Lecture 25:

Uniform Continuity:

Let $f : A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$. Let $x_0 \in A$.

Let us remember what it means that f is continuous at x_0: it means that, $\forall \varepsilon > 0$, there exists $\delta = \delta(x_0, \varepsilon) > 0$ (i.e., $\delta > 0$ potentially depending on x_0 and ε),

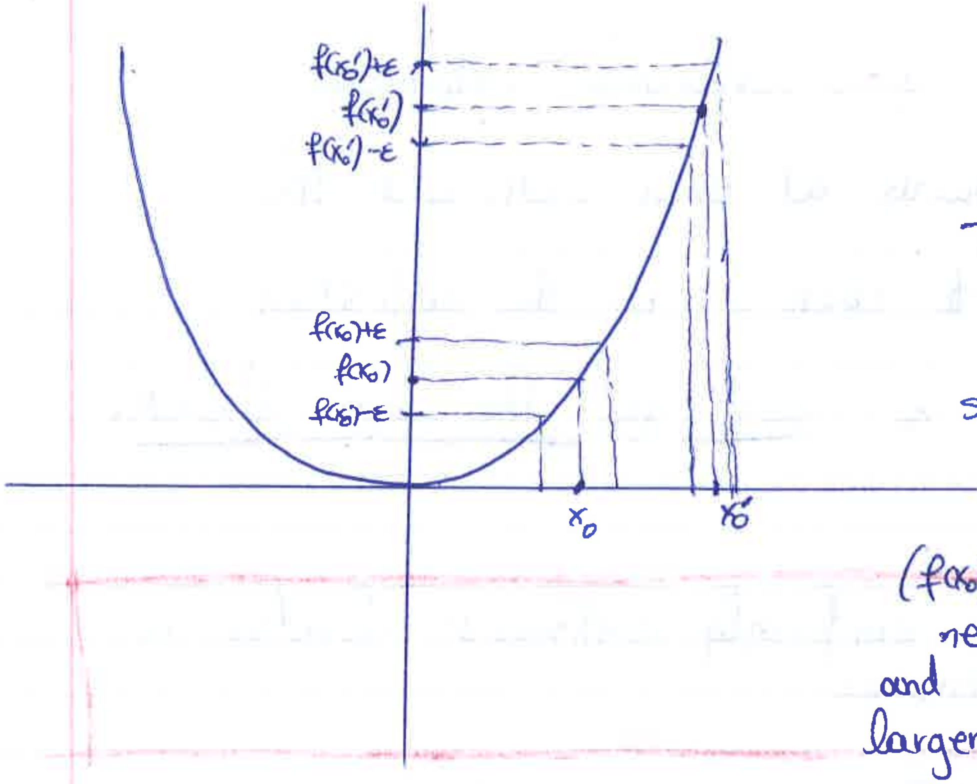
such that: if $x \in A$ with $|x - x_0| < \delta$,

then $|f(x) - f(x_0)| < \varepsilon$.

There are of course examples of functions (such as $f(x) = x$ $\forall x \in \mathbb{R}$, or $f(x) = c$ $\forall x \in \mathbb{R}$, see old notes on continuity)

where $\forall \varepsilon > 0$ we can pick a $\delta = \delta(\varepsilon) > 0$, i.e. a δ that only depends on ε, not on x_0, to satisfy the definition of continuity at x_0.

However, other functions (such as $f(x) = x^2$ $\forall x \in \mathbb{R}$) don't satisfy this:



The larger x_0 is, the faster x_0^2 grows.

Thus, for fixed $\varepsilon>0$,

while, when x_0 is small, quite a large neighbourhood of x_0 is sent inside $(f(x_0)-\varepsilon, f(x_0)+\varepsilon)$, this neighbourhood shrinks more and more as x_0 gets larger and larger.

So, for such a function, when we test for continuity at x_0, the $\delta>0$ that corresponds to each $\varepsilon>0$ has to depend on x_0, not just on ε.

