Maxwell Equations

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Abstract

This document describes the reduction from a 3-Dimensional Finite Difference set of Equations as derived in "Maxwell Equations" to a 1-Dimensional Form ready for use in numerical modeling algorithms.

Reduce to One-Dimension

We will reduce our Finite-Difference Equations to 1D. This means that we will have material that is uniform in two directions. This uniform material will cause the fields to be uniform as well. The change in material in our case we will define to be in the z-drection therefore the x anx y directions will be uniform. This means that the derivates in this direction will also equal zero.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

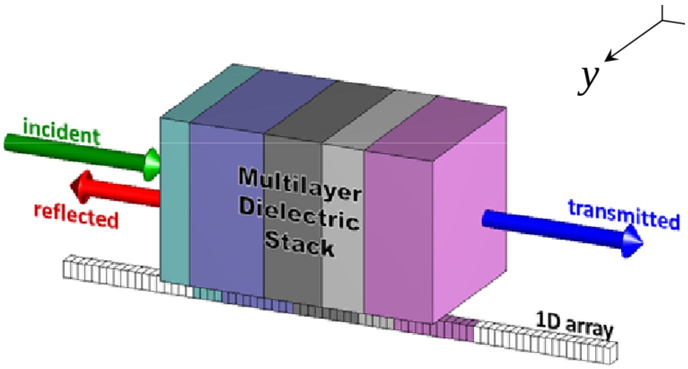

Figure 1: 1D Problem

From our Finite-Difference equations we need to cancel out the x and y derivates that are zero.

$$\cancel{\frac{\partial E_z}{\partial y}} - \frac{\partial E_y}{\partial z} = -\frac{\mu_{xx}}{c_0}\frac{\partial \tilde{H}_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \cancel{\frac{\partial E_z}{\partial x}} = -\frac{\mu_{yy}}{c_0}\frac{\partial \tilde{H}_y}{\partial t}$$

$$\cancel{\frac{\partial E_y}{\partial x}} - \cancel{\frac{\partial E_x}{\partial y}} = -\frac{\mu_{zz}}{c_0}\frac{\partial \tilde{H}_z}{\partial t}$$

$$\cancel{\frac{\partial \tilde{H}_z}{\partial y}} - \frac{\partial \tilde{H}_y}{\partial z} = \frac{\epsilon_{xx}}{c_0}\frac{\partial E_x}{\partial t}$$

$$\frac{\partial \tilde{H}_x}{\partial z} - \cancel{\frac{\partial \tilde{H}_z}{\partial x}} = \frac{\epsilon_{yy}}{c_0}\frac{\partial E_y}{\partial t}$$

$$\cancel{\frac{\partial \tilde{H}_y}{\partial x}} - \cancel{\frac{\partial \tilde{H}_x}{\partial y}} = \frac{\epsilon_{zz}}{c_0}\frac{\partial E_z}{\partial t}$$

$$\longrightarrow$$

$$\cancel{\frac{E_z^{i,j+1,k}\mid_t - E_z^{i,j,k}\mid_t}{\Delta y}} - \frac{E_y^{i,j,k+1}\mid_t - E_y^{i,j,k}\mid_t}{\Delta z} = -\frac{\mu_{xx}^{i,j,k}}{c_0}\frac{\tilde{H}_x^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k}\mid_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\frac{E_x^{i,j,k+1}\mid_t - E_x^{i,j,k}\mid_t}{\Delta z} - \cancel{\frac{E_z^{i+1,j,k}\mid_t - E_z^{i,j,k}\mid_t}{\Delta x}} = -\frac{\mu_{yy}^{i,j,k}}{c_0}\frac{\tilde{H}_y^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k}\mid_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\cancel{\frac{E_y^{i+1,j,k}\mid_t - E_y^{i,j,k}\mid_t}{\Delta x}} - \cancel{\frac{E_x^{i,j+1,k}\mid_t - E_x^{i,j,k}\mid_t}{\Delta y}} = -\frac{\mu_{zz}^{i,j,k}}{c_0}\frac{\tilde{H}_z^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j,k}\mid_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\cancel{\frac{\tilde{H}_z^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j-1,k}\mid_{t+\frac{\Delta t}{2}}}{\Delta y}} - \frac{\tilde{H}_y^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1}\mid_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{xx}^{i,j,k}}{c_0}\frac{E_x^{i,j,k}\mid_{t+\Delta t} - E_x^{i,j,k}\mid_t}{\Delta t}$$

