Moves required to solve Tower of Hanoi and 2 variants

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1 Introduction

The Tower of Hanoi is a mathematical game consisting of 3 rods and a number of disks with various diameters, which can slide into any rod. The puzzle begins with the disks stacked on the leftmost rod in order of decreasing size, the smallest at the top, thus approximating a conical shape. The objective of the puzzle is to move the entire stack to the rightmost rod, obeying the following rules:

1. Only one disk may be moved at a time.
2. Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack or on an empty rod.
3. No disk may be placed on top of a disk that is smaller than it.

With 3 disks, the puzzle can be solved in 7 moves. The minimal number of moves required to solve the Tower of Hanoi puzzle is $2^n - 1$ (a.k.a. the nth Mersenne Number), where n is the number of disks.

2 The three variations

For simplicity, we shall denote a_n the optimal number of moves to finish the puzzle with n pegs in that variation.

2.1 Normal Variation

Claim: $a_n = 2a_{n-1} + 1$
Proof: Clearly $a_1 = 1$. Then for us to move the nth plates to the rightmost rod, we must first shift the top $n - 1$ plates to another rod to "free up" the last plate, move the last plate to the rightmost rod, and then "cover up" with the $n - 1$ plates on the rightmost rod. It is obviously this is minimal. See Figure 1 in this section for a better understanding.

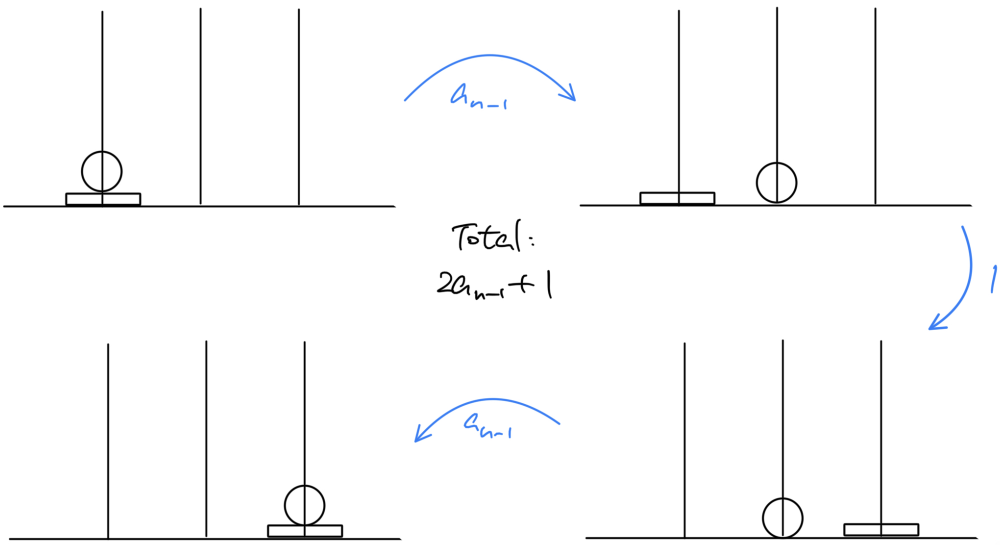

Figure 1: A picture showing the recursive relation between a_n and $a_n - 1$, where the ball refers to the top $n - 1$ plates

Now we will detail two ways to prove the minimum number of moves is $2^n - 1$:

1. Mathematical Induction(MI)
2. Generating Functions

Note: Generating Functions are used for solving general linear recurrences, but in this situation, Mathematical Induction will be sufficient, since we can easily guess the general formula by listing out the first few terms. But for more complex recurrence relations, we might not be able to guess the general term, and will have to turn to Generating Functions.

Mathematical Induction Framework:

1. Prove true for base case: n=1
2. Inductive step: Assume the formula (hypothesis) is true for some integer k, and prove the formula is true for $k+1$

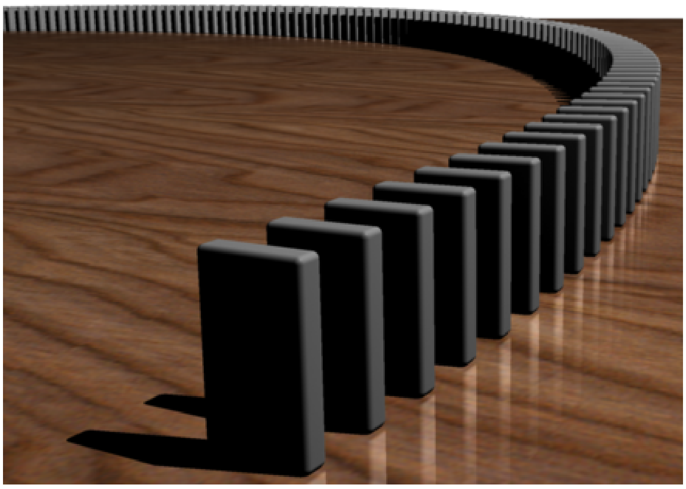

Figure 2: Mathematical Induction can also be thought of the falling of dominoes: the first case implies the second, the second implies the third, and so on.

