Modeling 1D Problems with FDTD

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Abstract

This document describes the reduction from a 3-Dimensional Finite Difference set of Equations as derived in "Maxwell Equations" to a 1-Dimensional Form ready for use in numerical modeling algorithms.

Reduce to One-Dimension

We will reduce our Finite-Difference Equations to 1D. This means that we will have material that is uniform in two directions. This uniform material will cause the fields to be uniform as well. The change in material in our case we will define to be in the z-drection therefore the x anx y directions will be uniform. This means that the derivates in this direction will also equal zero.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

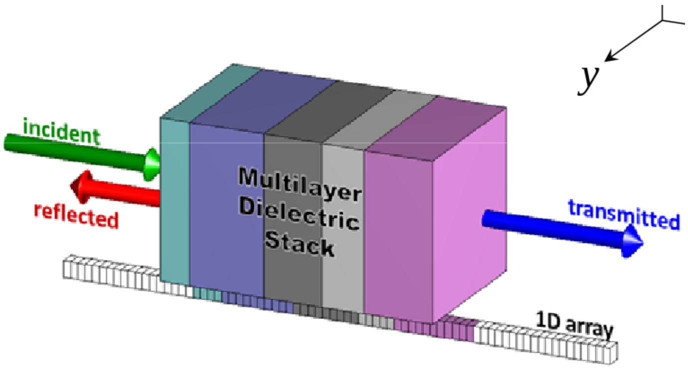


Figure 1: 1D Problem

From our Finite-Difference equations we need to cancel out the x and y derivates that are zero.

$$\cancel{\frac{\partial E_z}{\partial y}} - \frac{\partial E_y}{\partial z} = -\frac{\mu_{xx}}{c_0}\frac{\partial \tilde{H}_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \cancel{\frac{\partial E_z}{\partial x}} = -\frac{\mu_{yy}}{c_0}\frac{\partial \tilde{H}_y}{\partial t}$$

$$\cancel{\frac{\partial E_y}{\partial x}} - \cancel{\frac{\partial E_x}{\partial y}} = -\frac{\mu_{zz}}{c_0}\frac{\partial \tilde{H}_z}{\partial t}$$

$$\cancel{\frac{\partial \tilde{H}_z}{\partial y}} - \frac{\partial \tilde{H}_y}{\partial z} = \frac{\epsilon_{xx}}{c_0}\frac{\partial E_x}{\partial t}$$

$$\frac{\partial \tilde{H}_x}{\partial z} - \cancel{\frac{\partial \tilde{H}_z}{\partial x}} = \frac{\epsilon_{yy}}{c_0}\frac{\partial E_y}{\partial t}$$

$$\cancel{\frac{\partial \tilde{H}_y}{\partial x}} - \cancel{\frac{\partial \tilde{H}_x}{\partial y}} = \frac{\epsilon_{zz}}{c_0}\frac{\partial E_z}{\partial t}$$

$$\longrightarrow$$

$$\cancel{\frac{E_z^{i,j+1,k}\mid_t - E_z^{i,j,k}\mid_t}{\Delta y}} - \frac{E_y^{i,j,k+1}\mid_t - E_y^{i,j,k}\mid_t}{\Delta z} = -\frac{\mu_{xx}^{i,j,k}}{c_0}\frac{\tilde{H}_x^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k}\mid_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\frac{E_x^{i,j,k+1}\mid_t - E_x^{i,j,k}\mid_t}{\Delta z} - \cancel{\frac{E_z^{i+1,j,k}\mid_t - E_z^{i,j,k}\mid_t}{\Delta x}} = -\frac{\mu_{yy}^{i,j,k}}{c_0}\frac{\tilde{H}_y^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k}\mid_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\cancel{\frac{E_y^{i+1,j,k}\mid_t - E_y^{i,j,k}\mid_t}{\Delta x}} - \cancel{\frac{E_x^{i,j+1,k}\mid_t - E_x^{i,j,k}\mid_t}{\Delta y}} = -\frac{\mu_{zz}^{i,j,k}}{c_0}\frac{\tilde{H}_z^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j,k}\mid_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\cancel{\frac{\tilde{H}_z^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i,j-1,k}\mid_{t+\frac{\Delta t}{2}}}{\Delta y}} - \frac{\tilde{H}_y^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1}\mid_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{xx}^{i,j,k}}{c_0}\frac{E_x^{i,j,k}\mid_{t+\Delta t} - E_x^{i,j,k}\mid_t}{\Delta t}$$

