1 Master equation for a dissipative spin-chain

The project will mainly focus on a particular many-body quantum system, that of an array of optical cavities, with nearest-neighbour coupling. Additionally, these cavities will have strong non-linearities, restricting the local basis to a two-level approximation (see Figure **??**). Great interest lies in the non-equilibrium dynamics of driven-dissipative optical cavities. The reason for this is the rich non-equilibrium phase diagrams, with novel non-equilibrium phase transitions [**?, ?, ?**].

It can be shown (see [...]) that coupled non-linear optical cavities can replicate spin-chain physics. This comes about when the cavities have stong Kerr nonlinearities. Consider an array of N coupled non-linear optical cavities, decribed by the Hamiltonian

$$H_{cc} = \sum_i \omega_0 a_i^\dagger a_i - J \sum_i (a_i^\dagger a_{i+1} + a_i a_{i+1}^\dagger) + \frac{U}{2} \sum_i a_i^\dagger a_i^\dagger a_i a_i. \qquad (1)$$

Here, $a_i^\dagger$ and a_i create or destroy a photon in the ith cavity. The sum runs over all N cavity labels, i. The bare cavity frequency is ω_c and the non-linearity is decribed by an interaction strength U. Finally, neighbouring cavities may interact with one another, with a hopping strength J. The spin physics is emulated in the regime where the energy associated with the non-linearity is far above all other energy scales, namely $\omega_c, t \ll U$. This restricts the population of photons on each site to either 1 or 0. Correspondingly, the physics is effectively decribed by a chain of spins, with an effective Hamiltonian of the form

$$H_{eff} = \sum_i \omega_0 \sigma_i^z - J \sum_i (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^+ \sigma_{i+1}^-). \qquad (2)$$

In this Hamiltonian, we are now describing the occupation of a site in a spin language, with an up-spin corresponding to the presence of a photonic excitation and a down-spin corresponding to no excitation. σ_i^z is the Pauli operator acting in the z quadrature, and $\sigma_i^\pm$ is a spin raising(lowering) operator, written in terms of the x- and y-spin Pauli operators as $\sigma_i^\pm = \sigma_i^x \pm i\sigma_i^y$. Because this system has site-to-site coupling, the resulting eigenstates are inherently non-local.

The cavities also dissipate into their environment, and are driven such that the photon population is above zero. The exact nature of this dissipation depends on the physical processes, and there are many frameworks to treat dissipation. This report focuses on the application of a quantum master equation to describe the dissipation. A master equation is the governing equation of motion for the coupled-cavity quantum system, in the presence of environmental dissipation and driving. Typically, a phenominologically motivated master equation is written down, without considering a detailed model for the system-bath interaction. For example, it may be the case that each of the optical cavities is imperfect, and that they each leak light into their environment.

A rate of loss, Γ can be introduced, which characterises the dissipation from each site, independently of one another. This is justified in certain optical regimes [?], and has been shown to display photon crystallisation even in the presence of the losses and a coherent drive.

The non-equilibrium physics that emerges maintains long-range order in its steady states, despite the incoherent loss from each cavity. The question then is: *what results by considering all normal-modes in the dissipation, not simply site-independent dissipation?* This project investigates the result of correctly treating dissipation by expanding in a basis of the spins Hamiltonian, in the regime of a zero-temperature environment. We consider a special case of a system-environment coupling, in a rotating wave approximation, which preserves total excitation number in the composite system-environment quantum system.

In this section, we introduce a general model for a chain of spins, each coupled to their own environment. By considering a special form of the system-environment interaction, we derive a master equation which takes account of the full dissipative dynamics. This is acheived by not simply introducing independent decay terms, but by starting with a composite system-environment quantum system and calculating the master equation for the reduced system density matrix.