Proof with MI:

The case $f(1) = 1$ is trivial and follows the formula $a_n = 2^n - 1$: we directly move the plate to the last rod.

Assume the formula is true for some k, i.e. $a_k = 2^k - 1$. Then we can see $a_{k+1} = 2a_k + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 1$ as desired.

Solution using Generating Functions(for interest): Generating Functions

Let $G(a_n; x)$ be the generating function of the sequence. We have

$$G(a_n; x) = \sum_{n=1}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} a_n x^n$$

$$= x + \sum_{n=2}^{\infty} (2a_{n-1} + 1) x^n$$

$$= x + 2x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + \sum_{n=2}^{\infty} x^n + x + 1 - x - 1.$$

$$= x + 2x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} + \sum_{n=0}^{\infty} x^n - x - 1.$$

Since

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$A(x) = \cancel{x} + \sum_{n=2}^{\infty} a_{n-1}x^{n-1} + \frac{1}{1-x} \cancel{-x} - 1$$

$$G(a_n; x) = 2x\sum_{n=1}^{\infty} a_n x^n + \frac{1}{1-x} - 1$$

by shifting all terms from the previous summation by 1. We notice, in fact

$$\sum_{n=1}^{\infty} a_n x^n = G(a_n; x)$$

as defined from the start.

Thus we have

$$(1-2x)G(a_n; x) = \frac{1}{1-x} - 1$$

$$G(a_n; x) = \frac{1}{(1-x)(1-2x)} - \frac{1}{1-2x}$$

By breaking into partial fractions,

$$G(a_n; x) = \frac{-1}{1-x} + \frac{2}{1-2x} - \frac{1}{1-2x} = \frac{-1}{1-x} + \frac{1}{1-2x}$$

Converting them back into power series,

$$G(a_n; x) = -\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (2x)^n$$

$$= \sum_{n=0}^{\infty} (2^n - 1)x^n = \sum_{n=1}^{\infty} (2^n - 1)x^n.$$

Comparing with our initial assumption

$$G(a_n; x) = \sum_{n=1}^{\infty} a_n x^n \Rightarrow a_n = 2^n - 1.$$

2.2 Adjacent Pegs Variation

The different part of the Adjacent Pegs Variation from the Normal Version is as its name suggests, you can only move pegs to their adjacent rod. That being said, a_1 clearly is 2. With similar logic as above, $a_n = 3a_{n-1} + 2$.