$$\frac{\tilde{H}_x^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1}\mid_{t+\frac{\Delta t}{2}}}{\Delta z} - \cancel{\frac{\tilde{H}_z^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_z^{i-1,j,k}\mid_{t+\frac{\Delta t}{2}}}{\Delta x}} = \frac{\epsilon_{yy}^{i,j,k}}{c_0}\frac{E_y^{i,j,k}\mid_{t+\Delta t} - E_y^{i,j,k}\mid_t}{\Delta t}$$

$$\cancel{\frac{\tilde{H}_y^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i-1,j,k}\mid_{t+\frac{\Delta t}{2}}}{\Delta x}} - \cancel{\frac{\tilde{H}_x^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j-1,k}\mid_{t+\frac{\Delta t}{2}}}{\Delta y}} = \frac{\epsilon_{zz}^{i,j,k}}{c_0}\frac{E_z^{i,j,k}\mid_{t+\Delta t} - E_z^{i,j,k}\mid_t}{\Delta t}$$

With these cross outs we come out with two observations. The first is that the longitudinal field components Ez and Hz are always zero. The second is that the Maxwell's equations have decoupled into two sets of two equations Ex/Hy Mode and Ey/Hx Mode.

$$-\frac{\partial E_y}{\partial z} = -\frac{\mu_{xx}}{c_0}\frac{\partial \tilde{H}_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} = -\frac{\mu_{yy}}{c_0}\frac{\partial \tilde{H}_y}{\partial t}$$

$$0 = -\frac{\mu_{zz}}{c_0}\frac{\partial \tilde{H}_z}{\partial t}$$

$$-\frac{\partial \tilde{H}_y}{\partial z} = \frac{\epsilon_{xx}}{c_0}\frac{\partial E_x}{\partial t}$$

$$\frac{\partial \tilde{H}_x}{\partial z} = \frac{\epsilon_{yy}}{c_0}\frac{\partial E_y}{\partial t}$$

$$0 = \frac{\epsilon_{zz}}{c_0}\frac{\partial E_z}{\partial t}$$

$$\longrightarrow$$

$$-\frac{E_y^{i,j,k+1}\mid_t - E_y^{i,j,k}\mid_t}{\Delta z} = -\frac{\mu_{xx}^{i,j,k}}{c_0}\frac{\tilde{H}_x^{i,j,k}\mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k}\mid_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\frac{E_x^{i,j,k+1}\,|_t - E_x^{i,j,k}\,|_t}{\Delta z} = -\frac{\mu_{yy}^{i,j,k}}{c_0}\frac{\tilde{H}_y^{i,j,k}\,|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k}\,|_{t-\frac{\Delta t}{2}}}{\Delta t}$$

$$\tilde{H}_z^{i,j,k} = 0$$

$$-\frac{\tilde{H}_y^{i,j,k}\,|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{i,j,k-1}\,|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{xx}^{i,j,k}}{c_0}\frac{E_x^{i,j,k}\,|_{t+\Delta t} - E_x^{i,j,k}\,|_t}{\Delta t}$$

$$\frac{\tilde{H}_x^{i,j,k}\,|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{i,j,k-1}\,|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{yy}^{i,j,k}}{c_0}\frac{E_y^{i,j,k}\,|_{t+\Delta t} - E_y^{i,j,k}\,|_t}{\Delta t}$$

$$E_z^{i,j,k} = 0$$

Summary of 1D Finite-Difference Equations

The equations are now broken apart into their respective modes. These modes are physical and propage independently from each other. In the end though they are numerically the same and will exhibit the same electromagnetic behavior. This allows us to only have to solve one set of two equations. We can also note that since we only changing materials along the z axis we can drop the i and j Array Indices.

