Calculus

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Contents

1 Sequences **2**
- 1.1 Definition . . . 2
- 1.2 Monotonicity . . . 2
- 1.3 Convergence and divergence . . . 2
  - 1.3.1 $\varepsilon - N$ definition of convergence . . . 3
  - 1.3.2 Converging to ∞ and $-\infty$ . . . 4
  - 1.3.3 Divergence . . . 5
- 1.4 Sandwich theorem (squeeze theorem) . . . 6
- 1.5 Triangle inequality . . . 7
- 1.6 Bounds . . . 8
- 1.7 Cauchy sequences . . . 9
  - 1.7.1 Subsequences . . . 10
  - 1.7.2 Completeness . . . 11

2 Continuous functions **13**
- 2.1 Accumulation point . . . 13
- 2.2 $\varepsilon - \delta$ definition of limit of a function . . . 14
  - 2.2.1 Definitions for alternative use cases . . . 15
- 2.3 Continuity . . . 16
  - 2.3.1 Bolzano-Weierstrass theorem . . . 18
  - 2.3.2 Intermediate value theorem . . . 19
  - 2.3.3 Uniform continuity . . . 20
- 2.4 Exercises . . . 25

3 Series **27**
- 3.1 Definitions . . . 27
- 3.2 Tests . . . 28
  - 3.2.1 Divergence test . . . 28
  - 3.2.2 Comparison test . . . 28
  - 3.2.3 Limit comparison test . . . 29
  - 3.2.4 D'Alembert's (limit) ratio test . . . 30
  - 3.2.5 Integral test . . . 31
- 3.3 Absolute convergence . . . 33
  - 3.3.1 Absolute value comparison test . . . 34
  - 3.3.2 D'Alembert's (limit) ratio test, correct . . . 34
  - 3.3.3 n^{th} root test . . . 35

Chapter 1

Sequences

1.1 Definition

A sequence is a specific type of function. A function is a specific type of binary relations. All sequences are functions, and all functions are binary relations, but not all binary relations are functions. Are all functions sequences?

Definition 1.1. A sequence is a function $f : \mathbb{N} \mapsto \mathbb{R}$. For convenience, in this course we define a sequence to be $f : \mathbb{N}^+ \mapsto \mathbb{R}$ instead.

Notation. We can also denote a sequence as $(a_n)_{n\geq 1}$, where $a_n = f(n)$. Writing a_n or $a_1, a_2, ...$ is also fine as long as it is clear enough that they denote a sequence.

1.2 Monotonicity

Definition 1.2. A sequence is increasing $\Leftrightarrow a_{n+1} \geq a_n \forall n \geq 1$.

Definition 1.3. A sequence is decreasing $\Leftrightarrow a_{n+1} \leq a_n \forall n \geq 1$.

Definition 1.4. A sequence is monotonic $\Leftrightarrow$ it is either increasing or decreasing (or both).

Definition 1.5. A sequence is strictly increasing $\Leftrightarrow a_{n+1} > a_n \forall n \geq 1$.

Definition 1.6. A sequence is decreasing $\Leftrightarrow a_{n+1} < a_n \forall n \geq 1$.

1.3 Convergence and divergence

Definition 1.7. A sequence converges if it converges to a limit $l \in \mathbb{R}$, ∞, or $-\infty$. A sequence diverges if it does not converge.

1.3.1 $\varepsilon - N$ definition of convergence

Let $(a_n)_{n\geq 1}$ be a sequence.

Definition 1.8. $(a_n)_{n\geq 1}$ converges to a limit l if and only if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}(\forall n \in \mathbb{N}(n > N \Rightarrow |a_n - l| < \varepsilon))$$

We denote this as $\lim\limits_{n\to\infty} a_n = l$ or $(a_n)_{n\geq 1} \to l$. As above, $a_n \to l$ would also be fine.

Note that n should be strictly greater than N.

Intuition. If somebody gives us an arbitrary $\varepsilon > 0$, then we can always find an N such that every term after that will be within ε of the limit. So if ε is very close to 0, then this guarantees that after some point, all terms in the sequence will be very close to the limit.

The statement of the definition itself isn't very helpful when we try to prove limits with it. A general template for answering problems is as follows:

Template. Given some $\varepsilon > 0$, we want to find limit l and some $N \in \mathbb{N}$ such that $\forall n > N, |a_n - l| < \varepsilon$. We speculate that $l =$ [some value]. Now,

$$\begin{aligned} |a_n - l| < \varepsilon &\Leftrightarrow \text{[substitute } a_n \text{ and } l\text{]} \\ &\Leftrightarrow \text{[simplify]} \\ &\Leftrightarrow \text{[remove the absolute operators and justify]} \\ &\Leftrightarrow \ldots \\ &\Leftrightarrow n > \text{[some function of } \varepsilon\text{, let's call this } N(\varepsilon)\text{]} \end{aligned}$$

Choose $N \in \mathbb{N}$ such that $N > N(\varepsilon)$. Then $\forall n > N, n > N(\varepsilon)$.

$$\begin{aligned} |a_n - l| &= \ldots \\ &< \varepsilon \end{aligned}$$

Since ε arbitrary, $\forall \varepsilon > 0, \exists N \in \mathbb{N}(\forall n \in \mathbb{N}(n > N \Rightarrow |a_n - l| < \varepsilon))$, where $l =$ blah. So limit is blah.

If the question gives us the limit and asks us to prove it directly, just change around the sentences a bit.

Example. (MMT 1, 6(a)) Let α be a positive, real constant. Use a direct proof to show that the sequence $(a_n)_{n\geq 1} = (n^{-\alpha})$ converges to 0 as n tends to infinity.

Proof. Given some $\varepsilon > 0$, we want to find some $N \in \mathbb{N}$ such that $\forall n > N, |a_n - l| < \varepsilon$, where $l = 0$.

Now,

$$\begin{aligned}
|a_n - l| < \varepsilon &\Leftrightarrow \left|n^{-\alpha} - 0\right| < \varepsilon \\
&\Leftrightarrow \left|\frac{1}{n^\alpha}\right| < \varepsilon \\
&\Leftrightarrow \frac{1}{n^\alpha} < \varepsilon && \text{since } n > 0 \Rightarrow \frac{1}{n^\alpha} > 0 \\
&\Leftrightarrow n^\alpha > \frac{1}{\varepsilon} \\
&\Leftrightarrow n > \frac{1}{\varepsilon^\alpha} = \varepsilon^{-\alpha}
\end{aligned}$$

Choose $N \in \mathbb{N}$ such that $N > \varepsilon^{-\frac{1}{\alpha}}$. Then $\forall n > N, n > \varepsilon^{-\frac{1}{\alpha}}$.

$$\begin{aligned}
|a_n - l| &= \left|n^{-\alpha} - 0\right| \\
&= \frac{1}{n^\alpha} && \text{from above} \\
&< \frac{1}{(\varepsilon^{-\frac{1}{\alpha}})^\alpha} \\
&= \varepsilon
\end{aligned}$$

Since ε arbitrary, $\forall \varepsilon > 0, \exists N \in \mathbb{N}(\forall n \in \mathbb{N}(n > N \Rightarrow |a_n - l| < \varepsilon))$. So limit is 0. □

1.3.2 Converging to ∞ and $-\infty$

Definition 1.9. $(a_n)_{n\geq 1}$ converges to ∞ if and only if

$$\forall r \in \mathbb{R}, \exists N \in \mathbb{N}(\forall n \in \mathbb{N}(n \geq N \Rightarrow a_n > r))$$

Note that this time, the requirement is n greater than or equal to N, but not strictly greater than N. I don't know why this is the case, but we'll just accept it. Maybe the slides are wrong.

Intuition. If r is arbitrarily large, then this guarantees that after some point, all the terms will be larger than the arbitrarily large r, i.e. mega large.

Example. Use a direct proof to show that the sequence $(a_n)_{n\geq 1} = n!$ converges to ∞.

Proof. Given some $r \in \mathbb{R}$, we want to find some $N \in \mathbb{N}$ such that $\forall n \geq N, a_n > r$.

$$\begin{aligned}
a_n > r &\Leftrightarrow n! > r \\
&\Leftrightarrow n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 > r
\end{aligned}$$

Choose $N \in \mathbb{N}$ such that $N > r$. Then $\forall n \geq N, n > r$.

$$\begin{aligned}
a_n &= n! \\
&= n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 \\
&\geq n \cdot 1 \cdot 1 \cdot \ldots \cdot 1 \\
&= n \\
&> r
\end{aligned}$$

Since r arbitrary, $\forall r \in \mathbb{R}, \exists N \in \mathbb{N}(\forall n \in \mathbb{N}(n \geq N \Rightarrow a_n > r))$. So a_n converges to ∞. □

Definition 1.10. $(a_n)_{n\geq 1}$ converges to $-\infty$ if and only if

$$\forall r \in \mathbb{R}, \exists N \in \mathbb{N}(\forall n \in \mathbb{N}(n \leq N \Rightarrow a_n < r))$$

Also note that n is less than or equal to N, but not strictly less than N.

Example. Use a direct proof to show that the sequence $(a_n)_{n\geq 1} = \ln \frac{1}{n}$ converges to $-\infty$.

Proof. Given some $r \in \mathbb{R}$, we want to find some $N \in \mathbb{N}$ such that $\forall n \geq N, a_n < r$.

$$\begin{aligned} a_n < r &\Leftrightarrow \ln \frac{1}{n} < r \\ &\Leftrightarrow \frac{1}{n} < e^r \\ &\Leftrightarrow n > e^{-r} \end{aligned}$$

Choose $N \in \mathbb{N}$ such that $N > e^{-r}$. Then $\forall n \geq N, n > e^{-r}$.

$$\begin{aligned} a_n &= \ln \frac{1}{n} \\ &< \ln \frac{1}{e^{-r}} \\ &= \ln e^r \\ &= r \end{aligned}$$

Since r arbitrary, $\forall r \in \mathbb{R}, \exists N \in \mathbb{N}(\forall n \in \mathbb{N}(n \geq N \Rightarrow a_n < r))$. So a_n converges to $-\infty$. $\square$

1.3.3 Divergence

A sequence diverges if it does not converge to a limit $l \in \mathbb{R}$, does not converge to ∞, and does not converge to $-\infty$.

Example. Prove that the sequence $(a_n)_{n\geq 1} = (-1)^n$ diverges.

Proof. We will prove that a_n diverges by contradiction.

