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EEL 5390 – Special Topics (FDTD)
HW #5

Notes:

P1 – I have broken up my documents into three documents, Maxwell Equations Derivation, 1D Derivation, and Sources. I believe that this will help me most in the future. So the Next major document should be 2D Derivation then 3D. I am still battling image floating formatting, but got I have nice section headers, Yay!!!.

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Maxwell Equations

Christopher Stricklan

Revised: 'February 21, 2011

Abstract

This document describes the derivation of Maxwell's Equations from their Curl and Divergent forms to a final Finite Difference Form. A Yee Grid is used to derive the Finite Difference Equations.

MAXWELL'S EQUATIONS

$$\nabla \cdot \vec{D}(t) = \rho_v(t) \quad (1)$$

$$\nabla \cdot \vec{B}(t) = 0 \quad (2)$$

$$\nabla \times \vec{E}(t) = -\frac{\partial \vec{B}(t)}{\partial t} \quad (3)$$

$$\nabla \times \vec{H}(t) = \vec{J} + \frac{\partial \vec{D}(t)}{\partial t} \quad (4)$$

CONSTITUTIVE RELATIONS

$$\vec{D} = [\epsilon(t)] * \vec{E}(t) \quad (5)$$

$$\vec{B} = [\mu(t)] * \vec{H}(t) \quad (6)$$

ELIMINATE DIVERGENCE

Our use of the Yee Grid Scheme allows us to eliminate divergence via equations (7) and (8)

$$\nabla \cdot (\epsilon \vec{E}) = 0 \quad (7)$$

$$\nabla \cdot (\mu \vec{H}) = 0 \quad (8)$$

This also allows for the centering of the fields invoked by the curl equations Figure 1

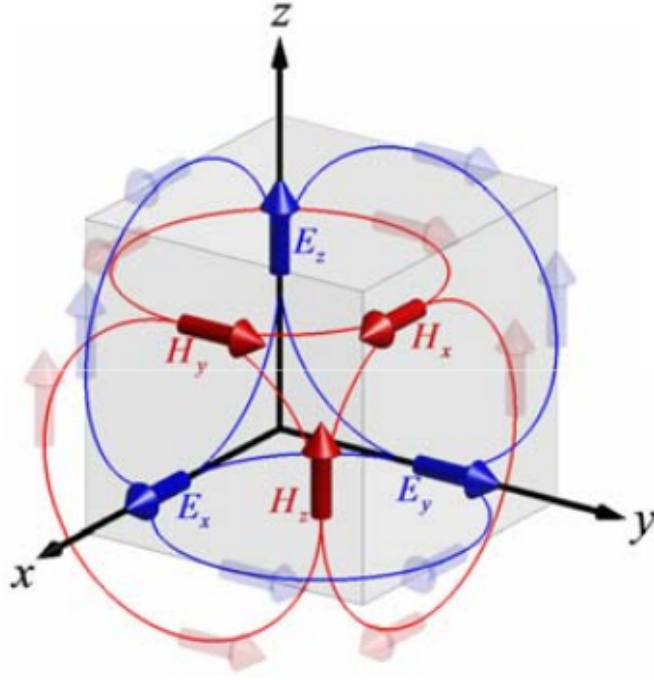


Figure 1: Yee Grid

SUBSTITUTION

Because we have satisfied the divergence equations by using the Yee Grid Scheme, we are now able to focus on the curl equations with our constitutive relation equations (5) and (6) substituted in. We are also not injecting a Source.

Where $\vec{J} = 0$

$$\nabla \times \vec{E}(t) = -[\mu] \frac{\partial \vec{H}(t)}{\partial t} \quad (9)$$

$$\nabla \times \vec{H}(t) = [\epsilon] \frac{\partial \vec{E}(t)}{\partial t} \quad (10)$$

NORMALIZE MAGNETIC FIELD

The Electric and Magnetic Fields are three orders of magnitude different. Rounding errors can propagate through our simulation, therefore we require that we normalize a Field. In this case we are normalizing the Magnetic Field.

$$\begin{aligned} \vec{\tilde{H}} = \sqrt{\frac{\mu_0}{\epsilon_0}} \vec{H} \implies \nabla \times \vec{E} &= -\frac{[\mu_r]}{c_0} \frac{\partial \vec{\tilde{H}}}{\partial t} \\ \nabla \times \vec{\tilde{H}} &= \frac{[\epsilon_r]}{c_0} \frac{\partial \vec{E}}{\partial t} \end{aligned} \quad (11)$$

EXPAND CURL EQUATIONS

$$\begin{aligned}\nabla \times \vec{E} &= -\frac{[\mu_r]}{c_0} \frac{\partial \vec{H}}{\partial t} \Rightarrow \\ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{1}{c_0} \left(\mu_{xx} \frac{\partial \tilde{H}_x}{\partial t} + \mu_{xy} \frac{\partial \tilde{H}_y}{\partial t} + \mu_{xz} \frac{\partial \tilde{H}_z}{\partial t} \right) \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{1}{c_0} \left(\mu_{yx} \frac{\partial \tilde{H}_x}{\partial t} + \mu_{yy} \frac{\partial \tilde{H}_y}{\partial t} + \mu_{yz} \frac{\partial \tilde{H}_z}{\partial t} \right) \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{1}{c_0} \left(\mu_{zx} \frac{\partial \tilde{H}_x}{\partial t} + \mu_{zy} \frac{\partial \tilde{H}_y}{\partial t} + \mu_{zz} \frac{\partial \tilde{H}_z}{\partial t} \right)\end{aligned}\tag{12}$$

$$\begin{aligned}\nabla \times \vec{H} &= \frac{[\epsilon_r]}{c_0} \frac{\partial \vec{E}}{\partial t} \Rightarrow \\ \frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} &= \frac{1}{c_0} \left(\epsilon_{xx} \frac{\partial E_x}{\partial t} + \epsilon_{xy} \frac{\partial E_y}{\partial t} + \epsilon_{xz} \frac{\partial E_z}{\partial t} \right) \\ \frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} &= \frac{1}{c_0} \left(\epsilon_{yx} \frac{\partial E_x}{\partial t} + \epsilon_{yy} \frac{\partial E_y}{\partial t} + \epsilon_{yz} \frac{\partial E_z}{\partial t} \right) \\ \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} &= \frac{1}{c_0} \left(\epsilon_{zx} \frac{\partial E_x}{\partial t} + \epsilon_{zy} \frac{\partial E_y}{\partial t} + \epsilon_{zz} \frac{\partial E_z}{\partial t} \right)\end{aligned}\tag{13}$$

CROSS OUT DIAGONAL TENSORS

$$\begin{aligned}\nabla \times \vec{E} &= -\frac{[\mu_r]}{c_0} \frac{\partial \vec{H}}{\partial t} \Rightarrow \\ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{1}{c_0} \left(\mu_{xx} \frac{\partial \tilde{H}_x}{\partial t} + \cancel{\mu_{xy} \frac{\partial \tilde{H}_y}{\partial t}} + \cancel{\mu_{xz} \frac{\partial \tilde{H}_z}{\partial t}} \right) \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{1}{c_0} \left(\cancel{\mu_{yx} \frac{\partial \tilde{H}_x}{\partial t}} + \mu_{yy} \frac{\partial \tilde{H}_y}{\partial t} + \cancel{\mu_{yz} \frac{\partial \tilde{H}_z}{\partial t}} \right) \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{1}{c_0} \left(\cancel{\mu_{zx} \frac{\partial \tilde{H}_x}{\partial t}} + \cancel{\mu_{zy} \frac{\partial \tilde{H}_y}{\partial t}} + \mu_{zz} \frac{\partial \tilde{H}_z}{\partial t} \right)\end{aligned}\tag{14}$$

$$\nabla \times \vec{H} = \frac{[\epsilon_r]}{c_0} \frac{\partial \vec{E}}{\partial t} \implies \quad (15)$$

$$\begin{aligned} \frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} &= \frac{1}{c_0} \left(\epsilon_{xx} \frac{\partial E_x}{\partial t} + \cancel{\epsilon_{xy} \frac{\partial E_y}{\partial t}} + \cancel{\epsilon_{xz} \frac{\partial E_z}{\partial t}} \right) \\ \frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} &= \frac{1}{c_0} \left(\cancel{\epsilon_{yx} \frac{\partial E_x}{\partial t}} + \epsilon_{yy} \frac{\partial E_y}{\partial t} + \cancel{\epsilon_{yz} \frac{\partial E_z}{\partial t}} \right) \\ \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} &= \frac{1}{c_0} \left(\cancel{\epsilon_{zx} \frac{\partial E_x}{\partial t}} + \cancel{\epsilon_{zy} \frac{\partial E_y}{\partial t}} + \epsilon_{zz} \frac{\partial E_z}{\partial t} \right) \end{aligned}$$

FINAL ANALYTICAL EQUATIONS

These equations are our final Maxwell Equations that will be used to derive the Finite-Difference Equations for all dimensions. We have one equation for each dimension for both the magnetic and electric fields. This gives a total of six equations that will need to be tracked.

Magnetic Field Equations

$$\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = \frac{\epsilon_{xx}}{c_0} \frac{\partial E_x}{\partial t} \quad (16)$$

$$\frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = \frac{\epsilon_{yy}}{c_0} \frac{\partial E_y}{\partial t} \quad (17)$$

$$\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = \frac{\epsilon_{zz}}{c_0} \frac{\partial E_z}{\partial t} \quad (18)$$

Electric Field Equations

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\mu_{xx}}{c_0} \frac{\partial \tilde{H}_x}{\partial t} \quad (19)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\mu_{yy}}{c_0} \frac{\partial \tilde{H}_y}{\partial t} \quad (20)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\mu_{zz}}{c_0} \frac{\partial \tilde{H}_z}{\partial t} \quad (21)$$

FINITE-DIFFERENCE DERIVED EQUATIONS

Each equation needs to be derived based on the Yee Grid Model. Remember that adjacent cells will need to be used to perform the calculation of the individual finite-difference equations for Magnetic and Electric fields in each direction x,y,z

Hx - Figure 2 Equation (22)

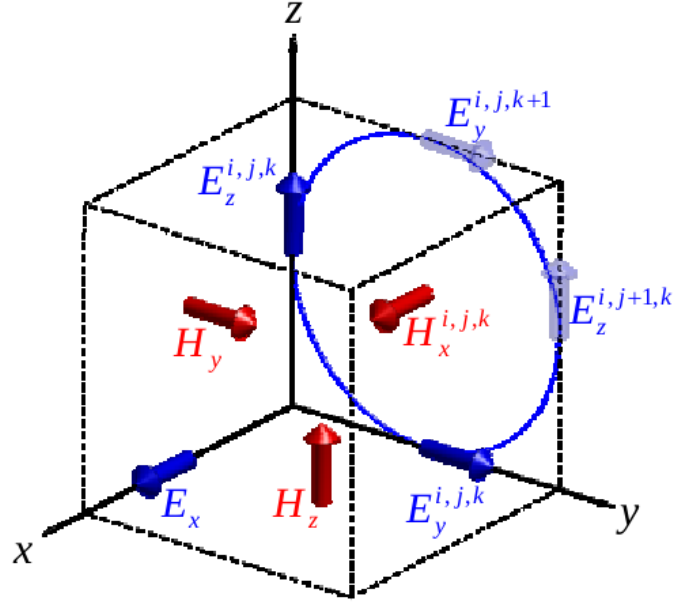


Figure 2: Yee Grid Hx

$$\frac{E_z^{i,j+1,k} |_{t+\frac{\Delta t}{2}} - E_z^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta y} - \frac{E_y^{i,j,k+1} |_{t+\frac{\Delta t}{2}} - E_y^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta z} = -\frac{\mu_{xx}}{c_0} \frac{\partial \tilde{H}_x}{\partial t} \longrightarrow \quad (22)$$

$$\frac{E_z^{i,j+1,k} |_{t+\frac{\Delta t}{2}} - E_z^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta y} - \frac{E_y^{i,j,k+1} |_{t+\frac{\Delta t}{2}} - E_y^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta z} = -\frac{\mu_{xx}^{i,j,k}}{c_0} \frac{\tilde{H}_x^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta t}$$

Hy - Figure 3 Equation (23)

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\mu_{yy}}{c_0} \frac{\partial \tilde{H}_y}{\partial t} \longrightarrow \quad (23)$$

$$\frac{E_x^{i,j,k+1} |_{t+\frac{\Delta t}{2}} - E_x^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta z} - \frac{E_z^{i+1,j,k} |_{t+\frac{\Delta t}{2}} - E_z^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta x} = -\frac{\mu_{yy}^{i,j,k}}{c_0} \frac{\tilde{H}_y^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta t}$$

Hz - Figure 4 Equation (24)

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\mu_{zz}}{c_0} \frac{\partial \tilde{H}_z}{\partial t} \longrightarrow \quad (24)$$

$$\frac{E_y^{i+1,j,k} |_{t+\frac{\Delta t}{2}} - E_y^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta x} - \frac{E_x^{i,j+1,k} |_{t+\frac{\Delta t}{2}} - E_x^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta y} = -\frac{\mu_{zz}^{i,j,k}}{c_0} \frac{\tilde{H}_z^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta t}$$

Ex - Figure 5 Equation (25)