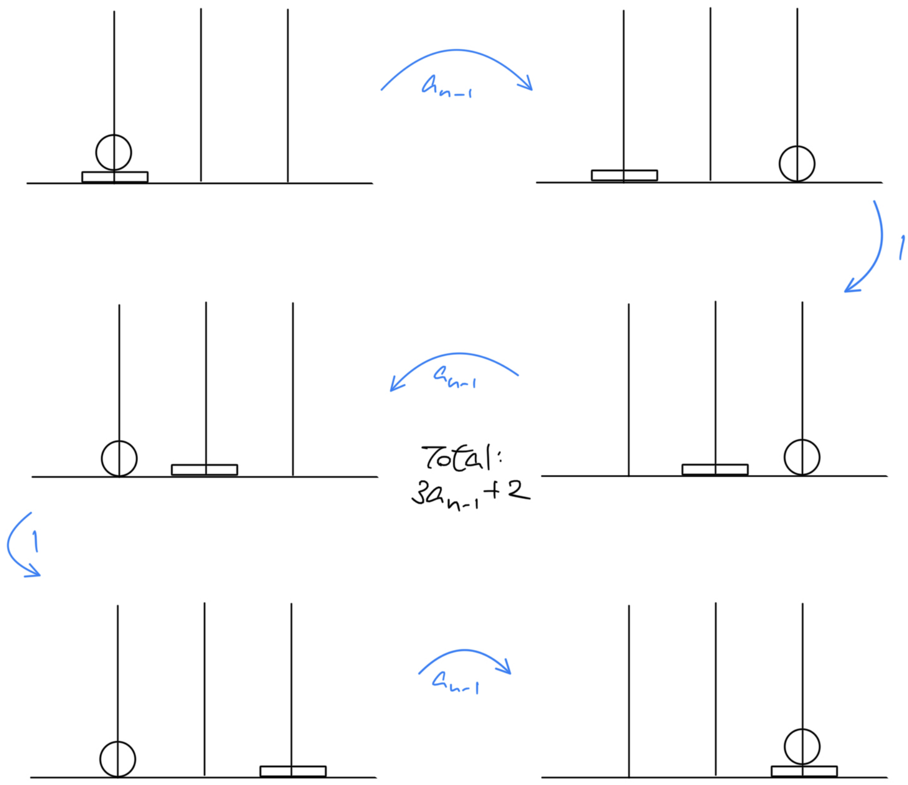

Figure 3: An illustration of the recursive relation in the adjacent pegs variation

A common question may be, why can we move the first $n-1$ disks directly to the rightmost rod? Isn't this the adjacent pegs variation? Notice a_{n-1}'s definition: it is the optimal number of moves required to finish the puzzle, which is equivalent to the optimal number of moves to move $n-1$ disks from the **leftmost rod to the rightmost rod**. By symmetry, we can also move all $n-1$ disks from the rightmost rod to the leftmost rod in a_{n-1} moves.

Note that this time, the general formula is $a_n = 3^n - 1$, which can be easily verified by using MI and Generating Functions with different values.

2.3 Cyclic Variation

Finally, we introduce the cyclic variation(Cyclic Hanoi). In Cyclic Hanoi, we are given three pegs (A, B, C), which are arranged as a circle with the clockwise and the counterclockwise directions being defined as $A \to B \to C \to A$ and $A \to C \to B \to A$ respectively. The moving direction of the disk must be clockwise. For this variation, we additionally define b_n as the number of moves required to move n plates to its next clockwise peg: $A \to B$ or $B \to C$ or $C \to A$. Notice a_n is **not** $2b_n$. An immediate counterexample is $a_2 = 7$, but $b_2 = 5$ (try it!).

We can try to get some of the first few numbers of the sequence a_n and b_n by trial:

n	a_n	b_n
1	2	1
2	7	5
3	21	15
4	59	43
5	163	118

Hmm... guessing isn't that easy this time. Let's try to form a recursive relationship, like the previous 2 examples.

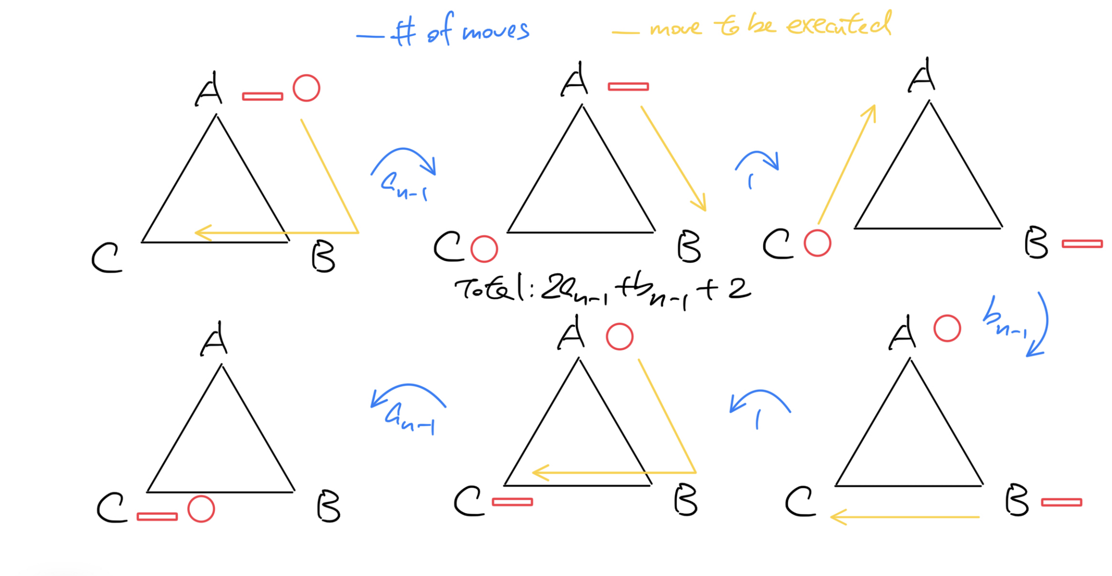

Figure 4: An illustration of the recursive relation a_n in the cyclic variation

So we have $a_n = 2a_{n-1} + b_{n-1} + 2$. How about for b_n?

