think of ϵ as the error

"perfect precision" finite precision

$\frac{1}{1} = 1$ $\frac{1}{1 \pm \epsilon} \approx 1$

Condition Number

How much of an impact does ϵ have on the result?

$$(\text{cond } f)(x) = \left| \frac{x f'(x)}{f(x)} \right| \approx \underline{1} \quad \underline{\text{is good}}$$

$$\gg \underline{1} \quad \text{is } \underline{\text{bad}}$$

Condition of a Problem

- Story time
  - Solar System
  - Foxes and Sheep (Fourth order Runge-Kutta)

Example - Page 9

$$y = f(x) = \sqrt{x+d} - \sqrt{x} \quad \text{for } |d| \approx 0$$

$$\rightarrow x+d \approx x$$

naively computed

$$f(x) = \sqrt{x+d} - \sqrt{x}$$

becomes $\approx \sqrt{x} - \sqrt{x}$

$$0$$

Fix the Problem

$$f(x) = \left(\sqrt{x+d} - \sqrt{x}\right) \frac{\left(\sqrt{x+d} + \sqrt{x}\right)}{\sqrt{x+d} + \sqrt{x}}$$

$$= \frac{x+d + \sqrt{x}\sqrt{x+d} - \sqrt{x}\sqrt{x+d} - x}{\sqrt{x+d} + \sqrt{x}}$$

$$= \frac{d}{\sqrt{x+d} + \sqrt{x}} \approx \frac{d}{2\sqrt{x}}$$

Consider

$$e^{-x} = \frac{1}{e^{x}}$$

map

$\underset{\text{input}}{x} \longrightarrow \boxed{f} \longrightarrow \underset{\text{output}}{y}$

$y = f(x) = f \qquad f: \mathbb{R}^m \to \mathbb{R}^n$

<u>Usually</u>

$f: \mathbb{R} \to \mathbb{R}$

e.g., sqrt

$f: \mathbb{R}^2 \to \mathbb{R}$

- distance in 2D
- addition
- multiplication

Condition Number

How sensitive is $f(x)$ to small changes in x?

Consider $f(x)$ vs $f(x+d)$ where $|d| \le eps$

How does $\dfrac{x^* - x}{x}$ compare to $\dfrac{y^* - y}{y}$?

$$(\text{cond } f)(x) = \left| \frac{x f'(x)}{f(x)} \right| \qquad \underline{\text{general form}}$$

$$(\text{cond } f)(x) = \left| \frac{f'(x)}{f(x)} \right| \qquad \text{if } x = 0 \wedge y \neq 0$$

$$(\text{cond } f)(x) = |x f'(x)| \qquad \text{if } x \neq 0 \wedge y = 0$$

Derivation of the Condition Number

$x \rightarrow$ true input
$x^* \rightarrow$ input with small pertubation

$x + \Delta x$ similiar to $x(1+\epsilon_x)$

$y = f(x) = f$

$y^* = f(x^*) = f(x+\Delta x)$

Some Quick Definitions

$\Delta x = x - x^*$ $\qquad$ $x^* = x + \Delta x$

$\Delta y = y - y^*$ $\qquad$ $y^* = y + \Delta y$

$$\Delta y = y^* - y$$
$$= f(x^*) - f(x)$$
$$\approx f'(x)\Delta x$$

Leads to:

$$\frac{\Delta x}{x} = \frac{x^* - x}{x} \qquad \frac{\Delta y}{y} = \frac{y^* - y}{y}$$

relative error?

Compare $\underbrace{\text{rel error }(y, y^*)}_{\frac{\Delta y}{y}}$ to $\underbrace{\text{rel error }(x, x^*)}_{\frac{\Delta x}{x}}$

$$\frac{\Delta y}{y} \quad \text{vs} \quad \frac{\Delta x}{x}$$

as $\Delta x \to 0$ 1 is ideal *

$$\lim_{\Delta x \to 0} \left| \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} \right| = \lim_{\Delta x \to 0} \left| \frac{x \Delta y}{y \Delta x} \right|$$

replace

$\Delta y = f'(x)\Delta x$

$y = f(x)$

$$= \lim_{\Delta x \to 0} \left| \frac{x \cancel{\Delta x} f'(x)}{f(x) \cancel{\Delta x}} \right|$$

$$= \lim_{\Delta x \to 0} \left| \frac{x f'(x)}{f(x)} \right|$$

$$(\text{cond } f)(x) = \left| \frac{x f'(x)}{f(x)} \right|$$

Thinking Exercise

Derive the other 2 forms