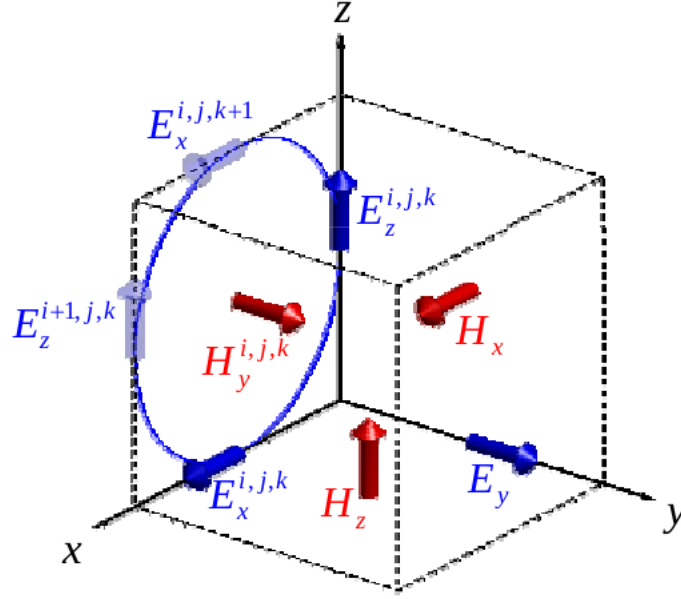


Figure 3: Yee Grid Hy

$$\frac{\tilde{H}_z^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j-1,k} \big|_{t+\frac{\Delta t}{2}}}{\Delta y} - \frac{\tilde{H}_y^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1} \big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = -\frac{\epsilon_{xx}}{c_0} \frac{\partial E_x}{\partial t} \longrightarrow \quad (25)$$

$$\frac{\tilde{H}_z^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j-1,k} \big|_{t+\frac{\Delta t}{2}}}{\Delta y} - \frac{\tilde{H}_y^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1} \big|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{xx}^{i,j,k}}{c_0} \frac{E_x^{i,j,k} \big|_{t+\Delta t} - E_x^{i,j,k} \big|_t}{\Delta t}$$

Ey - Figure 6 Equation (26)

$$\frac{\tilde{H}_x^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1} \big|_{t+\frac{\Delta t}{2}}}{\Delta z} - \frac{\tilde{H}_z^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i-1,j,k} \big|_{t+\frac{\Delta t}{2}}}{\Delta x} = \frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = -\frac{\epsilon_{yy}}{c_0} \frac{\partial E_y}{\partial t} \longrightarrow \quad (26)$$

$$\frac{\tilde{H}_x^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1} \big|_{t+\frac{\Delta t}{2}}}{\Delta z} - \frac{\tilde{H}_z^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i-1,j,k} \big|_{t+\frac{\Delta t}{2}}}{\Delta x} = \frac{\epsilon_{yy}^{i,j,k}}{c_0} \frac{E_y^{i,j,k} \big|_{t+\Delta t} - E_y^{i,j,k} \big|_t}{\Delta t}$$

Ez - Figure 7 Equation (27)

$$\frac{\tilde{H}_y^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i-1,j,k} \big|_{t+\frac{\Delta t}{2}}}{\Delta x} - \frac{\tilde{H}_x^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j-1,k} \big|_{t+\frac{\Delta t}{2}}}{\Delta y} = \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = -\frac{\epsilon_{zz}}{c_0} \frac{\partial E_z}{\partial t} \longrightarrow \quad (27)$$

$$\frac{\tilde{H}_y^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i-1,j,k} \big|_{t+\frac{\Delta t}{2}}}{\Delta x} - \frac{\tilde{H}_x^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j-1,k} \big|_{t+\frac{\Delta t}{2}}}{\Delta y} = \frac{\epsilon_{zz}^{i,j,k}}{c_0} \frac{E_z^{i,j,k} \big|_{t+\Delta t} - E_z^{i,j,k} \big|_t}{\Delta t}$$

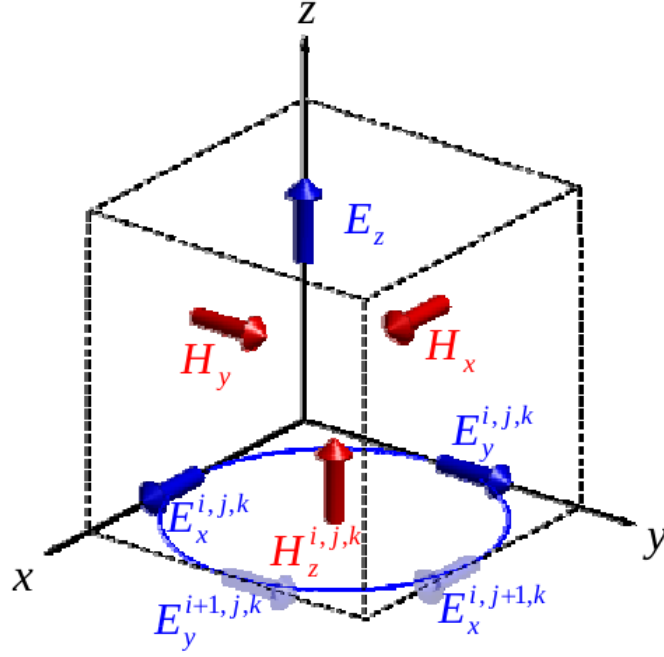


Figure 4: Yee Grid Hz

SUMMARY OF FINITE-DIFFERENCE EQUATIONS

The six sets equations below are our Finite-Difference equations derived from Maxwell's Equations. These equations we will use to derive equations for 1D, 2D, and 3D update equations for our FDTD algorithms.

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\mu_{xx}}{c_0} \frac{\partial \tilde{H}_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\mu_{yy}}{c_0} \frac{\partial \tilde{H}_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\mu_{zz}}{c_0} \frac{\partial \tilde{H}_z}{\partial t}$$

$$\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = -\frac{\epsilon_{xx}}{c_0} \frac{\partial E_x}{\partial t}$$

$$\frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = -\frac{\epsilon_{yy}}{c_0} \frac{\partial E_y}{\partial t}$$

$$\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = -\frac{\epsilon_{zz}}{c_0} \frac{\partial E_z}{\partial t}$$

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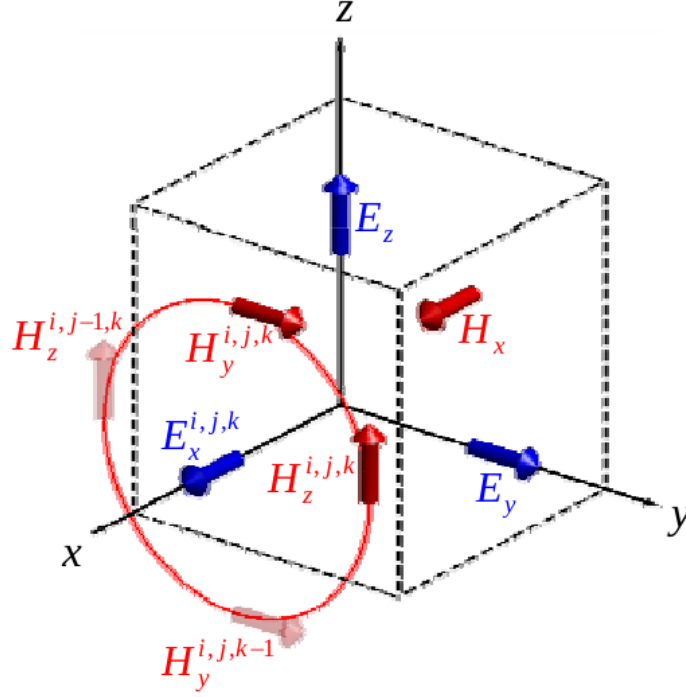


Figure 5: Yee Grid Ex

$$\begin{aligned}
\frac{E_z^{i,j+1,k} \big|_t - E_z^{i,j,k} \big|_t}{\Delta y} - \frac{E_y^{i,j,k+1} \big|_t - E_y^{i,j,k} \big|_t}{\Delta z} &= -\frac{\mu_{xx}^{i,j,k}}{c_0} \frac{\tilde{H}_x^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k} \big|_{t-\frac{\Delta t}{2}}}{\Delta t} \\
\frac{E_x^{i,j,k+1} \big|_t - E_x^{i,j,k} \big|_t}{\Delta z} - \frac{E_z^{i+1,j,k} \big|_t - E_z^{i,j,k} \big|_t}{\Delta x} &= -\frac{\mu_{yy}^{i,j,k}}{c_0} \frac{\tilde{H}_y^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k} \big|_{t-\frac{\Delta t}{2}}}{\Delta t} \\
\frac{E_y^{i+1,j,k} \big|_t - E_y^{i,j,k} \big|_t}{\Delta x} - \frac{E_x^{i,j+1,k} \big|_t - E_x^{i,j,k} \big|_t}{\Delta y} &= -\frac{\mu_{zz}^{i,j,k}}{c_0} \frac{\tilde{H}_z^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j,k} \big|_{t-\frac{\Delta t}{2}}}{\Delta t} \\
\frac{\tilde{H}_z^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j-1,k} \big|_{t+\frac{\Delta t}{2}}}{\Delta y} - \frac{\tilde{H}_y^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1} \big|_{t+\frac{\Delta t}{2}}}{\Delta z} &= \frac{\epsilon_{xx}^{i,j,k}}{c_0} \frac{E_x^{i,j,k} \big|_{t+\Delta t} - E_x^{i,j,k} \big|_t}{\Delta t} \\
\frac{\tilde{H}_x^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1} \big|_{t+\frac{\Delta t}{2}}}{\Delta z} - \frac{\tilde{H}_z^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i-1,j,k} \big|_{t+\frac{\Delta t}{2}}}{\Delta x} &= \frac{\epsilon_{yy}^{i,j,k}}{c_0} \frac{E_y^{i,j,k} \big|_{t+\Delta t} - E_y^{i,j,k} \big|_t}{\Delta t} \\
\frac{\tilde{H}_y^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i-1,j,k} \big|_{t+\frac{\Delta t}{2}}}{\Delta x} - \frac{\tilde{H}_x^{i,j,k} \big|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j-1,k} \big|_{t+\frac{\Delta t}{2}}}{\Delta y} &= \frac{\epsilon_{zz}^{i,j,k}}{c_0} \frac{E_z^{i,j,k} \big|_{t+\Delta t} - E_z^{i,j,k} \big|_t}{\Delta t}
\end{aligned}$$

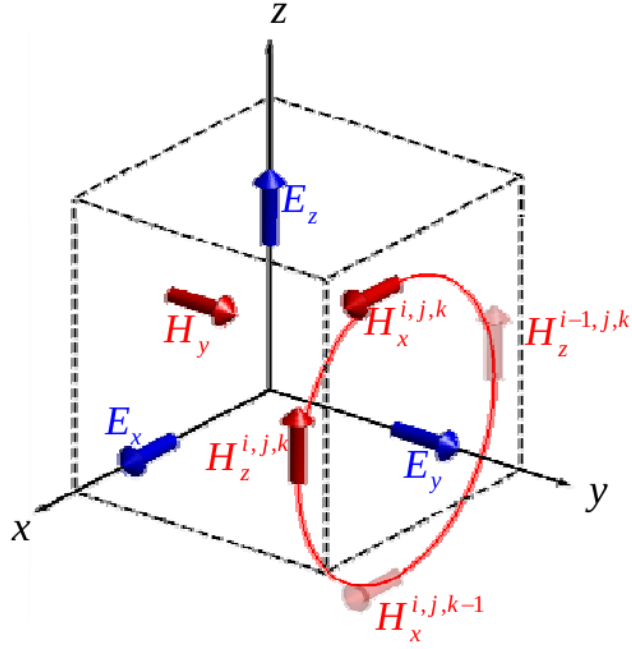


Figure 6: Yee Grid E_y

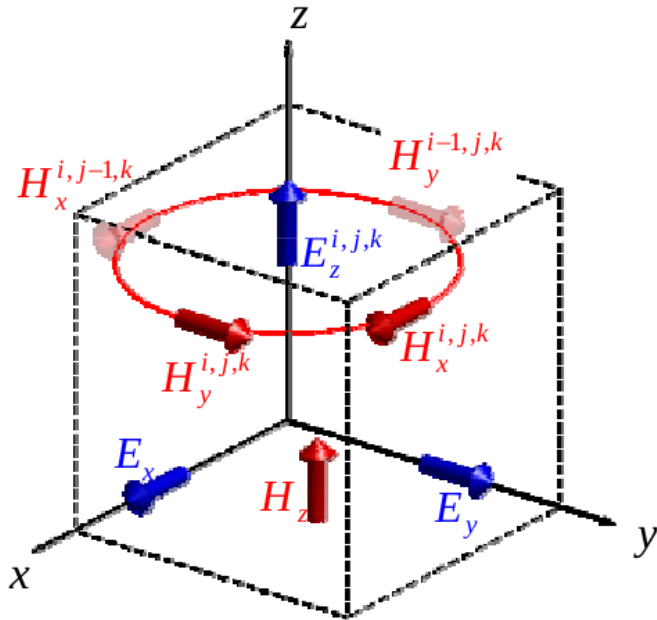


Figure 7: Yee Grid E_z

FDTD Sources

Christopher Stricklan

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Abstract

This document describes and formulates the various sources that can be used in FDTD models.

