# Total Roman Domination of Kneser Graphs

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## Abstract

The total Roman domination number of a graph is the smallest possible sum of the weights $0$, $1$, and $2$ applied to vertices of the graph that satisfy certain rules originating from Roman military strategy. In this poster, we discuss total Roman domination of Kneser graphs. We give exact counts for some graphs and demonstrate bounds for other graphs.

## Background

### Graphs

A *graph*, $G$ is a mathematical structure composed of a set of *vertices*, $V(G)$, and a set of *edges*, $E(G)$. The vertices correspond to objects (or sets of objects), and the edges represent relations between the objects (or sets of objects).

### Rules of total Roman domination

A total Roman dominating function on a graph $G$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the following conditions:

- each nonzero vertex must be adjacent to another nonzero vertex, and
- each vertex labeled $0$ must be adjacent to at least one vertex labeled $2$.

The weight of $f$ is $\sum_{v \in V(G)} f(v)$. The *total Roman domination number*, $\gamma_{tR}(G)$, is the minimum weight of a total Roman dominating function on $G$.

### Algebraic Structure of Kneser Graphs

Let $n$ and $k$ be integers with $n > k \geq 1$ and $[n] = \{1, 2, ..., n\}$. The Kneser graph $K(n, k)$ is the graph whose vertices represent $k$-subsets of $\{1, 2, ..., n\}$, and where two vertices are connected if and only if they correspond to disjoint subsets. Note that $K(n, k)$ has $\binom{n}{k}$ vertices and is regular of degree $\binom{n-k}{k}$.

### Examples

Consider $G = K(5, 2)$. The vertices are the 2-subsets of $\{1, 2, 3, 4, 5\}$ with edges appearing between two vertices who have no elements in common. Below we demonstrate three total Roman domination functions on $G$ with corresponding weights of $10$, $9$, and $7$ (the light blue vertices are 2's and the red vertices are 0's). Note: $\gamma_{tR}(G) = 7$.

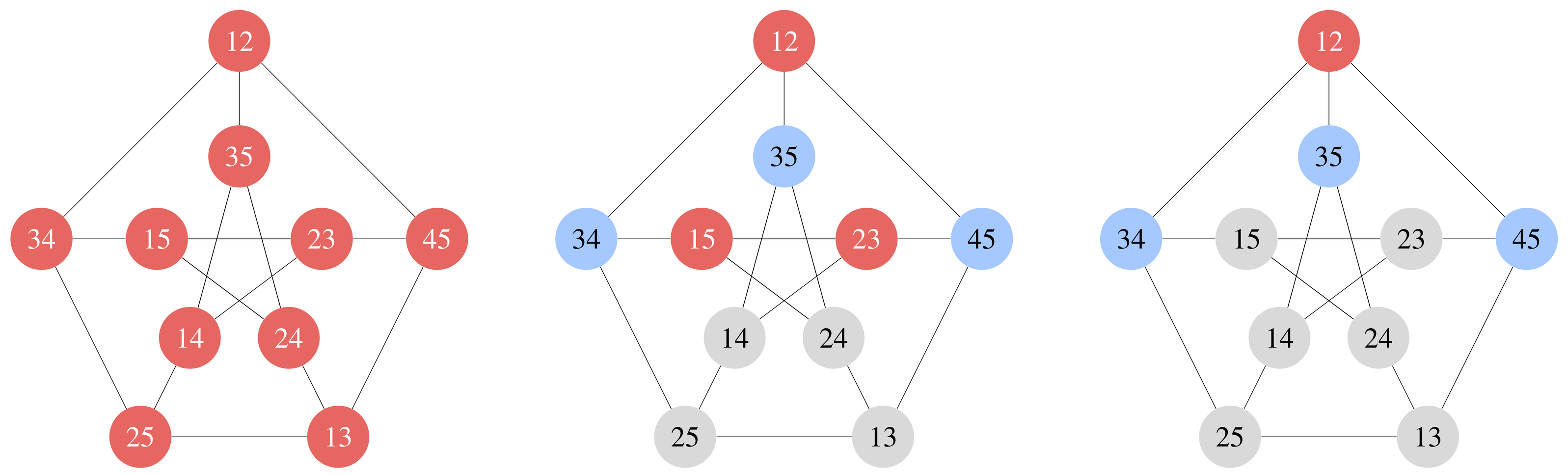


Figure 1: Total Roman domination function options on $G = K(5, 2)$.

Consider $K(6, 2)$. The vertices are the 2-subsets of $\{1, 2, 3, 4, 5, 6\}$ with edges appearing between two vertices who have no elements in common. The three vertices $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$ form a triangle and all other vertices are connected to that triangle. So we may label the triangle with 2's and all other vertices with 0's.