Model

The most general class of models one may write down to describe a chain of interacting spins is the fully anisotropic Heisenberg (or XYZ-) model. This is represented by the Hamiltonian:

$$
H_S = -J \sum_{i=1}^{N} \left(\Delta_x \sigma_i^x \sigma_{i+1}^x + \Delta_y \sigma_i^y \sigma_{i+1}^y + \Delta_z \sigma_i^z \sigma_{i+1}^z + \frac{g}{J} \sigma_i^z \right), \qquad (3)
$$

where σ_i^x, σ_i^y and σ_i^z are Pauli operators describing the spin degree of freedom at site i. The dimensionless parameters Δ_x, Δ_y and Δ_z describe the anisotropic, nearest-neighbour spin-spin interaction strength. Finally, an external field which tends to align the spins in the z-spin direction has be added with an associated energy scale of g. Finally, t is the hopping parameter, describing the strength of the internal system interactions.

Periodic boundary conditions (where we set $\sigma_1^\gamma = \sigma_{N+1}^\gamma$, for all $\gamma = x, y, z$) may or may not be adopted. In this report the choice will be stated before any relevant calculation.

The *closed* system dynamics of the system S are captured by the unitary evolution generated by the Hamiltonian. As such, states of system S, $|\psi\rangle_S$ obey the Schrödinger equation $i\partial_t |\psi\rangle_S = H_S |\psi\rangle_S$ (where $\hbar = 1$). However, this form of the dynamics is not the main concern of this report. The further influence on the system due to an array of environments, E is considered.
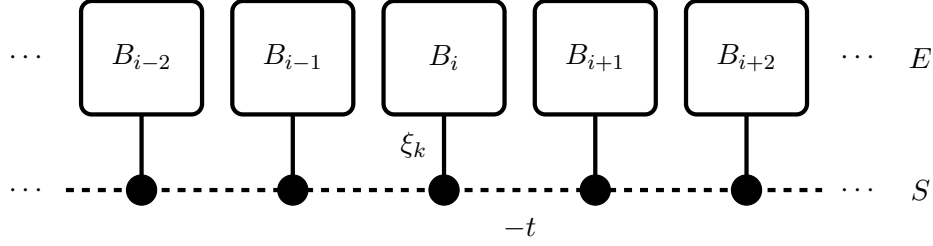
Figure 1: A schematic of the dissipative spin chain considered in this project. A chain of two-level systems (spin-$\frac{1}{2}$ particles) are coupled to their nearest-neighbours and to a local environment, with coupling strength parameters ξ_k. The environments are modelled as bosonic harmonic oscillators.

Suppose that each lattice site in the chain of spins is coupled to a local thermal reservoir, denoted B_i (see figure 1). Each reservoir is modelled as an harmonic oscillator, so that the free evolution of the environmental degrees of freedom is captured by the Hamiltonian (the terms bath, reservoir and environment are used interchangeably):

$$H_B = \sum_{i=1}^{N} \sum_k \omega_k b_{i,k}^\dagger b_{i,k}. \tag{4}$$

The $b_{i,k}$ are the bosonic annihilation operators for the ith site, k^{th} mode and ω_k is the energy of a particle in the kth mode. The baths are assumed to be identical (in that they each have the same ω_k) but are otherwise isolated from one another. This means that the canonical commutation relations for the bath operators are $[b_{i,k}, b_{j,q}] = [b_{i,k}^\dagger, b_{j,q}^\dagger] = 0$ and $[b_{i,k}, b_{j,q}^\dagger] = \delta_{i,j}\delta_{k,q}$. Within this framework, with a system of interest coupled to a large environment, the system of spins becomes an *open quantum system* (see Figure 2). In the theory of open quantum systems [**?**] one wants to determine the laws of motion for the reduced system dynamics. This involves tracing over the environments to determine $\dot\rho_S(t)$, the rate of change of the partially traced density matrix $\rho_S = \text{Tr}_B\{\rho_S\}$.