Assume a_n converges to a limit $l \in \mathbb{R}$. Then

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}(\forall n \in \mathbb{N}(n > N \Rightarrow |a_n - l| < \varepsilon))$$

Suppose $\varepsilon = 1$. Choose arbitrary $n > N$. Then $2n > N$ and $2n + 1 > N$.

$$\begin{aligned}
|a_{2n} - l| < \varepsilon &\Leftrightarrow \left|(-1)^{2n} - l\right| < 1 \\
&\Leftrightarrow |1 - l| < 1 \\
&\Leftrightarrow -1 < 1 - l < 1 \\
&\Leftrightarrow -2 < -l < 0 \\
&\Leftrightarrow 0 < l < 2 \\
|a_{2n+1} - 1| < \varepsilon &\Leftrightarrow \left|(-1)^{2n+1} - l\right| < 1 \\
&\Leftrightarrow |-1 - l| < 1 \\
&\Leftrightarrow -1 < -1 - l < 1 \\
&\Leftrightarrow 0 < -l < 2 \\
&\Leftrightarrow -2 < l < 0
\end{aligned}$$

Since no $l \in \mathbb{R}$ can satisfy both $0 < l < 2$ and $-2 < l < 0$, we have a contradiction. So a_n does not converge to a limit $l \in \mathbb{R}$.

Assume a_n converges to ∞. Then

$$\forall r \in \mathbb{R}, \exists N \in \mathbb{N}(\forall n \in \mathbb{N}(n \geq N \Rightarrow a_n > r))$$

Suppose $r = 2$. Since $a_n = -1$ or 1, and $-1 < 2$ and $1 < 2$, we have a contradiction. So a_n does not converge to ∞.

Assume a_n converges to $-\infty$. Then

$$\forall r \in \mathbb{R}, \exists N \in \mathbb{N}(\forall n \in \mathbb{N}(n \geq N \Rightarrow a_n < r))$$

Suppose $r = -2$. Since $a_n = -1$ or 1, and $-1 > -2$ and $1 > -2$, we have a contradiction. So a_n does not converge to $-\infty$.

Therefore, a_n diverges. □

1.4 Sandwich theorem (squeeze theorem)

Definition 1.11. Suppose $(l_n)_{n\geq 1} \to l$ and $(u_n)_{n\geq 1} \to l$, where $l \in \mathbb{R}$.

$$\exists N \in \mathbb{N} : \forall n \geq N, l_n \leq a_n \leq u_n \Rightarrow a_n \to l$$

Again, n is greater than or equal to, but somehow not strictly greater than, N.

Proof. Given some $\varepsilon > 0$, since $l_n \to l$,

$$\exists N_1 \in \mathbb{N} : \forall n > N_1, |l_n - l| < \varepsilon$$

Also, since $u_n \to l$,

$$\exists N_2 \in \mathbb{N} : \forall n > N_2, |u_n - l| < \varepsilon$$

Let $N' = \max(N_1, N_2)$. Then $\forall n > N'$,

$$\begin{aligned}
|l_n - l| < \varepsilon &\Leftrightarrow -\varepsilon < l_n - l < \varepsilon \\
&\Leftrightarrow l - \varepsilon < l_n < l + \varepsilon
\end{aligned}$$

and similarly, $l - \varepsilon < u_n < l + \varepsilon$.

Assume that

$$\exists N \in \mathbb{N} : \forall n \geq N, l_n \leq a_n \leq u_n$$

Then $\forall n > \max(N, N')$,

$$\begin{aligned} l - \varepsilon < l_n \leq a_n \leq u_n < l + \varepsilon &\Leftrightarrow l - \varepsilon < a_n < l + \varepsilon \\ &\Leftrightarrow -\varepsilon < a_n - l < \varepsilon \\ &\Leftrightarrow |a_n - l| < \varepsilon \end{aligned}$$

□

Note that it is possible to use a constant sequence for one of l_n or u_n.

Example. (MMT 1 6(b)) Use the sandwich theorem to show that

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Solution. Let $a_n = \frac{n!}{n^n}$. Let $l_n = 0$. Then $\lim\limits_{n\to\infty} l_n = 0$.

$$\forall n > 0, n! > 0 \text{ and } n^n > 0 \Rightarrow a_n > 0 \Rightarrow a_n > l_n$$

Let $u_n = \frac{1}{n}$. Then $\lim\limits_{n\to\infty} u_n = 0$. $\forall n > 0$,

$$\begin{aligned} a_n &= \frac{n!}{n^n} \\ &= \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot ... \cdot \frac{1}{n} \\ &= 1 \cdot (1 - \frac{1}{n}) \cdot (1 - \frac{2}{n}) \cdot ... \cdot \frac{1}{n} \end{aligned}$$

Since every term ≤ 1, we have $a_n \leq \frac{1}{n} = u_n$. So $\forall n > 0, l_n \leq a_n \leq u_n$. Since $l_n \to 0$ and $u_n \to 0$ as $n \to \infty$, by sandwich theorem, $a_n \to 0$.

1.5 Triangle inequality

Lemma 1.12.

$$\forall x, y > 0, x \leq y \Leftrightarrow x^2 \leq y^2$$

Proof. Assume $x \leq y$. Then

$$\begin{aligned} x \leq y \Rightarrow x^2 &\leq y * x && \text{since } x > 0 \\ &\leq y * y && \text{since } x \leq y \\ &= y^2 \end{aligned}$$

So $x \leq y \Rightarrow x^2 \leq y^2$.

Assume $x^2 \leq y^2$. Then

$$\begin{aligned} x^2 \leq y^2 &\Rightarrow x^2 - y^2 \leq 0 \\ &\Rightarrow (x-y)(x+y) \leq 0 \\ &\Rightarrow x - y \leq 0 && \text{since } x, y > 0 \Rightarrow x + y > 0 \\ &\Rightarrow x \leq y \end{aligned}$$

So $x^2 \leq y^2 \Rightarrow x \leq y$. □

Lemma 1.13 (Triangle inequality)**.**

$$|a+b| \leq |a| + |b|$$

Proof.

$$\begin{aligned} |a+b|^2 &= (a+b)^2 \\ &= a^2 + 2ab + b^2 \\ &\leq a^2 + 2|ab| + b^2 \\ &= |a|^2 + 2|ab| + |b|^2 \\ &= |a|^2 + 2|a||b| + |b|^2 \\ &= (|a| + |b|)^2 \end{aligned}$$

From lemma, $|a+b| \leq |a| + |b|$. □

Here is another proof of the triangle inequality:

Proof.

$$\begin{aligned} a + b \leq |a| + b \leq |a| + |b| \\ -a - b \leq |a| - b \leq |a| + |b| \Rightarrow a + b \leq -(|a| + |b|) \end{aligned}$$

So we have

$$-(|a| + |b|) \leq a + b \leq |a| + |b| \Leftrightarrow |a+b| \leq |a| + |b|$$

□

1.6 Bounds

Suppose $X \subseteq \mathbb{R}$.

- $u \in \mathbb{R}$ is an upper bound of X if $x \leq u$ for all $x \in X$.
- $s \in \mathbb{R}$ is the supremum (least upper bound) of X if $s \leq u$ for all upper bounds u of X.
- $l \in \mathbb{R}$ is a lower bound of X if $x \geq l$ for all $x \in X$.
- $i \in \mathbb{R}$ is the infimum (greatest lower bound) of X if all $i \geq l$ for all lower bounds l of X.

To describe the bounded conditions of X, we have

- X is bounded above if X has an upper bound.
- X is bounded below if X has a lower bound.
- X is bounded if X has an upper bound and a lower bound.

1.7 Cauchy sequences

Definition 1.14. A sequence $(a_n)_{n\geq 1}$ is a Cauchy sequence (in the real numbers) if and only if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}(\forall n, m \in \mathbb{N}(n > N \wedge m > N \Rightarrow |a_n - a_m| < \varepsilon))$$

The need to specify that the Cauchy sequence is in the real numbers will become apparent later.

Proposition 1.15. A sequence $(a_n)_{n\geq 1}$ converges to a limit $l \in \mathbb{R}$ (note $\mathbb{R}$) $\Rightarrow a_n$ is a Cauchy sequence (in the real numbers).

Proof. Since a_n converges to l,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}(\forall n \in \mathbb{N}(n > N \Rightarrow |a_n - l| < \varepsilon))$$

Given some $\varepsilon > 0$,

$$\exists N' \in \mathbb{N}(\forall n \in \mathbb{N}(n > N' \Rightarrow (|a_n - l| < \frac{\varepsilon}{2} < \varepsilon)))$$

Choose arbitrary $n, m > N'$. Then $|a_n - l| < \frac{\varepsilon}{2}$ and $|a_m - l| < \frac{\varepsilon}{2}$.

$$\begin{aligned}
|a_n - a_m| &= |a_n - l - (a_m - l)| \\
&\leq |a_n - l| + |-(a_m - l)| && \text{by triangle inequality} \\
&= |a_n - l| + |a_m - l| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Since n, m, ε arbitrary,

$$\forall \varepsilon > 0, \exists N' \in \mathbb{N}(\forall n, m \in \mathbb{N}(n > N' \wedge m > N' \Rightarrow |a_n - a_m| < \varepsilon))$$

So a_n is a Cauchy sequence. □

Using the contrapositive of this proposition, we present another explanation to why $(-1)^n$ does not converge to a limit $l \in \mathbb{R}$.

Explanation. We want to show that $(-1)^n$ is not a Cauchy sequence. Observe that

$$\forall N \in \mathbb{N}, \exists n, m > N : |a_n - a_m| = 2$$

Pick some $0 < \varepsilon < 2$. Then there does not exist $N \in \mathbb{N}$ such that $\forall n, m > N, |a_n - a_m| < \varepsilon$. So $(-1)^n$ is not a Cauchy sequence. By contrapositive, $(-1)^n$ does not converge to a limit $l \in \mathbb{R}$.

Proposition 1.16. Let $(a_n)_{n\geq 1}$ be a Cauchy sequence. Then a_n is bounded.

Proof. By definition,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \colon \forall n, m \in \mathbb{N}(n > N \wedge m > N \Rightarrow |a_n - a_m| < \varepsilon)$$

Let

$$M = \varepsilon + \max |a_1|, |a_2|, |a_3|, ..., |a_{N+1}|$$

Then $\forall n > N$,

$$\begin{aligned} |a_n| &= |a_n + a_{N+1} - a_{N+1}| \\ &\leq |a_n - a_{N+1}| + |a_{N+1}| && \text{by triangle inequality} \\ &< \varepsilon + |a_{N+1}| \\ &\leq M \end{aligned}$$

So M is an upper bound for a_n. □

1.7.1 Subsequences

Definition 1.17. A subsequence is an infinite ordered subset of a sequence.

Note that a subsequence, by default, has infinitely many terms. Since it is defined to be a subset of a sequence, the sequence from which the subsequence is taken from must also have infinitely many terms.

Notation. We denote a subsequence of a sequence $(a_n)_{n\geq 1}$ as $(a_{n_i})_{i\geq 1}$, where n_i is a strictly increasing sequence in $\mathbb{N}^+$.

Proposition 1.18. For any subsequence $(a_{n_i})_{i\geq 1}$, $n_i \geq i$.

Explanation. A sequence is a subsequence of itself, so in this case we have $n_i = i$ since we are not omitting any terms to form the subsequence (which is the sequence itself). For any other subsequence, suppose the first omitted term is a_k. Then

$$\begin{cases} n_i = i & i < k \\ n_i > i & i \geq k \end{cases}$$

so in general, for any subsequence, we have $n_i \geq i$.