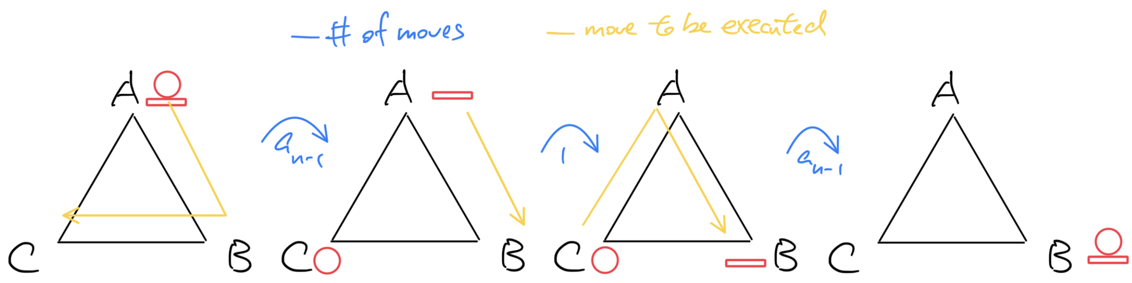

Figure 5: $b_n = 2a_{n-1} + 1$

Therefore,

$$a_n = 2a_{n-1} + b_{n-1} + 2 = 2a_{n-1} + 2a_{n-2} + 1 + 2 = 2a_{n-1} + 2a_{n-2} + 3.$$

A recurrence relation of the form

$$x_n = ax_{n-1} + bx_{n-2} + c$$

is called a **second order non-homogeneous linear recurrence relation.**

The general solution for the recurrence relation of this form is

$$x_n = x_n^{(s)} + x_n^{(g)},$$

where $x_n^{(s)}$ and $x_n^{(g)}$. denote the **specific solution**, and the **general solution** of the corresponding recurrence relation

$$x_n = ax_{n-1} + bx_{n-2}$$

respectively.

Theorem 2.3.1 *Let r_1 and r_2 be the roots of the characteristic equation of the second order linear recurrence relation $t^2 = at + b$. Then $x_n^{(s)} = A$ where A is a constant to be determined.*

We now have

$$a_n = 2a_{n-1} + 2a_{n-2} + 3 \Rightarrow A = 2A + 2A + 3 \Rightarrow A = x_n^{(s)} = -1.$$

For the general solution, we have to find the solution to

$$a_n = 2a_{n-1} + 2a_{n-2}$$

The characteristic equation is

$$\lambda^2 = 2\lambda + 2$$

$$\lambda = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2} = 1 \pm \sqrt{3}$$

Thus, the general solution is

$$x_n^{(g)} = A(1+\sqrt{3})^n + B(1-\sqrt{3})^n$$

With $a_1 = 2$ and $a_2 = 7$ computed by hand, we can substitute them into the general solution to find A and B.

$$\begin{cases} \text{A}(1+\sqrt{3}) + B(1-\sqrt{3}) - 1 = 2 \Rightarrow A(1+\sqrt{3}) + B(1-\sqrt{3}) = 3 \text{ — } ① \\ \text{A}(1+\sqrt{3})^2 + B(1-\sqrt{3})^2 - 1 = 7 \Rightarrow A(2+\sqrt{3}) + B(2-\sqrt{3}) = 4 \text{ — } ② \end{cases}$$

② - ①: $A + B = 1 \Rightarrow A = 1 - B$

Substituting $A = 1 - B$ into ②,

$$(1-B)(2+\sqrt{3}) + B(2-\sqrt{3}) = 3$$

$$B(2-\sqrt{3}-2-\sqrt{3}) = 2-\sqrt{3}$$

$$B = \frac{2-\sqrt{3}}{-2\sqrt{3}} = \frac{3-2\sqrt{3}}{6}$$

It follows that $A = 1 - B = \frac{3+2\sqrt{3}}{6}$. Hence the general formula for a_n is

$$\frac{3+2\sqrt{3}}{6}(1+\sqrt{3})^n + \frac{3-2\sqrt{3}}{6}(1-\sqrt{3})^n - 1$$

More on recurrences

3 Conclusion

To summarize, the number of moves required to solve Tower of Hanoi is:

- 2^{n-1} for the normal variation,
- 3^{n-1} for the adjacent pegs variation,
- and $\frac{3+2\sqrt{3}}{6}(1+\sqrt{3})^n + \frac{3-2\sqrt{3}}{6}(1-\sqrt{3})^n - 1$ for the cyclic variation.