Probabilistic Algorithms for Aerospace Autonomy
ASEN 6519 - Homework 1
Inference on Hidden Markov Models

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March 10, 2019

Contents

Problem 1 - Forward-Backward Algorithm

Implement the forward-backward algorithm for the `nominal_hmm_short_log.txt` sequence. Report the posterior probabilities $P(x_k|y_{1:T})$ for each timestep k.

Table 1: Posterior Probabilities, $P(x_k|y_{1:T})$

timestep, k	$P(x_k = x_1|y_{1:T})$	$P(x_k = x_2|y_{1:T})$	$P(x_k = x_3|y_{1:T})$	$P(x_k = x_4|y_{1:T})$
1	0	0.9512	0.0487	0
2	0	0.0123	0.9877	0
3	0	0.1538	0.8462	0
4	0	0.1057	0.8943	0
5	0	0.8982	0.1018	0
6	0	0.0022	0.9978	0
7	0	0.9418	0.0582	0
8	0.1833	0	0	0.8167
9	0.0100	0	0	0.9900
10	0.9768	0	0	0.0232
11	0.0012	0	0	0.9988
12	0.9310	0	0	0.0690
13	0.0023	0	0	0.9977
14	0.9509	0	0	0.0491
15	0.0549	0	0	0.9451

Report the data log-likelihood $\log P(y_{1:T})$ for all T observations. Starting with the definition of $\alpha(x_k)$, we have $\alpha(x_k) = P(x_k,\ y_{1:k})$. At the final timestep T, we have $\alpha(x_T) = P(x_T,\ y_{1:T})$. Marginalizing over x_T gives the data likelihood $P(y_{1:T})$

$$P(y_{1:T}) = \sum_{x_T} P(x_T,\ y_{1:T}) = \sum_{x_T} \alpha(x_T) = \sum_{i=1}^{n} \alpha_i(x_T) \tag{1}$$

where n is the number of discrete states. And thus the data *log*-likelihood, can be written as:

$$\log P(y_{1:T}) = \log \sum_{x_T} \alpha(x_T) = -28.138526 \tag{2}$$

Use the resulting posterior $P(x_k|y_{1:T})$ to classify the most likely state x_k for each timestep $k = 1 : T$ (plot these as a time trace).

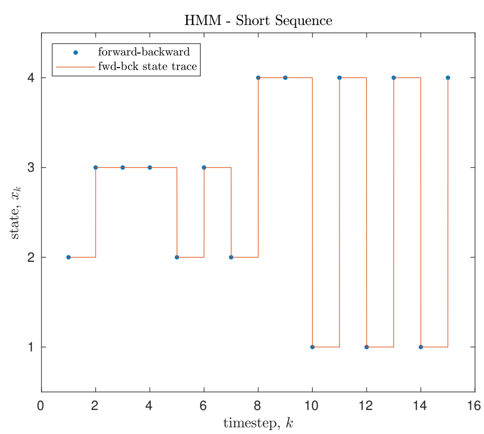


Figure 1: Forward-Backward Algorithm state trace - Short Sequence

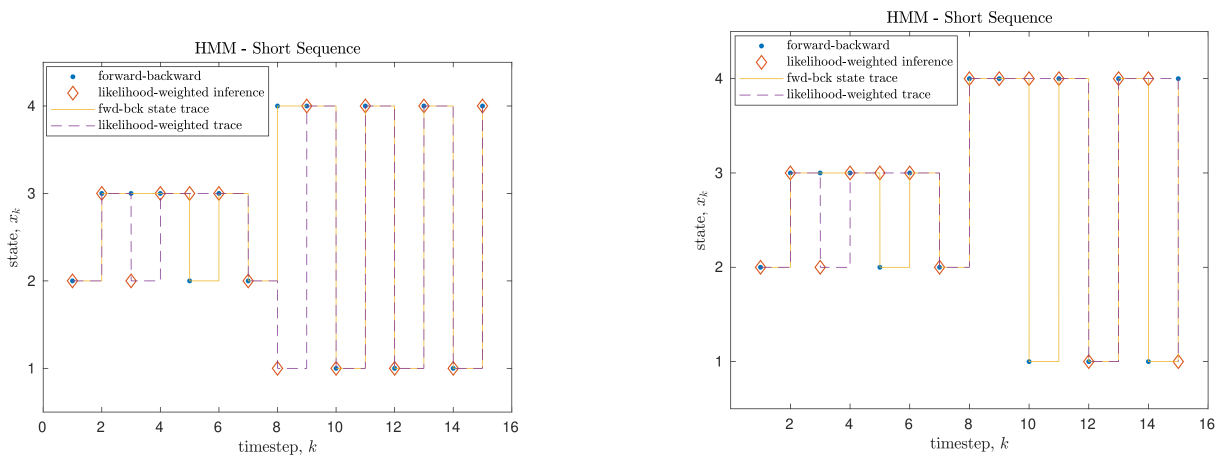
Problem 2 - Likelihood-Weighted Approximate Inference