1D SOURCES

Gaussian Pulse

The Gaussian Pulse is an impulse function that allows us to excite the problem with a broad range of frequencies all at the same time.

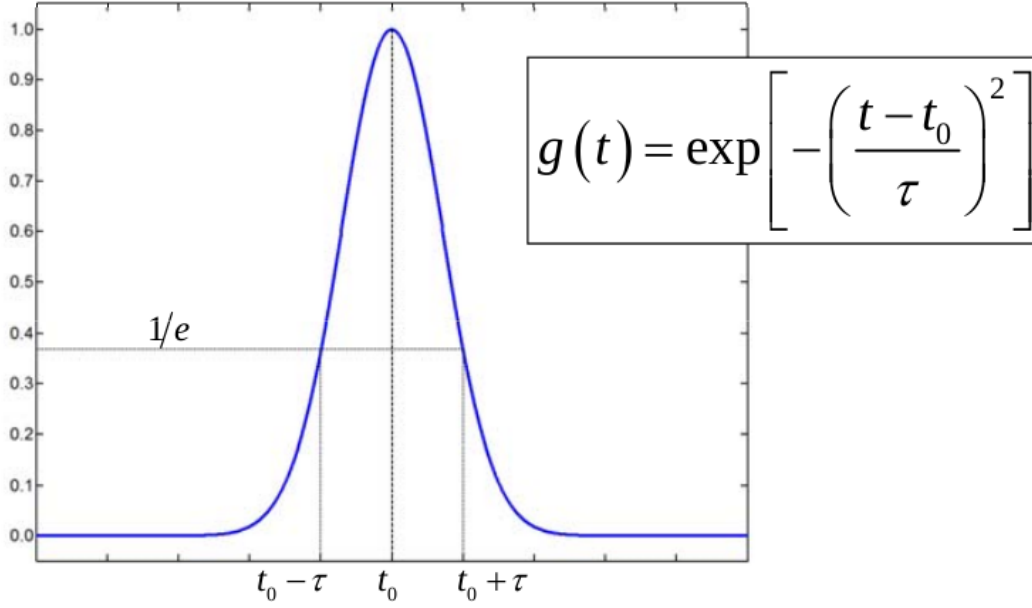


Figure 1: A Gaussian Pulse centered at t_0

To find the frequency content of the Pulse we must perform the Fourier Transform on the pulse.

$$g(t) = \exp\left(-\frac{t^2}{\tau^2}\right) \Rightarrow G(f) = \frac{1}{\sqrt{\pi}B} \exp\left[-\frac{f^2}{B^2}\right]$$

This shows that the frequency content of the Gaussian Pulse extends from DC up to B where

$$B = \frac{1}{\pi\tau} \quad (1)$$

Pulse Design

To design the pulse for our simulations we must first decide on the maximum frequency we are interested in f_{max} then compute the pulse width with this upper frequency

$$B = f_{max} = \frac{1}{\pi\tau} \Rightarrow \tau \leq \frac{1}{\pi f_{max}}$$

For simplification we can approximate τ as

$$\tau \cong \frac{0.5}{f_{max}} \quad (2)$$

In order to properly resolve our Gaussian pulse, which should be completed by at least 10 to 20 time steps, we need to recalculate Δt . We now need to re-evaluate this value in conjunction with the Courant Stability Condition. We will determine Δt based on the maximum frequency and pick the smallest Δt .

$$\Delta t \leq \frac{\tau}{10} \quad (3)$$

Finally to properly inject our source without any adverse reactions within our model we must include delay 1.1. This will allow the pulse to ease into the problem space without producing large field gradients.

$$t_0 \geq 6\tau \quad (4)$$

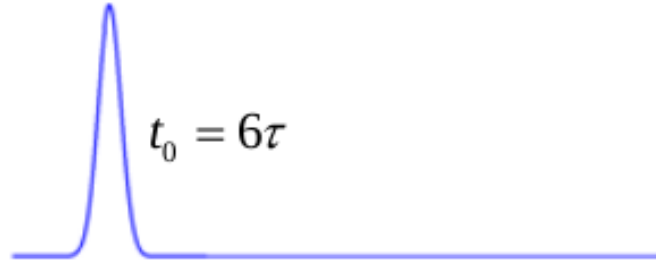


Figure 2: A Gaussian Pulse delayed by $t_0 = 6\tau$

Matlab Example

Program 1 The Gaussian Source Example

<i>% Parameters</i>	<i>%Max Frequency</i>
fmax = 1e9;	<i>%Time Step</i>
dt = 0.5*dz/c0	<i>%Pulse Duration</i>
tau = 0.5/fmax;	
<i>% Compute Gaussian Source</i>	<i>%position of source</i>
nzc = round(Nz/2)	
dt = 0.6*dz/c0	<i>%time axis</i>
ta = [0:STEPS-1]*dt;	<i>%Pulse Position</i>
t0 = 6*tau	<i>%Total delay between E and H</i>
s = dz/(2*c0) + dt/2	<i>%E field source</i>
Esrc = exp(-((ta-t0/tau).^2);	<i>%amplitude of H field</i>
A = - sqrt(ER(nzc)/UR(nzc));	<i>% H field source</i>
Hsrc = A * exp(-((ta-t0+s)/tau).^2);	

Modeling 1D Problems with FDTD

Christopher Stricklan

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Abstract

This document describes the reduction from a 3-Dimensional Finite Difference set of Equations as derived in “Maxwell Equations” to a 1-Dimensional Form ready for use in numerical modeling algorithms.

REDUCE TO ONE-DIMENSION

We will reduce our Finite-Difference Equations to 1D. This means that we will have material that is uniform in two directions. This uniform material will cause the fields to be uniform as well. The change in material in our case we will define to be in the z-direction therefore the x and y directions will be uniform. This means that the derivatives in this direction will also equal zero.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

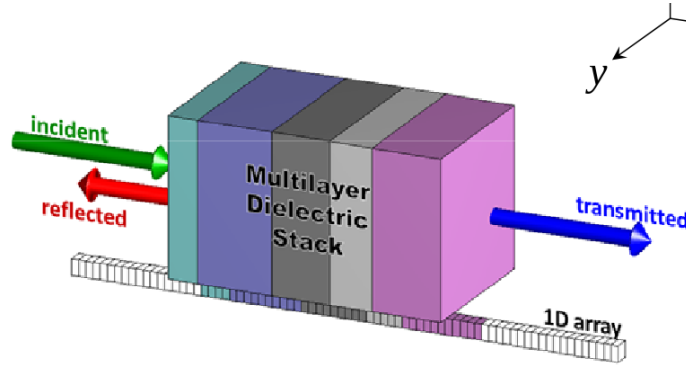


Figure 1: 1D Problem

From our Finite-Difference equations we need to cancel out the x and y derivatives that are zero.

$$\frac{\cancel{\partial E_z}}{\cancel{\partial y}} - \frac{\partial E_y}{\partial z} = -\frac{\mu_{xx}}{c_0} \frac{\partial \tilde{H}_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \frac{\cancel{\partial E_z}}{\cancel{\partial x}} = -\frac{\mu_{yy}}{c_0} \frac{\partial \tilde{H}_y}{\partial t}$$

$$\frac{\cancel{\partial E_y}}{\cancel{\partial x}} - \frac{\cancel{\partial E_x}}{\cancel{\partial y}} = -\frac{\mu_{zz}}{c_0} \frac{\partial \tilde{H}_z}{\partial t}$$

$$\frac{\cancel{\partial \tilde{H}_z}}{\cancel{\partial y}} - \frac{\partial \tilde{H}_y}{\partial z} = \frac{\epsilon_{xx}}{c_0} \frac{\partial E_x}{\partial t}$$

$$\frac{\partial \tilde{H}_x}{\partial z} - \frac{\cancel{\partial \tilde{H}_z}}{\cancel{\partial x}} = \frac{\epsilon_{yy}}{c_0} \frac{\partial E_y}{\partial t}$$

$$\frac{\cancel{\partial \tilde{H}_y}}{\cancel{\partial x}} - \frac{\cancel{\partial \tilde{H}_x}}{\cancel{\partial y}} = \frac{\epsilon_{zz}}{c_0} \frac{\partial E_z}{\partial t}$$

→

$$\begin{aligned}
& \frac{\cancel{\frac{E_z^{i,j+1,k} |_{t+\frac{\Delta t}{2}} - E_z^{i,j,k} |_{t+\frac{\Delta t}{2}}}{\Delta y}} - \frac{E_y^{i,j,k+1} |_{t+\frac{\Delta t}{2}} - E_y^{i,j,k} |_{t+\frac{\Delta t}{2}}}{\Delta z}}{\Delta y} = -\frac{\mu_{xx}^{i,j,k}}{c_0} \frac{\tilde{H}_x^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta t} \\
& \frac{\frac{E_x^{i,j,k+1} |_{t+\frac{\Delta t}{2}} - E_x^{i,j,k} |_{t+\frac{\Delta t}{2}}}{\Delta z} - \cancel{\frac{E_z^{i+1,j,k} |_{t+\frac{\Delta t}{2}} - E_z^{i,j,k} |_{t+\frac{\Delta t}{2}}}{\Delta x}}}{\Delta z} = -\frac{\mu_{yy}^{i,j,k}}{c_0} \frac{\tilde{H}_y^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta t} \\
& \frac{\cancel{\frac{E_y^{i+1,j,k} |_{t+\frac{\Delta t}{2}} - E_y^{i,j,k} |_{t+\frac{\Delta t}{2}}}{\Delta x}} - \frac{E_x^{i,j+1,k} |_{t+\frac{\Delta t}{2}} - E_x^{i,j,k} |_{t+\frac{\Delta t}{2}}}{\Delta y}}{\Delta y} = -\frac{\mu_{zz}^{i,j,k}}{c_0} \frac{\tilde{H}_z^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta t} \\
& \frac{\cancel{\frac{\tilde{H}_z^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j-1,k} |_{t+\frac{\Delta t}{2}}}{\Delta y}} - \frac{\tilde{H}_y^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1} |_{t+\frac{\Delta t}{2}}}{\Delta z}}{\Delta y} = \frac{\epsilon_{xx}^{i,j,k}}{c_0} \frac{E_x^{i,j,k} |_{t+\Delta t} - E_x^{i,j,k} |_{t+\frac{\Delta t}{2}}}{\Delta t} \\
& \frac{\frac{\tilde{H}_x^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1} |_{t+\frac{\Delta t}{2}}}{\Delta z} - \cancel{\frac{\tilde{H}_z^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i-1,j,k} |_{t+\frac{\Delta t}{2}}}{\Delta x}}}{\Delta z} = \frac{\epsilon_{yy}^{i,j,k}}{c_0} \frac{E_y^{i,j,k} |_{t+\Delta t} - E_y^{i,j,k} |_{t+\frac{\Delta t}{2}}}{\Delta t} \\
& \frac{\cancel{\frac{\tilde{H}_y^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i-1,j,k} |_{t+\frac{\Delta t}{2}}}{\Delta x}} - \frac{\tilde{H}_x^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j-1,k} |_{t+\frac{\Delta t}{2}}}{\Delta y}}{\Delta x} = \frac{\epsilon_{zz}^{i,j,k}}{c_0} \frac{E_z^{i,j,k} |_{t+\Delta t} - E_z^{i,j,k} |_{t+\frac{\Delta t}{2}}}{\Delta t}
\end{aligned}$$

With these cross outs we come out with two observations. The first is that the longitudinal field components E_z and H_z are always zero. The second is that the Maxwell's equations have decoupled into two sets of two equations E_x/H_y Mode and E_y/H_x Mode.

$$\begin{aligned}
-\frac{\partial E_y}{\partial z} &= -\frac{\mu_{xx}}{c_0} \frac{\partial \tilde{H}_x}{\partial t} \\
\frac{\partial E_x}{\partial z} &= -\frac{\mu_{yy}}{c_0} \frac{\partial \tilde{H}_y}{\partial t} \\
0 &= -\frac{\mu_{zz}}{c_0} \frac{\partial \tilde{H}_z}{\partial t} \\
-\frac{\partial \tilde{H}_y}{\partial z} &= \frac{\epsilon_{xx}}{c_0} \frac{\partial E_x}{\partial t} \\
\frac{\partial \tilde{H}_x}{\partial z} &= \frac{\epsilon_{yy}}{c_0} \frac{\partial E_y}{\partial t} \\
0 &= \frac{\epsilon_{zz}}{c_0} \frac{\partial E_z}{\partial t} \\
&\longrightarrow \\
-\frac{E_y^{i,j,k+1} |_{t+\frac{\Delta t}{2}} - E_y^{i,j,k} |_{t+\frac{\Delta t}{2}}}{\Delta z} &= -\frac{\mu_{xx}^{i,j,k}}{c_0} \frac{\tilde{H}_x^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta t}
\end{aligned}$$

$$\begin{aligned}
\frac{E_x^{i,j,k+1} |_{t+\Delta t} - E_x^{i,j,k} |_t}{\Delta z} &= -\frac{\mu_{yy}^{i,j,k}}{c_0} \frac{\tilde{H}_y^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k} |_{t-\frac{\Delta t}{2}}}{\Delta t} \\
\tilde{H}_z^{i,j,k} &= 0 \\
-\frac{\tilde{H}_y^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1} |_{t+\frac{\Delta t}{2}}}{\Delta z} &= \frac{\epsilon_{xx}^{i,j,k}}{c_0} \frac{E_x^{i,j,k} |_{t+\Delta t} - E_x^{i,j,k} |_t}{\Delta t} \\
\frac{\tilde{H}_x^{i,j,k} |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1} |_{t+\frac{\Delta t}{2}}}{\Delta z} &= \frac{\epsilon_{yy}^{i,j,k}}{c_0} \frac{E_y^{i,j,k} |_{t+\Delta t} - E_y^{i,j,k} |_t}{\Delta t} \\
E_z^{i,j,k} &= 0
\end{aligned}$$

SUMMARY OF 1D FINITE-DIFFERENCE EQUATIONS

The equations are now broken apart into their respective modes. These modes are physical and propagate independently from each other. In the end though they are numerically the same and will exhibit the same electromagnetic behavior. This allows us to only have to solve one set of two equations. We can also note that since we only changing materials along the z axis we can drop the i and j Array Indices.