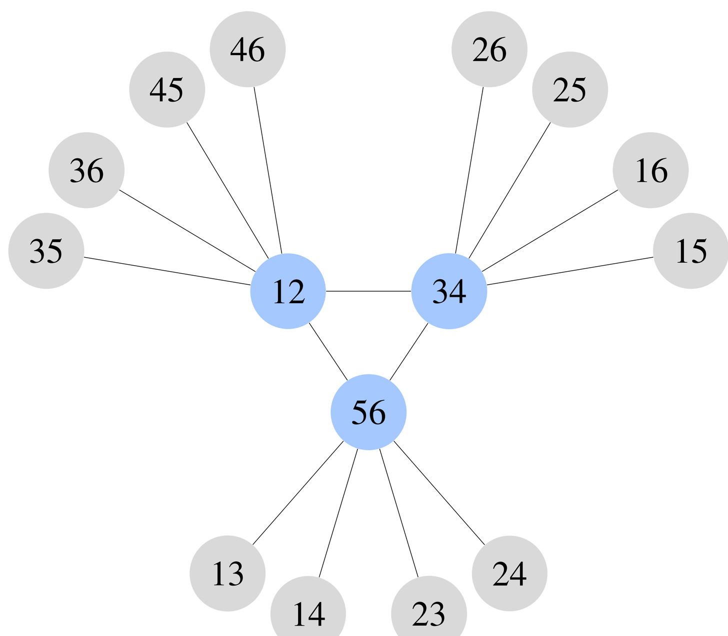


Figure 2: Total Roman domination function option for $K(6, 2)$.

## Upper Bound

In this section we discuss an upper bound for the total Roman domination number of $K(n, k)$, where $n > 2k \geq 6$ using a disconnected dominating set. Any dominating set $D$ for $G$ can possibly be disconnected. We label each vertex in a dominating set with 2's. For each element in the dominating set, we label one adjacent vertex with a 1. This gives an upper bound of $\sum_{v \in V(G)} f(v) = 2t + w \leq 3|D|$.

### Example

Here we have a dominating set of $K(7, 3)$ shown in light blue. The vertices in red are the additional 1's needed to make the 2's not isolated. The upper bound given by this labeling is $f(t) = 21$.

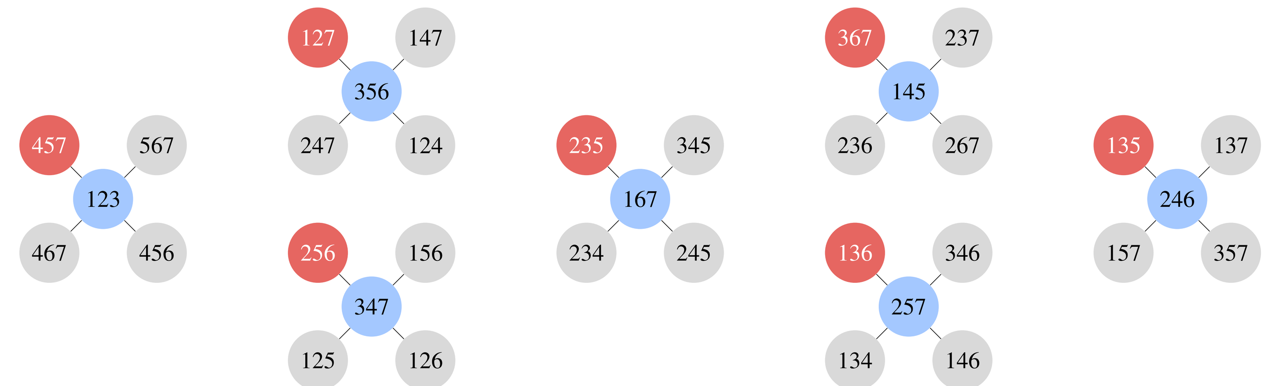


Figure 3: Total Roman domination function for $K(7, 3)$.

## Lower Bound

Fix a Kneser graph $G = K(n, k)$ where $n > 2k \geq 6$ and a total Roman dominating function $f$. The number of vertices in $G$ is $|V| = \binom{n}{k}$ and for each vertex in the graph the degree is $d = \binom{n-k}{k}$. Let $Z = \{v \in V(G) \mid f(v) = 0\}$, $W = \{v \in V(G) \mid f(v) = 1\}$, and $T = \{v \in V(G) \mid f(v) = 2\}$; and let $z = |Z|$, $w = |W|$, and $t = |T|$. Since $Z \cup W \cup T = V$ is a disjoint 3-partition of $V$, then