System-bath interaction

We are now in position to put the all the elements of a dissipative array of optical cavities. All that is left is to describe the specific interaction between the system and environment. When the array dissipates into an environment, there needs to be a drive to keep the population non-zero. We follow [**?, ?, ?**] and consider a coherent drive, which creates pairs of photons in neighbouring cavities. The driven Hamiltonian has a time dependence, and in the spin

language is given by

$$H = \sum_i \omega_0 \sigma_i^z - J \sum_i (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^+ \sigma_{i+1}^-)$$
$$- \Omega \sum_i \left(\sigma_i^+ \sigma_{i+1}^+ e^{-2i\omega_P t} + \sigma_i^- \sigma_{i+1}^- e^{2i\omega_P t} \right). \tag{5}$$

The additional non-number-conserving terms $\sigma_i^+ \sigma_{i+1}^+ e^{-2i\omega_P t}$ and $\sigma_i^- \sigma_{i+1}^- e^{2i\omega_P t}$ represent the external coherent drive, which has strength Ω. Finally, we model dissipation by introducing an interaction with a set of environments. This interaction is assumed to be of the form

$$H_I = \sum_i \sum_k \xi_k \sigma_i^x \otimes (b_{i,k} + b_{i,k}^\dagger). \tag{6}$$

It is convenient to move into a rotating frame with repect to the driving frequency. When we do this the system operator in the coupling, σ_i^x gains a time-dependence and the interaction can we written

$$H_I = \sum_i \sum_k \xi_k \left(\sigma_i^+ b_{i,k} + \sigma_i^- b_{i,k}^\dagger + \sigma_i^+ b_{i,k}^\dagger e^{-2i\omega_P t} + \sigma_i^- b_{i,k} e^{2i\omega_P t} \right). \tag{7}$$

The common assumption, and the assumption we make in this work is that the pump frequency is well above the bare cavity frequency ω_0, and the hopping strength J. This means that the fast oscillating terms $\sigma_i^+ b_{i,k}^\dagger e^{-2i\omega_P t}$ and $\sigma_i^- b_{i,k} e^{2i\omega_P t}$ can be neglected due to averaging. The final interaction in the rotating frame has the simple form

$$H_I = \sum_i \sum_k \xi_k (\sigma_{+i} b_{i,k} + \sigma_i^- b_{i,k}^\dagger). \tag{8}$$

It is straight-forward to demonstrate that in the rotating frame the Hamiltonian above becomes

$$H = \sum_i (\omega_0 - \omega_P) \sigma_i^z - J \sum_i (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^+ \sigma_{i+1}^-), \tag{9}$$

with the environments' Hamitlonian being similarly transformed:

$$H_B = \sum_{i=1}^N \sum_k (\omega_k - \omega_P) b_{i,k}^\dagger b_{i,k}. \tag{10}$$

The total system-environment Hamiltonian takes the final form

$$H = H_S + H_B + H_I$$
$$= \sum_i (\omega_0 - \omega_P) \sigma_i^z - J \sum_{i=1}^N \left(\sigma_i^+ \sigma_{i+1}^- + \sigma_i^+ \sigma_{i+1}^- \right)$$
$$+ \sum_{i=1}^N \sum_k (\omega_k - \omega_P) b_{i,k}^\dagger b_{i,k} + \sum_i \sum_k \xi_k (\sigma_{+i} b_{i,k} + \sigma_{-i} b_{i,k}^\dagger). \tag{11}$$
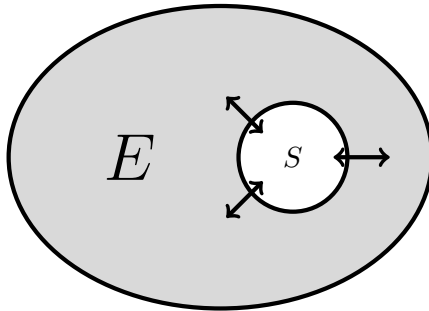
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Figure 2: A representation of an open quantum system. A system of interest S is influenced by its surrounding environment, E. We assume that this influence can be represented by an interaction term in the Hamiltonian, H_I.

This model now contains all the elements to develop an open quantum theory. All that remains is that we calculate the dynamical equation of motion for the system of cavities.