Theorem 1.19. Let a_n be a sequence and a_{n_i} be a subsequence of a_n. If a_n converges to limit $l \in \mathbb{R}$, then a_{n_i} also converges to l.

Proof. Since a_n converges to l, given some $\varepsilon > 0$,

$$\exists N \in \mathbb{N} : \forall i > N, |a_i - l| < \varepsilon$$

Since $n_i \geq i$, $i > N$ implies $n_i > N$, so

$$\forall i > N, |a_{n_i} - l| < \varepsilon$$

□

Theorem 1.20. Every subsequence of real numbers has a monotonic subsequence.

Proof. Let a_n be a peak if

$$a_n > a_m \qquad \forall m > n$$

If the sequence has infinitely many peaks at $n_1 < n_2 < n_3 < ...$, then $a_{n_1} > a_{n_2} > a_{n_3} > ...$ is a strictly decreasing subsequence.

Otherwise, the sequence has only finitely many peaks at $n_1 < n_2 < ... < n_k$, or no peaks. Consider a_{n_k+1}. Since $a_{n_{k+1}}$ is not a peak,

$$\exists i > k+1 : a_{n_i} \geq a_{n_k+1}$$

and since a_{n_i} is not a peak, we can construct a (not strictly) increasing subsequence in this way. □

1.7.2 Completeness

We define two notions of completeness:

Definition 1.21. A metric space (X, d) is Cauchy complete if and only if all Cauchy sequences in (X, d) converges to an element in X.

Example. The set of rational numbers $\mathbb{Q}$ is not Cauchy complete.

Consider the sequence

$$a_1 = 1, a_{n+1} = \frac{x_n + \frac{2}{x_n}}{2}$$

Every term in the sequence is in $\mathbb{Q}$, but the sequence converges to $\sqrt{2}$, which is not in $\mathbb{Q}$.

Consider another sequence in the rationals

$$a_1 = 3.1, a_2 = 3.14, a_3 = 3.142, a_4 = 3.1416, ...$$

where a_n represents the nth decimal approximation of π. The sequence converges to $\pi \notin \mathbb{Q}$.

Definition 1.22 (Axiom of Dedekind completeness)**.** A partially ordered set X is Dedekind complete if and only if it has the least-upper-bound property, i.e. every non-empty subset of X with an upper bound has a least upper bound (i.e. supremum).

Theorem 1.23 (Fundamental theorem of analysis)**.** Every increasing sequence of real numbers a_n with an upper bound:

- has a supremum s, and
- converges to s.

Proof. A non-empty subset of the partially ordered set $\mathbb{R}$ is equivalent to an increasing sequence of real numbers. So, the axiom of Dedekind completeness tells us that every increasing sequence of real numbers with an upper bound has a supremum.

We prove the second point in two stages.

In the first stage, we claim that

$$\exists N \in \mathbb{N}(|a_N - s| < \varepsilon)$$

We prove this by contradiction. Suppose $\forall n \in \mathbb{N}, |a_n - s| \geq \varepsilon$. Since s is the supremum, we have $s \geq a_n \forall n \in \mathbb{N}$, so $s - a_n \geq \varepsilon \forall n \in \mathbb{N}$. Rearranging the inequality gives

$$a_n \leq s - \varepsilon \qquad \forall n \in \mathbb{N}$$

By definition, $s - \varepsilon$ is an upper bound. However, since $\varepsilon > 0$, $s - \varepsilon < s$, which contradicts the fact that s is the supremum, i.e. least upper bound. So the claim is true.

In the second stage, we claim that

$$\forall n \in \mathbb{N}(n > N \Rightarrow |a_n - s| < \varepsilon)$$

where N satisfies the criterion in our first claim. Since a_n increasing, $n > N \Rightarrow a_n \geq a_N$. So

$$\begin{aligned} s - a_n &\leq s - a_N \\ &< \varepsilon && \text{from above} \end{aligned}$$

Since $s - a_n \geq 0$, we have $|a_n - s| < \varepsilon$. $\square$

Chapter 2

Continuous functions

2.1 Accumulation point

Definition 2.1 (Neighbourhood). A set $A \subseteq \mathbb{R}$ is a neighbourhood of a point a, if there exists an open interval I such that

$$a \in I \subseteq A$$

Note that an interval is a set.

Definition 2.2 (Accumulation point). A real number ζ is an accumulation point of a set $A \in \mathbb{R}$ if every neighbourhood of ζ contains an infinite number of elements in A.

Note that ζ does not have to be an element of A.

- $\mathbb{Z}$ has no accumulation point since there exist a neighbourhood which does not contain an infinite number of members of $\mathbb{Z}$. For every $x \in \mathbb{Z}$, consider the neighbourhood

$$(x - \frac{1}{2},\ x + \frac{1}{2})$$

  which contains no members of $\mathbb{Z}$ other than x.

- Every point $x \in \mathbb{R}$ is an accumulation point, since all neighbourhoods of x contain an infinite number of members of $\mathbb{R}$.

- Similarly, every rational number $x \in Q$ is an accumulation point of $\mathbb{Q}$. Every irrational number is also an accumulation point of $\mathbb{Q}$. So every real number is an accumulation point of $\mathbb{Q}$.

- The set

$$\{\frac{1}{n} : n \in \mathbb{N}^{+}\}$$

  has only one accumulation point, 0, which is not in the set. 0 is an accumulation point because every neighbourhood of 0 must have no upper bound, so something in the form of

$$(n, \infty)$$

  which contains an infinite number of members of the set.

- Every point in the closed interval $[a, b]$ is an accumulation point of $[a, b]$ (intervals are sets). Every point outside the closed interval is not an accumulation point of $[a, b]$, since there must exist a neighbourhood of the point which lies entirely outside of the interval, and therefore contains no elements of the interval (so does not contain an infinite number of elements in the interval).

- Every point in the open interval (a,b) is an accumulation point of (a,b). The endpoints, a and b, are also accumulation points.

2.2 $\varepsilon - \delta$ definition of limit of a function

Basically the most important thing in this chapter:

Definition 2.3 ($\varepsilon - \delta$ definition of limit of a function)**.** A function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$, has a limit $l \in \mathbb{R}$ at the accumulation point x_0 of A if and only if

$$\forall \varepsilon > 0,\ \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon$$

We denote this as

$$\lim_{x \to x_0} f(x) = l$$

An intuitive translation of the definition reads whenever the input is sufficiently close to the target input, then the output must be within the specified range of the limit.

Example. Prove that

$$\lim_{x \to 2} x^2 + x - 2 = 4$$

Solution. Let $f(x) = x^2 + x - 2$. Given some $\varepsilon > 0$, we want to find some $\delta > 0$ such that

$$|x - 2| < \delta \Rightarrow |f(x) - 4| < \varepsilon.$$

$$\begin{aligned} |f(x) - 4| < \varepsilon &\Leftrightarrow \left|(x^2 + x - 2) - 4\right| < \varepsilon \\ &\Leftrightarrow \left|x^2 + x - 6\right| < \varepsilon \\ &\Leftrightarrow |(x+3)(x-2)| < \varepsilon \\ &\Leftrightarrow |x - 2| < \frac{\varepsilon}{|x+3|} \end{aligned}$$

Suppose, arbitrarily, we let $|x - 2| < 1$. Then $|x + 3| \leq |x - 2| + 5 < 6$. We want the above inequality to be true as well, so we have

$$|x - 2| < \frac{\varepsilon}{|x+3|} < \frac{\varepsilon}{6}$$

Now, we have two constraints that we want to both satisfy:

$$|x - 2| < 1 \text{ and } |x - 2| < \frac{\varepsilon}{6}$$

so we choose $\delta = \min(1, \frac{\varepsilon}{6})$ and both will be satisfied.

The actual proof. Choose $\delta = \min(1, \frac{\varepsilon}{6})$. Then $\delta \leq 1$ and $\delta \leq \frac{\varepsilon}{6}$. Suppose $|x - 2| < \delta$. Then we have

$$|x - 2| < 1 \Rightarrow |x + 3| < 6$$

and

$$|x - 2| < \frac{\varepsilon}{6}$$

$$
\begin{aligned}
\left|x^2+x-6\right| &= |(x-2)(x+3)| \\
&= |x-2||x+3| \\
&< \frac{\varepsilon}{6}|x+3| \\
&< \frac{\varepsilon}{6}\cdot 6 \\
&= \varepsilon
\end{aligned}
$$

which completes the proof.

Example. (MMT 1.5, Q1) Prove that

$$\lim_{x\to 0}\frac{1}{x}$$

does not exist.

Solution. Suppose the limit $= l \in \mathbb{R}$. Then either $l > 0$, $l < 0$, or $l = 0$.

$\mathbf{l > 0.}$ Suppose $\varepsilon = l$. There exists $\delta > 0$ such that

$$|x| < \delta \Rightarrow \left|\frac{1}{x} - l\right| < l$$

Choose $\delta < x < 0$. Clearly the LHS of the implication is true. But

$$\left|\frac{1}{x} - l\right| < l \Leftrightarrow 0 < \frac{1}{x} < 2l$$

so the RHS is false, and so the implication is false, contradiction.

$\mathbf{l < 0.}$ Suppose $\varepsilon = -l$ (since $\varepsilon > 0$). There exists $\delta > 0$ such that

$$|x| < \delta \Rightarrow \left|\frac{1}{x} - l\right| < -l$$

Choose $0 < x < \delta$. Clearly the LHS of the implication is true. But

$$\left|\frac{1}{x} - l\right| < -l \Leftrightarrow 2l < \frac{1}{x} < 0$$

so the RHS is false, and so the implication is false, contradiction.

$\mathbf{l = 0.}$ Suppose $\varepsilon = 1$. There exists $\delta > 0$ such that

$$|x| < \delta \Rightarrow \left|\frac{1}{x}\right| < 1$$

Choose $0 < x < \min(\delta, 1)$. The LHS of the implication is true. But

$$|x| < 1 \Leftrightarrow \left|\frac{1}{x}\right| > 1$$

so the RHS is false, and so the implication is false, contradiction.

In all three cases, we reach a contradiction. So the limit does not exist.

2.2.1 Definitions for alternative use cases

Here, we present a few definitions for some common use cases:

Definition 2.4.

$$\lim_{x\to\infty} f(x) = l$$

if and only if

$$\forall\varepsilon > 0,\ \exists c \in \mathbb{R} : \forall x > c,\ |f(x) - l| < \varepsilon$$

Definition 2.5.

$$\lim_{x\to-\infty} f(x) = l$$

if and only if

$$\forall\varepsilon > 0,\ \exists c \in \mathbb{R} : \forall x < c,\ |f(x) - l| < \varepsilon$$

Definition 2.6.

$$\lim_{x\to x_0} f(x) = \infty$$

if and only if

$$\forall r > 0,\ \exists\delta > 0 : |x - x_0| < \delta \Rightarrow f(x) > r$$

Definition 2.7.

$$\lim_{x\to\infty} f(x) = \infty$$

if and only if

$$\forall r > 0,\ \exists c \in \mathbb{R} : x > c \Rightarrow f(x) > r$$

2.3 Continuity

Definition 2.8. A function f is continuous at x_0 if

$$\lim_{x\to x_0} f(x) = f(x_0)$$

Definition 2.9. A function f is continuous on the interval $[a, b]$ if it is continuous at all $x_0 \in [a, b]$.

