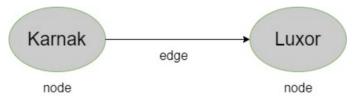
Comp 460 - Algorithms & Complexity

Spring Semester 2020 - Week 13 Dr Nick Hayward

graphs - intro recap

- graph data structure in computer science
- a way to model a given set of connections
- commonly use a graph to model patterns and connections for a given problem
- e.g. connections may infer relationships within data
- graph includes nodes and edges
- help us define such connections
- e.g. we have two nodes with a single edge

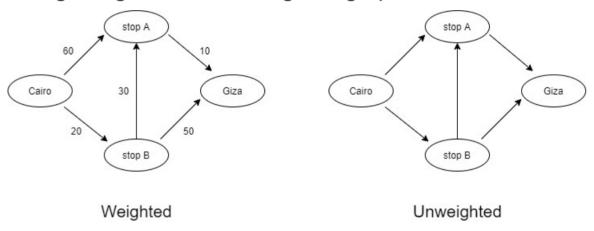


Graph Nodes and Edge

- each node may be connected to many other nodes in the graph
 - commonly referenced as neighbour nodes

graphs - Dijkstra's algorithm - consideration of terminology - part 1

- key concept for working with Dijkstra's algorithm is the association of values,
- e.g. numbers for each edge in the graph
- valyes are the weights assigned to the edge in the graph
- when we assign weights to an edge
- creating a weighted graph
- if we do not assign weights to edges
- defining an unweighted graph
- e.g. weighted and unweighted graphs



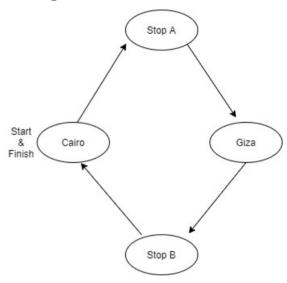
Graphs - Weighted and Unweighted

graphs - Dijkstra's algorithm - consideration of terminology - part 2

- choice of weighted versus unweighted
- · also affecta our choice of algorithm
- and the context of its usage
- e.g. if we want to calculate the shortest path in an unweighted graph
- we may use the breadth-first search algorithm
- to perform a similar calculation for a weighted graph
 - we may use Dijkstra's algorithm...

graphs - Dijkstra's algorithm - graph cycles - part 1

- may also encounter a graph with cycles
- i.e. we can cycle from Stop A back around to Stop A
- e.g.



Graph - Cycle

we may start and finish at the same node in the graph

graphs - Dijkstra's algorithm - graph cycles - part 2

- if we consider a graph with a cycle segment
- need to calculate shortest path between two defined nodes
- for most calculations commonly choose a path that avoids the cycle
- cycle will usually add greater weight to the calculation
- if we then follow cycle more than once
- simply adding extra weight to calculation for each completed cycle
- if we consider an undirected graph
 - now working with a cycle
- connected nodes in an undirected graph point to each other
 - effectively a cycle
- each edge will add another cycle to an undirected graph
- Dijkstra's algorithm only works with directed acyclic graphs (DAGs)

graphs - Dijkstra's algorithm - shortest path - part 1

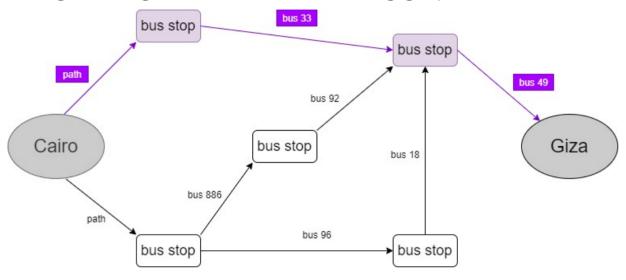
- common requirement for working with graphs
- a consideration of weighted and unweighted edges
- with weighted graphs
- interested in options for assigning more or less weight
- i.e. to edges within the graph
- Dijkstra's algorithm helps us work with queries for paths in our graphs
- e.g. we might need to answer the question

which path is the shortest to a node?