Ex/Hy Mode

$$-\frac{\tilde{H}_y^k |_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k-1} |_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{xx}^k}{c_0} \frac{E_x^k |_{t+\Delta t} - E_x^k |_t}{\Delta t} \quad (1)$$

$$\frac{E_x^{k+1} |_{t+\Delta t} - E_x^k |_t}{\Delta z} = -\frac{\mu_{yy}^k}{c_0} \frac{\tilde{H}_y^k |_{t+\frac{\Delta t}{2}} - \tilde{H}_y^k |_{t-\frac{\Delta t}{2}}}{\Delta t} \quad (2)$$

Ey/Hx Mode

$$\frac{\tilde{H}_x^k |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} |_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{yy}^k}{c_0} \frac{E_y^k |_{t+\Delta t} - E_y^k |_t}{\Delta t} \quad (3)$$

$$\frac{E_y^{k+1} |_{t+\Delta t} - E_y^k |_t}{\Delta z} = \frac{\mu_{xx}^k}{c_0} \frac{\tilde{H}_x^k |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k |_{t-\frac{\Delta t}{2}}}{\Delta t} \quad (4)$$

UPDATE EQUATIONS

We are now going to derive the update equations used during in FDTD algorithm. We have arbitrarily chosen to use Ey/Hx Mode found in Equations (3) and (4)

Update Equation for Ey

$$\frac{\tilde{H}_x^k |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} |_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{yy}^k}{c_0} \frac{E_y^k |_{t+\Delta t} - E_y^k |_t}{\Delta t}$$

We want to solve for the E-Field at the future time value.

$$\begin{aligned} \frac{\epsilon_{yy}^k}{c_0} \frac{E_y^k |_{t+\Delta t} - E_y^k |_t}{\Delta t} &= \frac{\tilde{H}_x^k |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} |_{t+\frac{\Delta t}{2}}}{\Delta z} \\ E_y^k |_{t+\Delta t} - E_y^k |_t &= \frac{c_0 \Delta t}{\epsilon_{yy}^k} \left(\frac{\tilde{H}_x^k |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} |_{t+\frac{\Delta t}{2}}}{\Delta z} \right) \\ E_y^k |_{t+\Delta t} &= E_y^k |_t + \left(\frac{c_0 \Delta t}{\epsilon_{yy}^k} \right) \left(\frac{\tilde{H}_x^k |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} |_{t+\frac{\Delta t}{2}}}{\Delta z} \right) \end{aligned} \quad (5)$$

Update Equation for Hx

$$\frac{E_y^{k+1} |_t - E_y^k |_t}{\Delta z} = \frac{\mu_{xx}^k}{c_0} \frac{\tilde{H}_x^k |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k |_{t-\frac{\Delta t}{2}}}{\Delta t}$$

We want to solve for the H-Field at the future time value

$$\begin{aligned} \frac{\mu_{xx}^k}{c_0} \frac{\tilde{H}_x^k |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k |_{t-\frac{\Delta t}{2}}}{\Delta t} &= \frac{E_y^{k+1} |_t - E_y^k |_t}{\Delta z} \\ \tilde{H}_x^k |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k |_{t-\frac{\Delta t}{2}} &= \frac{c_0 \Delta t}{\mu_{xx}^k} \left(\frac{E_y^{k+1} |_t - E_y^k |_t}{\Delta z} \right) \\ \tilde{H}_x^k |_{t+\frac{\Delta t}{2}} &= \tilde{H}_x^k |_{t-\frac{\Delta t}{2}} + \left(\frac{c_0 \Delta t}{\mu_{xx}^k} \right) \left(\frac{E_y^{k+1} |_t - E_y^k |_t}{\Delta z} \right) \end{aligned} \quad (6)$$

UPDATE COEFFICIENTS

Since the update coefficients don't change during the simulation. We can compute them once before our FDTD Algorithm is actually implemented in loop.

$$\tilde{H}_x^k |_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k |_{t-\frac{\Delta t}{2}} + m_{H_x}^k \left(\frac{E_y^{k+1} |_t - E_y^k |_t}{\Delta z} \right) \quad (7)$$

$$E_y^k |_{t+\Delta t} = E_y^k |_t + m_{E_y}^k \left(\frac{\tilde{H}_x^k |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} |_{t+\frac{\Delta t}{2}}}{\Delta z} \right) \quad (8)$$

$$m_{E_y}^k = \frac{c_0 \Delta t}{\epsilon_{yy}^k} \quad (9)$$

$$m_{H_x}^k = \frac{c_0 \Delta t}{\mu_{yy}^k} \quad (10)$$

BOUNDARY CONDITIONS

We have two types of boundary conditions that can be incorporated into this 1-Dimensional Simulation. The Dirichlet and Perfectly Absorbing Boundary Conditions are methods that allow us to handle the fields that lie at the boundary of the model. Specifically these points lie at $k = 0$ and $k = N_z + 1$. At these points there is no grid to pull our Electric and Magnetic fields from, therefore to maintain the stability of the Finite Difference Maxwell's Equations we must handle these separately.

Dirichlet Boundary Condition

This boundary condition is the easiest to implement. We simply assume the fields that lie outside the grid are zero. This is the most crude method and will cause reflection back into our model as energy reaches the boundary.

$$\text{For } k < N_z \quad \tilde{H}_x^k |_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k |_{t-\frac{\Delta t}{2}} + m_{H_x}^k \left(\frac{E_y^{k+1} |_t - E_y^k |_t}{\Delta z} \right)$$

$$\text{For } k = N_z \quad \tilde{H}_x^k |_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k |_{t-\frac{\Delta t}{2}} + m_{H_x}^k \left(\frac{0 - E_y^k |_t}{\Delta z} \right)$$

$$\text{For } k > 1 \quad E_y^k |_{t+\Delta t} = E_y^k |_t + m_{E_y}^k \left(\frac{\tilde{H}_x^k |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} |_{t+\frac{\Delta t}{2}}}{\Delta z} \right)$$

$$\text{For } k = 1 \quad E_y^k |_{t+\Delta t} = E_y^k |_t + m_{E_y}^k \left(\frac{\tilde{H}_x^k |_{t+\frac{\Delta t}{2}} - 0}{\Delta z} \right)$$

Periodic Boundary Condition

This boundary condition assumes that the device being modeled is periodic along a particular direction. In our case the periodicity is along the z-axis.

$$\text{For } k < N_z \quad \tilde{H}_x^k |_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k |_{t-\frac{\Delta t}{2}} + m_{H_x}^k \left(\frac{E_y^{k+1} |_t - E_y^k |_t}{\Delta z} \right)$$

$$\text{For } k = N_z \quad \tilde{H}_x^k |_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k |_{t-\frac{\Delta t}{2}} + m_{H_x}^k \left(\frac{E_y^1 |_t - E_y^{N_z} |_t}{\Delta z} \right)$$

$$\text{For } k > 1 \quad E_y^k |_{t+\Delta t} = E_y^k |_t + m_{E_y}^k \left(\frac{\tilde{H}_x^k |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} |_{t+\frac{\Delta t}{2}}}{\Delta z} \right)$$

$$\text{For } k = 1 \quad E_y^k |_{t+\Delta t} = E_y^k |_t + m_{E_y}^k \left(\frac{\tilde{H}_x^1 |_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{N_z} |_{t+\frac{\Delta t}{2}}}{\Delta z} \right)$$

Perfectly Absorbing Boundary Condition

The Perfectly Absorbing Boundary Condition provides a method to absorb almost all of the energy out of the model. There is approximately 10 orders of magnitude less energy reflected back into the problem space. This is very minor and should not affect our results. There are three conditions though that must be met in order to use this boundary condition.

- Waves at the boundaries are only traveling outward.
- Materials at the boundaries are linear, homogeneous, isotropic and non-dispersive.
- Time step is chosen so physical waves travel 1 cell in two time steps $\Delta t = \frac{\Delta z}{2c_0}$

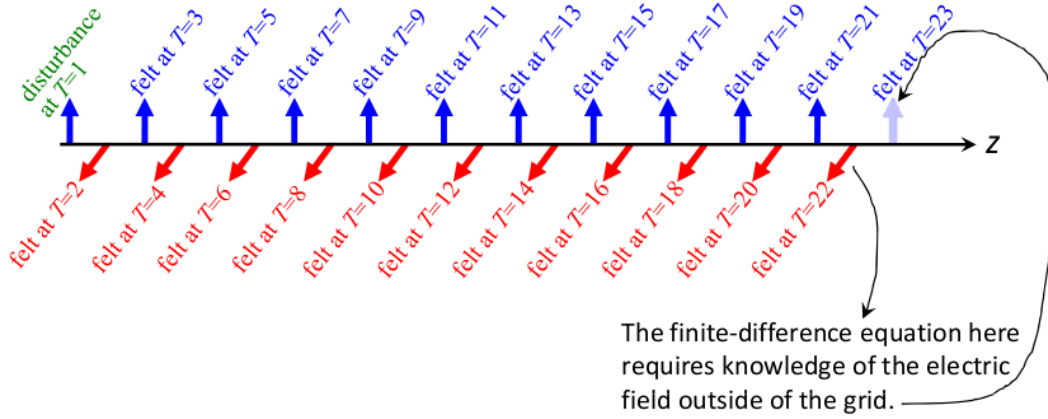


Figure 2: Perfect Boundary Time Steps

As can be seen in 5.3 the last time step is magically off the grid. In order to implement the perfect boundary condition we must use a field that would have propagated past our last grid cell $2\Delta t$ time steps ago. In equations (11) and (12) the field value that exists outside our cells needs to be used when the wave reaches it at the $2\Delta t$ time frame

$$\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k|_{t-\frac{\Delta t}{2}} + m_{H_x}^k \left(\frac{E_y^{N_z+1}|_t - E_y^{N_z}|_t}{\Delta z} \right) \quad (11)$$

$$E_y^k|_{t+\Delta t} = E_y^k|_t + m_{E_y}^k \left(\frac{\tilde{H}_x^1|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^0|_{t+\frac{\Delta t}{2}}}{\Delta z} \right) \quad (12)$$

Since the field values did exist $2\Delta t$ ago in our boundary cell in the problem space. We can then infer the value is the same as the energy leaves the grid.

$$E_y^{N_z+2}|_t = E_y^{N_z}|_{t-2\Delta t}$$

$$\tilde{H}_x^0|_t = \tilde{H}_x^1|_{t-2\Delta t}$$

To implement the boundary condition we keep track of previous history in variables. At the z-low boundary we only modify the E-Field update equations. At the z-high boundary we only modify the H-Field update equations.

$$h_3 = h_2 \quad h_2 = h_1 \quad h_1 = \tilde{H}_x^1$$

$$E_y^k|_{t+\Delta t} = E_y^1|_t + m_{E_y}^1 \left(\frac{\tilde{H}_x^1|_{t+\frac{\Delta t}{2}} - h_3}{\Delta z} \right)$$

$$e_3 = e_2 \quad e_2 = e_1 \quad e_1 = E_y^{N_z}$$

$$\tilde{H}_x^{N_z}|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^{N_z}|_{t-\frac{\Delta t}{2}} + m_{H_x}^{N_z} \left(\frac{e_3 - E_y^{N_z}|_t}{\Delta z} \right)$$

SOURCES

There are two ways to incorporate a source into our models, Hard Source and Soft Source. The hard source is the simplest to implement. After we have performed our update equations over the entire grid we overwrite some fields with our source. This injects the source energy into our model. This does cause a problem because the point where the source was inject behaves like a perfect electric or magnetic conductor to scattered fields.

$$\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} = g_H|_k \quad E_y^k|_{t+\Delta t} = g_E|_k$$

The soft source on the otherhand is the best way to inject energy into our model. After performing the update equations across the entire grid, the source is added to the current value of our source injection points. This injects energy into the model, but allows scattered waves to pass through the source region like it was transparent.

$$\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k|_{t+\frac{\Delta t}{2}} + g_H|_k \quad E_y^k|_{t+\Delta t} = E_y^k|_{t+\Delta t} + g_E|_k$$

MODEL PARAMETERS

Grid Resolution

There are multiple considerations that need to be addressed in order to properly calculate the grid resolution.

