# Algebraic mechanics of connected rigid bodies

**Hervé Capart**

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Adapted by CPS.

## Structure as a graph

Consider a mechanical structure (building) represented by a set of raw vertices $u \in \mathcal{U}$ with positions $p(u)$ and a topology in which each vertex is connected to at least two other vertices. This representation of the structure is a graph $\mathcal{B}$. We use the qualifier "raw" for these vertices and their graph, because they need to be restructured if we are to allow the mechanical structure they represent to deform.

Within this graph $\mathcal{B}$, identify and split off each sub-graph $s_i \in \mathcal{S}$ formed by the union of those sets of raw vertices belonging to loops of three that share at least one loop edge.

These sub-graphs need to be further categorized into members, applied loads, and ground. For each sub-graph whose vertex count is $\#u(s_i) = 3$ (i.e., a triangle, for 2D structures), assume that it represents an applied load: label it $l_i \in \mathcal{L}$. Find the sub-graph with the largest area, and assume that it represents ground; label it $g = \mathcal{G}$. The remaining sub-graphs $S_{\backslash \mathcal{L} \backslash \mathcal{G}}$ are called members and are labeled $m_j \in \mathcal{M}$.

By splitting graph $\mathcal{B}$ into the set of sub-graphs $\mathcal{S}$, a new set of vertices $v_i \in \mathcal{V}$ is formed. Find each of these vertices that share the same geometric position as at least one other vertex, i.e., locate all the vertices at which the graph was broken into sub-graphs. Usually groups of overlapping vertices will come in pairs (where a split into two sub-graphs took place), but there may be more per group if three or more sub-graphs were split at the same raw vertex. From now on, all references to vertices will mean to elements of the set $\mathcal{V}$ and not to the raw set.

Label the overlapping vertices as $w_j \in \mathcal{W}$, and call them "connecting vertices" Geometrically, the connecting vertices all lie at the same position as the original shared raw vertex. The set of connecting vertices is a subset of all vertices: $w_j \subset v_i$ or $\mathcal{W} \subset \mathcal{V}$.

At each connecting vertex $w_j$, create a linking node and label it as $n_k \in \mathcal{N}$. Now join the connecting vertices to this linking node; each of these connections is labeled $c_{jk} \in \mathcal{C}$ and consists of pairs $(w_j, n_k)$ of connecting vertices and nodes. Topologically, each node lies between connecting vertices, but shares the same geometric position as them (for now).

Denote the respective numbers of members, vertices, connecting vertices, nodes, and connections as $\#m$, $\#v$, $\#w$, $\#n$, and $\#c$.

Some clarifications. The set of all vertices and nodes is

$$\mathcal{V} \cup \mathcal{N} = \bigcup_j v_j \cup \bigcup_k n_k$$

The subgraphs categorized as the ground, applied loads, and members, together comprise the set of all sub-graphs of the structure

$$\mathcal{S} = \mathcal{G} \cap \mathcal{L} \cap \mathcal{M}$$

Each member has a set of vertices which we can write as

$$\mathcal{V}_m = \bigcup_j v_j(m)$$

Since we split the initial graph members into disjoint sets of vertices,

$$\mathcal{V}_{m_i} \cap \mathcal{V}_{m_j} = \emptyset \quad \text{for} \quad i \neq j$$

## System equilibrium

Consider the initially unloaded geometry. The locations of connecting vertices coincide with that of their linking node

$$\forall\, i, k \;:\; \exists\, c_{ik}\,, \quad \mathbf{p}_0(w_i) = \mathbf{p}_0(n_k)$$

which means that all vertices that share a linking node have the some position

$$\forall\, i, j, k \;:\; \exists\, c_{ik}, c_{jk}\,, \quad \mathbf{p}_0(w_i) = \mathbf{p}_0(w_j)$$

which was how the connecting vertices, connections and nodes were defined in any case.

Now let's go beyond the initial geometry by applying the loads and examining the consequent displacements and rotations of the members.

Force balance for each member $m_i$ requires the sum of forces at each connecting node to be the zero vector

$$\forall\, i \;:\; \sum_{j,k} \mathbf{f}_{jk} = \mathbf{0} \qquad \forall\, j, k \;:\; w_j \in m_i\,, \;\; \exists\, c_{jk}$$

Assuming small displacements, for the purpose of moment balances make the approximation

$$\mathbf{p}(w_j) \;\approx\; \mathbf{p}_0(w_j)$$

so that the moment balance for each member $m_i$ requires

$$\forall\, i \;:\; \sum_{j,k} \mathbf{p}_0(w_j) \cdot \mathbf{f}_{jk} = 0 \quad \forall\, j, k \;:\; w_j \in m_i\,, \;\; \exists\, c_{jk}$$

Let's convert to matrix form. First, write out the components of the force balance equation for the whole structure (all members)

$$\sum_{j,k} f_{jk}^x = 0 \tag{1}$$

$$\sum_{j,k} f_{jk}^y = 0 \tag{2}$$

$$\sum_{j,k} \left( p_0^x(w_j) f_{jk}^x + p_0^y(w_j) f_{jk}^y \right) = 0 \tag{3}$$

The matrix representation is then

$$
\begin{matrix}
m_1 \left\{ \rule{0pt}{24pt}\right. \\
\\
m_2 \left\{ \rule{0pt}{24pt}\right. \\
\\
m_3 \left\{ \rule{0pt}{24pt}\right. \\
m_4, \ldots \left\{ \rule{0pt}{12pt}\right.
\end{matrix}
\left[
\begin{array}{cccccc}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
p_0^x(w_1) & p_0^y(w_1) & p_0^x(w_2) & p_0^y(w_2) & p_0^x(w_3) & p_0^y(w_3) \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & p_0^x(w_3) & p_0^y(w_3) \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
p_0^x(w_1) & p_0^y(w_1) & 0 & 0 & 0 & 0 \\
\vdots & & \vdots & & \vdots &
\end{array}
\begin{array}{c} \cdots \\ \\ \cdots \\ \\ \cdots \\ \end{array}
\right]
\begin{bmatrix}
f_{1k}^x \\ f_{1k}^y \\ f_{2k}^x \\ f_{2k}^y \\ f_{3k}^x \\ f_{3k}^y \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots
\end{bmatrix}
$$