- e.g. which is the shortest path to node A?
- this path may not be fastest
- · but it will be the shortest in the graph
- shortest because it will include least number of edges between nodes

graphs - Dijkstra's algorithm - shortest path - part 2

e.g. we might consider the following graph for the shortest route

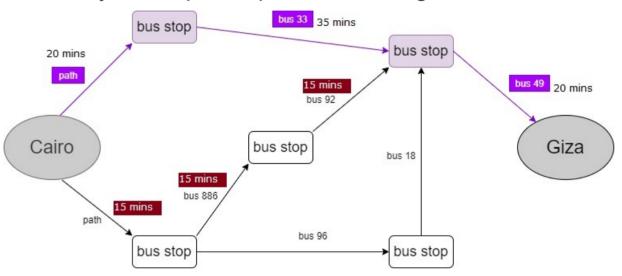


Graph Routes - shortest

- clearly see shortest path with least number of segments, three
- may use breadth-first search to find path with fewest segments

graphs - Dijkstra's algorithm - shortest path - part 3

- if we then added travel times to edges for competing routes
- i.e. costs for each edge
- we may find a quicker path for traveling



Graph Routes - shortest

- e.g. we can see time difference between purple and red paths
- from Cairo to Giza
- red path is faster, in spite of one more segment
- to find the fastest path
- may use a different option to breadth-first search
- we may use *Dijkstra's algorithm* to help us query graph for fastest path

Video - Algorithms and Data Structures

graphs - Dijkstra's algorithm - part 1

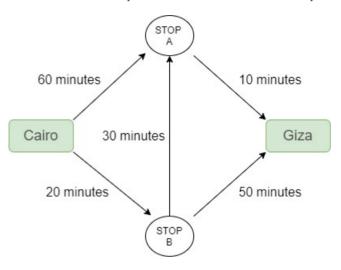


Graphs - Dijkstra's algorithm - intro - UP TO 1:01

Source - Dijkstra's algorithm - YouTube

graphs - Dijkstra's algorithm - working example 1 - part 1

- if we consider the following basic weighted graph
- may initially see how Dijkstra's algorithm works
- i.e. to help us find the fastest path



Dijkstra - example graph

- each segment in this graph has a corresponding cost
- time in minutes
- may use cost with Dijkstra's algorithm
- calculate shortest possible time from Cairo to Giza...

- may initially use the following steps with Dijkstra's algorithm
- i.e. to calculate fastest path from a defined start to finish in the graph
- for the current start node
- 1. identify the cheapest node
- i.e. next node we can reach in least amount of time
- simply check neighbour nodes for current node
- identify path with shortest time
- e.g. from our starting point of Cairo there are two options
 - either Stop A with a time of 60 minutes
- or Stop B with a time of 20 minutes
- we don't know values of other nodes at this point of search
- as we don't yet know how long it will take to get to finish
 - Giza defined with an overall time of infinity...

- we know closest node from start, Cairo
- Stop B with a time of 20 minutes

node	time
Stop A	60 minutes
Stop B	20 minutes
Giza	infinity

- 2. update any costs for neighbours of this node
- need to calculate all times from Stop B to available neighbour nodes
- in current example
- includes Stop A and our finish node of Giza
- may now update our times from start, Cairo
- update to each node currently known in the graph

node	time
Stop A	50 minutes
Stop B	20 minutes
Giza	70 minutes

- first improvement is a faster time from Cairo to Stop A
- even though we have to go through node Stop B
- with current known neighbour nodes
- may also follow a path from start node, Cairo,
- follow to finish in Giza
- path takes 70 minutes
- currently have a shorter path from Cairo to Stop A
- plus a shorter path to finish from start...

- 3. repeat this pattern for each node in the graph
- may now repeat pattern to check for other neighbour nodes
- potentially faster routes from start to finish...
- repeat first step again
- need to find next node with shortest travel time
- checked all of the neighbour nodes for Stop B
- we can now check next fastest neighbour of start node Cairo
- in current example
- this will be node Stop A...