→ Def: (Uniform Continuity):

Let $f: A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$. We say that f is uniformly continuous if:

For all $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$,

s.t. if $x, y \in A$, with $|x-y|<\delta$,

then $|f(x)-f(y)|<\varepsilon$.

In other words, f is uniformly continuous if it is continuous at each $x_0 \in A$, and the δ corresponding to each ε in the definition of continuity at x_0 can be the same $\forall x_0 \in A$.

Observation: Each uniformly continuous function is continuous.

Proof: Let $f: A^{\subseteq \mathbb{R}} \longrightarrow \mathbb{R}$ be uniformly continuous.

Let $x_0 \in A$. Let $\varepsilon > 0$.

Since f is uniformly continuous, for this $\varepsilon > 0$ there exists $\delta\,(= \delta(\varepsilon)) > 0$, such that:

If $x, y \in A$, and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

In particular, if $x \in A$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, f is continuous at x_0.

Since $x_0 \in A$ was arbitrary, f is continuous on A. ∎

Lecture 26

Examples:

- $f(x)=x, \forall x\in\mathbb{R}$: uniformly continuous.
- $f(x)=x^2, \forall x\in\mathbb{R}$: not uniformly continuous.
- $f(x)=x^2, \forall x\in[-M,M]$: uniformly continuous.
  M fixed, >0

Let's prove the third bullet point:

Let $x,y\in[-M,M]$. Then,

$$\boxed{\begin{aligned}|f(x)-f(y)| &= |x^2-y^2| = |(x-y)\cdot(x+y)| = \\ &= |x-y|\cdot\underbrace{|x+y|}_{\leq |x|+|y|\leq 2M} \leq 2M\cdot|x-y|.\end{aligned}}$$

Let $\varepsilon>0$. If $x,y\in[-M,M]$, with $|x-y|<\frac{\varepsilon}{2M}$, then, by the above,

$$|f(x)-f(y)| < \cancel{2M}\cdot\frac{\varepsilon}{\cancel{2M}} = \varepsilon.$$

Since $\delta(\varepsilon)=\frac{\varepsilon}{2M}$ only depends on ε, f is uniformly continuous.

We generalise this in <u>both</u> the Propositions that follow:

Def: (Lipschitz continuity):

Let $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$. We say that f is Lipschitz continuous if there exists $M > 0$ such that

$$|f(x) - f(y)| \leq M \cdot |x-y|, \quad \forall x, y \in A.$$

$$\boxed{\text{Prop: Every Lipschitz continuous function is uniformly continuous.}}$$

Proof: Let $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, be Lipschitz continuous. Then, there exists $M > 0$ such that:

$$|f(x) - f(y)| \leq M \cdot |x-y|, \quad \forall x, y \in A.$$

Let $\varepsilon > 0$. We choose $\delta = \frac{\varepsilon}{M}$ (depends only on ε).

If $x, y \in A$, and $|x-y| < \delta$, then

$$|f(x) - f(y)| \leq M \cdot |x-y| < M \cdot \delta = \cancel{M} \cdot \frac{\varepsilon}{\cancel{M}} = \varepsilon,$$

i.e. $|f(x) - f(y)| < \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, f is uniformly continuous. ∎

works for open intervals (as long as f' bounded).

Prop: Let $f: I \to \mathbb{R}$, I an interval, continuous on I, differentiable in the interior of I.

If f' is bounded, then f is Lipschitz continuous (and thus uniformly continuous).

Proof: By our assumptions, there exists $M>0$ s.t.

$$\boxed{|f'(\xi)| \leq M, \ \forall \xi \text{ in the interior of } I.} \quad (*)$$

Let $x, y \in I$, with $x<y$ (WLOG). Since f is continuous on $[x,y]$ and differentiable on (x,y), we can apply the mean value theorem on $[x,y]$:

$$f(y) - f(x) = f'(\xi) \cdot (y-x),$$

thus $|f(y) - f(x)| = |f'(\xi)| \cdot |y-x|$, for some ξ between x and y, thus in the interior of I.

