Vector Norms

L_1-Norm $\|\vec{x}\|_1 = \sum_{i=1}^{N} |x_i|$

L_2-Norm $\|\vec{x}\|_2 = \left(\sum_{i=1}^{N} |x_i|^2 \right)^{\frac{1}{2}}$

Similiar to $r = \sqrt{x^2 + y^2 + z^2}$

∞-Norm $\|\vec{x}\|_\infty = \max_{1 \le i \le n} |x_i|$

... I prefer $\max \{ |x_1|, |x_2| \ldots |x_n| \}$

Properties of Vector Norms

$\|\vec{x}\| \geq 0$ if $\vec{x} \neq 0$

$\|c\vec{x}\| = |c| \cdot \|\vec{x}\|$ for $c \in \mathbb{R}$

$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

$\|\vec{x} - \vec{y}\| \geq \|\vec{x}\| - \|\vec{y}\|$

Condition Number of a Vector Function

$$(\text{cond } f)(\vec{x}) = \frac{\|\vec{x}\|_\infty \, \|f'(\vec{x})\|_\infty}{\|f(\vec{x})\|_\infty}$$

$f'(\vec{x}) \rightarrow$ Jacobian Matrix

$$\frac{\partial f}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & & \vdots \\ & & \ddots & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Two Well-Known Vector Functions

1- $f(\vec{x}) = \sqrt{x^2+y^2} = \left[\sqrt{x^2+y^2}\right]$

Technically... it is really

$$f(\vec{x}) = \begin{bmatrix} \sqrt{x^2+y^2} \\ \arctan\left(\frac{y}{x}\right) \end{bmatrix}$$

Note $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ in this case

2- $f(\vec{r}) = \begin{bmatrix} |r| \cos\theta \\ |r| \sin\theta \end{bmatrix}$

Note $\vec{r} = \begin{bmatrix} r \\ \theta \end{bmatrix}$ in this case

$$(\operatorname{cond} f)(\vec{x}) = \frac{\|\vec{x}\|_\infty \, \|f'(\vec{x})\|_\infty}{\|f(\vec{x})\|_\infty}$$

Example

$$f(\vec{x}) = \left[\sqrt{x^2+y^2} \right]$$

$$\frac{\partial f}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \frac{2x}{\sqrt{x^2+y^2}} & \frac{1}{2} \frac{2y}{\sqrt{x^2+y^2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}$$

$$\|f\|_\infty = \left\| \sqrt{x^2+y^2} \right\|_\infty$$

$$\|f'\|_\infty = \max\left(\left| \frac{x}{\sqrt{x^2+y^2}} \right|, \left| \frac{y}{\sqrt{x^2+y^2}} \right| \right)$$

$$= \frac{\max(|x|, |y|)}{\sqrt{x^2+y^2}}$$

we know that $\left|\sqrt{x^2+y^2}\right| = \sqrt{x^2+y^2}$ for all real numbers

$$\|\vec{x}\|_\infty = \max(|x|, |y|)$$

$$(\text{cond } f)(\vec{x}) = \left(\frac{\max(|x|,|y|) \cdot \frac{\max(|x|,|y|)}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} \right)$$

$$(\text{cond} f)(\vec{x}) = \frac{\left(\max(|x|,|y|)\right)^2}{x^2+y^2}$$

This is our final "general" solution

Ex: Suppose $x=4$, $y=5$

$$(\text{cond } f)(\vec{x}) = \frac{\left(\max(4,5)\right)^2}{16+25}$$

$$= \frac{25}{41} \approx \frac{25}{40} = \frac{5}{8} < \underline{1}$$