- don't need to update time from start node
 - i.e. from Cairo to node Stop A
 - already identified a faster route...
- we may check times for quickest route to finish
- now have an extra path to check
- i.e. from Stop A to finish in Giza
- gives us a shorter time from start to finish
- due to its time of 10 minuutes...

graphs - Dijkstra's algorithm - working example 1 - part 8

now update our times as follows

node	time
Stop A	50 minutes
Stop B	20 minutes
Giza	60 minutes

- i.e. define fastest routes for following paths
 - Cairo to Stop A = 50 minutes
 - Cairo to Stop B = 20 minutes
 - Cairo to Giza = 60 minutes
- able to identify a quicker path from start to Stop A
- and a quicker path from start to finish...

- calculate the time for the final path
- may currently define final path
- calculated fastest time of 60 minutes...
- if we compare this calculation with a search using breadth-first
- may see that it would not have found that path
- breadth-first would have found shorter path
- but a slower path in this example graph...

- in current example, we may see an initial benefit
- benefit relative to this context
- e.g. for a search with Dijkstra's algorithm compared with a breadth-first search
- may use breadth-first search to find shortest path
- e.g. between defined nodes in graph
- may use Dijkstra's algorithm to assign weights to graph
- use to find path with smallest total of calculated weights...

Video - Algorithms and Data Structures

graphs - Dijkstra's algorithm - part 2

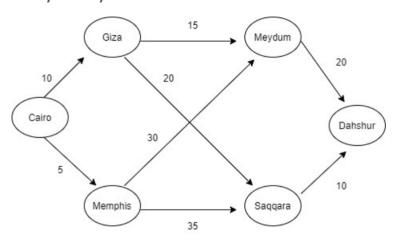


Graphs - Dijkstra's algorithm - example outline - UP TO 3:57

Source - Dijkstra's algorithm - YouTube

graphs - Dijkstra's algorithm - working example 2 - part 1

- consider another working example for a graph with weighted edges
- e.g. a graph with values for cost to travel from one node to another
- perhaps from Cairo to Giza or Giza to Saggara...



Graph Weighted

- in this graph
- define weights for associated costs of travel along each edge
- i.e. travel from *Memphis* to *Meydum* for 30
- or, perhaps, from Giza to Meydum for only 15...

graphs - Dijkstra's algorithm - working example 2 - part 2

- if we consider this graph
- may need to calculate cheapest route from Cairo
- e.g. start point to an end point of Dahshur...
- may use Dijkstra's algorithm to perform this calculation
- follow defined four steps for this algorithm
- initial costings may be defined as follows

parent	node	cost
Cairo	Giza	10
Cairo	Memphis	5
N/A	Meydum	infinity
N/A	Saqqara	infinity
N/A	Dahshur	infinity

and set an initial parent for each node...

- then update this table as we execute the algorithm
- 1. start by finding cheapest node
- in this graph, from a starting node of Cairo cheapest edge is 5 to Memphis
- we can't make this initial path any cheaper
- cheapest node will be Cairo to Memphis
- 2. then, calculate cost to neighbours of this cheapest node, i.e. from Memphis
- we now have costs for Meydum, 30, and Saqqara, 35
- we can update our table of costs from our starting point to each neighbour
- Cairo to Meydum and Cairo to Saggara
- Cairo -> Memphis -> Meydum
- Cairo -> Memphis -> Saggara

parent	node	cost
Cairo	Giza	10
Cairo	Memphis	5
Memphis	Meydum	35
Memphis	Saqqara	40
N/A	Dahshur	infinity

- now have costs for Meydum and Saqqara
- may define their costs as we travel through Memphis node
- as we can see in the table
- their parent node may also be updated to Memphis...