Example. Let f be defined by

$$f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Prove that f is continuous at 0.

Proof. We want to show that

$$\lim_{x\to 0} f(x) = f(0) = 0$$

$$\begin{aligned} \left|x\sin\frac{1}{x} - 0\right| < \varepsilon &\Leftrightarrow \left|x\sin\frac{1}{x}\right| < \varepsilon \\ &\Leftrightarrow |x|\left|\sin\frac{1}{x}\right| < \varepsilon \\ &\Leftrightarrow |x| < \frac{\varepsilon}{\left|\sin\frac{1}{x}\right|} \leq \frac{\varepsilon}{1} = \varepsilon \end{aligned}$$

Let $\delta = \varepsilon$. Then

$$\begin{aligned}|x| < \delta &\Rightarrow |x| < \varepsilon \\ &\Rightarrow |x|\left|\sin\frac{1}{x}\right| < \varepsilon \cdot \left|\sin\frac{1}{x}\right| < \varepsilon \\ &\Rightarrow \left|x\sin\frac{1}{x}\right| < \varepsilon\end{aligned}$$

□

Example. Consider the sign function $sgn : \mathbb{R} \to \mathbb{R}$ defined by

$$sgn(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Prove that sgn is continuous at all points except 0.

Proof. Take arbitrary $x_0 < 0$. Take arbitrary $\varepsilon > 0$. For this ε, we want to find some $\delta > 0$ such that for all $x \in \mathbb{R}$,

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

Intuition. When we have $x_0 < 0$, we want to make sure the x that we can choose are close enough to x_0 such that $x < 0$, then both $f(x)$ and $f(x_0)$ evaluate to -1.

Let $\delta = \frac{|x_0|}{2}$. Suppose $|x - x_0| < \delta$ (because that's how we prove implications – we assume the antecedent is true, then proceed to show that the consequent is true as well). Then we have

$$\begin{aligned}-\frac{|x_0|}{2} &< x - x_0 < \frac{|x_0|}{2} \\ \Leftrightarrow x_0 - \frac{|x_0|}{2} &< x < x_0 + \frac{|x_0|}{2}\end{aligned}$$

We focus on the upper inequality.

$$\begin{aligned}x &< x_0 + \frac{|x_0|}{2} \\ &= x_0 - \frac{x_0}{2} \text{ since } x_0 < 0 \\ &= \frac{x_0}{2} \\ &< 0\end{aligned}$$

so $f(x) = -1$. Then

$$|f(x) - f(x_0)| = |-1 - (-1)| = 0 < \varepsilon$$

Since $\varepsilon > 0$ arbitrary, f is continuous at x_0. Since $x_0 < 0$ arbitrary, f is continuous at all $x_0 < 0$.

The case where $x_0 > 0$ follows similarly (but is still included here for completeness). Take arbitrary

$x_0 > 0$. Take arbitrary $\varepsilon > 0$. Let $\delta = \frac{|x_0|}{2}$. Suppose $|x - x_0| < \delta$. Then we have

$$\begin{aligned} -\frac{|x_0|}{2} &< x - x_0 < \frac{|x_0|}{2} \\ \Leftrightarrow x_0 - \frac{|x_0|}{2} &< x < x_0 + \frac{|x_0|}{2} \end{aligned}$$

We focus on the lower inequality.

$$\begin{aligned} x &> x_0 - \frac{|x_0|}{2} \\ &= x_0 - \frac{x_0}{2} \text{ since } x_0 > 0 \\ &= \frac{x_0}{2} \\ &> 0 \end{aligned}$$

so $f(x) = 1$. Then

$$|f(x) - f(x_0)| = |1 - 1| = 0 < \varepsilon$$

Since $\varepsilon > 0$ arbitrary, f is continuous at x_0. Since $x_0 > 0$ arbitrary, f is continuous at all $x_0 > 0$.

Let $x_0 = 0$. Since we want to show that *sgn* is not continuous at 0, we want to show

$$\exists \varepsilon > 0 : \forall \delta > 0,\ \exists x \in \mathbb{R} : |x - 0| < \delta \wedge |f(x) - f(0)| \geq \varepsilon$$

Intuition. We know that all the points around 0 either evaluate to -1 or 1, so they are of distance 1 away from $f(0) = 0$. We can find points that are infinitesmally close to 0 (i.e. satisfy $|x - x_0| < \delta$), but are still of distance 1 away from $f(0)$. So we pick some $\varepsilon < 1$, then the distance must be greater than ε.

Remark. It doesn't really matter if we use *take arbitrary* or *let* to show *for all*, but things can get messy if we want to show *there exists* and *for all* at the same time, so it's best to distinguish them. Also, we can't use *take arbitrary* to show *there exists*.

Let $\varepsilon = \frac{1}{2}$ (existence). Take arbitrary $\delta > 0$ (for all). Let $x = \frac{\delta}{2}$ (existence). Then

$$\begin{aligned} |x| &= \left|\frac{\delta}{2}\right| \\ &= \frac{\delta}{2} \text{ since } \delta > 0 \\ &< \delta \end{aligned}$$

Since $x = \frac{\delta}{2}$, we know that $x > 0$, so $f(x) = 1$. But then

$$|f(x) - f(0)| = |1 - 0| = 1 > \varepsilon$$

and we are done. □

2.3.1 Bolzano-Weierstrass theorem

Definition 2.10 (Bolzano-Weierstrass theorem)**.** Every bounded sequence has a convergent subsequence.

We will see how the theorem can be used to prove several properties of continuous functions.

Proposition 2.11. If $f : [a, b] \to \mathbb{R}$ is a continuous function, then there exist $r, s \in [a, b]$ such that

$$f(r) = \sup\{f(x) : x \in [a, b]\} \in \mathbb{R}$$
$$f(s) = \inf\{f(x) : x \in [a, b]\} \in \mathbb{R}$$

In other words, we want to show two things:

- f attains its maximum and minimum in $[a, b]$ in $[a, b]$ (confusing, yes)
- The maximum and minimum are finite

Proof. We will only show the first half of both statements, i.e. the parts for maximum. The parts for minimum follow similarly. Let $M = \sup\{f(x) : x \in [a, b]\}$.

By definition of supremum, for all $n \geq 1$, there exists x_n such that

$$M \geq f(x_n) \geq M - \frac{1}{n}$$

Remark. Suppose there exists $n \geq 1$ such that for all $x \in [a, b]$, the inequality does not hold. Then either $f(x_n) > M$ or $f(x_n) < M - \frac{1}{n}$. For the former, this violates the fact that M is the supremum, and hence an upper bound. For the latter, this also violates the fact that M is the supremum, since we now have a smaller upper bound (namely, $M - \frac{1}{n}$) and so M is not the least upper bound.

The sequence x_n is bounded (by $[a, b]$), so by Bolzano-Weierstrass, there exists a convergent subsequence x_{n_i}.

Remark. x_n is not necessarily convergent.

Suppose $x_{n_i} \to v_M$. Since $a \leq x_{n_i} \leq b \; \forall i$, we have $v_M \in [a, b]$. By continuity of f, we have

$$\lim_{i \to \infty} f(x_{n_i}) = f(v_M)$$

But then $f(x_{n_i})$ is a subsequence of $f(x_n)$, so they converge to the same limit. By sandwich theorem, $f(x_n) \to M$, so also $f(x_{n_i}) \to M$. Therefore,

$$f(v_M) = \lim_{i \to \infty} f(x_{n_i}) = M$$

Suppose M is not finite, i.e. f is not bounded above. Then for all $n \geq 1$, there exists $x_n \in [a, b]$ such that $f(x_n) > n$. Since x_n is taken from $[a, b]$, x_n is bounded. By Bolzano-Weierstrass, there exists a convergent subsequence x_{n_i}.

Suppose $x_{n_i} \to x_0$. Since $a \leq x_{n_i} \leq b$ for all i, $x_0 \in [a, b]$, so f is continuous at x_0. Therefore, we have

$$\lim_{i \to \infty} f(x_{n_i}) = f(x_0)$$

which is finite (??????). However, since $f(x_{n_i})$ is a subsequence of $f(x_n)$, they tend to the same limit. Since (the rest of the proof seems wrong.) □

2.3.2 Intermediate value theorem

Theorem 2.12 (Intermediate value theorem)**.** Consider a continuous function $f : [a, b] \to \mathbb{R}$. If s is between $f(a)$ and $f(b)$, i.e.

$$\min\{f(a), f(b)\} < s < \max\{f(a), f(b)\}$$

then there exists $c \in (a, b)$ such that $f(c) = s$.

Proof. Suppose $f(a) < s < f(b)$. The case where $f(b) < s < f(a)$ can be handled similarly. Consider the set

$$S = \{x \in [a, b] : f(x) \leq s\}$$

Since S is bounded above and non-empty, it must have a supremum, say c.

(Why must $c \in [a, b]$?)

Since $c \in [a, b]$, f is continuous at c. By continuity, we have

$$\forall \varepsilon > 0,\ \exists \delta > 0 : |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

Take arbitrary $\varepsilon > 0$. Since c is the supremum, there must exist some $x_1 \in (c - \delta, c]$ such that $x_1 \in S$.

Remark. Otherwise, this means that the largest element is some finite distance δ smaller than c, so the supremum would be at most $c - \delta$, but not c.

Since $x_1 \in S$, we have $f(x_1) \leq s$. As

$$|f(x) - f(c)| < \varepsilon \Leftrightarrow f(c) - \varepsilon < f(x) < f(c) + \varepsilon$$

we get

$$\begin{aligned} & f(c) - \varepsilon < f(x_1) < f(c) + \varepsilon \\ & \Rightarrow f(c) - \varepsilon < f(x_1) \leq s \\ & \Rightarrow f(c) - \varepsilon < s \\ & \Rightarrow f(c) < s + \varepsilon \end{aligned} \tag{2.1}$$

Since c is the supremum, for all $x \in (c, c + \delta)$, $x \notin S$.

Remark. Otherwise, we have an element greater than c, so c is not an upper bound.

