Small and Sweet MapReduce Algorithms

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MapReduce

A platform for massive parallel computation.

- First (formal) paper in 2004.
- Now a large collection of algorithms on this platform.

This talk:

Principles in the design and analysis of algorithms in MapReduce (and other similar platforms).

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Ultimate goals:

- Small: Simple enough for practical implementation
- Sweet: With non-trivial theoretical guarantees.

Computation Model

This is the starting point before any meaningful analysis.

1. MRC model [SODA'10]
2. BSP-like model [ISSAC'11]

3. Minimality Model [SIGMOD'13]
4. Massively Parallel Computation (MPC) [PODS'13].

Computation Model

This talk:

3. Minimality Model [SIGMOD'13]
    - A very stringent model – suitable for studying "easy" problems.
4. Massively Parallel Computation (MPC) [PODS'13].
    - A more relaxed model – suitable for studying "hard" problems.

Minimality Model

Input and Machines

p = number of machines.
n = number of elements in the input (i.e., dataset)
Assumption: $p \leq \sqrt{\frac{n}{2 \ln n}}$

At the beginning, $O(n/p)$ elements per machine.

- The algorithm has no control over how the elements are initially distributed.

Minimality Model

Superstep

Two phases:

1. Machines message each other.

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Π is minimal if it admits an algorithm satisfying all:

1. It has $O(1)$ supersteps.
2. It uses $O(n/p)$ space at all times.
3. Every machine sends $O(n/p)$ words in total.
4. Every machine incurs $O(f(n/p))$ CPU time in total.

The algorithm is called a minimal algorithm for Π.

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Balanced during the entire algorithm.

Requirement 3: $O(n/p)$ words sent and received.
Asymptotically the same time for each machine to look at its portion of the input.

Minimality Model

Requirement 4: $O(f(n/p))$ CPU time for every machine.

- The MapReduce algorithm must improve automatically whenever a faster sequential algorithm is discovered!
  - Implies that the MapReduce algorithm must utilize a sequential algorithm as a black box!

Minimality Model

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  - Implies that the MapReduce algorithm must utilize a sequential algorithm as a black box!

- The best you can do!
  - If you can achieve $o(f(n/p))$ on MapReduce, you can simulate the same algorithm in the RAM model in $o(p \cdot f(n/p))$ time.

Minimal Algorithm for Sorting

At the beginning: n elements on p machines.

At the end: it is possible to label the machines from 1 to p such that all elements on machine i are smaller than those on machine j, for any $i < j$.

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> **Remark (illustration of Requirement 4)**
>
> - Comparison-based sort: $\Theta(n \log n)$ sequentially $\Rightarrow$ $O(\frac{n}{p} \log \frac{n}{p})$ MapReduce .
> - Sorting n integers: $O(n \log \log n)$ sequentially $\Rightarrow$ $O(\frac{n}{p} \log \log \frac{n}{p})$ MapReduce.
>
> All by the same algorithm.

Superstep 1

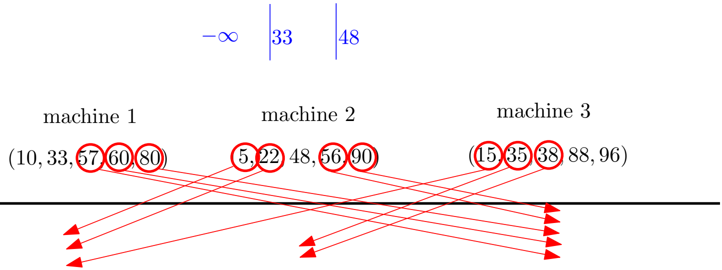
- **Phase 1:** Each machine samples each object with probability $\rho = \frac{p}{n} \ln(nt)$ independently, and sends the samples to all other machines.
- **Phase 2:** Each machine sorts the samples received, and identify the split elements that partition the list into t segments evenly.
  - Machine i will get all the elements in the i-th segment at the end.

machine 1	machine 2	machine 3
$(10, 33, 57, 60, 80)$	$(5, 22, 48, 56, 90)$	$(15, 35, 38, 88, 96)$
$(5, \boxed{33}, 38, \boxed{48}, 80, 88)$	$(5, \boxed{33}, 38, \boxed{48}, 80, 88)$	$(5, \boxed{33}, 38, \boxed{48}, 80, 88)$

Minimal Algorithm for Sorting

Superstep 2

- **Phase 1:** Each machines sends the elements in the i-th segment to machine i.
- **Phase 2:** Machine i sorts the elements in the i-th segment.