Equation of motion for S

In the Schrödinger picture, the equation of motion for the density matrix of the compsite system $S + E$ is given by the equation:

$$\dot{\rho} = -i[H, \rho]. \tag{12}$$

In an open quantum system, one traces over the environment, and is left with an equation of motion for the reduced system density matrix ρ_S of the form

$$\dot{\rho}_S = -i[H_S, \rho_S] + \mathcal{D}\rho_S. \tag{13}$$

The term $\mathcal{D}\rho_S$ is known as a dissipator of the dynamics and decribes a generalised relaxation process. We are intersted in deriving a form for $\mathcal{D}\rho_S$ from *first principles*. Often, what is done to arrive at an equation like the one above is that a form for the dissipator is chosen based of phenomenological reasoning. In particular, often a dissipator in Lindblad form

$$\mathcal{D}[a]\rho_S = a\rho_S a^\dagger - \frac{1}{2}\{a^\dagger a, \rho_S\} \tag{14}$$

is chosen. The operator a is a system operator, and is refered to as a *jump* operator. A dissipator in Lindblad form generates a form of dynamics which is trace preserving, and crucially, completely positive.

To derive the form of $\mathcal{D}$ from scratch we take the standard approach of deriving a Born-Markov master equation in a weak system-environment coupling regime. Let us first move into an interaction picture, so that kets have a time

dependence generated by the 'free' Hamiltonian $H_S + H_E$, with operator's time dependence being generated by the interaction Hamiltonian $H_I(t)$ (see [?] for a complete description of the interaction picture). In the interaction picture, the equation of motion for the density matrix of $S + E$ is the von Neumann equation:

$$\dot{\rho}(t) = -i[H_I(t), \rho(t)]. \tag{15}$$

The trace over the environment then may be taken, to obtain an equation of motion for just the system of interest, S:

$$\dot{\rho}_S(t) = -i\,\mathrm{Tr}_B[H_I(t), \rho(t)]. \tag{16}$$

It is in general exceedingly difficult to work with the von Neumann equation directly, as the step of tracing the environment has a dependence on the full composite system density matrix, $\rho(t)$. It is thus necessary to look for approximations to the RHS of Equation (16).

One common set of approximations is the combination of the Born approximation and the Markov approximation which, when combined, convert the RHS of Equation (16) into a Born-Markov master equation. Firstly, we formally integrate Equation (15). This then gives us a closed form for the density matrix $\rho(t) = -i\int^t ds[H_I(s), \rho(s)]$, which then may be substitured back into the von Neumann equation, resulting in

$$\dot{\rho}(t) = -\int^t ds[H_I(t), [H_I(s), \rho(s)]]. \tag{17}$$

This process can of course be repeated *ad infinitum*, by formally integrating and substituting, resulting in higher and higher orders of perturbation theory. As was mentioned above, we are going to work in the weak coupling regime, and so stop this process after one formal integration and subtitution.

The Born approximation says that the system and environment are sufficiently weakly coupled such that the total density matrix may be written as a product $\rho(t) = \rho_S(t) \otimes \tilde{\rho}_B$. The environment is assumed to be sufficiently large such that the weak interaction with system does not perturb its state. We denote the time-constant environmental density matrix $\tilde{\rho}_B$. This can take many forms, for example a thermal state

$$\tilde{\rho}_B = \prod_{i,k} \frac{e^{-\beta\omega_k b_{i,k}^\dagger b_{i,k}}}{\mathcal{Z}}, \tag{18}$$

where $\beta = 1/k_B T$ is the inverse temperature and $\mathcal{Z} = \sum_{i,k} e^{-\beta\omega_k}$ is the partition function. Under asssumption the master equation (17) takes the form, after tracing over the baths

$$\dot{\rho}_S(t) = -\int^t ds\,\mathrm{Tr}_B[H_I(t), [H_I(s), \rho_S(s) \otimes \tilde{\rho}_B]]. \tag{19}$$

Now the Markov approximation is made, assuming there is no 'history dependence' of the system on its previous states (time locality). The following modifications to the above master equation are made. First, we remove the history dependence by replacing $\rho_S(s) \to \rho_S(t)$. Next the limit on the integral is taken to infinity, while the replacement $H_I(s) \to H_I(t-s)$, so that the integral has no dependence on the initial time. Without loss of generality, we let $t = 0$ be the time at which the system is set up. So the final Born-Markov master equation

$$\dot{\rho}_S(t) = -\int_0^\infty ds \, \mathrm{Tr}_B[H_I(t), [H_I(t-s), \rho_S(t) \otimes \tilde{\rho}_B]]. \tag{20}$$

This equation thus determines the dynamics of the system S, whilst taking account of the irreversible losses to an environment. Keeping track of all the environmental degrees of freedom is no longer required, an assumption about the state of the environment is all the remains. This would then allow one to evaluate the trace Tr_B and time integral, as we will now demonstrate.