Wavelength

The grid resolution must be sufficient to resolve the shortest wavelength. First we must determine the smallest wavelength then we must divide this by a value that can be found in the chart. These values are used as a guideline depending on the material with the highest refractive index.

$$\lambda_{min} = \frac{c_0}{f_{max} n_{max}}$$

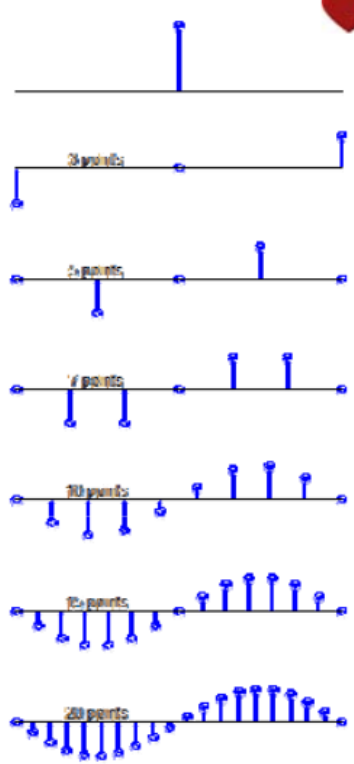


Figure 3: Building up wavelength resolution

$$\Delta_\lambda \approx \frac{\lambda_{min}}{N_\lambda} \quad N_\lambda \geq 10$$

Approximate N_λ Values		
N_λ	Refractive Index	Comment
10 to 20	$n = 1 - 10$	Low contrast dielectrics
10 to 30	$n = 10 - 30$	High contrast dielectrics
40 to 60	$n = 40 - 60$	Most metallic structures
100 to 200	$n \geq 60$	Plasmonic devices

Mechanical Features

The grid resolution must be sufficient to resolve the smallest physical feature of the device. First, we must determine the smallest feature d_{min} . then we must divide this by 1 to 4 to accurately manage the width of our smallest feature.

$$\Delta_d \approx \frac{d_{min}}{N_d} \quad N_d \geq 1$$

Calculate Grid Resolution

We will first calculate an initial grid resolution by taking the minimum between the model feature Δ_d and wavelength Δ_λ .

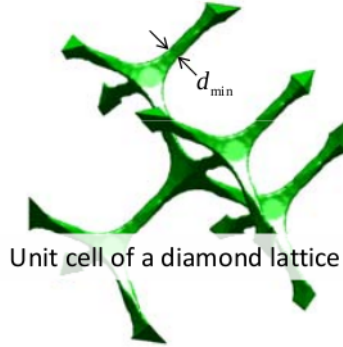


Figure 4:

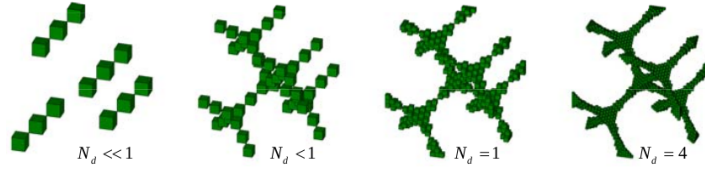


Figure 5:

$$\Delta_x = \Delta_y = \min[\Delta_\lambda, \Delta_d]$$

Finally we need to “Snap” grid to critical dimensions. In 7.1 we have modified delta to not fall halfway through a grid cell. We can’t fill half a cell in a discrete environment. First we Decide what dimensions along each axis are critical. d_x and d_y . We then compute how many grid cells comprise d_c and round up (13). Finally adjust grid resolution to fit this dimension in grid exactly (14).

$$\begin{aligned} M_x &= \text{ceil}\left(\frac{d_x}{\Delta_x}\right) \\ M_y &= \text{ceil}\left(\frac{d_y}{\Delta_y}\right) \end{aligned} \tag{13}$$

$$\begin{aligned} \Delta_x &= \frac{d_x}{M_x} \\ \Delta_y &= \frac{d_y}{M_y} \end{aligned} \tag{14}$$

Courant Stability Condition

For each iteration of our update equations a currently existing electric field will not immediatly affect adjacent fields. A disturbance in the electric field at one point will only be felt by the immediately adjacent magnetic fields. For an adjacent electric field to feel the disturbance it takes at least two time steps.

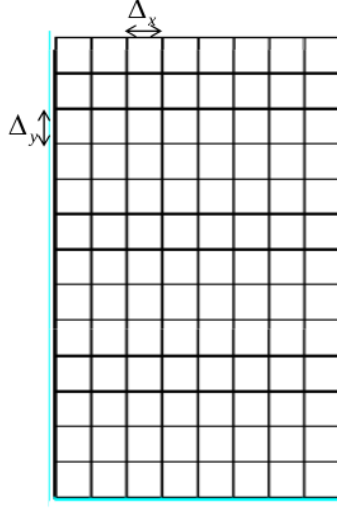


Figure 6: Grid X/Y

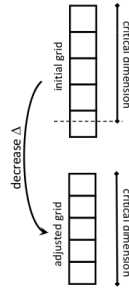


Figure 7: Critical Dimension Expansion

Keeping in mind that Electromagnetic waves propagate at the speed of light in a vacuum. Inside a material, a wave propagates at a reduce speed based on the refractive index (15)

$$v = \frac{c_0}{n} \quad (15)$$

Because we are dealing with a numerical algorithm it is not possible for a disturbance to travel farther than one unit cell in a single time step. So we need to force a condition that would force a wave not to propagate farther than a single unit cell in one time step.

$$\frac{c_0 \Delta t}{n} < \Delta z$$

So now we can place an upper limit on a time steps

$$\Delta t < \frac{n \Delta z}{c_0}$$

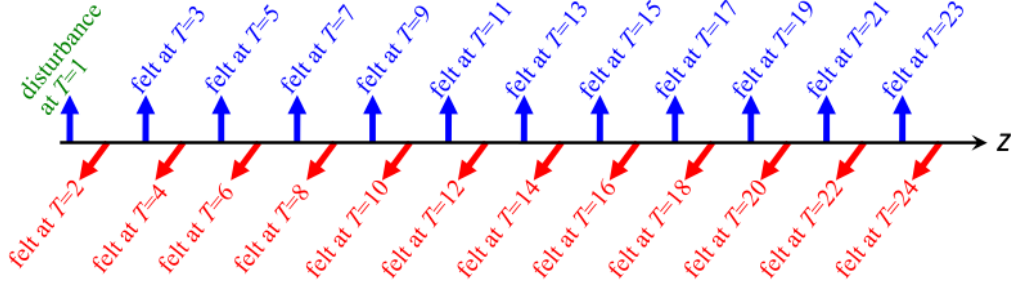


Figure 8: Timeline of electric field propagation

To force our wave to travel the distance of one grid cell in exactly two time steps we finish up with (16)

$$\Delta t = \frac{n\Delta z}{2c_0} \quad (16)$$

LEARNING FROM OUR MODEL

Efficient Fourier Transform

In order to be able to learn from our model we must be able to somehow collect the data and put it in a form that is useful. In our case we can use the Fourier Transform to calculate the transmitted and reflected energy. In order to save CPU and Memory consumption instead of using the traditional method of recording the fields as functions of time. Then perform the FFT on these functions. We are instead going to perform a numerical approximation of the Fourier Transform as the model is running. In each iteration of the model we can sum up parts of the energy that exist as time goes by.

The standard Fourier transform is defined in (17)

$$F(f) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft} \quad (17)$$

Since we only know the $f(t)$ at discrete points, the Fourier transform is approximated numerically in (18)

$$F(f) \cong \sum_{m=1}^M \Delta t f(m\Delta t) e^{-j2\pi f m\Delta t} \quad (18)$$

We can write this in a form that is beneficial for numerical modeling (19)

$$F(f) \cong \Delta t \sum_{m=1}^M (e^{-j2\pi f \Delta t})^m f(m\Delta t) \quad (19)$$

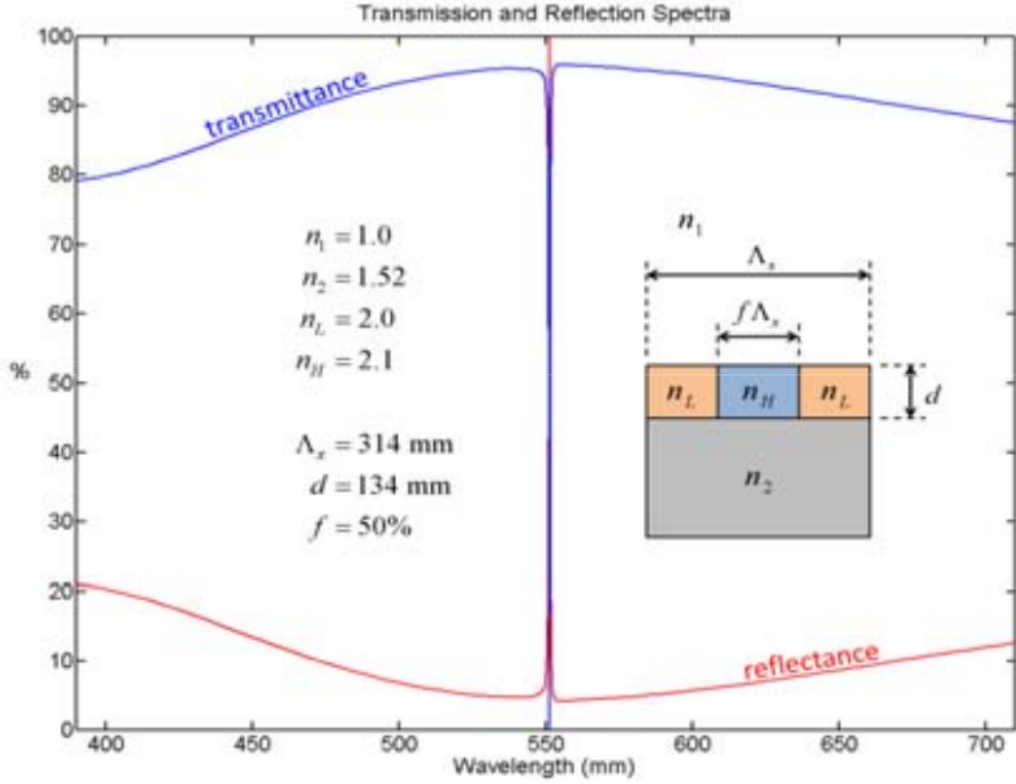
Finally we can break this down into parts

$$F(f) \cong \Delta t \sum_{m=1}^M (e^{-j2\pi f \Delta t})^m f(m\Delta t) \quad (2)$$

Δt Multiplication can be done after the main FDTD loop in a post-processing step (2)
 $e^{-j2\pi f \Delta t}$ This “kernel” can be computed prior to the main FDTD loop for each frequency of interest (2)
 $f(m\Delta t)$ This is simply the field value of interest at the current time step. (2)

Reflectance and Transmittance

When simulating passive device we generally want to plot an output of reflectance and transmittance as a function of frequency. Reflectance is the fraction of power reflected from a device. Transmittance is the fraction of power transmitted through the device.



We typically start the computation of power by Fourier transforming the reflected and transmitted field using our approximated fourier transform (19). Our results will typically look like figure 8.2

Finally we must normalize the spectra to calculate transmittance and reflectance. We do this by dividing the reflection and transmission by the source spectrum. We check for energy conservation by adding the reflectance and transmittance and ensuring the sum equals 100%. This is assuming the device itself as no loss or gain.