By $(*)$, $|f(y) - f(x)| \leq M \cdot |y-x|$.

Since $x, y \in I$ were arbitrary, f is Lipschitz continuous.

Therefore, f is uniformly continuous. ∎

ex: $f(x) = \sin x$, ~~$\forall x \in \mathbb{R}$:~~ $|f'(x)| = |\cos x| \leq 1 \ \forall x \in \mathbb{R}$, thus f Lipschitz continuous an interval, thus f uniformly continuous.

Thm: Characterisation of uniform continuity via sequences:

Let $f : A \to \mathbb{R}$. ($A \subset \mathbb{R}$)

f is uniformly continuous

$$\Longleftrightarrow$$

for any pair of sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in A,

with $x_n - y_n \longrightarrow 0$,

we have $f(x_n) - f(y_n) \longrightarrow 0$.

Proof: ($\Longrightarrow$) Let $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ be in A,

with $x_n - y_n \longrightarrow 0$. We want to show

that $f(x_n) - f(y_n) \longrightarrow 0$:

Let $\varepsilon > 0$. Since f is uniformly continuous, we have

that: for this $\varepsilon > 0$, there exists $\delta (= \delta(\varepsilon)) > 0$,

such that: $\boxed{\text{if } |x_n - y_n| < \delta, \text{ then } |f(x_n) - f(y_n)| < \varepsilon.}$ (*)

Since $x_n - y_n \longrightarrow 0$, there exists $n_0 \in \mathbb{N}$ s.t.:

$\forall n \geq n_0, \quad |x_n - y_n| < \delta.$

By $(*)$: $\underline{\forall n \geq n_0, \ |f(x_n) - f(y_n)| < \varepsilon.}$

Since $\varepsilon > 0$ was arbitrary, $f(x_n) - f(y_n) \longrightarrow 0.$

$(\Longleftarrow)$ Suppose that f is not uniformly continuous.

We will construct sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in A,

with $x_n - y_n \longrightarrow 0$, but $f(x_n) - f(y_n) \not\longrightarrow 0.$

even better: $|f(x_n) - f(y_n)| \geq \varepsilon$, for some fixed $\varepsilon > 0$

Indeed: Since f doesn't satisfy the definition of uniform continuity:

there exists $\varepsilon > 0$ s.t. : $\forall \delta > 0,$

there exist $x_\delta, y_\delta \in A$, with $|x_\delta - y_\delta| < \delta$, yet $|f(x_\delta) - f(y_\delta)| \geq \varepsilon$. $(*')$

We apply $(*')$ for $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$:

for $\delta = 1$: $\exists x_1, y_1 \in A$, with $|x_1 - y_1| < 1$, but $|f(x_1) - f(y_1)| \geq \varepsilon$.

for $\delta = 1/2$: $\exists x_2, y_2 \in A$, with $|x_2 - y_2| < \frac{1}{2}$, but $|f(x_2) - f(y_2)| \geq \varepsilon$.

$\vdots$

for $\delta = 1/n$: $\exists x_n, y_n \in A$, with $|x_n - y_n| < \frac{1}{n}$, but $|f(x_n) - f(y_n)| \geq \varepsilon$

$\vdots$

Since $|x_n - y_n| < \frac{1}{n}$ $\forall n \in \mathbb{N}$,

we have $x_n - y_n \longrightarrow 0$ (by the sandwich lemma).

Thus, by our assumption, we should have

$$f(x_n) - f(y_n) \longrightarrow 0.$$

However, $|f(x_n) - f(y_n)| \geq \underbrace{\varepsilon}_{\text{fixed}} > 0$, $\forall n \in \mathbb{N}$,

thus $f(x_n) - f(y_n) \not\longrightarrow 0$.

This is a contradiction. Therefore, f is uniformly continuous.

$\blacksquare$

This characterisation of uniform continuity can prove particularly useful when we want to show that a function f is not uniformly continuous.