Dissipation from first principles

One particularly useful and general approach to the evaluation of the RHS of the Born-Markov master equation is to introduce a set of eigenoperators of the system Hamiltonian H_S. (This method is outlined in the text book *The Theory of Open Quantum Systems* by Breuer and Petruccione.)

The aim in this project is to calculate the full dissipation the interaction picture we have that

$$H_I(t) = e^{i(H_S+H_E)t} H_I(0) e^{-i(H_S+H_E)t}. \tag{21}$$

Given a general H_I, this is not easy to calculate explicitly the time dependence without the help of the system eigenoperators, which we now introduce.

Suppose we fully understand the system and environment seperately, that is, we know how to calculate the system and environment eigenspectrums (this may be always achieved in principle by numerical diagonalisation) then we may express a resolution of identity in the following manner (the system alone for simplicity):

$$\mathcal{I}_S = \sum_{\substack{\forall \epsilon \text{ s.t.} \\ H_S|\epsilon\rangle = \epsilon|\epsilon\rangle}} |\epsilon\rangle\langle\epsilon|. \tag{22}$$

The equivalent expression for the baths is simply the sum of outer products of Fock basis states (since we model the environment as a system of oscillators). With the assumption that the interaction is seperable: $H_I = \sum_i A_i \otimes B_i$, this then allows direct calculation of the interaction Hamiltonian in the interaction

picture:

$$H_I(t) = \sum_i \sum_{\forall \epsilon, \forall \upsilon} e^{iH_S t} |\epsilon\rangle \langle\epsilon| A_i |\upsilon\rangle \langle\upsilon| e^{-iH_S t} \otimes B_i(t)$$

$$= \sum_i \sum_{\forall \epsilon, \forall \upsilon} e^{i(\epsilon-\upsilon)t} \langle\epsilon| A_i |\upsilon\rangle |\epsilon\rangle \langle\upsilon| \otimes B_i(t)$$

$$\equiv \sum_i \sum_{\forall \epsilon, \forall \upsilon} e^{i(\epsilon-\upsilon)t} A^i_{\epsilon,\upsilon} \otimes B_i(t), \tag{23}$$

where $A^i_{\epsilon,\upsilon} = \langle\epsilon| A_i |\upsilon\rangle |\epsilon\rangle \langle\upsilon|$. Sometimes it is convenient to rewrite this as a single sum over possible energy-*differences*, as opposed to an explicit double sum over energy eigenvalues. To this end, we introduce the energy eigenoperators of S:

$$A^i_\omega = \sum_{\forall \epsilon, \forall \upsilon} \delta_{\omega, \epsilon-\upsilon} A^i_{\epsilon,\upsilon}. \tag{24}$$

With this notation we may then write a single sum to denote sum over all frequency differences, ω which gives:

$$H_I(t) = \sum_i \sum_\omega e^{i\omega t} A^i_\omega \otimes B_i(t). \tag{25}$$

We have thus expressed the interaction picture Hamiltonian in terms of operators we may readily calculate with. In this technique it is generally assumed that the environments are vast and can be represented by harmonic oscillators. All expectations and correlations of operators acting on such a bath written in terms of the bath's creation and annihilation operators may be easily calculated. So the complexity of such expressions as (25) is due the unknown form of the operators A^i_ω. However, calculation of these operators is necessary if we want to assure that the full dynamics due to the system-environment coupling is accounted for.