The posterior $P(x_k|y_{1:T})$ can be constructed from the Monte Carlo sample sequences as follows

$$P(x_k = x_i|y_{1:T}) = \frac{\sum_{s=1}^{Ns} w_s \cdot \text{ind}(x_k = x_i)}{\sum_{s=1}^{Ns} w_s} \tag{3}$$

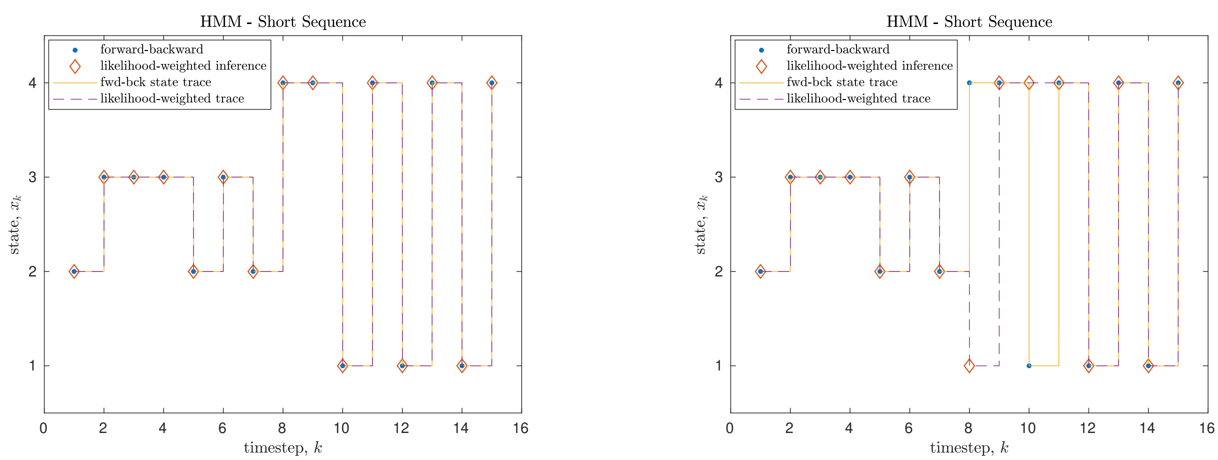
where w_s is the Monte Carlo sequence weight generated according to Algorithm 2.5 in Kochenderfer, and ind() is the indicator function, and simply returns one when $x_k = x_i$.

For Monte Carlo samples sizes N_s of 100, 1000 and 10,000 the following results are achieved. Each MC sample sequence is a sequence of discrete states chosen randomly according to the state transition probability table, $P(x_k|x_{k-1})$. For sample sizes of less than 10,000 samples, the results (the predicted discrete states) of the likelihood-weighted approximate inference method are unreliable. Given a sample size of 10,000 or greater, the likelihood-weighted approximate inference agrees with the forward-backward algorithm.