But then either $f(x) \leq s$ or $f(x) > s$. So there exists $x_2 \in (c, c + \delta)$ such that $f(x_2) > s$. Then

$$\begin{aligned} & f(c) - \varepsilon < f(x_2) < f(c) + \varepsilon \\ & \Rightarrow s < f(x_2) < f(c) + \varepsilon \\ & \Rightarrow s < f(c) + \varepsilon \\ & \Rightarrow s - \varepsilon < f(c) \end{aligned} \tag{2.2}$$

Combining Equation (2.1) and Equation (2.2), we get

$$s - \varepsilon < f(c) < s + \varepsilon$$

Since $\varepsilon > 0$ arbitrary, this is true for all $\varepsilon > 0$. So we have $f(c) = s$. □

2.3.3 Uniform continuity

Definition 2.13 (Uniform continuity). A function $f : A \to \mathbb{R}$ is uniformly continuous if

$$\forall\varepsilon > 0,\ \exists\delta > 0 : \forall x, y \in A,\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

How is this different from the definition of continuity? The difference is that for continuity,

$$\forall\varepsilon > 0\ \forall y \in A,\ \exists\delta > 0 : \forall x \in A,\ |x - y| \Rightarrow |f(x) - f(y)| < \varepsilon$$

the value of δ may depend on both ε and the accumulation point y, whereas in uniform continuity, the value of δ only depends on ε and is independent of the choice of y. We illustrate the difference in the following two examples.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Show that f is not uniformly continuous.

Proof. We want to show that

$$\exists\varepsilon > 0 : \forall\delta > 0,\ \exists x, y \in \mathbb{R} : |x - y| < \delta \wedge |f(x) - f(y)| \geq \varepsilon$$

Let $\varepsilon = 1$. Take arbitrary $\delta > 0$. How might we find the right x, y? Firstly, we want

$$|x - y| < \delta$$

so we can choose $y = x + \frac{\delta}{2}$, then the difference will be $\frac{\delta}{2} < \delta$. Then we try to expand $f(x) - f(y)$:

$$\begin{aligned}
|f(x) - f(y)| &= \left|x^2 - (x + \frac{\delta}{2})^2\right| \\
&= \left|(x + \frac{\delta}{2})^2 - x^2\right| \\
&= \left|x^2 + \delta x + \frac{\delta^2}{4} - x^2\right| \\
&= \left|\delta x + \frac{\delta^2}{4}\right| \\
&= \delta(x + \frac{\delta}{4})
\end{aligned}$$

With this, we can choose $x = \frac{1}{\delta}$ so that the expanded product becomes $1+$ something positive $> \varepsilon$.

Let $x = \frac{1}{\delta} + \frac{\delta}{2}$, $y = \frac{1}{\delta}$ (x and y are swapped from the above selection for easier manipulation.) Then

$$|x - y| = \left|\frac{1}{\delta} + \frac{\delta}{2} - \frac{1}{\delta}\right| = \frac{\delta}{2} < \delta$$

Also,

$$\begin{aligned}
|f(x) - f(y)| &= \left|(\frac{1}{\delta} + \frac{\delta}{2})^2 - (\frac{1}{\delta})^2\right| \\
&= \left|\frac{1}{\delta^2} + 1 + \frac{\delta^2}{4} - \frac{1}{\delta^2}\right| \\
&= \left|1 + \frac{\delta^2}{4}\right| \\
&= 1 + \frac{\delta^2}{4} \\
&> 1 \\
&= \varepsilon
\end{aligned}$$

Since $\delta > 0$ arbitrary, we are done. □

Example. Let $f : [-M, M] \to \mathbb{R}$ be defined by $f(x) = x^2$. Show that f is uniformly continuous.

Proof. We want to show that

$$\forall \varepsilon > 0,\ \exists \delta > 0 : \forall x, y \in [-M, M],\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Take arbitrary $\varepsilon > 0$, and take arbitrary $x, y \in [-M, M]$. For $|f(x) - f(y)| < \varepsilon$ to hold for this ε, we want

$$\begin{aligned}
|f(x) - f(y)| < \varepsilon &\Leftrightarrow \left|x^2 - y^2\right| < \varepsilon \\
&\Leftrightarrow |(x + y)(x - y)| < \varepsilon \\
&\Leftrightarrow |x + y||x - y| < \varepsilon \\
&\Leftrightarrow |x - y| < \frac{\varepsilon}{|x + y|} \leq \frac{\varepsilon}{2M}
\end{aligned}$$

since $-2M < x + y < 2M$. Then we let $\delta = \frac{\varepsilon}{2M}$. We are basically done here because we have just found a δ that only depends on ε, but not x or y, but we will proceed with the definition just to be cautious.

$$\begin{aligned}
|x - y| < \delta &\Rightarrow |x - y| < \frac{\varepsilon}{2M} \\
&\Leftrightarrow |x - y||x + y| < \frac{\varepsilon}{2M}|x + y| \\
&\Leftrightarrow \left|x^2 - y^2\right| < \frac{\varepsilon}{2M}|x + y| \leq \frac{\varepsilon}{2M} \cdot 2M = \varepsilon
\end{aligned}$$

□

Now, we will show a few (useful?) properties involving uniform continuity.

Proposition 2.14. If $f : A \to \mathbb{R}$ is uniformly continuous, then it is continuous on A.

Proof. We want to show that

$$\forall \varepsilon > 0\ \forall x_0 \in A,\ \exists \delta > 0 : \forall x \in A,\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

Assume f is uniformly continuous. Then we have

$$\forall \varepsilon > 0,\ \exists \delta > 0 : \forall x, y \in A,\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Take arbitrary $\varepsilon > 0$ and take arbitrary $x_0 \in A$. By assumption, we know that

$$\exists \delta > 0 : \forall x \in A,\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

Since $\varepsilon > 0$ arbitrary, f is continuous at x_0. Since $x_0 \in A$ arbitrary, f is continuous on A.

Remark. In this proof, it is important to be precise in what is arbitrary and what is not, and the implications of generalising an arbitrary variable to all cases.

□

Proposition 2.15 (Characterisation of uniform continuity via sequences)**.** Let $f : A \to \mathbb{R}$. f is uniformly continuous *if and only if* for any pair of sequences $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \in A$ with $x_n - y_n \to 0$, we have $f(x_n) - f(y_n) \to 0$.

Proof. $\Rightarrow$ Suppose f is uniformly continuous. Then

$$\forall \varepsilon > 0,\ \exists \delta > 0 : \forall x, y \in A,\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Take arbitrary $\varepsilon > 0$ and let $\delta > 0$ satisfy the above for this ε. Since $x_n - y_n \to 0$, we know that

$$\exists N \in N : n > N \Rightarrow |x_n - y_n| < \delta$$

But then

$$\exists N \in N : n > N \Rightarrow |f(x_n) - f(y_n)| < \varepsilon$$

so $f(x_n) - f(y_n) \to 0$.

$\Leftarrow$. Suppose for any pair of sequences $x_n, y_n \in A$ with $x_n - y_n \to 0$, we have $f(x_n) - f(y_n) \to 0$. Assume that f is not uniformly continuous. Then this means

$$\exists \varepsilon > 0 : \forall \delta > 0,\ \exists x, y \in A : |x - y| < \delta \wedge |f(x) - f(y)| \geq \varepsilon$$

Intuition. We are assuming towards a contradiction. One possible way, which is how this proof will follow, to reach a contradiction is to contradict the initial implication assumption, i.e. $x_n - y_n \to 0$ but $f(x_n) - f(y_n) \not\to 0$.

The definition of uniform continuity (or rather, its negation), as it is presented now, has nothing to do with sequences. What do we do? We construct δ specifically such that it forms a sequence that converges to 0, and so by the sandwich theorem, $x_n - y_n \to 0$. Now we have introduced n into the inequality. But then $\varepsilon > 0$, so $f(x_n) - f(y_n) \not\to 0$.

Remark. When in doubt, set $\delta = \frac{1}{n}$.

Then we have

$$\forall n \in \mathbb{N},\ \exists x_n, y_n \in A : |x_n - y_n| < \frac{1}{n} \wedge |f(x_n) - f(y_n)| \geq \varepsilon$$

Since

$$|x_n - y_n| < \frac{1}{n}\ \forall n \in \mathbb{N}$$

we have $x_n - y_n \to 0$. But then for every n,

$$f(x_n) - f(y_n) \geq \varepsilon > 0$$

so $f(x_n) - f(y_n) \not\to 0$, which contradicts our initial assumption. So f must be uniformly continuous. $\square$

This characterisation is particularly useful when we want to show that some function is *not* uniformly continuous, i.e. we find some pair of sequences x_n and y_n such that $x_n - y_n \to 0$ but $f(x_n) - f(y_n) \not\to 0$.

Example. Show that $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x}$$

is not uniformly continuous.

Solution. Let $x_n = \frac{1}{n}$ and $y_n = \frac{1}{2n}$. We know that

$$x_n - y_n = \frac{1}{2n} \to 0$$

but

$$f(x_n) - f(y_n) = n - 2n = -n \to -\infty$$

so by Proposition 2.15, f is not uniformly continuous.

Theorem 2.16. Every continuous function $f : [a, b] \to \mathbb{R}$ is uniformly continuous.

Proof. Suppose f is not uniformly continuous. Then we have

$$\exists\varepsilon > 0 : \forall\delta > 0,\ \exists x, y \in [a, b] : |x - y| < \delta \wedge |f(x) - f(y)| \geq \varepsilon$$

which implies (again, the same trick)

$$\exists\varepsilon > 0 : \forall n \geq 1,\ \exists x_n, y_n \in [a, b] : |x_n - y_n| < \frac{1}{n} \wedge |f(x_n) - f(y_n)| \geq \varepsilon$$

Now we have two sequences x_n and y_n to work with. From the above, we get

$$-\frac{1}{n} < x_n - y_n < \frac{1}{n}$$

and so by the sandwich theorem, we get $x_n - y_n \to 0$.

Since x_n is bounded, by Bolzano-Weierstrass, there exists a convergent subsequence x_{n_i}. Suppose $x_{n_i} \to x_0$. Since $a \leq x_n \leq b$ for all $n \geq 1$, we know that the limit $x_0 \in [a, b]$. Since f is continuous on $[a, b]$, it is continuous on every point in the interval, and so it is continuous on x_0. By definition of continuity, we have $f(x_{n_i}) \to f(x_0)$.

Similarly, since y_n is bounded, there exists a convergent subsequence y_{n_i}. As $x_n - y_n \to 0$, we get $x_{n_i} - y_{n_i} \to 0$ as well.

Remark. I don't know how sound it is to directly jump to this conclusion, but loosely speaking we have

$$x_n - y_n \to 0 \equiv \lim_{n\to\infty} x_n - y_n = 0$$

Since $n_i \to \infty$ as $i \to \infty$, we have

$$\lim_{i \to \infty} x_{n_i} - y_{n_i} = 0$$

Then by the linearity of limits we get $y_{n_i} \to x_0$. By continuity, we have

$$f(y_{n_i}) \to f(x_0)$$

and again, by the linearity of limits we get

$$f(x_{n_i}) - f(y_{n_i}) \to f(x_0) - f(x_0) = 0$$

However, this contradicts the initial assumption that $f(x_{n_i}) - f(y_{n_i}) \not\to 0$. □

Observe that in the theorem we have just proved, f is continuous on a *closed* interval $[a, b]$.