Ex/Hy Mode

$$-\frac{\tilde{H}_y^{k}\,|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k-1}\,|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{xx}^{k}}{c_0}\frac{E_x^{k}\,|_{t+\Delta t} - E_x^{k}\,|_t}{\Delta t} \tag{1}$$

$$\frac{E_x^{k+1}\,|_t - E_x^{k}\,|_t}{\Delta z} = -\frac{\mu_{yy}^{k}}{c_0}\frac{\tilde{H}_y^{k}\,|_{t+\frac{\Delta t}{2}} - \tilde{H}_y^{k}\,|_{t-\frac{\Delta t}{2}}}{\Delta t} \tag{2}$$

Ey/Hx Mode

$$\frac{\tilde{H}_x^{k}\,|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}\,|_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{yy}^{k}}{c_0}\frac{E_y^{k}\,|_{t+\Delta t} - E_y^{k}\,|_t}{\Delta t} \tag{3}$$

$$\frac{E_y^{k+1}\,|_t - E_y^{k}\,|_t}{\Delta z} = \frac{\mu_{xx}^{k}}{c_0}\frac{\tilde{H}_x^{k}\,|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k}\,|_{t-\frac{\Delta t}{2}}}{\Delta t} \tag{4}$$

Update Equations

We are now going to derive the update equations used during in FDTD algorithm. We have arbitrarily chosen to use Ey/Hx Mode found in Equations (3) and (4)

Update Equation for Ey

$$\frac{\tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} \mid_{t+\frac{\Delta t}{2}}}{\Delta z} = \frac{\epsilon_{yy}^k}{c_0} \frac{E_y^k \mid_{t+\Delta t} - E_y^k \mid_t}{\Delta t}$$

We want to solve for the E-Field at the future time value.

$$\frac{\epsilon_{yy}^k}{c_0} \frac{E_y^k \mid_{t+\Delta t} - E_y^k \mid_t}{\Delta t} = \frac{\tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} \mid_{t+\frac{\Delta t}{2}}}{\Delta z}$$

$$E_y^k \mid_{t+\Delta t} - E_y^k \mid_t = \frac{c_0 \Delta t}{\epsilon_{yy}^k} \left(\frac{\tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} \mid_{t+\frac{\Delta t}{2}}}{\Delta z} \right)$$

$$E_y^k \mid_{t+\Delta t} = E_y^k \mid_t + \left(\frac{c_0 \Delta t}{\epsilon_{yy}^k} \right) \left(\frac{\tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} \mid_{t+\frac{\Delta t}{2}}}{\Delta z} \right) \tag{5}$$

Update Equation for Hx

$$\frac{E_y^{k+1} \mid_t - E_y^k \mid_t}{\Delta z} = \frac{\mu_{xx}^k}{c_0} \frac{\tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k \mid_{t-\frac{\Delta t}{2}}}{\Delta t}$$

We want to sove for the H-Field at the future time value

$$\frac{\mu_{xx}^k}{c_0} \frac{\tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k \mid_{t-\frac{\Delta t}{2}}}{\Delta t} = \frac{E_y^{k+1} \mid_t - E_y^k \mid_t}{\Delta z}$$

$$\tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^k \mid_{t-\frac{\Delta t}{2}} = \frac{c_0 \Delta t}{\mu_{xx}^k} \left(\frac{E_y^{k+1} \mid_t - E_y^k \mid_t}{\Delta z} \right)$$

$$\tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k \mid_{t-\frac{\Delta t}{2}} + \left(\frac{c_0 \Delta t}{\mu_{xx}^k} \right) \left(\frac{E_y^{k+1} \mid_t - E_y^k \mid_t}{\Delta z} \right) \tag{6}$$