$$R(f) = \left(\frac{FFT[E_{ref}(t)]}{FFT[E_{src}(t)]} \right)^2 \quad (24)$$

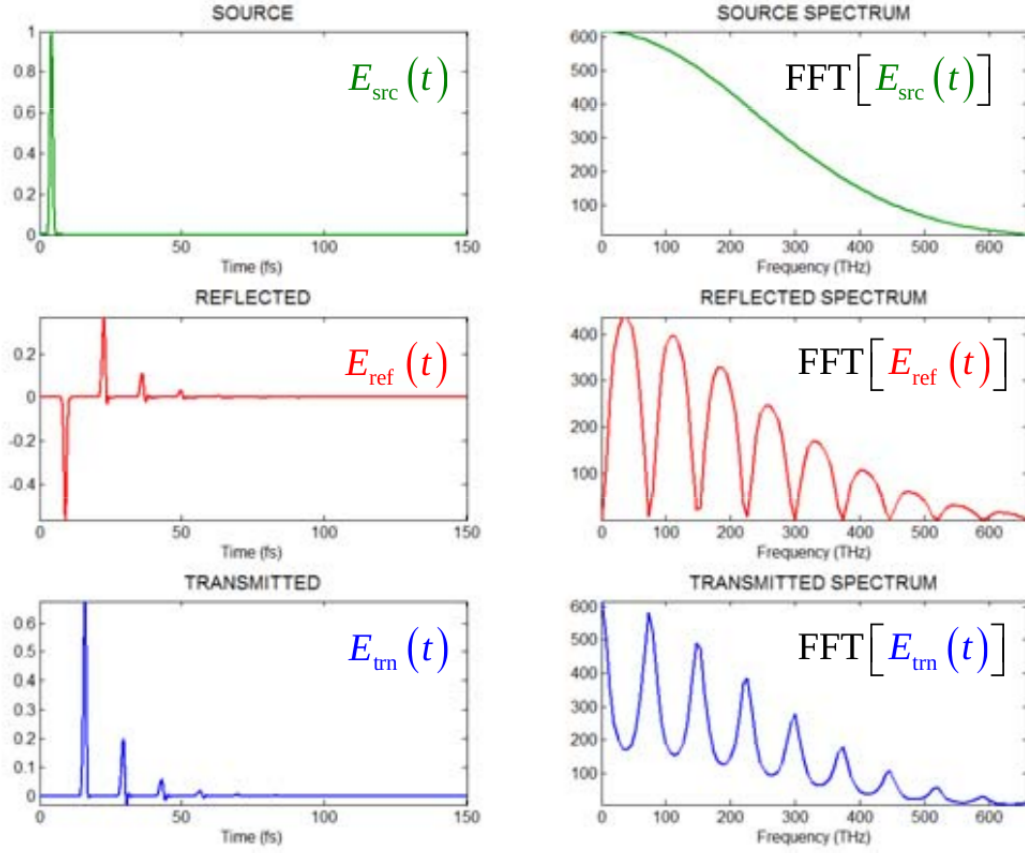


Figure 9: FDTD typical Reflected and Transmitted Power results

$$T(f) = \left(\frac{FFT[E_{trn}(t)]}{FFT[E_{src}(t)]} \right)^2 \quad (25)$$

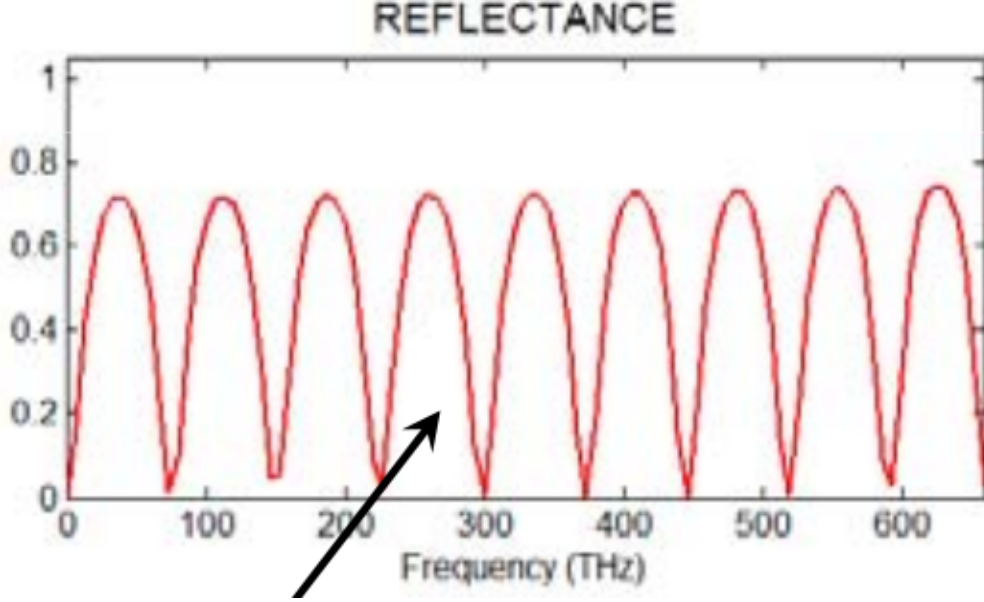
$$C(f) = R(f) + T(f) \quad (26)$$

Total-Field / Scattered-Field

The total-field/scattered-field (TF/SF) is a technique to inject a “one-way” source. This eliminates backward propagating energy, ensures waves at the boundaries are only traveling outward, and 100% of energy injected by the source is incident on the device being simulated. In figure 8.3 shows the typical grid layout with the TF/SF interface included.

On the scattered field side of the TF/SF interface, the finite-difference equation contains a term from the total-field side. This is due to the staggered nature of the yee grid, this only occurs in the update equation for a magnetic field.

$$\tilde{H}_x \Big|_{t+\frac{\Delta t}{2}}^{k_{src}-1} = \tilde{H}_x \Big|_{t-\frac{\Delta t}{2}}^{k_{src}-1} + m_{H_x}^{k_{src}-1} \left(\frac{E_y \Big|_t^{k_{src}} - E_y \Big|_t^{k_{src}-1}}{\Delta z} \right)$$



This equation that is trying to calculate a Total-Field value contains a Scattered-Field value $E_y^{k_{src}}$. To fix this issue we must now subtract out the source from this Scattered-Field value.

$$\tilde{H}_x |_{t+\frac{\Delta t}{2}}^{k_{src}-1} = \tilde{H}_x |_{t-\frac{\Delta t}{2}}^{k_{src}-1} + m_{H_x}^{k_{src}-1} \left(\frac{(E_y |_{t}^{k_{src}} - E_y^{src} |_{t}^{k_{src}}) - E_y |_{t}^{k_{src}-1}}{\Delta z} \right)$$

Which can be simplified to

$$\tilde{H}_x |_{t+\frac{\Delta t}{2}}^{k_{src}-1} = \tilde{H}_x |_{t-\frac{\Delta t}{2}}^{k_{src}-1} + m_{H_x}^{k_{src}-1} \left(\frac{E_y |_{t}^{k_{src}} - E_y |_{t}^{k_{src}-1}}{\Delta z} \right) - \frac{m_{H_x}^{k_{src}-1}}{\Delta z} E_y^{src} |_{t}^{k_{src}} \quad (27)$$

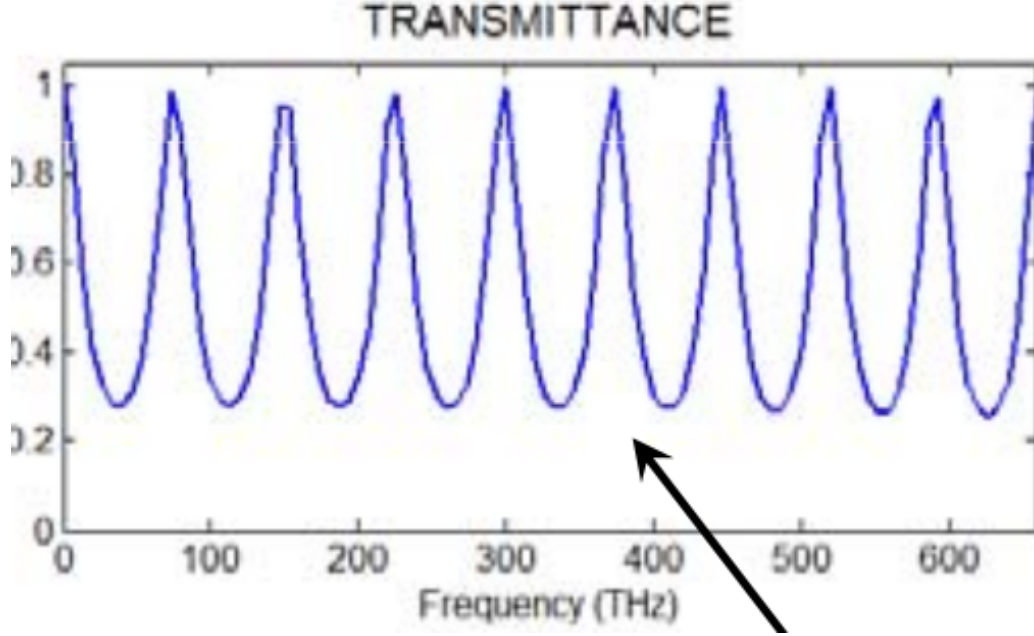
This has to be accomplished for electric field update equation as well.

$$E_y |_{t+\Delta t}^{k_{src}} = E_y |_{t}^{k_{src}} + m_{E_y}^{k_{src}} \left(\frac{\tilde{H}_x^{k_{src}} |_{t+\frac{\Delta t}{2}} - \tilde{H}_x |_{t+\frac{\Delta t}{2}}^{k_{src}-1}}{\Delta z} \right) \quad (28)$$

$$E_y |_{t+\Delta t}^{k_{src}} = E_y |_{t}^{k_{src}} + m_{E_y}^{k_{src}} \left(\frac{\tilde{H}_x^{k_{src}} |_{t+\frac{\Delta t}{2}} - (\tilde{H}_x |_{t+\frac{\Delta t}{2}}^{k_{src}-1} + \tilde{H}_x^{src} |_{t+\frac{\Delta t}{2}}^{k_{src}-1})}{\Delta z} \right) \quad (29)$$

$$E_y |_{t+\Delta t}^{k_{src}} = E_y |_{t}^{k_{src}} + m_{E_y}^{k_{src}} \left(\frac{\tilde{H}_x^{k_{src}} |_{t+\frac{\Delta t}{2}} - \tilde{H}_x |_{t+\frac{\Delta t}{2}}^{k_{src}-1}}{\Delta z} \right) - \frac{m_{E_y}^{k_{src}}}{\Delta z} \tilde{H}_x^{src} |_{t+\frac{\Delta t}{2}}^{k_{src}-1} \quad (30)$$

Source Terms



Now we know we need to calculate two source functions $\tilde{H}_x^{src} |_{t+\frac{\Delta t}{2}}^{k_{src}-1}$ and $E_y^{src} |_t^{k_{src}}$. We need to make a few observations that must be accounted for before we can calculate these source functions correctly.

- The amplitude of these functions can be different as E and H are related through the material impedance.
- These functions are a half grid cell apart and have a small time delay between them.
- These functions exist at different time steps.

Assuming the source is injected inside a homogeneous material, the fields must be plane waves.

$$E_y(t) = \sin(\omega t - \beta z) \quad (31)$$

We now must find the amplitude A and phase ϕ of the magnetic field related to the electric field

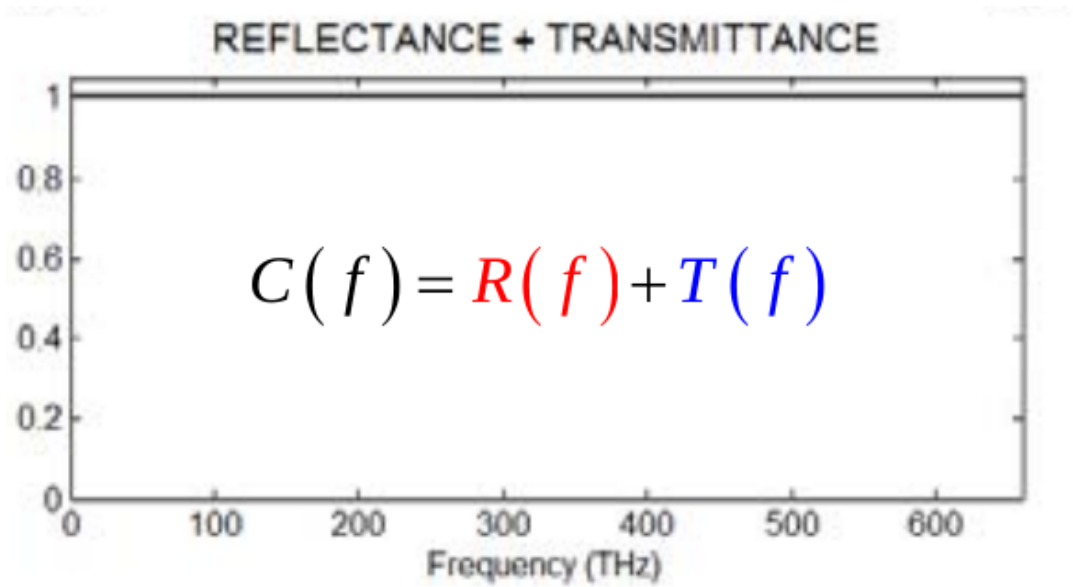
$$\tilde{H}_x(t) = A \sin(\omega t - \beta z - \phi) \quad (32)$$

The Amplitude and phase depend on the mode

$$A_{E_y H_x} = -\sqrt{\frac{\epsilon_r^{src}}{\mu_r^{src}}} \quad \phi_{E_y H_x} = 0 \quad (33)$$

$$A_{E_x H_y} = \sqrt{\frac{\epsilon_r^{src}}{\mu_r^{src}}} \quad \phi_{E_x H_y} = 0 \quad (34)$$

Therefore Ey/Hx mode



$$E_y^{src} |_{t^{k_{src}}} = g(t) \quad (35)$$

$$\tilde{H}_x^{src} |_{t+\frac{\Delta t}{2}}^{k_{src}-1} = -\sqrt{\frac{\epsilon_r^{src}}{\mu_r^{src}}} g \left(t + \frac{n\Delta Z}{2C_0} + \frac{\Delta t}{2} \right) \quad (36)$$

FDTD ALGORITHM

We first must do some initialization work

- Compute Grid Resolution
- Compute time step
- Compute Source
- Compute Update Coefficients
- Initialize Fields