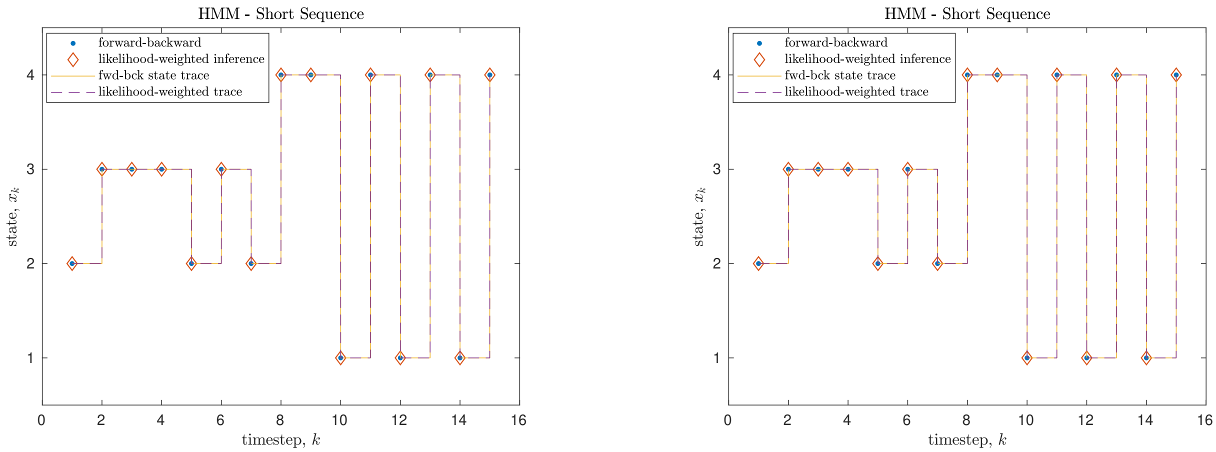
(a) (b)

Figure 2: Monte Carlo sample size, N_s = 100



(a) (b)

Figure 3: Monte Carlo sample size, N_s = 1000



(a) (b)

Figure 4: Monte Carlo sample size, N_s = 10,000

Problem 3 - Mann Extended-Logarithm Forward-Backward Algorithm

Implement the forward-backward algorithm for the `nominal_hmm_long_log.txt` sequence. Use the resulting posterior $P(x_k|y_{1:T})$ to classify the most likely state x_k for each timestep $k = 1 : T$ (plot these as a time trace). In the state trace shown below, the states predicted by the standard Forward-Backward algorithm are compared against the states predicted by the Mann Extended-Logarithm Forward Backward algorithm.

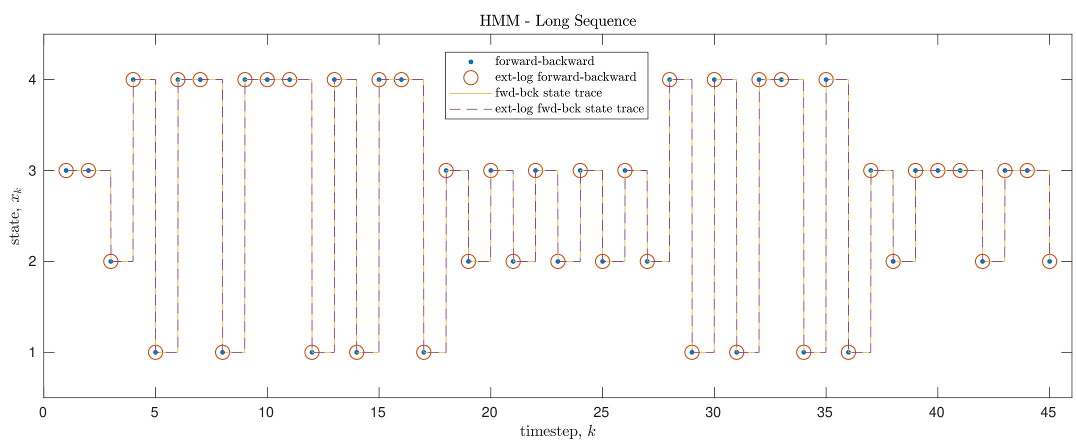


Figure 5: Mann Extended-Logarithm Forward-Backward Algorithm state trace - Long Sequence