Example: Show that $f : (0,1) \longrightarrow \mathbb{R}$, with $f(x) = \frac{1}{x}$, $\forall x \in (0,1)$,

! METHOD !

is not uniformly continuous.

Solution: We will find $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in $(0,1)$, with $x_n - y_n \longrightarrow 0$, but $f(x_n) - f(y_n) \not\longrightarrow 0$.

Indeed: Let $x_n = \frac{1}{n}$ $\forall n\in\mathbb{N}$, and $y_n = \frac{1}{2n}$ $\forall n\in\mathbb{N}$. The sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ are both in $(0,1)$.

And: $x_n - y_n = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n} \longrightarrow 0$.

But: $f(x_n) - f(y_n) = \frac{1}{x_n} - \frac{1}{y_n} = n - 2n = -n \longrightarrow -\infty \neq 0$

Thus, f is not uniformly continuous.

→ You can try a similar trick to show that

$f(x) = x^2$ $\forall x\in\mathbb{R}$ and $g(x) = \cos(x^2)$ $\forall x\in\mathbb{R}$

are not uniformly continuous.

(Exercise).

⇝ ⚠ • A bounded and continuous function is not necessarily uniformly continuous (ex.: $g(x) = \cos(x^2)$, $\forall x\in\mathbb{R}$).

• $f: I \longrightarrow \mathbb{R}$ (I: open interval) continuous is not necessarily uniformly continuous. (ex.: $f(x) = \frac{1}{x}$, $\forall x\in(0,1)$).

works only for closed intervals.

Thm: Every continuous function $f : [a,b] \longrightarrow \mathbb{R}$ is uniformly continuous.

Proof: Suppose that f is not uniformly continuous.

Then, there exist $\varepsilon > 0$ and $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in $[a,b]$, such that $x_n - y_n \longrightarrow 0$, but $|f(x_n) - f(y_n)| \geq \varepsilon, \forall n\in\mathbb{N}$.

follows from proof of characterisation via sequences. Stronger than just $f(x_n) - f(y_n) \not\to 0$!

We will show that this is a contradiction.

Indeed:

- $(x_n)_{n\in\mathbb{N}}$ is bounded, thus, by Bolzano-Weierstrass, it has a convergent subsequence $(x_{k_n})_{n\in\mathbb{N}}$.
- That is, $x_{k_n} \longrightarrow x_0 \in \mathbb{R}$, for this $(x_{k_n})_{n\in\mathbb{N}}$.

  Since $a \leq x_{k_n} \leq b \quad \forall n\in\mathbb{N}$,

  we have $x_0 \in [a,b]$. Therefore, f continuous at x_0 $\longrightarrow f(x_{k_n}) \longrightarrow f(x_0)$.
- Since $x_n - y_n \longrightarrow 0$, we have $x_{k_n} - y_{k_n} \longrightarrow 0$ as well.

  And since in addition $x_{k_n} \longrightarrow x_0$, we have $y_{k_n} \longrightarrow x_0$.

$\left(y_{k_n} = x_{k_n} - (x_{k_n} - y_{k_n}) \longrightarrow x_0 - x_0 = 0 \right)$

Since f is continuous at x_0, it follows that

$$f(y_{k_n}) \longrightarrow f(x_0).$$

Therefore:

$$\left.\begin{array}{l} f(x_{k_n}) \longrightarrow f(x_0) \\ f(y_{k_n}) \longrightarrow f(x_0) \end{array}\right\} \longrightarrow f(x_{k_n}) - f(y_{k_n}) \longrightarrow f(x_0) - f(x_0) = 0.$$

However, this is a contradiction: we picked $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ such that $\left| f(x_{k_n}) - f(y_{k_n}) \right| \geqslant \underbrace{\varepsilon}_{\text{fixed}} > 0$, $\forall n \in \mathbb{N}$, thus

$$f(x_{k_n}) - f(y_{k_n}) \not\longrightarrow 0.$$

Therefore, f is uniformly continuous.

$\blacksquare$