- may now repeat these two steps for next cheapest node from Cairo
- i.e. Giza at a cost of 10
- update its values in the table as well

parent	node	cost
Cairo	Giza	10
Cairo	Memphis	5
Giza	Meydum	25
Giza	Saqqara	30
N/A	Dahshur	infinity

- the costs for travel from starting point, Cairo, to Meydum and Saqqara updated
 - they are cheaper now
 - so we update the values in the table
- i.e. it's now cheaper to travel to Meydum and Saqqara via Giza...

- we may check cost to travel to end point, Dahshur
- check cheapest node from Giza
- currently Meydum at 15
- may update its neighbours
 - gives us an initial cost for Dahshur of 20
- if we update table at this point
 - we get the following travel cost from Cairo to Dahshur

parent	node	cost
Cairo	Giza	10
Cairo	Memphis	5
Giza	Meydum	25
Giza	Saqqara	30
Meydum	Dahshur	45

- we finally have an initial travel cost from start to finish of 45
 - i.e. from Cairo -> Giza -> Meydum -> Dahshur

- we may also check next cheapest node from Giza, Saqqara
- then, we may travel from Saqqara to Dahshur
- for a total of 40 from Cairo
- i.e. may now update cost of travel from start to finish
- update to a lower overall cost of 40

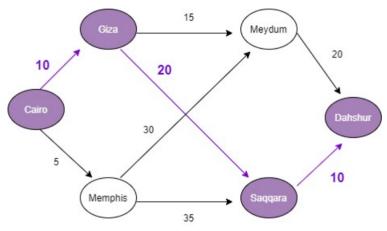
parent	node	cost
Cairo	Giza	10
Cairo	Memphis	5
Giza	Meydum	25
Giza	Saqqara	30
Saqqara	Dahshur	40

parent	node	cost
Cairo	Giza	10
Cairo	Memphis	5
Giza	Meydum	25
Giza	Saqqara	30
Saqqara	Dahshur	40

- we may see that shortest path costs 40
- using this overall cost
 - now define path for travel from start to finish in this graph
- to help with this path definition
- may check parent node set in last table
- i.e. ended up with Saggara as parent for end point Dahshur

graphs - Dijkstra's algorithm - working example 2 - part 9

- we know that we need to travel from Saqqara to Dahshur
- may then follow path to parent of Saqqara, set to Giza
- i.e. to travel to Saqqara we need to begin at Giza
- we follow Giza back to its parent
- starting point at Cairo
- now have a complete route for traveling from start point to end point in least cost, 40



Graph Weighted - Final Costs

Video - Algorithms and Data Structures

graphs - Dijkstra's algorithm - part 3



Graphs - Dijkstra's algorithm - example usage - UP TO 8:48

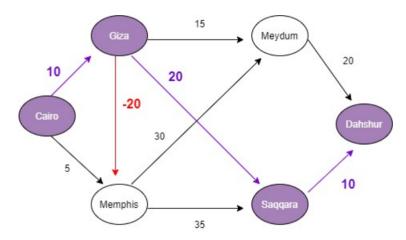
Source - Dijkstra's algorithm - YouTube

graphs - Dijkstra's algorithm - edges with negative weight - part 1

- in last example
 - we have weighted edges from Cairo to Giza, and Cairo to Memphis
- each of these routes has a cost involved,
- i.e. the weight of the edge
- we may now add a path directly from Giza to Memphis
- in current example
- this edge will pay us 20
- we're able to claim the cost back...s

graphs - Dijkstra's algorithm - edges with negative weight - part 2

- edge may be defined with a negative weight of -20
- now have two routes to consider to allow us to travel from Cairo to Memphis
- might take previous route
 - · direct from Cairo to Memphis
 - route will cost 5
- or we might consider updated route via Giza
- second route, Cairo -> Giza -> Memphis
 - now costs -10