- If f were continuous on an *open* interval, uniform continuity is not guaranteed (e.g. $f : (0, 1) \to \mathbb{R}$ with $f(x) = \dfrac{1}{x}$).
- If f were bounded (with regards to its outputs) and continuous, uniform continuity is also not guaranteed (e.g. $f(x) = \cos(x^2)$).

2.4 Exercises

1. Prove the following limits using the $\varepsilon - \delta$ definition.
   a. $\lim_{x \to 4} x = 4$
   b. $\lim_{x \to \infty} \sqrt{x} = \infty$
   c. $\lim_{x \to 0} x(\cos x)(x^2 + 1) = 0$
   d. $\lim_{x \to 8} \sqrt[3]{x} = 2$
   e. $\lim_{x \to 1} x^3 - 2x = -1$
2. Using the definition of uniform continuity, show that the following functions are not uniformly continuous.
   a. $f : (0, 1] \to \mathbb{R}$ with $f(x) = \dfrac{1}{x}$
   b. $f(x) = x^2 \ \forall x \in \mathbb{R}$
   c. $f(x) = \cos(x^2) \ \forall x \in \mathbb{R}$
3. Using Proposition 2.15, show that the following functions are not uniformly continuous.
   a. $f(x) = x^2 \ \forall x \in \mathbb{R}$
   b. $f(x) = \cos(x^2) \ \forall x \in \mathbb{R}$

Proof. We want to show that

$$\exists \varepsilon > 0 : \forall \delta > 0, \ \exists x, y \in (0, 1] : |x - y| < \delta \wedge |f(x) - f(y)| < \varepsilon$$

Let $\varepsilon = 1$. Take arbitrary δ such that $0 < \delta < 1$ (we will deal with $\delta \geq 1$ later). Since we want

$$|x - y| < \delta$$

we choose $y = x + \frac{\delta}{2}$. Then we expand:

$$\begin{aligned}
|f(x) - f(y)| \geq \varepsilon &= \left|\frac{1}{x} - \frac{1}{x + \frac{\delta}{2}}\right| \\
&= \left|\frac{1}{x} - \frac{2}{2x + \delta}\right| \\
&= \left|\frac{2x + \delta - 2x}{x(2x + \delta)}\right| \\
&= \frac{\delta}{x(2x + \delta)}
\end{aligned}$$

We want this to $\geq \varepsilon$:

$$\begin{aligned}
\frac{\delta}{x(2x + \delta)} \geq \varepsilon &\Leftrightarrow \frac{\delta}{x(2x + \delta)} \geq 1 \\
&\Leftrightarrow \delta \geq x(2x + \delta) \\
&\Leftrightarrow 2x^2 + \delta x - \delta \leq 0 \\
&\Leftrightarrow 2(x + \frac{\delta}{4}) - \delta - \frac{\delta^2}{8} \leq 0
\end{aligned}$$

So we choose $x = \frac{\delta}{4}$, $y = \frac{\delta}{4} + \frac{\delta}{2} = \frac{3\delta}{4}$. Since $0 < \delta < 1$, we know that $x, y \in (0, 1]$. Then

$$|x - y| = \left|\frac{\delta}{4} - \frac{3\delta}{4}\right| = \left|-\frac{\delta}{2}\right| = \frac{\delta}{2} < \delta$$

Also,

$$\begin{aligned}
|f(x) - f(y)| &= \left|\frac{4}{\delta} - \frac{4}{3\delta}\right| \\
&= \left|\frac{8\delta}{3\delta}\right| \\
&= \frac{8}{3} \\
&> 1 \\
&= \varepsilon
\end{aligned}$$

so we are done for all $\delta \in (0, 1)$.

What about $\delta \geq 1$? We simply take $\varepsilon = 1$ (as before), $x = \frac{1}{4}$ and $y = \frac{3}{4}$. Then

$$|x - y| = \left|\frac{1}{4} - \frac{3}{4}\right| = \frac{1}{2} < 1 \leq \delta$$

and the other half of the proof follows as above. $\square$

Chapter 3

Series

3.1 Definitions

Definition 3.1. A series is defined to be an infinite sum of real numbers:

$$\sum_{i=1}^{\infty} a_i$$

This definition isn't very insightful. Instead,

Definition 3.2. For a series (an infinite series) $\sum_{i=1}^{\infty} a_i$, the partial sum S_n is defined as

$$S_n = \sum_{i=1}^{n} a_i$$

We will use partial sums to define the convergence and divergence of a series.

Definition 3.3. A sequence converges to some $l \in \mathbb{R}$ if and only if

$$\lim_{n\to\infty} S_n = l$$

where $(S_n)_{n\geq 1}$ represents the sequence of partial sums.

A sequence diverges if it does not converge to some $l \in \mathbb{R}$.

It is very important to note the differences in the definitions of convergence and divergence for sequences and series:

Limit	$l \in \mathbb{R}$	∞	$-\infty$	No limit
Sequence a_n	Converges	Converges	Converges	Diverges
Series $\sum a_n$	Converges	Diverges	Diverges	Diverges

Nobody knows why we are defining them like this. Anyway, we proceed.

3.2 Tests

The following section is not strictly laid out according to the slides, but how a student outside of DoC would most likely be learning them.

3.2.1 Divergence test

Test 3.4 (Divergence test).

$$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

In the notes, the contrapositive of this test is stated instead (and it is not stated as a test). Note that unlike in sequences, if a series does not diverge, then it converges to some $l \in \mathbb{R}$, which enables this contrapositive statement in the slides. We will prove the contrapositive here.

Proof. We want to prove that

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Rightarrow \lim_{n\to\infty} a_n = 0$$

Assume $\sum_{n=1}^{\infty} a_n$ converges. Then the sequence S_n converges to some limit $l \in \mathbb{R}$. So S_n is a Cauchy sequence.

Given some $\varepsilon > 0$, using the $\varepsilon - N$ definition of convergence, we want find $N \in \mathbb{N}$ such that for all $n > N$, $|a_n| < \varepsilon$. Since S_n is a Cauchy sequence,

$$\exists N' \in \mathbb{N} : \forall n, m > N, |S_m - S_n| < \varepsilon$$

We pick $m = n+1$. Then $|S_{n+1} - S_n| < \varepsilon$. Since $S_{n+1} - S_n = a_{n+1}$, we have $|a_{n+1}| < \varepsilon$ for all $n > N'$. So let $N = N' + 1$. Then for all $n > N, n-1 > N'$, so $|a_n| < \varepsilon$. □

3.2.2 Comparison test

We write the following notations:

- $\sum_{i=1}^{\infty} c_i$ represents a (c)onverging series.
- $\sum_{i=1}^{\infty} d_i$ represents a (d)iverging series.

Test 3.5 (Comparison test). If there exists $N \in \mathbb{N}$ such that $\forall i > N$,

- $a_i \leq c_i$, then $\sum_{i=1}^{\infty} a_i$ converges.
- $d_i \geq a_i$, then $\sum_{i=1}^{\infty} a_i$ diverges.

Note that there is no need for every term of $a_i \leq c_i$, or $a_i \geq d_i$ – we just need to show that *after some point*, all terms in the sequence will fulfill the requirement. This enables us to apply the comparison test to determine convergence of series that we would otherwise not be able to determine using the more commonly seen version, which necessitates that *every* term fulfill the requirement.

Example. (MMT 2, 1(b), adapted) Use the comparison test to determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{4}{3n^2 - 4}$$

Solution. Let $a_n = \frac{4}{5n^2-4}$ and $c_n = \frac{2}{n^2}$.

$$\begin{aligned} a_n \le c_n &\Leftrightarrow \frac{4}{3n^2-4} \le \frac{2}{n^2} \\ &\Leftrightarrow 4n^2 \le 6n^2 - 4 \\ &\Leftrightarrow 2n^2 \ge 4 \end{aligned}$$

So $\forall n \ge 2$, $2n^2 \ge 4$, $a_n \le c_n$.

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{2}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since $\sum \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} c_n$ also converges. So $\sum_{n=1}^{\infty} a_n$ converges.

Example. (MMT 2, 1(c)) Use the comparison test to determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n-4}$$

Solution. Let $a_n = \frac{1}{n-4}$ and $d_n = \frac{1}{n}$.

$$\begin{aligned} d_n \le a_n &\Leftrightarrow \frac{1}{n} \le \frac{1}{n-4} \\ &\Leftrightarrow n - 4 \le n \\ &\Leftrightarrow -4 \le 0 \text{ which is true } \forall n. \end{aligned}$$

So $\forall n \ge 1$, $d_n \le a_n$. Since d_n diverges, a_n diverges.

3.2.3 Limit comparison test

Test 3.6 (Limit comparison test)**.**

$$\lim_{n\to\infty} \frac{a_n}{c_n} = l \in \mathbb{R} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges.}$$

$$\lim_{n\to\infty} \frac{d_n}{a_n} = l \in \mathbb{R} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Note that in this statement of the test, a_n is in the numerator when we want to show convergence, but a_n is in the denominator when we want to show divergence. This is very odd as putting a_n in the denominator does not change anything, but we will accept it for the purposes of passing this module. The more commonly seen version of the test is as follows:

Test 3.7 (Limit comparison test, better)**.** If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l \in \mathbb{R}$$

then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

Example. (MMT 2, 2(c)) Use the limit comparison test to determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{1}{3n^2+4n-2}$$

Solution. Let $a_n = \frac{1}{3n^2+4n-2}$ and $c_n = \frac{1}{3n^2}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{c_n} &= \frac{\frac{1}{3n^2+4n-2}}{\frac{1}{3n^2}} \\
&= \lim_{n\to\infty} \frac{3n^2}{3n^2+4n-2} \\
&= \lim_{n\to\infty} \frac{3}{3+\frac{4}{n}-\frac{2}{n^2}} \\
&= \frac{3}{3+0-0} \\
&= 1
\end{aligned}$$

Also,

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{3n^2} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} c_n$ also converges. Since $\lim_{n\to\infty} \frac{a_n}{c_n} \in \mathbb{R}$ exists and $\sum_{n=1}^{\infty} c_n$ converges, a_n converges.

3.2.4 D'Alembert's (limit) ratio test

Test 3.8 (D'Alembert's ratio test). If there exists some $N \in \mathbb{N}$ such that

- $\frac{a_{n+1}}{a_n} \geq 1$ for all $n \geq N$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- there exists some $k \in \mathbb{R}$ such that $k < 1$ and $\frac{a_{n+1}}{a_n} \leq k$ for all $n \geq N$, then $\sum_{n=1}^{\infty} a_n$ converges.

This test isn't very useful, but it can be used to prove the correctness of the more useful *limit* ratio test:

Test 3.9 (D'Alembert's limit ratio test). Suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l \in \mathbb{R}$.

- $l > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.
- $l < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ converges.
- $l = 1 \Rightarrow$ inconclusive.