Non-local dissipation to a zero-temperature environment

In this section a master equation with all non-local contributions included is derived. To evaluate the RHS of the Born-Markov master equation (20), the time dependence of the interaction Hamiltonian $H_I(t)$ in the interaction picture must be determined. One method of choice is to choose a basis of eigenoperators of the system Hamiltonian H_S, and this is the approach utilised in this report. Recall the number-conserving interaction Hamiltonian between each spin-site and its unique environment

$$H_I = \sum_{i=1}^N \sum_k \xi_k \left(\sigma_i^+ b_{i,k} + \sigma_i^- b_{i,k}^\dagger \right). \tag{26}$$

This is can be expressed in terms of the eigenoperators of H_S as explained in the preceding section. This then gives the interaction Hamiltonian:

$$H_I(t) = \sum_{i,k} \sum_{\omega} \xi_k A_{\omega}^i b_{i,k}^{\dagger}(t) e^{i\omega t} + \text{h.c.}, \qquad (27)$$

where $A_{\omega}^i = \sum_{\epsilon,v} \delta_{\omega,\epsilon-v} A_{\epsilon,v}^i$ and $A_{\epsilon,v}^i = \langle \epsilon | \sigma_i^- | v \rangle | \epsilon \rangle \langle v |$. The corresponding quantum master equation is obtained upon evaluation of the right hand side of (20). This involves a double commutator, which, since the interaction Hamiltonian has two terms, will in general yield 16 terms in the master equation. We are going to neglect three quarters of these terms and calculate the expectations with the assumption of empty baths (zero temperature). Expectations with respect to a thermal density matrix, such as (18), are (where $\langle \cdot \rangle = \text{Tr}\{\cdot \, \tilde{\rho}_B\}$):

$$\left\langle b_{i,k} b_{j,q} \right\rangle = \left\langle b_{i,k}^{\dagger} b_{j,q}^{\dagger} \right\rangle = 0,$$
$$\left\langle b_{i,k}^{\dagger} b_{j,q} \right\rangle = \delta_{i,j} \delta_{k,q} n(\omega_k),$$
$$\left\langle b_{i,k} b_{j,q}^{\dagger} \right\rangle = \delta_{i,j} \delta_{k,q} \left(n(\omega_k) + 1 \right). \qquad (28)$$

The following replacement is made for *empty thermal reservoirs*: $n(\omega_k) \to 0$, for all modes k. The simplification is clear: since the only non-zero two-point correlator is $\langle b_{i,k} b_{j,q}^{\dagger} \rangle = \delta_{i,j} \delta_{k,q}$, many terms on the RHS of (20) are zero. With this in mind the calculation of the dissipator will now be undertaken.

Before turning the focus to the integral, the first step is to evaluate the traced commutator $\text{Tr}_B[H_I(t), [H_I(t-s), \rho_S(t) \otimes \tilde{\rho}_B]]$ by inserting the eigenoperator representation of $H_I(t)$ as given in (27).

$$\text{Tr}_B[H_I(t), [H_I(t-s), \rho_S(t) \otimes \tilde{\rho}_B]],$$
$$= \sum_{i,j} \sum_{k,q} \sum_{\omega,\omega'} \text{Tr}_B[\xi_k A_{\omega}^i b_{i,k}^{\dagger}(t) e^{i\omega t} + \text{h.c.}, [\xi_q A_{\omega'}^j b_{j,q}^{\dagger}(t-s) e^{i\omega'(t-s)} + \text{h.c.}, \rho_S(t) \otimes \tilde{\rho}_B]],$$
$$= \sum_{i,j} \sum_{k,q} \sum_{\omega,\omega'} \text{Tr}_B([\xi_k A_{\omega}^i b_{i,k}^{\dagger}(t) e^{i\omega t}, [\xi_q A_{\omega'}^{j,\dagger} b_{j,q}(t-s) e^{-i\omega'(t-s)}, \rho_S(t) \otimes \tilde{\rho}_B]]$$
$$+ [\xi_k A_{\omega}^{i,\dagger} b_{i,k}(t) e^{-i\omega t}, [\xi_q A_{\omega'}^j b_{j,q}^{\dagger}(t-s) e^{i\omega'(t-s)}, \rho_S(t) \otimes \tilde{\rho}_B]]). \qquad (29)$$

The second to third line step in (29) follows by dropping terms which generate traces of the form $\text{Tr}\{b_{i,k}^{\dagger}(t) b_{j,q}^{\dagger}(t-s) \tilde{\rho}_B\}$ and $\text{Tr}\{b_{i,k}(t) b_{j,q}(t-s) \tilde{\rho}_B\}$, which are each exactly zero for bosonic particles.