Graph - Negative Weighted Edges

graphs - Dijkstra's algorithm - Dijkstra and negative weights - part 1

- if we continue path through graph to end point at Dahshur
- might consider following this route with a negative weighted edge
- if we try to perform our usual calculation with *Dijkstra's* algorithm
- end up following more expensive route
- i.e. negative-weighted edges will break use of Dijkstra's algorithm
- issue may not be final predicted route, as seen above
- issue is commonly with defined calculations
 - performed at various stages during algorithm's execution

graphs - Dijkstra's algorithm - Dijkstra and negative weights - part 2

- if we run Dijkstra's algorithm again on this graph
 - this time with negative weighted edge
- we get a false definition for cheapest route to Memphis
- e.g. following standard pattern of calculation
- we get the following table of costs

node	cost
Giza	10
Memphis	5
Saqqara	infinity

- then, we find lowest-cost node
- and update costs for each of its neigbours
- Memphis is initial lowest code node from Cairo with a cost of 5

graphs - Dijkstra's algorithm - Dijkstra and negative weights - part 3

- according to the Dijkstra algorithm
- there is no cheaper path to travel from Cairo to Memphis
- due to negative-weighted edge from Giza to Memphis
- we know this calculation and assertion is incorrect
- if we continue to follow Dijkstra's algorithm
- we update the table as follows

node	cost
Giza	10
Memphis	5
Saqqara	40

- then, we get next lowest cost node from Cairo
- Giza with a cost of 10
- and update cost of its neighbours

graphs - Dijkstra's algorithm - Dijkstra and negative weights - part 4

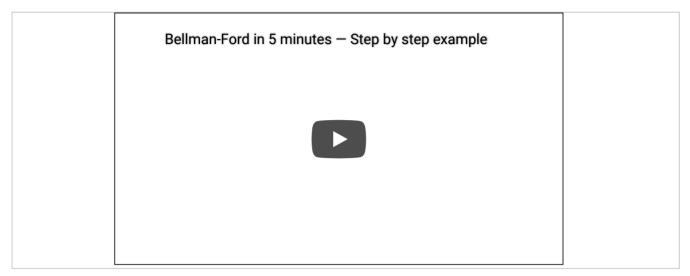
- if we consider neighbours of Giza in updated graph
 - we have a negative weighted edge from Giza to Memphis
- issue is trying to update cost for Memphis node
- a clear sign that something is not right with use of algorithm
- already processed Memphis node
- i.e. there should not now be a cheaper route to that node
- due to negative weighted edge
 - we've actually found a cheaper route
- if we check the cost up to the node Saggara
- algorithm will return already calculated cost of 40...

graphs - Dijkstra's algorithm - Dijkstra and negative weights - part 5

- due to negative weighted edge
- we know there is a cheaper route
- but Dijkstra's algorithm did not find this route
- algorithm makes an assumption about processing of nodes
- due to initial costs of weighted edge...
- i.e. as we were processing the Memphis node
- Dijkstra's algorithm assumes there is now no faster way to that node
- this assumption only holds true
- e.g. if we do not have negative weighted edges
- n.b. we can't use negative weighted edges with Dijkstra's algorithm
- to calculate shortest path in a graph with negative weighted edges
 - instead, use Bellman-Ford algorithm...

Video - Algorithms and Data Structures

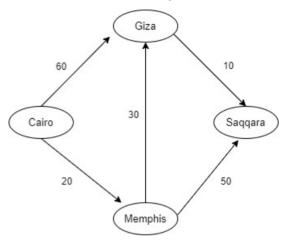
graphs - Bellman-Ford algorithm



Graphs - Bellman-Ford algorithm example - UP TO 4:51

Source - Bellman-Ford algorithm - simple example - YouTube

- now consider a basic coded example of implementing Dijkstra's algorithm in Python
- for this example, we'll start with following graph