The meat of the algorithm is the loop

- Update H from E
- Update E from H
- Inject Source
- Visualize Fields

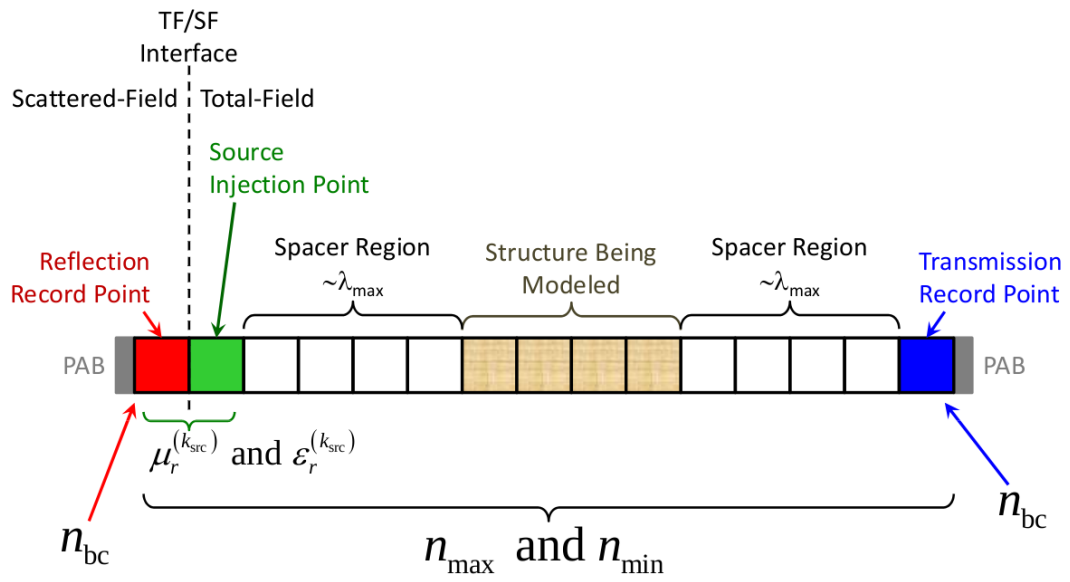


Figure 10: 1D Grid Layout including TF/SF

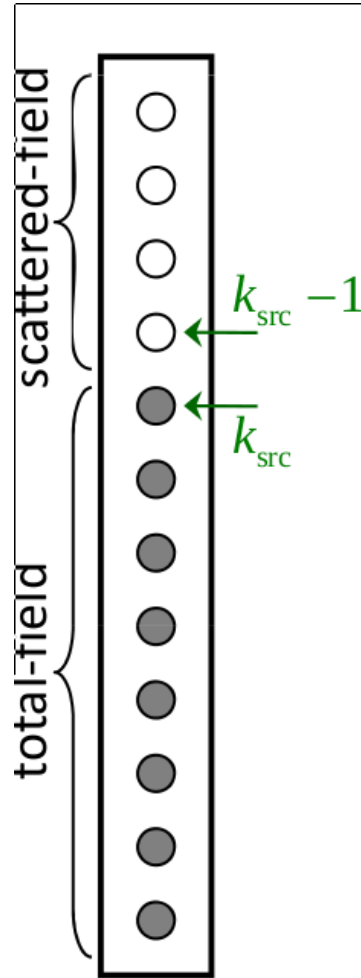
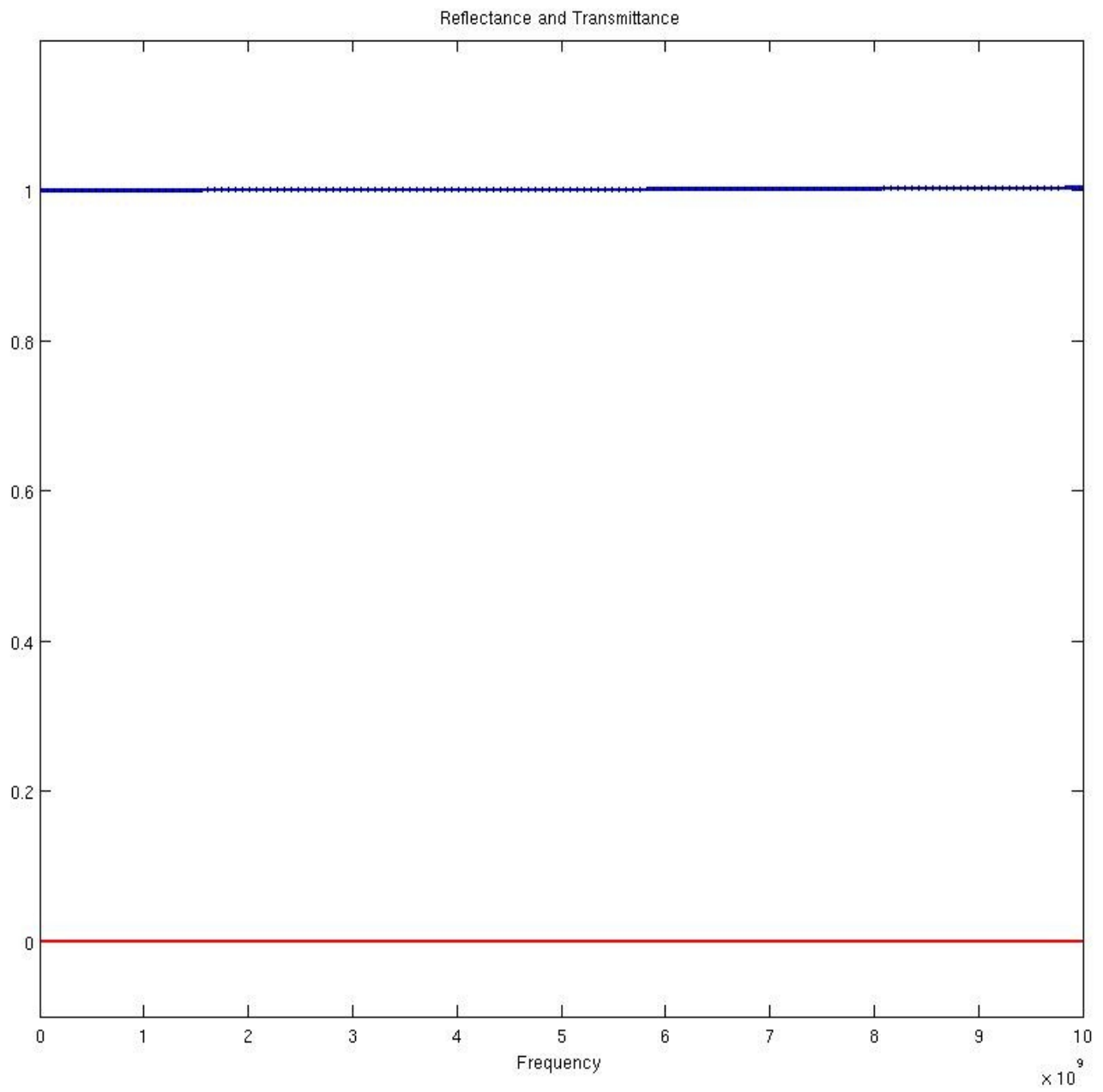
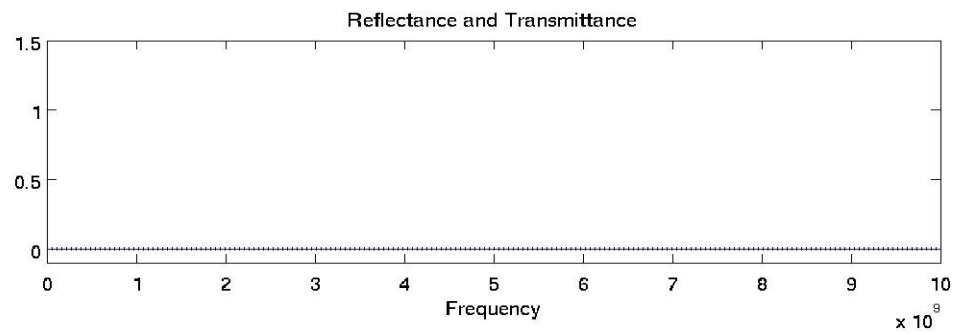
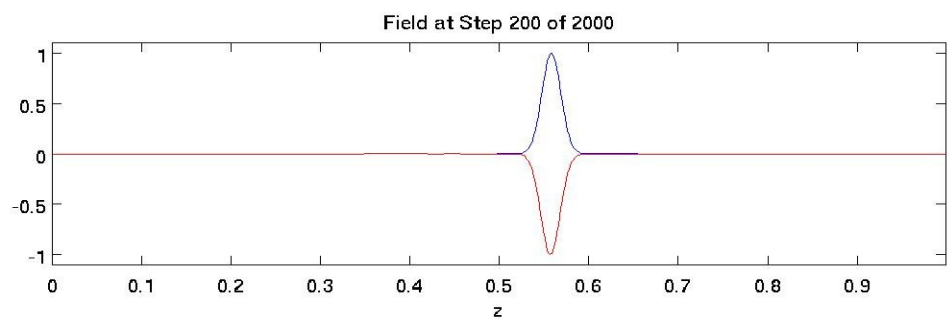
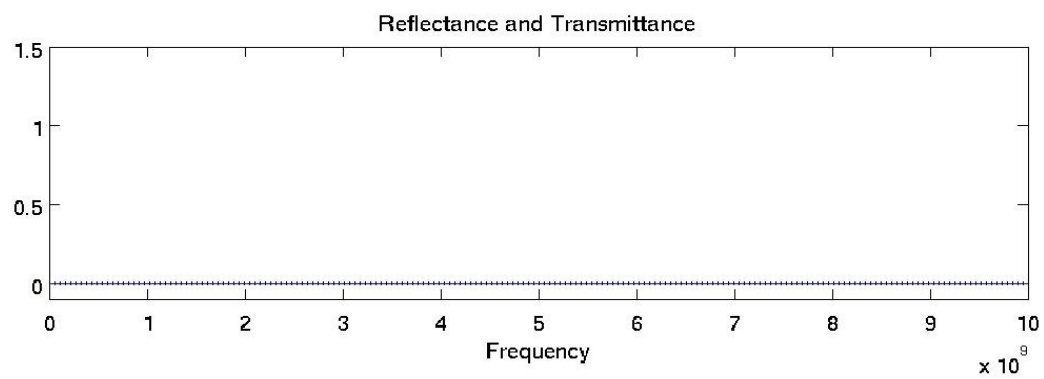
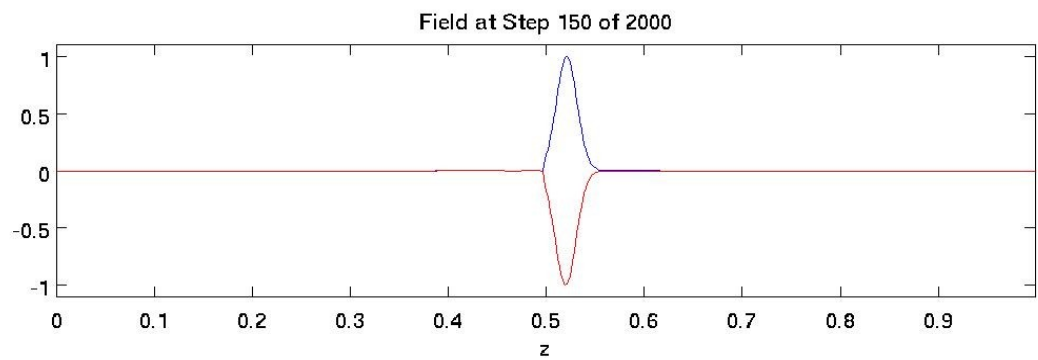
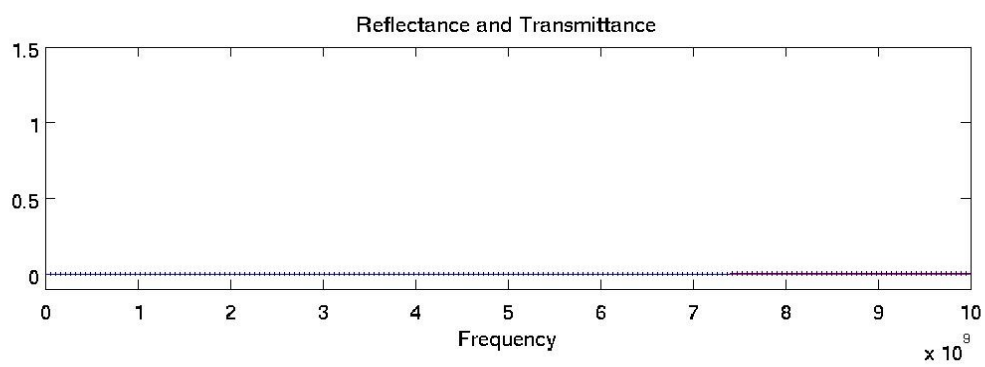
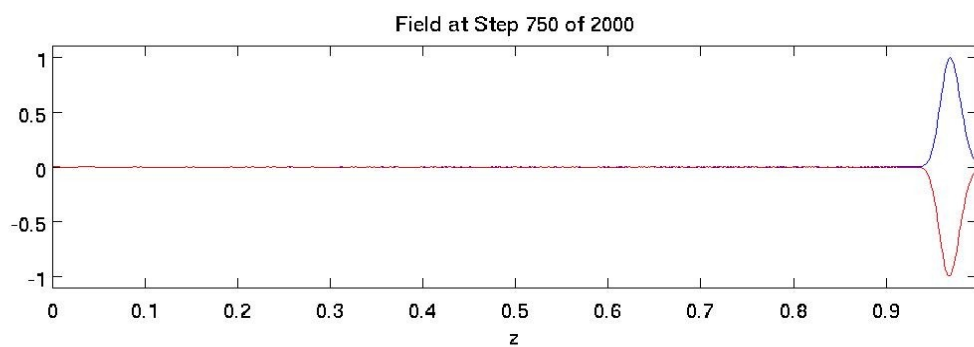
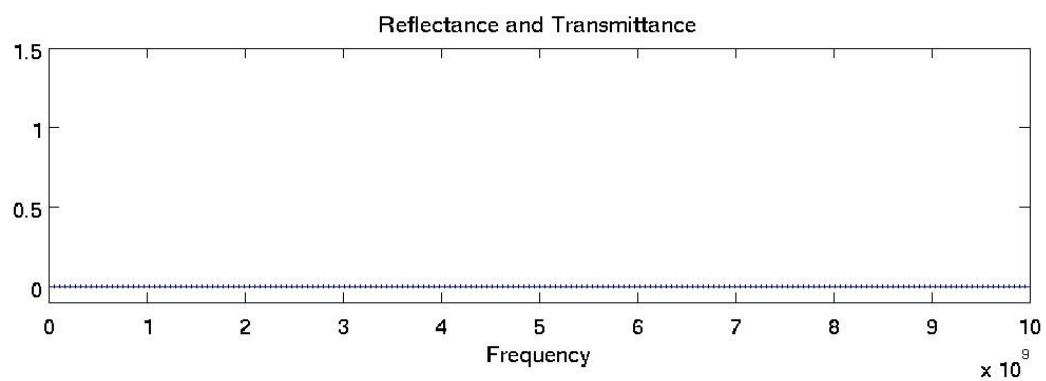
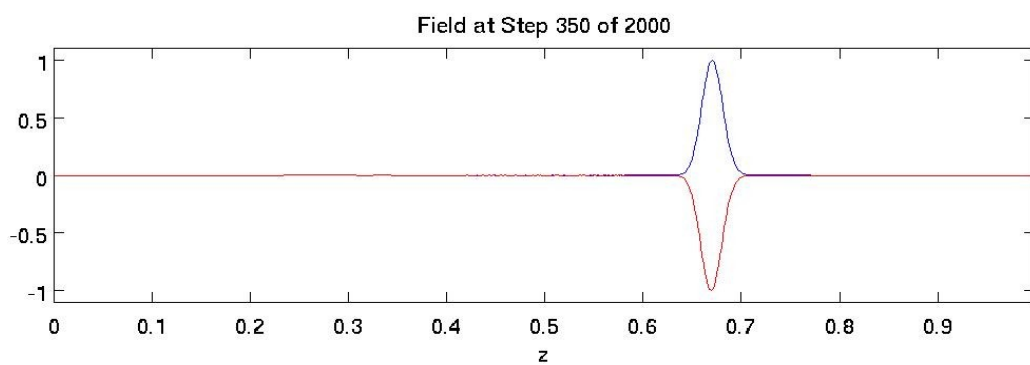