$$\frac{\tilde{H}_x^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1}\mid_{t+\frac{\Delta t}{2}}}{\Delta z} - \cancel{\frac{\tilde{H}_z^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i-1,j,k}\mid_{t+\frac{\Delta t}{2}}}{\Delta x}} = \frac{\epsilon_{yy}^{i,j,k}}{c_0}\frac{E_y^{i,j,k}\mid_{t+\Delta t} - E_y^{i,j,k}\mid_t}{\Delta t}$$

$$\cancel{\frac{\tilde{H}_y^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i-1,j,k}\mid_{t+\frac{\Delta t}{2}}}{\Delta x}} - \cancel{\frac{\tilde{H}_x^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j-1,k}\mid_{t+\frac{\Delta t}{2}}}{\Delta y}} = \frac{\epsilon_{zz}^{i,j,k}}{c_0}\frac{E_z^{i,j,k}\mid_{t+\Delta t} - E_z^{i,j,k}\mid_t}{\Delta t}$$

With these cross outs we come out with two observations. The first is that the longitudinal field components Ez and Hz are always zero. The second is that the Maxwell's equations have decoupled into two sets of two equations Ex/Hy Mode and Ey/Hx Mode.

$$-\frac{\partial E_y}{\partial z} = -\frac{\mu_{xx}}{c_0}\frac{\partial \tilde{H}_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} = -\frac{\mu_{yy}}{c_0}\frac{\partial \tilde{H}_y}{\partial t}$$

$$0 = -\frac{\mu_{zz}}{c_0}\frac{\partial \tilde{H}_z}{\partial t}$$

$$-\frac{\partial \tilde{H}_y}{\partial z} = \frac{\epsilon_{xx}}{c_0}\frac{\partial E_x}{\partial t}$$

$$\frac{\partial \tilde{H}_x}{\partial z} = \frac{\epsilon_{yy}}{c_0}\frac{\partial E_y}{\partial t}$$

$$0 = \frac{\epsilon_{zz}}{c_0}\frac{\partial E_z}{\partial t}$$

$$\longrightarrow$$

$$-\frac{E_y^{i,j,k+1}\mid_t - E_y^{i,j,k}\mid_t}{\Delta z} = -\frac{\mu_{xx}^{i,j,k}}{c_0}\frac{\tilde{H}_x^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k}\mid_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\frac{E_x^{i,j,k+1}\mid_t - E_x^{i,j,k}\mid_t}{\Delta z} = -\frac{\mu_{yy}^{i,j,k}}{c_0}\frac{\tilde{H}_y^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k}\mid_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\tilde{H}_z^{i,j,k} = 0$$

$$-\frac{\tilde{H}_y^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1}\mid_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{xx}^{i,j,k}}{c_0}\frac{E_x^{i,j,k}\mid_{t+\Delta t} - E_x^{i,j,k}\mid_t}{\Delta t}$$

$$\frac{\tilde{H}_x^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1}\mid_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{yy}^{i,j,k}}{c_0}\frac{E_y^{i,j,k}\mid_{t+\Delta t} - E_y^{i,j,k}\mid_t}{\Delta t}$$

$$E_z^{i,j,k} = 0$$

Summary of 1D Finite-Difference Equations

The equations are now broken apart into their respective modes. These modes are physical and propage independently from each other. In the end though they are numerically the same and will exhibit the same electromagnetic behavior. This allows us to only have to solve one set of two equations. We can also note that since we only changing materials along the z axis we can drop the i and j Array Indices.

Ex/Hy Mode

$$-\frac{\tilde{H}_y^{k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k-1}\mid_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{xx}^{k}}{c_0}\frac{E_x^{k}\mid_{t+\Delta t} - E_x^{k}\mid_t}{\Delta t} \tag{1}$$

$$\frac{E_x^{k+1}\mid_t - E_x^{k}\mid_t}{\Delta z} = -\frac{\mu_{yy}^{k}}{c_0}\frac{\tilde{H}_y^{k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k}\mid_{t-\frac{\Delta t}{2}}}{\Delta t} \tag{2}$$

Ey/Hx Mode

$$\frac{\tilde{H}_x^{k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}\mid_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{yy}^{k}}{c_0}\frac{E_y^{k}\mid_{t+\Delta t} - E_y^{k}\mid_t}{\Delta t} \tag{3}$$

$$\frac{E_y^{k+1}\mid_t - E_y^{k}\mid_t}{\Delta z} = \frac{\mu_{xx}^{k}}{c_0}\frac{\tilde{H}_x^{k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k}\mid_{t-\frac{\Delta t}{2}}}{\Delta t} \tag{4}$$

Update Equations

We are now going to derive the update equations used during in FDTD algorithm. We have arbitrarily chosen to use Ey/Hx Mode found in Equations (3) and (4)