$$z + w + t = \binom{n}{k}.$$

Total Roman domination rules require each vertex in $T$ must be adjacent to a vertex in $W \cup T$. So for every $v \in V$, if $v \in T$, letting $N_G(v)$ denote the set of vertices incident to $v$,

$$|N_G(v) \cap Z| \leq d - 1.$$

Yet, each vertex in $Z$ must be adjacent to a vertex in $T$ so $z \leq (d-1)t$. Using this inequality, we see $\binom{n}{k} = z + w + t \leq (d-1)t + w + t = dt + w$. This means $w \geq \binom{n}{k} - dt$. Since $w \geq 0$ we know $w \geq \max\left\{\binom{n}{k} - dt, 0\right\}$. Therefore, for any total Roman domination function $f : V(G) \to \{0, 1, 2\}$,

$$\sum_{v \in V(G)} f(v) = 2t + w \geq \varphi(t)$$

where

$$\varphi(t) = \begin{cases} 2t + \binom{n}{k} - dt, & \text{if } \binom{n}{k} - dt > 0 \\ 2t, & \text{if } \binom{n}{k} - dt \leq 0. \end{cases}$$

It therefore follows that

$$\gamma_{tR}(G) = \min_f \sum_{v \in V(G)} f(v) \geq \min_{t \geq 0} \varphi(t).$$

**Example:** Let $G = K(7, 3)$. Then $|V| = \binom{7}{3} = 35$ and $d = \binom{4}{3} = 4$. So,

$$\varphi(t) = \begin{cases} 2t + 35 - 4t, & \text{if } 35 - 4t > 0 \\ 2t, & \text{if } 35 - 4t \leq 0 \end{cases} = \begin{cases} 35 - 2t, & \text{if } t < 9 \\ 2t, & \text{if } t \geq 9. \end{cases}$$

Therefore, $\gamma_{tR}(G) \geq \min_{t \geq 0} \varphi(t) = 18$.

## References

[1 ] Ahangar, Hossein Abdollahzadeh, et al. *Total Roman domination in graphs*, Applicable Analysis and Discrete Mathematics, 10.2: 501-517, 2016.

[1 ] Reinhard Diestel. *Graph theory 3rd ed.* Graduate texts in mathematics, 173:33, 2005.

[2 ] Igor Gorodezky. *Dominating Sets in Kneser Graphs*, Master's Thesis, University of Waterloo, 2007.

## Improving a Lower Bound

We already showed that $\gamma_{tR}(G) \leq 21$ for $G = K(7, 3)$. To show that $\gamma_{tR}(G) = 21$ it therefore suffices to show that there is no valid total Roman domination function $f : V(G) \to \{0, 1, 2\}$ such that $\sum_{v \in V(G)} f(v) \leq 20$. To this end we will consider a weight-budget of $20$ and demonstrate that every construction of a valid total Roman domination function exceeds our weight-budget. Observe that if every vertex is labeled $1$ then the sum of the weights would be $35$. Hence, some of our vertices are labeled $2$. If none of the 2's share an edge then we have a dominating set such as the one used to find the upper-bound of weight $21$. It follows that we must consider that there are at least one pair of 2's that are adjacent.

Without loss of generality, we assume those vertices are $123$ and $456$. The neighbors of these vertices are $467$, $457$, $567$, $237$, $127$, and $137$ and can now be labeled with any number, including $0$ since they are adjacent to a $2$. We will call these vertices the *initial null* set. Labeling the *initial null* set with 0's will force a lot of structure on the remainder of the graph. The diagram below on the left shows a partition of the remainder of the vertices into to sets: the *inner* set which is comprised of eighteen vertices adjacent to the *initial null* set, and the *outer* set which are the nine remaining vertices. Each vertex in the *inner* set has one edge within the *inner* set, one edge to the *initial null* set, and two edges to the *outer* set. Each vertex in the *outer* set has all four edges to the *inner* set. Using this, we are able to create cases based on how much of the weight-budget is allocated to the *outer* set to contradict our assumption that a weight-budget of $20$ is possible.