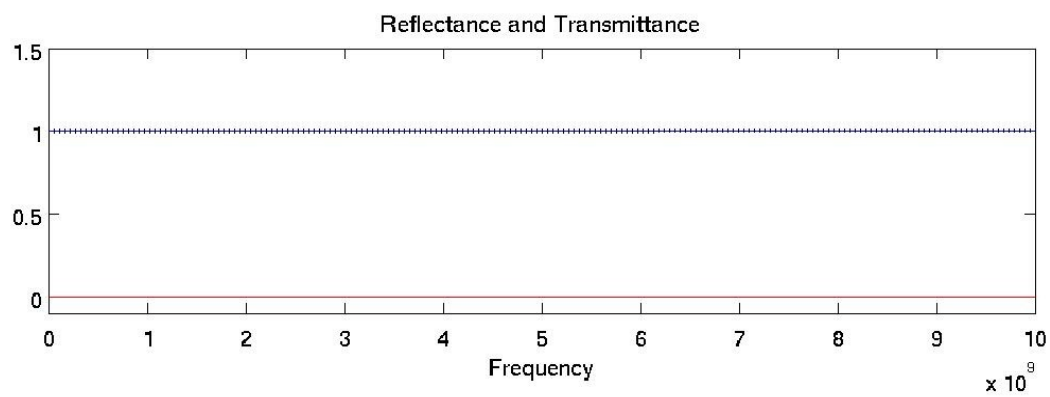
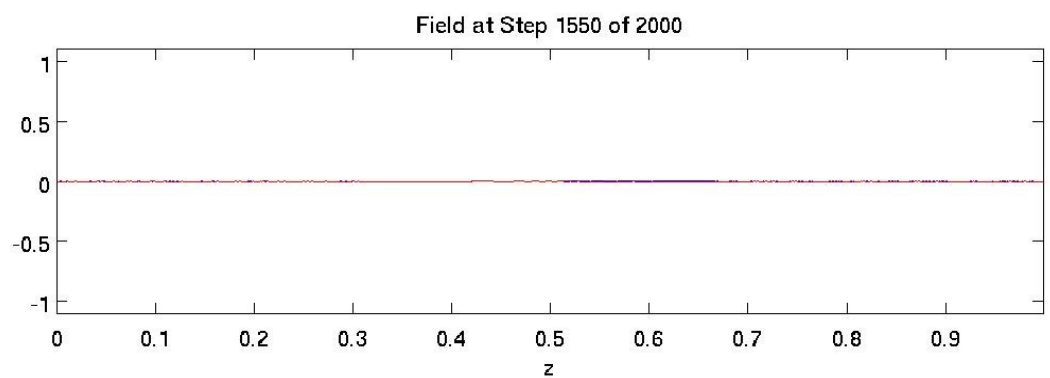
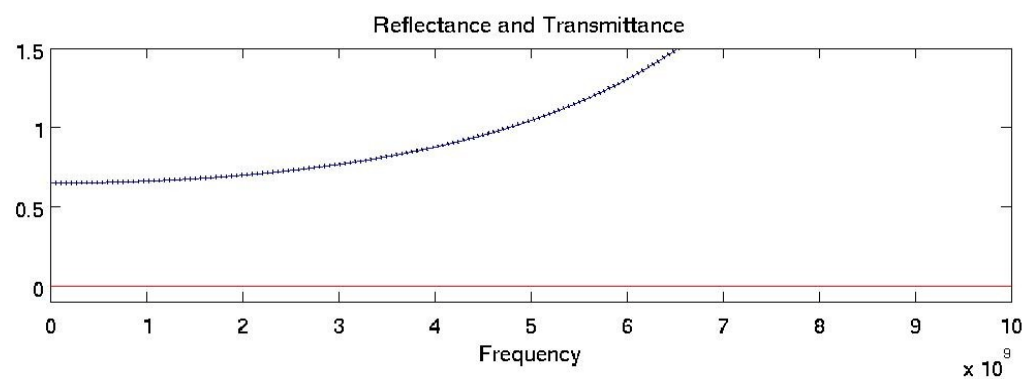
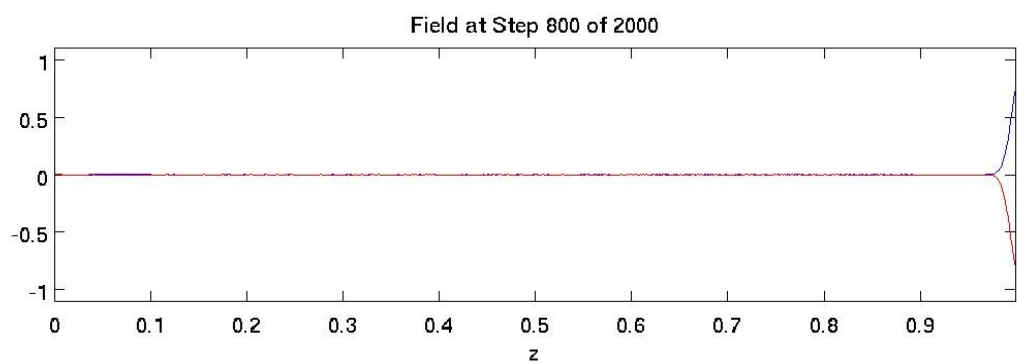
Figure 11: 1D Grid Source Boundary

Problem 2









Appendix – P2 Code

%FDTD1D

Pre-Program Work

```
% Initialize MATLAB
```

```
close all; clc;
clear all;
```

%Constants

```
c0 = 299792458; %m/s
e0 = 8.854187817*10^-12; %F/m
u0 = 1.256637061*10^-6; %H/m
```

```

% Initialization

```

%Simulated Environment Settings

```
STEPS = 2000;  
Nz = 180;  
dz = 0.02;  
f_max = 10e9; % 10Ghz
```

```
%Compute Grid Resolution
```

```
N_lambda = 20;
lambda_min = c0 / (f_max);
d_wl = lambda_min/N_lambda;
N_d = 4;
d_d = 1/4; % since we are only working with freespace we will set d to 1;
dz = min(d_wl, d_d);
Nz = ceil(1/dz);
dz = 1/Nz;
```

```
%Grid Axis
```

```
za=[0:Nz-1]*dz;
```

%Compute Time Steps

```
dt = dz/(2*c0); %secs
ta = [0:STEPS-1]*dt; % Time Axis;
```

% Source Parameters

```
nzc = round (Nz/2); %Position of Sources
NFREQ = f_max / 10e6; %Frequencies every 100Mhz upto 10Ghz
FREQ = linspace(0, f_max, NFREQ); %FREQ List
tau = 0.5/f_max; % tau parameter
t0 = 6*tau; % Delay/Pulse Position
```

```
% Model
```

```

%Material Vectors
ER = ones([1 Nz]);
UR = ones([1 Nz]);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Source
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
s = dz/(2*c0) + dt/2; % Delay between E and H
Esrc = exp(-((ta-t0)/tau).^2); % E Source
A = -sqrt(ER(nzc)/UR(nzc)); % H Amplitude
Hsrc = A*exp(-((ta-t0+s)/tau).^2); % H Source

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%FDTD Initialization
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Compute Update Coefficients
mER = (c0*dt/dz)./ER;
mHR = (c0*dt/dz)./UR;

% Initialize Feilds
Ey = zeros([1 Nz]);
Hx = zeros([1 Nz]);

%PAB Parameters
h1 = 0; h2 = 0; h3 = 0;
e1 = 0; e2 = 0; e3 = 0;

%Power Measurements
REF = zeros(1, NFREQ);
TRN = zeros(1, NFREQ);
SRC = zeros(1, NFREQ);
K = exp(-1i*2*pi*dt*FREQ);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Execute Simulation
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for t = 1:STEPS

    % Calculate H
    for nz = 1:Nz-1
        Hx(nz) = Hx(nz) + mHR(nz)*(Ey(nz+1)-Ey(nz));
    end

    Hx(Nz) = Hx(Nz) + mHR(Nz)*(e3 - Ey(Nz));

    %H Sources
    Hx(nzc-1) = Hx(nzc-1) - mHR(nzc-1)*Esrc(t);

    h3 = h2; h2 = h1; h1 = Hx(1); % Boundary Params;

    % Calculate E
    Ey(1) = Ey(1) + mER(1)*(Hx(1) - h3);
    for nz = 2:Nz
        Ey(nz) = Ey(nz) + mER(nz)*(Hx(nz)-Hx(nz-1));
    end
end

```

```

end

%Inject Source
Ey(nzc) = Ey(nzc) - mER(nzc)*Hsrc(t);

e3=e2; e2=e1; e1=Ey(Nz); % Boundary Params;

%Update Fourier Transforms
for nf = 1: NFREQ
    REF(nf) = REF(nf) + (K(nf)^t)*Ey(1)*dt;
    TRN(nf) = TRN(nf) + (K(nf)^t)*Ey(Nz)*dt;
    SRC(nf) = SRC(nf) + (K(nf)^t)*Esrc(t)*dt;
end

if(mod(t,10) == 0)
    h = subplot(11,1,1:4);
    plot(za, Ey, '-b'); hold on;
    plot(za, Hx, '-r'); hold off;
    axis([za(1) za(Nz) -1.1 1.1]);
    xlabel('z');
    title(['Field at Step ' num2str(t) ' of ' num2str(STEPS)]);

    R = abs(REF./SRC).^2;
    T = abs(TRN./SRC).^2;

    subplot(11,1,8:11)
    plot(FREQ, R, '-r'); hold on;
    plot(FREQ, T, '-b');
    plot(FREQ, R+T, ':k', 'LineWidth', 2); hold off;
    axis([FREQ(1) FREQ(NFREQ) -0.1 1.5]);
    xlabel('Frequency');
    title('Reflectance and Transmittance');
end

drawnow();

%if(mod(t,50) == 0)
%    saveas(h, ['images/' num2str(t) '.jpg'], 'jpg');
%end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Compute Values
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

REF = abs(REF./SRC).^2;
TRN = abs(TRN./SRC).^2;
CON = REF+TRN;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plot Fields
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fig = figure;
SetFigure(fig, 'HW#5-P2', [500 274 965 826]);

plot(FREQ, REF, '-r', 'LineWidth', 2); hold on;

```



```
plot(FREQ, TRN, '-b', 'LineWidth', 2);  
plot(FREQ, CON, ':k', 'LineWidth', 3); hold off;  
axis([FREQ(1) FREQ(NFREQ) -0.1 1.2]);  
xlabel('Frequency');  
title('Reflectance and Transmittance');
```