Additionally, the data log-likelihood $\log P(y_{1:T})$, can be reported as follows

$$\log P(y_{1:T}) = \log \sum_{x_T} \alpha(x_T) = -84.743390 \tag{4}$$

and for the Mann extended-log implementation:

$$\log P(y_{1:T}) = \text{eln}\left(\sum_{x_T} \text{eexp}\left(eln\alpha_T\right)\right) = -84.743390 \tag{5}$$

In order for the Extended-Log Forward-Backward state trace to completely match standard Forward-Backward state trace, the following correction had to be made to lines 8-9 of Algorithm 6 in the Mann paper.

$$logbeta \leftarrow elnsum(logbeta, elnproduct(eln(a_{ij}), elnproduct(\mathbf{eln(b_j(O_{t+1}))}, eln\beta_{t+1}(j))))$$

Report the posterior probabilities for only the first five and last five steps in the sequence.

Table 2: Forward-Backward Posterior Probabilities, $P(x_k|y_{1:T})$

timestep, k	$P(x_k = x_1\|y_{1:T})$	$P(x_k = x_2\|y_{1:T})$	$P(x_k = x_3\|y_{1:T})$	$P(x_k = x_4\|y_{1:T})$
1	0	0.1723	0.8277	0
2	0	0.0058	0.9942	0
3	0	0.9951	0.0049	0
4	0.0062	0	0	0.9938
5	0.7125	0	0	0.2875
$\vdots$				
41	0	0.0132	0.9868	0
42	0	0.9435	0.0565	0
43	0	0.0134	0.9866	0
44	0	0.1575	0.8425	0
45	0	0.8313	0.1687	0

The Tobias Mann Extended-Logarithm posterior probabilities are shown below for comparison. *They match!*

Table 3: Mann Ext-Log Forward-Backward Posterior Probabilities, $P(x_k|y_{1:T})$

timestep, k	$P(x_k = x_1\|y_{1:T})$	$P(x_k = x_2\|y_{1:T})$	$P(x_k = x_3\|y_{1:T})$	$P(x_k = x_4\|y_{1:T})$
1	0	0.1723	0.8277	0
2	0	0.0058	0.9942	0
3	0	0.9951	0.0049	0
4	0.0062	0	0	0.9938
5	0.7125	0	0	0.2875
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41	0	0.0132	0.9868	0
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44	0	0.1575	0.8425	0
45	0	0.8313	0.1687	0

Appendix - MATLAB Code

The following MATLAB code was used to generate the data presented above.