Minimal Algorithm for Sorting

Theorem: The above algorithm is minimal with probability at least $1 - O(1/n)$.

Minimality $\Rightarrow$ MPC

For certain problems, if $O(1)$ supersteps are required, it is impossible to ensure that each machine sends $O(n/p)$ words only.
$\Rightarrow$ No minimal algorithms.

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$\Rightarrow$ No minimal algorithms.

Phrased differently, if $O(1)$ supersteps are required, we must allow a machine to send more than $O(n/p)$ words – but how much more?
$\Rightarrow$ The MPC model.

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Assumption: $p \leq n^{1/c}$ for some constant $c > 1$.

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Remark 1: CPU time for free.

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Aim: Minimize load.

Remark 1: CPU time for free.
Remark 2: Every minimal algorithm must have load $O(n/p)$.
Remark 3: Any problem admits an algorithm of load $O(n)$.

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Focus of this talk: **Cartesian Product**

R: a set of n **red** elements
S: a set of n **blue** elements

At the beginning: The $2n$ elements are stored on the p machines (each $O(n/p)$ elements).
At the end: Each element in $R \times S$ must appear on some machine.

MPC Lower Bound for Cartesian Product

To warm up, we will prove that, when $p = 2$, any algorithm solving the problem must have load $\Omega(n)$.

$\Rightarrow$ Implication: No better algorithm than simply sending everything over to the other machine.

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Suppose that an algorithm solves the problem with load L.

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which solves to $L \geq \frac{2-\sqrt{2}}{2\sqrt{2}} n$.

MPC Lower Bound for Cartesian Product

Next, we will extend the algorithm to prove that, any cartesian product algorithm must have load $\Omega(n/\sqrt{p})$.

$\Rightarrow$ Implication: No minimal algorithm!

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pairs.
$\Rightarrow$ Putting together p machines means

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$\Rightarrow$ Putting together p machines means

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which solves to $L = \Omega(n/\sqrt{p})$.

MPC Lower Bound for Pairwise Natural Join

Pairwise Natural Join

$R = (A, B)$: a table of n tuples
$S = (B, C)$: a table of n tuples

At the beginning: The $2n$ tuples are stored on the p machines.
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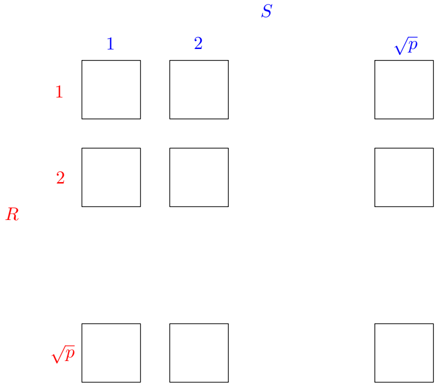
LOWER BOUND: Load $\Omega(n/\sqrt{p})$!

MPC Cartesian Product

An optimal algorithm—named cube—with load $O(n/\sqrt{p})$ is known. The algorithm actually computes the cartesian product.



MPC Model: Understood Problems

Problems whose communication complexities (i.e., load) have been well understood:

- Enumeration of constant-sized subgraphs.
- Restricted natural joins.
- Linear programming.
- ...

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Future work 2: Lower bound for problems with small result sizes!

- From the theory of communication complexity, it is easy to prove that when $p = 2$, $\Omega(n)$ communication is necessary for detecting whether a natural join has an empty result!

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Future work 2: Lower bound for problems with small result sizes!

- From the theory of communication complexity, it is easy to prove that when $p = 2$, $\Omega(n)$ communication is necessary for detecting whether a natural join has an empty result!

Future work 3: $o(n/p)$-load algorithms!