Update Coefficients

Since the update coefficients don't change during the simulation. We can compute them once before our FDTD Algorithm is actually implemented in loop.

$$\tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k \mid_{t-\frac{\Delta t}{2}} + m_{H_x}^k \left(\frac{E_y^{k+1} \mid_t - E_y^k \mid_t}{\Delta z} \right) \tag{7}$$

$$E_y^k \mid_{t+\Delta t} = E_y^k \mid_t + m_{E_y}^k \left(\frac{\tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1} \mid_{t+\frac{\Delta t}{2}}}{\Delta z} \right) \tag{8}$$

$$m_{E_y}^k = \frac{c_0 \Delta t}{\epsilon_{yy}^k} \tag{9}$$

$$m_{H_x}^k = \frac{c_0 \Delta t}{\mu_{yy}^k} \tag{10}$$

Boundary Conditions

We have two types of boundary conditions that can be incorporated into this 1-Dimensional Simulation. The Dirichlet and Perfectly Absorbing Boundary Conditions are methods that allow us to handle the fields that lie at the boundary of the model. Specifically these points lie at $k = 0$ and $k = N_z + 1$. At these points there is no grid to pull our Electric and Magnetic fields from, therefore to maintain the stability of the Finite Difference Maxwell's Equations we must handle these seperatly.

Dirichlet Boundary Condition

This boundary condition is the easiest to implement. We simply assume the fields that lie outside the grid are zero. This is the most crude method and will cause reflection back into our model as energy reaches the boundary.

$$\text{For } k < N_z \quad \tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k \mid_{t-\frac{\Delta t}{2}} + m_{H_x}^k \left(\frac{E_y^{k+1}|_t - E_y^k|_t}{\Delta z} \right)$$

$$\text{For } k = N_z \quad \tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k \mid_{t-\frac{\Delta t}{2}} + m_{H_x}^k \left(\frac{0 - E_y^k|_t}{\Delta z} \right)$$

$$\text{For } k > 1 \quad E_y^k \mid_{t+\Delta t} = E_y^k \mid_t + m_{E_y}^k \left(\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} \right)$$

$$\text{For } k = 1 \quad E_y^k \mid_{t+\Delta t} = E_y^k \mid_t + m_{E_y}^k \left(\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} - 0}{\Delta z} \right)$$

Periodic Boundary Condition

This boundary condition assumes that the device being modeled is periodic along a particular direction. In our case the periodicity is along the z-axis.

$$\text{For } k < N_z \quad \tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k \mid_{t-\frac{\Delta t}{2}} + m_{H_x}^k \left(\frac{E_y^{k+1}|_t - E_y^k|_t}{\Delta z} \right)$$

$$\text{For } k = N_z \quad \tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k \mid_{t-\frac{\Delta t}{2}} + m_{H_x}^k \left(\frac{E_y^1|_t - E_y^{N_z}|_t}{\Delta z} \right)$$

$$\text{For } k > 1 \quad E_y^k \mid_{t+\Delta t} = E_y^k \mid_t + m_{E_y}^k \left(\frac{\tilde{H}_x^k|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{k-1}|_{t+\frac{\Delta t}{2}}}{\Delta z} \right)$$

$$\text{For } k = 1 \quad E_y^k \mid_{t+\Delta t} = E_y^k \mid_t + m_{E_y}^k \left(\frac{\tilde{H}_x^1|_{t+\frac{\Delta t}{2}} - \tilde{H}_x^{N_z}|_{t+\frac{\Delta t}{2}}}{\Delta z} \right)$$

Perfectly Absorbing Boundary Condition

The Perfectly Absorbing Boundary Condition provides a method to absorb almost all of the energy out of the model. There is approximately 10 orders of magnitude less energy reflected back into the problem space. This is very minor and should not affect our results. There are three conditions though that must be met in order to use this boundary conditionl.