hw1_hmm.m - main script

```
%% Header

% Filename:     hw1_ForwardBackward.m
% Author:       Carl Stahoviak
% Date Created:

clc;
clear;
close ALL;

%% Load Data

load('nominal_hmm_params.mat')
trans_prob = pxk_xkm1;  % column-stochastic
obs_prob = pyk_xk;      % column-stochastic

% load('nominal_hmm_short_log.mat')
load('nominal_hmm_long_log.mat')

%% The Forward-Backward Algorithm

%%% standard forward-backward algorithm
[alpha, alpha2] = forward( px0, trans_prob, obs_prob, y_obs );  % Txn
[beta, beta2] = backward( trans_prob, obs_prob, y_obs );        % Txn

% get the posterior distribution (exact inference)
posterior = posterior_fb( alpha(2:end,:)', beta' );             % Txn

% calculate data log-likelihood for forward-backward alg.
data_ll_fb = log(sum(alpha(end,:)));
fprintf('\nFB data log-likelihood = %f\n\n', data_ll_fb);

%% Mann Extended-Logarithm Forward-Backward Algorithm

% use Mann notation
trans_prob = trans_prob';   % row-stochastic (Rabiner/Mann convention)

%%% Mann log-weighted (numerically-stable) forward-backward alg.
eln_alpha = forward_eln( px0, trans_prob, obs_prob, y_obs );
eln_beta = backward_eln( trans_prob, obs_prob, y_obs );

% get the log-posterior distribution (exact inference)
% [ eln_gamma, gamma ] = posterior_elnfb( eln_alpha(:,2:end), ...
%     eln_beta(:,2:end) );
[ eln_gamma, gamma ] = posterior_elnfb( eln_alpha, eln_beta );
eln_posterior = gamma;  % true posterior, not log-posterior

% calculate data log-likelihood for ext-log forward-backward alg.
nonNaN_idx = ¬isnan(eln_alpha(:,end));
data_ll_elnfb = eln(sum(eexp(eln_alpha(nonNaN_idx,end))));
fprintf('\nExt-Log FB data log-likelihood = %f\n\n', data_ll_elnfb);

%% Liklihood-Weighted Sampling

n = size(px0,1);     % number of states
Ns = 10000;          % number of Monte Carlo sample sequences

% get Ns Monte Carlo sample sequences of length T, and
% corresponding sequence weights
T = size(y_obs,1);
```

```
[ lw_samples, weights ] = lw_sampling( Ns, px0, trans_prob, obs_prob, y_obs );

% get likelihood-weighted approximate inference posterior
lw_posterior = lw_inference( n, lw_samples, weights );

%% Plot Data

% create timeseries
tspan = linspace(1,size(y_obs,1),5000)';

% get continuous state trace
% NOTE: assumes posterior to be an nxT matrix
[ state, trace ] = getStateTrace( tspan, posterior' );
[ eln_state, eln_trace ] = getStateTrace( tspan, eln_posterior(:,2:end) );
[ lw_state, lw_trace ] = getStateTrace( tspan, lw_posterior' );

figure(1)
plot(state,'.','MarkerSize',10); hold on;
plot(eln_state,'o','MarkerSize',10);
% plot(lw_state,'diamond','MarkerSize',7);
plot(tspan,trace);
plot(tspan,eln_trace,'--');
% plot(tspan,lw_trace,'--');
xlim([0,size(y_obs,1)+1])
ylim([0.5,4.5]); yticks([1 2 3 4 5])
title('HMM - Long Sequence','Interpreter','latex');
xlabel('timestep, $k$','Interpreter','latex');
ylabel('state, $x_k$','Interpreter','latex');
% for problem 2
% hdl = legend('forward-backward','likelihood-weighted inference', ...
%     'fwd-bck state trace', 'likelihood-weighted trace');
% for problem 3
hdl = legend('forward-backward','ext-log forward-backward', ...
'fwd-bck state trace', 'ext-log fwd-bck state trace');
% hdl = legend('forward-backward','ext-log forward-backward', ...
%     'likelihood-weighted inference','fwd-bck state trace', ...
%     'ext-log fwd-bck state trace');
set(hdl,'Interpreter','latex','Location','Northwest')
```

forward.m - forward pass of the forward-backward algorithm

```
function [ alpha, alpha2 ] = forward( px0, trans_prob, obs_prob, y_obs )

n = size(px0,1);
T = size(y_obs,1);

% initialization - alpha(x0) = prior
alpha = zeros(n,T+1);
alpha(:,1) = px0;

alpha2 = zeros(n,T+1);
alpha2(:,1) = px0;

% forward pass
for k=1:T
    % matrix math version... works!
    alpha2(:,k+1) = alpha2(:,k)'*trans_prob'*diag(obs_prob(y_obs(k),:));
    for i=1:n
        for j=1:n
            alpha(i,k+1) = alpha(i,k+1) + ( alpha(j,k) * ...
                trans_prob(i,j) * obs_prob(y_obs(k),i) );
        end
    end
    % normalize (do NOT normalize alpha values!)
    % alpha(:,k+1) = alpha(:,k+1)./sum(alpha(:,k+1));
end