Update Equation for Ey

$$\frac{\tilde{H}_x^{k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}\mid_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{yy}^{k}}{c_0}\frac{E_y^{k}\mid_{t+\Delta t} - E_y^{k}\mid_t}{\Delta t}$$

We want to solve for the E-Field at the future time value.

$$\frac{\epsilon_{yy}^k}{c_0}\frac{E_y^k|_{t+\Delta t}-E_y^k|_t}{\Delta t}=\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}}-\tilde{H}_x^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z}$$

$$E_y^k|_{t+\Delta t}-E_y^k|_t=\frac{c_0\Delta t}{\epsilon_{yy}^k}\left(\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}}-\tilde{H}_x^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z}\right)$$

$$E_y^k|_{t+\Delta t}=E_y^k|_t+\left(\frac{c_0\Delta t}{\epsilon_{yy}^k}\right)\left(\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}}-\tilde{H}_x^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z}\right) \tag{5}$$

Update Equation for Hx

$$\frac{E_y^{k+1}|_t-E_y^k|_t}{\Delta z}=\frac{\mu_{xx}^k}{c_0}\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}}-\tilde{H}_x^k|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

We want to sove for the H-Field at the future time value

$$\frac{\mu_{xx}^k}{c_0}\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}}-\tilde{H}_x^k|_{t-\frac{\Delta t}{2}}}{\Delta t}=\frac{E_y^{k+1}|_t-E_y^k|_t}{\Delta z}$$

$$\tilde{H}_x^k|_{t+\frac{\Delta t}{2}}-\tilde{H}_x^k|_{t-\frac{\Delta t}{2}}=\frac{c_0\Delta t}{\mu_{xx}^k}\left(\frac{E_y^{k+1}|_t-E_y^k|_t}{\Delta z}\right)$$

$$\tilde{H}_x^k|_{t+\frac{\Delta t}{2}}=\tilde{H}_x^k|_{t-\frac{\Delta t}{2}}+\left(\frac{c_0\Delta t}{\mu_{xx}^k}\right)\left(\frac{E_y^{k+1}|_t-E_y^k|_t}{\Delta z}\right) \tag{6}$$

Update Coefficients

Since the update coefficients don't change during the simulation. We can compute them once before our FDTD Algorithm is actually implemented in loop.

$$\tilde{H}_x^k|_{t+\frac{\Delta t}{2}}=\tilde{H}_x^k|_{t-\frac{\Delta t}{2}}+m_{H_x}^k\left(\frac{E_y^{k+1}|_t-E_y^k|_t}{\Delta z}\right) \tag{7}$$

$$E_y^k|_{t+\Delta t}=E_y^k|_t+m_{E_y}^k\left(\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}}-\tilde{H}_x^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z}\right) \tag{8}$$

$$m_{E_y}^k=\frac{c_0\Delta t}{\epsilon_{yy}^k} \tag{9}$$

$$m_{H_x}^k=\frac{c_0\Delta t}{\mu_{yy}^k} \tag{10}$$

Boundary Conditions

We have two types of boundary conditions that can be incorporated into this 1-Dimensional Simulation. The Dirichlet and Perfectly Absorbing Boundary Conditions are methods that allow us to handle the fields that lie at the boundary of the model. Specifically these points lie at

$$k = 0$$

and

$$k = N_z + 1$$

. At these points there is no grid to pull our Electric and Magnetic fields from, therefore to maintain the stability of the Finite Difference Maxwell's Equations we must handle these seperatly.

Dirichlet Boundary Condition This boundary condition is the easiest to implement. We simply assume the fields that lie outside the grid are zero. This is the most crude method and will allow for reflection back into our model as energy reaches the boundary.

$$\begin{aligned}
&\text{For } k < N_z \quad \tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k \mid_{t-\frac{\Delta t}{2}} + m_{H_x}^k \left(\frac{E_y^{k+1}|_t - E_y^k|_t}{\Delta z} \right) \\
&\text{For } k = N_z \quad \tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k \mid_{t-\frac{\Delta t}{2}} + m_{H_x}^k \left(\frac{0 - E_y^k|_t}{\Delta z} \right)
\end{aligned}$$

$$\begin{aligned}
&\text{For } k > 1 \quad E_y^k \mid_{t+\Delta t} = E_y^k \mid_t + m_{E_y}^k \left(\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} \right) \\
&\text{For } k = 1 \quad E_y^k \mid_{t+\Delta t} = E_y^k \mid_t + m_{E_y}^k \left(\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} - 0}{\Delta z} \right)
\end{aligned}$$