The master equation can now be expressed as [1]

$$\dot{\rho}_S(t) = -\int_0^\infty ds \sum_{i,j} \sum_{k,q} \sum_{\omega,\omega'} \mathrm{Tr}_B[\xi_k A_\omega^i b_{i,k}^\dagger(t)e^{i\omega t}, [\xi_q A_{\omega'}^{j,\dagger} b_{j,q}(t-s)e^{-i\omega'(t-s)}, \rho_S(t) \otimes \tilde{\rho}_B]]$$

$$+ \mathrm{h.\,c.} \tag{30}$$

We now expand the double commutator as $[A, [B, C]] = ABC - ACB - CBA + BCA$ and separate the system operators from the bath operators:

$$\dot{\rho}_S(t) = -\int_0^\infty ds \sum_{i,j} \sum_{k,q} \sum_{\omega,\omega'} e^{i(\omega-\omega')t} e^{i\omega's} \xi_k \xi_q$$

$$\times \left(A_\omega^i A_{\omega'}^{j,\dagger} \rho(t) \, \mathrm{Tr}_B\{b_{i,k}^\dagger(t)b_{j,q}(t-s)\tilde{\rho}_B\} \right.$$

$$- A_\omega^i \rho(t) A_{\omega'}^{j,\dagger} \, \mathrm{Tr}_B\{b_{i,k}^\dagger(t)\tilde{\rho}_B b_{j,q}(t-s)\}$$

$$- A_{\omega'}^{j,\dagger} \rho(t) A_\omega^i \, \mathrm{Tr}_B\{b_{j,q}(t-s)\tilde{\rho}_B b_{i,k}^\dagger(t)\}$$

$$\left. + \rho(t) A_{\omega'}^{j,\dagger} A_\omega^i \, \mathrm{Tr}_B\{\tilde{\rho}_B b_{j,q}(t-s)b_{i,k}^\dagger(t)\} \right)$$

$$+ \mathrm{h.\,c.,} \tag{31}$$

and we now cyclically permute the traces and use the formulae

$$\mathrm{Tr}_B\{b_{i,k}^\dagger(t)b_{j,q}(t-s)\tilde{\rho}_B\} = \delta_{i,j}\delta_{k,q}n(\omega_k)e^{-i\omega_k s} \to 0, \text{and}$$

$$\mathrm{Tr}_B\{b_{j,q}(t-s)b_{i,k}^\dagger(t)\tilde{\rho}_B\} = \delta_{i,j}\delta_{k,q}(n(\omega_k)+1)e^{-i\omega_k s}$$

$$\to \delta_{i,j}\delta_{k,q}e^{-i\omega_k s}, \tag{32}$$

to express the master equation as

$$\dot{\rho}_S(t) = \sum_{i,j} \sum_{k,q} \xi_k \xi_q \delta_{i,j}\delta_{k,q} \sum_{\omega,\omega'} e^{i(\omega-\omega')t} \int_0^\infty ds\, e^{i(\omega'-\omega_k)s}$$

$$\times \left(A_\omega^i \rho(t) A_{\omega'}^{j,\dagger} - \rho(t) A_{\omega'}^{j,\dagger} A_\omega^i \right) + \mathrm{h.\,c.,}$$

$$= \sum_i \sum_{\omega,\omega'} e^{i(\omega-\omega')t} \sum_k \xi_k^2 \int_0^\infty ds\, e^{i(\omega'-\omega_k)s}$$

$$\times \left(A_\omega^i \rho(t) A_{\omega'}^{i,\dagger} - \rho(t) A_{\omega'}^{i,\dagger} A_\omega^i \right) + \mathrm{h.\,c.,}$$

$$= \sum_i \sum_{\omega,\omega'} e^{i(\omega-\omega')t} \tilde{J}^*(\omega') \left(A_\omega^i \rho(t) A_{\omega'}^{i,\dagger} - \rho(t) A_{\omega'}^{i,\dagger} A_\omega^i \right) + \mathrm{h.\,c.,} \tag{33}$$

where we have denoted $\tilde{J}(\omega')$ as the following integral:

$$\tilde{J}(\omega') = \sum_k \xi_k^2 \int_0^\infty ds\, e^{i(\omega_k - \omega')s}, \tag{34}$$

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[1]Note that the hermitian conjugate can be written since $[A, [A^\dagger, h]]^\dagger = [A^\dagger, [A, h]]$ if $h = h^\dagger$.

and now we will discuss the evaluation of this integral.