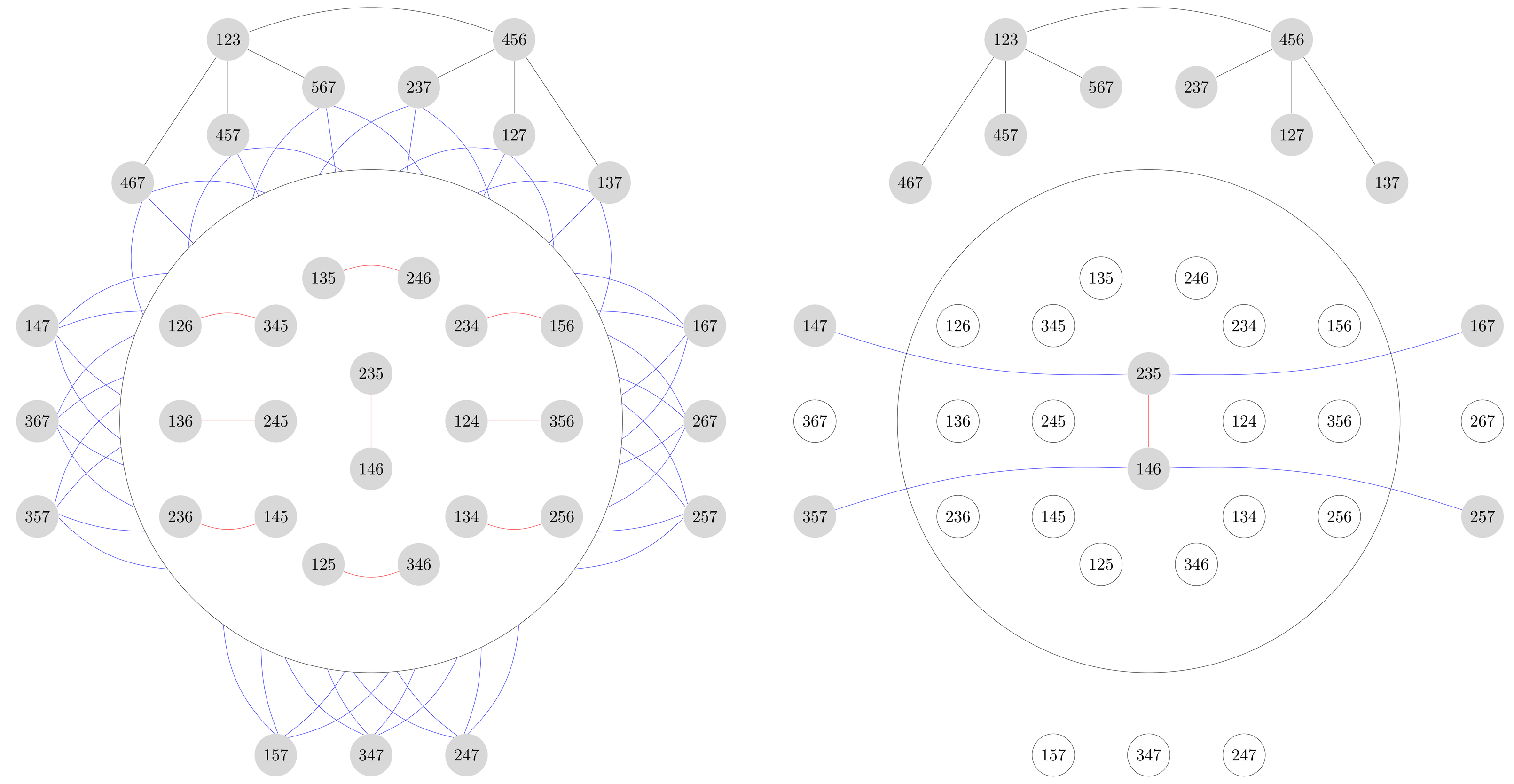

Figure 4: Two representations of $K(7, 3)$ showing that a weight-budget of $20$ is impossible.

## Results

| n\k | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | N/A | N/A | N/A | N/A | N/A |
| 2 | 2 | Disconnected | N/A | N/A | N/A | N/A |
| 3 | 3 | Disconnected | Disconnected | N/A | N/A | N/A |
| 4 | 3 | 6 | Disconnected | Disconnected | N/A | N/A |
| 5 | 3 | 7 | Disconnected | Disconnected | Disconnected | N/A |
| 6 | 3 | 6 | 20 | Disconnected | Disconnected | Disconnected |
| 7 | 3 | 6 | 21 | Disconnected | Disconnected | Disconnected |
| 8 | 3 | 6 | $12 \leq \cdots \leq 21$ | 70 | Disconnected | Disconnected |
| 9 | 3 | 6 | $10 \leq \cdots \leq 21$ | $51 \leq \cdots \leq 81$ | Disconnected | Disconnected |
| 10 | 3 | 6 | $8 \leq \cdots \leq 18$ | $28 \leq \cdots \leq 60$ | 252 | Disconnected |
| 11 | 3 | 6 | $6 \leq \cdots \leq 15$ | $20 \leq \cdots \leq 51$ | $154 \leq \cdots \leq 198$ | Disconnected |
| 12 | 3 | 6 | 8 | $16 \leq \cdots \leq 36$ | $76 \leq \cdots \leq 177$ | 924 |

Figure 5: Total Roman domination numbers - Known values and bounds for $\gamma_{tR}(G)$ of $G = K(n, k)$.

## Acknowledgments

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