, the output $u(1)$ is not a bijective function of τ_1 and τ_2. Multiple parameter combinations can yield identical outputs. Moreover, the ill-defined nature of higher-order derivatives (such as $\delta(A) \cdot \delta(B)$) highlights the fundamental limitations of analytic inversion.

Conclusion. Recursive Heaviside functions exhibit structural non-invertibility. The nesting of discontinuities prevents smooth reconstruction of internal parameters from outputs or derivative information. This supports the broader thesis that *recursive cognitive architectures are inherently non-invertible.*

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