Graph - Coded Example

- to help implement a working example for this graph
- define three hash tables for use with this implemented working example
- 1. graph
 - parent node
 - neighbour node
 - cost of weighted edge
- 2. costs
- node
- current cost from start point
- 3. parents
- node
- current parent node

graphs - Dijkstra's algorithm - working implementation - part 3

graph

parent	node	cost
cairo	giza	60
	memphis	20
giza	saqqara	10
memphis	giza	10
	saqqara	50
saqqara		

graphs - Dijkstra's algorithm - working implementation - part 4

costs

node	current cost	
giza	60	
memphis	20	
saqqara	infinity	

graphs - Dijkstra's algorithm - working implementation - part 5

parents

node	current parent
giza	cairo
memphis	cairo
saqqara	-

- as we execute the algorithm
- update values for costs and parents tables...

graphs - Dijkstra's algorithm - working implementation - part 6 implement the graph

- need to implement graph for this coded example
 - we'll use a hash table for graph

 $graph = \{\}$

- in this hash table
- need to store multiple values for neighbours
- then set cost for travel along that edge
- e.g. for current graph
 - we can see that Cairo has two neighbours, Giza and Memphis

graphs - Dijkstra's algorithm - working implementation - part 7

- a number of options we might consider for structuring this pattern of data
 - including nested hash tables for each node relative to the parent
- e.g.

```
graph["cairo"] = {}
graph["cairo"]["giza"] = 60
graph["cairo"]["memphis"] = 20
```

creates following structure for our data

```
{'cairo': {'giza': 60, 'memphis': 20}}
```

 corresponds to structure and values defined in above table for graph

graphs - Dijkstra's algorithm - working implementation - part 8

we might, of course, check its values as follows

```
print(graph["cairo"].keys())
```

- if we need to find weights for edges from Cairo
- we may call the following

```
print(graph["cairo"]["giza"])
print(graph["cairo"]["memphis"])
```

- following this pattern
- add remaining nodes and neighbours to hash table for graph

```
# update other graph nodes and weights
graph["giza"] = {}
graph["giza"]["saqqara"] = 10
graph["memphis"] = {}
graph["memphis"]["giza"] = 30
graph["memphis"]["saqqara"] = 50
# no current neighbour nodes for saqqara - graph end point
graph["saqqara"] = {}
```

- hash table now represents graph with defined neighbour nodes and weighted edges
- e.g.

```
{
  'cairo': {'giza': 60, 'memphis': 20},
  'giza': {'saqqara': 10},
  'memphis': {'giza': 30, 'saqqara': 50},
  'saqqara': {}
}
```

- next structure we need to create is a hash table for costs of each node
- i.e. using cost to define value of weighted edge from one node to another
- cost of node will represent calculated total
 - i.e. for weighted edges from start, Cairo, to a given node
- e.g. we know it will cost 60 to get from Cairo to Giza
- and 20 to get from Cairo to Memphis...

graphs - Dijkstra's algorithm - working implementation - part 11

- we can represent currently unknown costs as *infinity*
- we may represent this hash table as follows,

```
# cost table - weighted edges from start node
infinity = float("inf")
cost = {}
cost["giza"] = 60
cost["memphis"] = 20
cost["saqqara"] = infinity
```

this may be represented as follows

```
{'giza': 60, 'memphis': 20, 'saqqara': inf}
```

graphs - Dijkstra's algorithm - working implementation - part 12

then, we may add our third table for parent nodes in graph

```
# parents table - parent nodes in graph
parents = {}
# define inital parents
parents["giza"] = "cairo"
parents["memphis"] = "cairo"
# parent for end point - updated during execution...
parents["saqqara"] = None
```

- update such values as we work through algorithm and its execution
- also need to maintain a record of nodes already processed in graph
- i.e. to avoid duplicated effort...

```
nodes_checked = []
```

- need to implement the following pattern for the algorithm
- while nodes exist to continue processing
- o get the node closest to the start node
- update any costs for the node's neighbours
- o if any costs for the neighbours have been updated
- · update the parents
- mark node as processed
- repeat this process as necessary...
- we may implement this pattern in Python to add Dijkstra's algorithm to an app