For $l = 1$, consider $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$. The limits $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ and $\lim_{n\to\infty} \frac{b_{n+1}}{a_b}$ are both 1, but a_n diverges and b_n converges.

Also, it is very painful that we do not introduce the absolute value version (i.e. the correct version) of d'Alembert's (limit) ratio test, since as we will show later, absolute convergence implies unconditional convergence. I am of the strong opinion that this non-absolute value version is wrong, but for the purposes of passing this module, this incorrect version is in this set of notes, as aligned with the slides. The correct version is stated in the next section.

Example. (MMT 2, 3(d)) Using d'Alembert's limit ratio test and $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$, determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Solution. Let $a_n = \frac{n!}{n^n}$. Since $a_n > 0$ for all $n > 0$, we can apply d'Alembert's limit ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \\
&= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} (n+1) \cdot \frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \\
&= \lim_{n\to\infty} (\frac{n}{n+1})^n \\
&= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} \\
&= \frac{1}{e} \\
&< 1
\end{aligned}$$

So $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

3.2.5 Integral test

Test 3.10 (Integral test)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, decreasing, and positive on the interval $[1, \infty)$, such that $a_n = f(n)$. If there exists $N \in \mathbb{N}$ such that

- $\int_N^{\infty} f(x)\,\mathrm{d}x$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- $\int_N^{\infty} f(x)\,\mathrm{d}x$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example. We consider the upper Riemann approximation to show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. It is clear from the diagram that $\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{n}$. Then,

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x}\,\mathrm{d}x &= \lim_{b\to\infty} [\ln x]_1^b \\
&= \lim_{b\to\infty} [\ln b - \ln 1] \\
&= \lim_{b\to\infty} \ln b
\end{aligned}$$

which diverges since $\ln x$ is strictly increasing and not bounded above. So $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges.

Example. We consider the lower Riemann approximation to show that $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges (note that we are summing from 2, not 1). It is clear from the diagram that $\sum_{n=2}^{\infty} \frac{1}{n} < \int_1^{\infty} \frac{1}{n^2}$. Then,

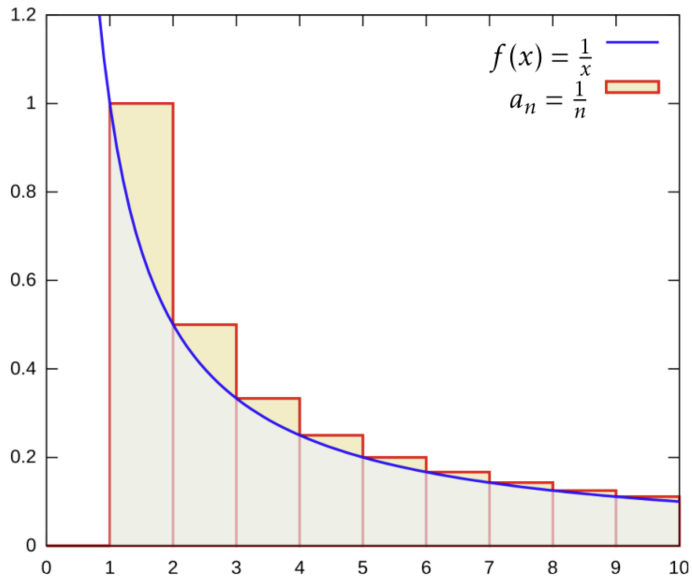

Figure 3.1: Upper Riemann approximation of $\frac{1}{n}$

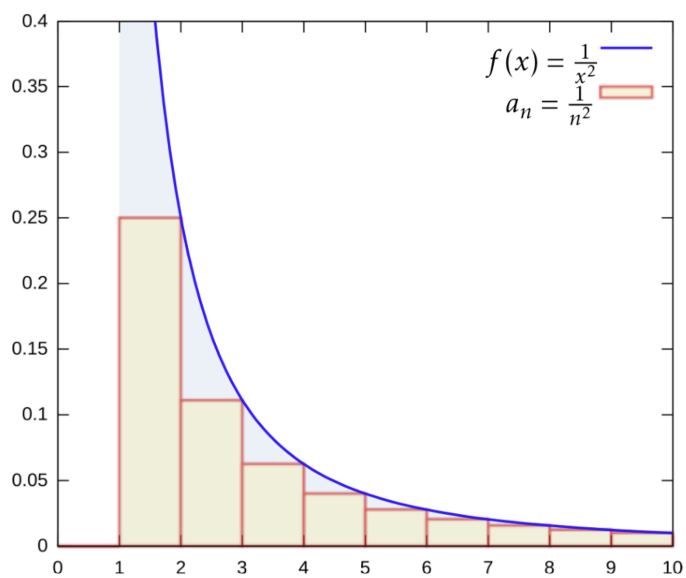

Figure 3.2: Lower Riemann approximation of $\frac{1}{n^2}$

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x &= \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b \\ &= \lim_{b\to\infty}(-\frac{1}{b}+\frac{1}{1}) \\ &= \lim_{b\to\infty}(1-\frac{1}{b}) \\ &= 1-0 \\ &= 1\end{aligned}$$

so the integral converges. Then $\sum_{n=2}^{\infty}\frac{1}{n^2}$ also converges, so $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

Example. (MMT 2, 4) Use the integral test to determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty}\frac{1}{e^n}$$

Solution. Let $f(x)=\frac{1}{e^x}$. $f(x)$ is continuous, decreasing, and positive on the interval $[1,\infty)$.

$$\begin{aligned}\int_1^\infty f(x)\,\mathrm{d}x &= \lim_{b\to\infty}\int_1^b \frac{1}{e^x}\,\mathrm{d}x \\ &= \lim_{b\to\infty}\left[-\frac{1}{e^x}\right]_1^b \\ &= \lim_{b\to\infty}(-\frac{1}{e^b}+\frac{1}{e}) \\ &= 0+\frac{1}{e} \\ &= \frac{1}{e}\end{aligned}$$

so the integral converges. So $\sum_{n=1}^{\infty}\frac{1}{e^n}$ also converges.

3.3 Absolute convergence

Before we write down the correct version of the (limit) ratio test, we need to define a few things.

Definition 3.11 (Absolute convergence)**.** A series $\sum_{n=1}^{\infty}a_n$ is absolutely convergent if and only if

$$\sum_{n=1}^{\infty}|a_n|$$

converges also.

Definition 3.12 (Unconditional convergence)**.** We define a permutation π over the natural numbers by $\pi:\mathbb{N}\to\mathbb{N}$, such that π is bijective. A series $\sum_{n=1}^{\infty}a_n$ is unconditionally convergent if and only if it converges, and

$$\sum_{n=1}^{\infty}a_{\pi(n)}$$

converges to the same limit, for all $\pi:\mathbb{N}\to\mathbb{N}$.

Lemma 3.13. Absolute convergence implies (unconditional) convergence.

3.3.1 Absolute value comparison test

Test 3.14 (Absolute value comparison test)**.** Let b_n be a non-decreasing sequence such that $\sum_{n=1}^{\infty} b_n$ converges. Let a_n be a sequence such that $|a_n| \leq b_n$ for all $n \geq 1$. Then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Since $|a_n| \leq b_n$ for all $n \geq 1$, we know that b_n is a non-negative, non-decreasing (given) sequence. Let S_n and S'_n denote the partial sums for $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, respectively. Since $\sum_{n=1}^{\infty} b_n$ converges, by contrapositive of divergence test, $\lim\limits_{n\to\infty} b_n = 0$. Since the sequence b_n converges to some real number, S'_n is a Cauchy sequence. Then given some $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $m, n > N$, $|S'_m - S'_n| < \varepsilon$. We let $m \geq n > N$. Since b_n is non-decreasing, we have $S'_m \geq S'_n$, so $S'_m - S'_n < \varepsilon$.

$$\begin{aligned}
|S_m - S_n| &= \left|\sum_{i=n+1}^{m} a_i\right| \\
&\leq \sum_{i=n+1}^{m} |a_i| \text{ by triangle inequality} \\
&\leq \sum_{i=n+1}^{m} b_i \\
&= S'_m - S'_n \\
&< \varepsilon
\end{aligned}$$

So S_n is a Cauchy sequence as well, and converges to some $l \in \mathbb{R}$. So $\sum_{n=1}^{\infty} a_n$ converges. □

3.3.2 D'Alembert's (limit) ratio test, correct

Test 3.15 (D'Alembert's (limit) ratio test, correct)**.** Suppose $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l$.

- $l > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.
- $l < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ converges absolutely, so by lemma, converges unconditionally.
- $l = 1 \Rightarrow$ inconclusive.

Proof. l > 1. By $\varepsilon - N$ definition of convergence, given some $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - l\right| < \varepsilon$$

Suppose $\varepsilon = \frac{l-1}{2}$. Since $l > 1$, $\varepsilon > 0$. Then we have

$$\begin{aligned}
\left|\left|\frac{a_{n+1}}{a_n}\right| - l\right| < \frac{l-1}{2} &\Leftrightarrow \frac{1-l}{2} < \left|\frac{a_{n+1}}{a_n}\right| - l < \frac{l-1}{2} \\
&\Leftrightarrow \frac{1+l}{2} < \left|\frac{a_{n+1}}{a_n}\right| < \frac{3l-1}{2}
\end{aligned}$$

So

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &> \frac{1+l}{2} \\ &> 1 \text{ since } l > 1\end{aligned}$$

Let $c = \frac{1+l}{2}$. Then $c > 1$.

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| > c \Leftrightarrow \frac{|a_{n+1}|}{|a_n|} > c \\ \Leftrightarrow |a_{n+1}| > c|a_n| \\ \Rightarrow |a_{m+N+2}| > c^m|a_{N+1}| \text{ for all } m \geq 1\end{aligned}$$

Since $c > 1$, the geometric sequence $c^m|a_n|$ converges to ∞, so the sequence a_n also converges to ∞. (Sound reasoning?) By divergence test, since $\lim_{n\to\infty}|a_n| \neq 0$, $\sum_{n=1}^{\infty}|a_n|$ diverges. So $\sum_{n=1}^{\infty} a_n$ diverges.

l < 1. Given some $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - l\right| < \varepsilon$$

Suppose $\varepsilon = \frac{1-l}{2}$. Since $l < 1$, $\varepsilon > 0$. Then we have

$$\begin{aligned}\left|\left|\frac{a_{n+1}}{a_n}\right| - l\right| < \frac{1-l}{2} &\Leftrightarrow \frac{l-1}{2} < \left|\frac{a_{n+1}}{a_n}\right| - l < \frac{1-l}{2} \\ &\Leftrightarrow \frac{3l-1}{2} < \left|\frac{a_{n+1}}{a_n}\right| < \frac{1+l}{2}\end{aligned}$$

So

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &< \frac{1+l}{2} \\ &< 1 \text{ since } l < 1\end{aligned}$$

Let $c = \frac{1+l}{2}$. Then $c < 1$.