```

```
% return alpha with dmensions Txn
alpha = alpha';
alpha2 = alpha2';

end
```

backward.m - backward pass of the forward-backward algorithm

```
function [ beta, beta2 ] = backward( trans_prob, obs_prob, y_obs )

n = size(trans_prob,1);
T = size(y_obs,1);

% initialization
beta = zeros(n,T);
beta(:,end) = ones(n,1);

beta2 = zeros(n,T);
beta2(:,end) = ones(n,1);

% backward pass
for k=(T-1):-1:1
    % matrix math version... not working
    % beta2(:,k) = beta2(:,k+1)'*trans_prob*diag(obs_prob(y_obs(k+1),:));
    beta2(:,k) = obs_prob(y_obs(k+1),:)*trans_prob*diag(beta2(:,k+1));

    for i=1:n
        for j=1:n
            beta(i,k) = beta(i,k) + ( beta(j,k+1) * ...
                trans_prob(j,i) * obs_prob(y_obs(k+1),j) );
        end
    end
    % normalize (do NOT normalize beta values!)
    % beta(:,k) = beta(:,k)./sum(beta(:,k));
end

% return beta with dimensions Txn
beta = beta';
beta2 = beta2';

end
```

fb_posterior.m - forward-backward algorithm posterior calculation

```
function [ posterior ] = fb_posterior( alpha, beta )

if size(alpha,2) ≠ size(beta,2)
    error('alpha, beta size mismatch')

else
    n = size(alpha,1);
    T = size(alpha,2);
    posterior = zeros(n,T);

    for k=1:T
        % sanity check
        fprintf('sum(alpha(:,%d).*beta(:,%d)) = %e\n', k, ...
            k, sum(alpha(:,k).*beta(:,k)))

        posterior(:,k) = (alpha(:,k).*beta(:,k)) / ...
        sum( alpha(:,k).*beta(:,k) );

        % normalize
        posterior(:,k) = posterior(:,k)./sum(posterior(:,k));
    end

```

```
    % return a Txn matrix
    posterior = posterior';
end
```

forward_eln.m - extended-logarithm forward pass

```
function [ eln_alpha ] = forward_eln( px0, trans_prob, obs_prob, y_obs )

% NOTE: in accordance with Rabiner/Mann notation, the trans_prob matrix
% must be a row-stochastic matrix

n = size(px0,1);
T = size(y_obs,1);
eln_alpha = zeros(n,T+1);

% initialization
for i=1:n
%     eln_alpha(i,1) = elnprod( eln(px0(i,1)), ...
%         eln(obs_prob(y_obs(1),i)) );
    eln_alpha(i,1) = eln(px0(i,1));
end

for k=2:T+1
    for j=1:n
        logalpha = NaN;
        for i=1:n
            logalpha = elnsum( logalpha, ...
                elnprod( eln_alpha(i,k-1), eln(trans_prob(i,j)) ));
        end
        eln_alpha(j,k) = elnprod( logalpha, ...
            eln(obs_prob(y_obs(k-1),j)) );
    end
end

end
```

backward_eln.m - extended-logarithm backward pass

```
function [ eln_beta ] = backward_eln( trans_prob, obs_prob, y_obs )

% NOTE: in accordance with Rabiner/Mann notation, the trans_prob matrix
% must be a row-stochastic matrix

n = size(trans_prob,1);
T = size(y_obs,1);

% initialization
eln_beta = zeros(n,T+1);

for k=T:-1:1
    for i=1:n
        logbeta = NaN;
        for j=1:n
            logbeta = elnsum( logbeta, ...
                elnprod( eln(trans_prob(i,j)), ...
                elnprod( eln(obs_prob(y_obs(k),j)), eln_beta(j,k+1) )));
        end
        eln_beta(i,k) = logbeta;
    end
end
```

elnfb_posterior.m - extended-logarithm posterior calculation

```
function [ eln_gamma, gamma ] = posterior_elnfb( eln_alpha, eln_beta )

if size(eln_alpha,2) ≠ size(eln_beta,2)
    error('eln_alpha, eln_beta size mismatch')

else
    n = size(eln_alpha,1);
    T = size(eln_alpha,2);

    % initialization
    eln_gamma = zeros(n,T);     % log-posterior
    gamma = zeros(n,T);         % true posterior

    for k=1:T
        normalizer = NaN;
        for i=1:n
            eln_gamma(i,k) = elnprod(eln_alpha(i,k),eln_beta(i,k));
            normalizer = elnsum(normalizer,eln_gamma(i,k));
        end

        for i=1:n
            eln_gamma(i,k) = elnprod(eln_gamma(i,k),-normalizer);
            gamma(i,k) = eexp(eln_gamma(i,k));
        end
        % sanity check
        fprintf('1 - sum(gamma(:,%d)) = %e\n', k, ...
        1-nansum(eexp(eln_gamma(:,k))))
    end
end
```