- initially define while loop
- and checks we need to perform for each iteration of the loop

```
# execute check for lowest cost node that has not been processed...
node = find low cost node(cost)
# loop through nodes to check - exit when all nodes are processed
while node is not None:
    node cost = cost[node]
    # add neighbour nodes to hash table
    neighbours = graph[node]
    # loop through all neighbours of current node
    for neighbour in neighbours.keys():
        # update cost where available
        new node cost = node cost + neighbours[neighbour]
        # check updated cost to see if it's now cheaper
        if cost[neighbour] > new_node_cost:
            # update cost for this node
            cost[neighbour] = new_node_cost
            # current node becomes new parent for this neighbour
            parents[neighbour] = node
    # mark node as now processed...
    nodes_checked.append(node)
    # find next node to process - then loop through again...
    node = find_low_cost_node(cost)
```

- start by checking passed cost table
- check for lowest cost node in defined graph...

graphs - Dijkstra's algorithm - working implementation - part 15

custom function find_low_cost_node() may be implemented as follows

- simple implementation to check for current lowest common node
- use this custom function to get lowest common node
- we may then use with the while loop...

graphs - Dijkstra's algorithm - working implementation - part 16

- loop itself may be considered as follows
- helps us further understand how implemented algorithm will work with a sample graph

code breakdown

- e.g. begin by checking for node with lowest cost
- i.e. from start point in graph, Cairo

```
# execute check for Lowest cost node that has not been processed...
node = find_low_cost_node(cost)
```

- in the hash table
- this check will return Memphis with a cost of 20
- we can now get cost for this node
 - and its neighbour nodes as well...

graphs - Dijkstra's algorithm - working implementation - part 17

then add these neighbour nodes to their own hash table

```
neighbours = graph[node]
```

use this structure to loop through stored neighbours

```
# loop through all neighbours of current node
for neighbour in neighbours.keys():
```

- each of these neighbour nodes will have their own cost
- detail cost from start node, Cairo, to that node

- in effect, we're calculating cost of node from start node
- i.e. if we went through the current node
- e.g Cairo -> Memphis -> Giza with an updated cost of 50
- updated cost is lower than current cost
- for a route from start Cairo to Giza
- cost was previously 60
- we can update the cost as follows

```
# update cost where available
new_node_cost = node_cost + neighbours[neighbour]
```

- this is calculated as
- cost of Memphis, 20, plus cost from Memphis to Giza, 30
- now have an updated lowest cost of 50 for a path from Cairo to Giza

graphs - Dijkstra's algorithm - working implementation - part 19

new cost is now updated in cost hash table as well

```
# update cost for this node
cost[neighbour] = new_node_cost
```

may also update parent node for Giza in parents hash table

```
# current node becomes new parent for this neighbour
parents[neighbour] = node
```

- now back at start of while loop
 - we may now move on to next neighbour
- Saggara for current graph
- we repeat above pattern
- checking and updating hash tables for cost of path to current node Saqqara
- the finish node in the current graph...

graphs - Dijkstra's algorithm - working implementation - part 20

- if we execute this algorithm with the current graph
- we get the following initial output

```
initial costs
{'giza': 60, 'memphis': 20, 'saqqara': inf}
```

updated as follows after we run Dijkstra's algorithm

```
updated lowest cost from start to each node:
{'giza': 50, 'memphis': 20, 'saqqara': 60}
```

- once we've processed each node in graph
- algorithm is complete
- we have an output for lowest cost from start node Cairo to finish node Saqqara.

Video - Algorithms and Data Structures

graphs - Dijkstra's algorithm - part 4



Graphs - Dijkstra's algorithm - improve usage - UP TO END

Source - Dijkstra's algorithm - YouTube

Resources

various

- A* search algorithm
- Bellman-Ford algorithm
- Dijkstra's algorithm
- Graph abstract data type

videos

- A* (A star) search algorithm
- Bellman-Ford algorithm simple example
- Dijkstra's algorithm