The environment has been implemented into our model by considering N independent quantum harmonic oscillators. One way to characterise the interaction between the system and environment is to write a spectral density function, which for this bosonic environment, is given by $\gamma(\omega) = \pi \sum_k \xi_k \delta(\omega - \omega_k)$. It follows from this equation that the sum over k-modes of a function evaluated at the mode frequencies ω_k can be rewritten:

$$\pi \sum_k \xi_k^2 f(\omega_k) = \int d\omega \, \gamma(\omega) f(\omega). \tag{35}$$

This rewriting allows the expression of $\tilde{J}(\omega')$ in terms of the spectral density of each of the identical baths, $\gamma(\omega')$:

$$\begin{aligned}
\tilde{J}(\omega') &= \sum_k \xi_k^2 \int_0^\infty ds \, e^{i(\omega_k - \omega')s}, \\
&= \int \frac{d\omega}{\pi} \gamma(\omega) \int_0^\infty ds \, e^{i(\omega - \omega')s}, \\
&= \int \frac{d\omega}{\pi} \gamma(\omega) \left(\frac{i}{\omega - \omega' + i0} \right), \\
&= \gamma(\omega') + i \int \frac{d\omega}{\pi} \gamma(\omega) \mathcal{P} \frac{1}{\omega - \omega'}.
\end{aligned} \tag{36}$$

In going from line one to two, the property of $\gamma(\omega')$ has been utilised, followed by the standard technique of regularising the integrand and expressing the result in terms of the principle value $\mathcal{P}$. For what following in the rest of the report, we are going to assume that $\gamma(\omega')$ is real and that it has a functional form such that the principal value integral is exactly zero. This allows us to make the simplification $\tilde{J}(\omega') = \tilde{J}^*(\omega') = \gamma(\omega')$.

Putting this all together gives us the master equation in the interaction picture:

$$\begin{aligned}
\dot{\rho}_S(t) = \sum_i \sum_{\omega,\omega'} \gamma(\omega') e^{i(\omega - \omega')t} \\
\times \left(A_\omega^i \rho(t) A_{\omega'}^{i,\dagger} + A_{\omega'}^i \rho(t) A_\omega^{i,\dagger} - A_\omega^{i,\dagger} A_{\omega'}^i \rho(t) - \rho(t) A_{\omega'}^{i,\dagger} A_\omega^i \right).
\end{aligned} \tag{37}$$

To eliminate the explicit time dependence, encoded in the oscillatory term $e^{i(\omega - \omega')t}$, we move back into the Schrödinger picture, which re-introduces the unitary evolution and results in

$$\dot{\rho}(t) = -i[H_S, \rho(t)] + \sum_i \mathcal{D}_i \rho(t), \tag{38}$$

where the dissipator $\mathcal{D}_i\rho(t)$ is given in terms of the eigenoperators of H_S as

$$\mathcal{D}_i\rho(t) = \sum_{\omega,\omega'} J(\omega')\Big(A_\omega^{i,\dagger} A_{\omega'}^i \rho(t) + \rho(t) A_{\omega'}^{i,\dagger} A_\omega^i - A_{\omega'}^i \rho(t) A_\omega^{i,\dagger} - A_\omega^i \rho(t) A_{\omega'}^{i,\dagger} \Big)$$

(39)

Talk about secular.

Talk about the structured baths in various papers and known interesting result. E.g. bath-induced coherence, coherence protection.

Non-Markovianity and the idea for the numerics.

With the derivation of the master equation complete, we now wish to investigate the physics which results.