lw_sampling.m - likelihood-weighted sampling

```
function [ lw_samples, weights ] = lw_sampling( Ns, T, px0, trans_prob, obs_prob, y_obs )
%LW_SAMPLING Liklihood Weighted Sampling

lw_samples = zeros(Ns,T);   % V_E - likelihood weighted sample
weights = ones(Ns,1);       % W_E - likelihood weights

% get intial state from prior distribution, px0
[¬,idx] = max(px0);

for s=1:Ns
    for k=1:T
    if k==1
        % draw random sample according to initial (known) state, x0
        lw_samples(s,k) = randsample(4,1,true, trans_prob(:,idx));
    else
        % draw random sample according to state transtion probabilities
        lw_samples(s,k) = randsample(4,1,true, trans_prob(:,lw_samples(s,k-1)));
    end

    % update sequence weight:
    % weight = weight * P( y_k | Pa(y_k) = x_k )
    % NOTE: parent of each observation y_k is the state x_k
    weights(s,1) = weights(s,1) * obs_prob(y_obs(k),lw_samples(s,k));
    end
end

end
```

lw_inference.m - likelihood-weighted approximate inference

```
function [ posterior ] = lw_inference( n, lw_samples, weights )
%LW_INFERENCE Likelihood-weighted approximate inference

T = size(lw_samples,2);
posterior = zeros(n,T);

for k=1:T
    for i=1:n
    % get index location of where x_k = x_i across all sequences
    idx = (lw_samples(:,k) == i);

    % the posterior for each state i at timestep k is the weighted sum
    % of the relaizations (indicator function) of state x_i at
    % timestep k across all MC sample sequences
    posterior(i,k) = sum(weights(idx))/sum(weights);
    end
end

posterior = posterior';

end
```

Mann Extended-Logarithm Functions Library

```
function [ out ] = eexp( x )
% EEXP - Extended Exponential function

    % if x == 'LOGZERO'
    if isnan(x)
        out = 0;
    else
        out = exp(x);
    end

end

function [ out ] = eln( x )
% ELN - Extended Natural Logarathm function
%   Computes the extended natural logarithm as defined by Mann, 2006

    if x == 0
        % out = 'LOGZERO';
        out = NaN;
    elseif x > 0
        out = log(x);
    else
        error('eln() negative input error')
    end

end

function [ prod ] = elnprod( eln_x, eln_y )
% ELNSUM - Extended Logartithm Product function
%   Computes the extended logarithm of the product of x and y given as
%   given as inputs the extended logarithm of x and y, as defined by Mann, 2006

    % if strcmp(eln(x),'LOGZERO') || strcmp(eln(y),'LOGZERO')
    if isnan(eln_x) || isnan(eln_y)
        % prod = 'LOGZERO';
        prod = NaN;
    else
        prod = eln_x + eln_y;
    end

```

```
end

function [ sum ] = elnsum( eln_x, eln_y )
% ELNSUM - Extended Logartithm Sum function
%   Computes the extended logarithm of the sum of x and y given as inputs
%   the extended logarithm of x and y, as defined by Mann, 2006

    % if strcmp(eln(x),'LOGZERO') || strcmp(eln(y),'LOGZERO')
    if isnan(eln_x) || isnan(eln_y)
        % if strcmp(eln(x),'LOGZERO')
        if isnan(eln_x)
            sum = eln_y;
        else
            sum = eln_x;
        end
    else
        if eln_x > eln_y
            sum = eln_x + eln(1 + exp(eln_y-eln_x));
        else
            sum = eln_y + eln(1 + exp(eln_x-eln_y));
        end
    end

end
```