

think of  $\epsilon$  as the error

"perfect precision"    *finite precision*

$$\frac{1}{1} = 1$$

$$\frac{1}{1 \pm \epsilon} \approx 1$$

### Condition Number

How much of an impact does  $\epsilon$  have on the result?

$$(\text{cond } f)(x) = \left| \frac{x f'(x)}{f(x)} \right| \approx \underline{1} \quad \underline{\text{is good}}$$

$\gg \underline{1}$  is bad

## Condition of a Problem

- Story time
  - Solar System
  - Foxes and Sheep (Fourth Order Runge-Kutta)

Example - Page 9

$$y = f(x) = \sqrt{x+d} - \sqrt{x} \quad \text{for } |d| \approx 0$$
$$\rightarrow x+d \approx x$$

naively computed

$$f(x) = \sqrt{x+d} - \sqrt{x}$$

becomes  $\approx \sqrt{x} - \sqrt{x}$

0

Fix the Problem

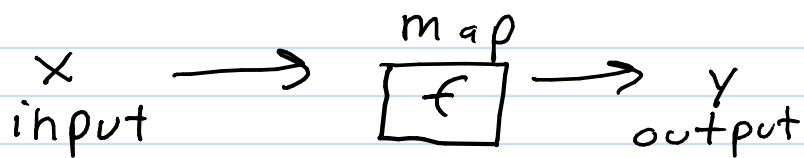
$$f(x) = (\sqrt{x+d} - \sqrt{x}) \frac{(\sqrt{x+d} + \sqrt{x})}{\sqrt{x+d} + \sqrt{x}}$$

$$= \frac{x+d + \sqrt{x}\sqrt{x+d} - \sqrt{x}\sqrt{x+d} - x}{\sqrt{x+d} + \sqrt{x}}$$

$$= \frac{d}{\sqrt{x+d} + \sqrt{x}} \approx \frac{d}{2\sqrt{x}}$$

Consider

$$e^{-x} = \frac{1}{e^x}$$



usually

$f: \mathbb{R} \rightarrow \mathbb{R}$   
 e.g.,  $\text{sqrt}$

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$   
 - distance in 2D  
 - addition  
 - multiplication

$y = f(x) = f$

$f: \mathbb{R}^m \rightarrow \mathbb{R}^n$

Condition Number

How sensitive is  $f(x)$  to small changes in  $x$ ?

Consider  $f(x)$  vs  $f(x+d)$  where  $|d| \leq \epsilon$

How does  $\frac{x^* - x}{x}$  compare to  $\frac{y^* - y}{y}$ ?

$(\text{cond } f)(x) = \left| \frac{x f'(x)}{f(x)} \right|$  general form

$(\text{cond } f)(x) = \left| \frac{f'(x)}{f(x)} \right|$  if  $x=0 \wedge y \neq 0$

$(\text{cond } f)(x) = |x f'(x)|$  if  $x \neq 0 \wedge y=0$

## Derivation of the Condition Number

$x \rightarrow$  true input

$x^* \rightarrow$  input with small perturbation

$x + \Delta x$  similar to  $x(1 + \epsilon_x)$

$$y = f(x) = f$$

$$y^* = f(x^*) = f(x + \Delta x)$$

### Some Quick Definitions

$$\Delta x = x - x^*$$

$$x^* = x + \Delta x$$

$$\Delta y = y - y^*$$

$$y^* = y + \Delta y$$

$$\Delta y = y^* - y$$

$$= f(x^*) - f(x)$$

$$\approx f'(x) \Delta x$$

Leads to:

$$\frac{\Delta x}{x} = \frac{x^* - x}{x}$$

$$\frac{\Delta y}{y} = \frac{y^* - y}{y}$$

relative error?

Compare rel error  $(y, y^*)$  to rel error  $(x, x^*)$   
 $\frac{\Delta y}{y}$  vs  $\frac{\Delta x}{x}$

as  $\Delta x \rightarrow 0$  | is ideal \*

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \left| \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} \right| &= \lim_{\Delta x \rightarrow 0} \left| \frac{x \Delta y}{y \Delta x} \right| \quad \begin{array}{l} \text{replace} \\ \Delta y = f'(x) \Delta x \end{array} \\ &= \lim_{\Delta x \rightarrow 0} \left| \frac{x \cancel{\Delta x} f'(x)}{f(x) \cancel{\Delta x}} \right| \quad \begin{array}{l} y = f(x) \end{array} \\ &= \lim_{\Delta x \rightarrow 0} \left| \frac{x f'(x)}{f(x)} \right| \end{aligned}$$

$$(cond f)(x) = \left| \frac{x f'(x)}{f(x)} \right|$$

Thinking Exercise

Derive the other 2 forms