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| < c \Leftrightarrow |a_{n+1}| < c|a_n| \\ \Rightarrow |a_{m+N+2}| < c^m|a_{N+1}| \text{ for all } m \geq 1\end{aligned}$$

Since $c < 1$, the geometric series

$$\sum_{n=1}^{\infty} c^m|a_{N+1}| = |a_{N+1}|\sum_{n=1}^{\infty} c^m$$

converges. Then by comparison,

$$\sum_{n=N+3}^{\infty} |a_n|$$

converges as well, so $\sum_{n=1}^{\infty}|a_n|$ converges, and $\sum_{n=1}^{\infty} a_n$ converges too. $\square$

3.3.3 n^{th} root test

Definition 3.16 (Limit superior). Let $(a_n)_{n\geq 1}$ be a sequence. Let $(b_n)_{n\geq 1}$ be a sequence defined by

$$b_n = \sup\{a_m \mid m \geq n\}$$

Then

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Semantically, b_n represents the supremum of the set of all the terms after and including a_n. Note that b_n is a decreasing (non-increasing) sequence, since we may be removing the maximum from the set, in which case the supremum may decrease; otherwise, the supremum remains unchanged.

Definition 3.17 (Limit inferior). Let $(a_n)_{n\geq 1}$ be a sequence. Let $(b_n)_{n\geq 1}$ be a sequence defined by

$$b_n = \inf\{a_m \mid m \geq n\}$$

Then

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Note that b_n is an increasing (non-decreasing) sequence, since we may be removing the smallest number in the set, and therefore increase the infimum; or we may not be removing the smallest number, in which case the infimum remains unchanged.

Lemma 3.18. For any sequence $(a_n)_{n\geq 1}$, there exists a subsequence $(a_{n_i})_{i\geq 1}$ such that

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_{n_i}$$

Proof. Warning: this proof is fairly involved and reasonably long.

The cases where the limit equals ∞ or $-\infty$ are trivial. This proof will focus on the case when the limit is equal to some $t \in \mathbb{R}$. Recall the following definition from the chapter on sequences:

> **Definition 3.19.** a_m is a peak if and only if $a_m > a_n \; \forall n > m$.

Case 1: there are infinitely many peaks. Suppose these peaks are at $n_1 < n_2 < \dots$. By definition of peak,

$$a_{n_1} > a_{n_2} > \dots$$

so we have

$$a_{n_k} = \sup\{a_n \mid n \geq n_k\}$$

Let $b_k = a_{n_k}$.

$$\begin{aligned}
\lim_{k\to\infty} b_k &= \lim_{n_k\to\infty} \sup\{a_n \mid n \geq n_k\} \\
&= \lim_{n\to\infty} \sup\{a_m \mid m \geq n\} \\
&= \limsup_{n\to\infty} a_n, \text{ by definition}
\end{aligned}$$

Case 2: there are finitely many peaks. Suppose the last peak is at N_0.

Proposition 3.20.

$$\forall n > N_0, b_n = \sup\{a_m \mid m > n\} = t$$

In other words, the sequence b_n is eventually constant after a certain point N_0.

Proof. Assume b_n is not eventually constant. Since b_n is non-increasing and it is not eventually constant, b_n is strictly decreasing. Then there exists $N > N_0$ such that

$$b_N > b_{N+1} \Leftrightarrow \sup\{b_m \mid m > N\} > \sup\{b_m \mid m > N+1\}$$

Since the LHS set has only one extra element b_{N+1} than the RHS set,

$$b_{N+1} > \sup\{b_m \mid m > N+1\}$$

so b_{N+1} is greater than all of $b_{N+1}, b_{N+2}, \ldots$, so b_{N+1} is a peak. However, we picked N such that $N > N_0$, so $N+1$ is past the last peak, contradiction. So b_n is eventually constant. Since $\lim_{n\to\infty} b_n = t$, b_n is eventually constant and equal to t. □

Case (a): there are infinitely many terms = t. Construct a subsequence from these terms, done.

Case (b): there are finitely many terms = t. Suppose the last term equal to t (if such term exists) is at N'. Then we pick N_1 such that $N_1 > \max(N', N_0)$, so N_1 is beyond both the last peak and the last term equal to t. It is easy to check that

$$\forall n > N_1,\ a_n < t$$

It cannot be the case that $a_n = t$ since n is beyond the last term equal to t, and it cannot be the case that $a_n > t$ since from the above proposition, we get

$$a_n > t = \sup\{a_m \mid m \geq n\}$$

which contradicts the definition of supremum. As every term is strictly less than t, the only hope for an infinite subsequence that converges to t is one that is increasing.

Proposition 3.21. There exists a subsequence a_{n_k} such that $\forall k$,

$$t - \frac{1}{k} < a_{n_k} < t$$

Proof. We construct this subsequence inductively. Clearly, we want to pick n_k such that $n_k > N_1$, as $a_{n_k} < t$ immediately follows from above.

For the base case, we have

$$a_{n_1} > t - 1$$

which is true as by the definition of supremum, there must exist $n_1 > N_1$ which satisfies the inequality. (Intuitively, since the lim sup is t, there must exist some term that is arbitrarily close to t.)

For the inductive case, suppose we have constructed a finite increasing subsequence with $a_{n_1} > t-1$, $a_{n_2} > t - \frac{1}{2}$, ..., $a_{n_k} > t - \frac{1}{k}$. We want to show two things:

- $n_{k+1} > t - \frac{1}{k+1}$
- $a_{n_k} < a_{n_{k+1}}$ and $n_k < n_{k+1}$, i.e. the sequence is increasing

For the first point, using the same argument as above, there exists some $n_{k+1} > N_1$ such that n_{k+1} is arbitrarily close to t. Since $\frac{1}{k+1}$ is finite and positive, the inequality is satisfied.

For the second point,

$$\max(a_{N_1+1}, a_{N_1+2}, ..., a_{n_k}) < a_{n_{k+1}}$$

using the same arbitrarily close argument. Since $a_{n_{k+1}}$ is greater, and therefore different from all the other terms after N_1, we have $n_{k+1} > n_k$. □

By the sandwich theorem, since the lower and upper bounds both approach t as $k \to \infty$, we have

$$\lim_{k \to \infty} a_{n_k} = t$$

as required. □

Proposition 3.22. A sequence a_n converges to $l \in \mathbb{R}$ if and only if

$$\limsup_{n \to \infty} a_n = \liminf_{n \to \infty} = l$$

Proof. If. Suppose

$$\limsup_{n \to \infty} a_n = \liminf_{n \to \infty} = l$$

Let

$$b_n = \sup\{a_m \mid m \geq n\}, \qquad c_n = \inf\{a_m \mid m \geq n\}$$

By definition of infimum and supremum, for all $n \in N$,

$$c_n \leq a_n \leq b_n$$

so by squeeze theorem, $a_n \to l$ as $n \to \infty$.

Only if. Conversely, suppose $a_n \to l$ as $n \to \infty$. Given some $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$l - \varepsilon < a_n < l + \varepsilon$$

(Intuitively, this means every term beyond N is within ε of the limit l.) It follows that

$$l - \varepsilon \leq c_n \leq b_n \leq l + \varepsilon$$

(since loosely speaking, the infimum c_n and the supremum b_n are both arbitrarily close to some term in the sequence a_n, which is within the finite and positive distance of ε from the limit.) So we have

$$|c_n - l| < \varepsilon \qquad\qquad |b_n - l| < \varepsilon$$

and both $c_n \to l$ and $b_n \to l$. So

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = l$$

as required. □

Test 3.23 (n^{th} **root test).** Suppose $\limsup_{n\to\infty}|a_n|^{\frac{1}{n}} = l$.

- $l > 1 \Rightarrow$ diverges.
- $l < 1 \Rightarrow$ converges.
- $l = 1 \Rightarrow$ inconclusive: may converge absolutely, converge, or diverge.

Proof. l > 1. Since $\limsup_{n\to\infty}|a_n|^{\frac{1}{n}} = l$, there exists a subsequence a_{n_i} such that

$$\lim_{n\to\infty}|a_{n_i}|^{\frac{1}{n_i}} = l$$

Then given some $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n_i > N \Leftrightarrow i > N$,

$$\left||a_{n_i}|^{\frac{1}{n_i}} - l\right| < \varepsilon$$

Suppose $\varepsilon = \frac{l-1}{2}$. Since $l > 1$, $\varepsilon > 0$.

$$\begin{aligned}\left||a_{n_i}|^{\frac{1}{n_i}} - l\right| < \frac{l-1}{2} &\Leftrightarrow \frac{1-l}{2} < |a_{n_i}|^{\frac{1}{n_i}} - l < \frac{l-1}{2}\\ &\Leftrightarrow \frac{1+l}{2} < |a_{n_i}|^{\frac{1}{n_i}} < \frac{3l-1}{2}\end{aligned}$$

Let $c = \frac{1+l}{2}$. Then $c > 1$.

$$|a_{n_i}|^{\frac{1}{n_i}} > c \Leftrightarrow |a_{n_i}| > c^{n_i}$$

Since the sequence c^{n_i} converges to ∞, the sequence $|a_{n_i}|$ also converges to ∞, so the sequence $|a_n|$ also converges to ∞. Then $\lim\limits_{n\to\infty}|a_n| \neq 0$, so $\sum_{n=1}^{\infty}|a_n|$ diverges, and $\sum_{n=1}^{\infty} a_n$ diverges.

l < 1. Consider the sequence b_n defined by

$$b_n = \sup\{|a_m|^{\frac{1}{m}} \mid m \geq n\}$$

By definition of limit superior, $\lim\limits_{n\to\infty} b_n = l$. Then given some $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n > N$,

$$\begin{aligned}|b_n - l| < \varepsilon &\Leftrightarrow l - \varepsilon < b_n < l + \varepsilon\\ &\Leftrightarrow l - \varepsilon < \sup\{|a_m|^{\frac{1}{m}} \mid m \geq n\} < l + \varepsilon\\ &\Rightarrow |a_n|^{\frac{1}{n}} < l + \varepsilon\end{aligned}$$

Let $\varepsilon = \frac{l-1}{2}$, and let $c = l + \varepsilon = \frac{l+1}{2}$. Since $l < 1$, $c < 1$.

$$|a_n|^{\frac{1}{n}} < c \Leftrightarrow |a_n| < c^n$$

Since $\sum_{n=1}^{\infty} c^n$ converges, $\sum_{n=1}^{\infty} |a_n|$ also converges, so $\sum_{n=1}^{\infty} a_n$ converges.

l = 1. (Finish this off when done with L'Hopital and limits.) $\square$

Example. Use the n^{th} root test to determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2} 4^n$$

Solution. Let

$$a_n = (\frac{n}{n+1})^{n^2} 4^n$$

$$\begin{aligned}
\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} &= \limsup_{n\to\infty} (\frac{n}{n+1})^n \cdot 4 \\
&= \limsup_{nto\infty} \frac{4}{(1+\frac{1}{n})^n} \\
&= \frac{4}{e} \\
&> 1
\end{aligned}$$

so the series diverges.