- Waves at the boundaries are only traveling outward.
- Materials at the boundaries are linear, homogeneous, isotropic and non-dispersive.
- Time step is chosen so physical waves travel 1 cell in two time steps $\Delta t = \frac{\Delta z}{2c_o}$

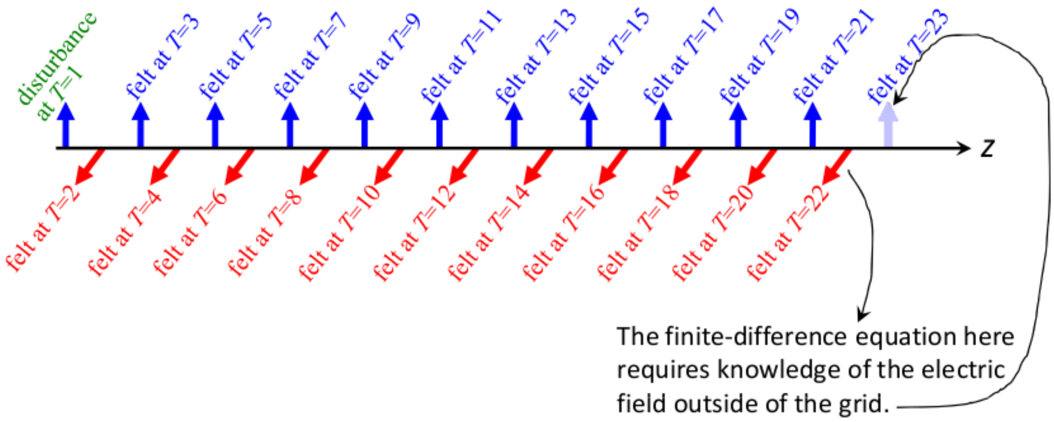


Figure 2: Perfeect Boundary Time Steps

As can be seen in 5.3 the last time step is magically off the grid. In order to implement the perfect boundary condition we must use a field that would have propagated past our last grid cell $2\Delta t$ time steps ago. In equations (11) and (12) the field value that exists outside our cells needs to be used when the wave reaches it at the $2\Delta t$ time frame

$$\tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k \mid_{t-\frac{\Delta t}{2}} + m_{H_x}^k \left(\frac{E_y^{N_z+1} \mid_t - E_y^{N_z} \mid_t}{\Delta z} \right) \tag{11}$$

$$E_y^k \mid_{t+\Delta t} = E_y^k \mid_t + m_{E_y}^k \left(\frac{\tilde{H}_x^1 \mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x^0 \mid_{t+\frac{\Delta t}{2}}}{\Delta z} \right) \tag{12}$$

Since the field values did exist $2\Delta t$ ago in our boundary cell in the problem space. We can then infer the value is the same as the energy leaves the grid.

$$E_y^{N_z+2} \mid_t = E_y^{N_z} \mid_{t-2\Delta t}$$

$$\tilde{H}_x^0 \mid_t = \tilde{H}_x^1 \mid_{t-2\Delta t}$$

To implement the boundary condition we keep track of previous history in variables. At the z-low boundary we only modify the E-Field update equations. At the z-high boundary we only modify the H-Field update equations.

$$h_3 = h_2 \quad h_2 = h_1 \quad h1 = \tilde{H}_x^1$$

$$E_y^k \mid_{t+\Delta t} = E_y^1 \mid_t + m_{E_y}^1 \left(\frac{\tilde{H}_x^1 \mid_{t+\frac{\Delta t}{2}} - h_3}{\Delta z} \right)$$

$$e_3 = e_2 \quad e_2 = e_1 \quad e_1 = E_y^{N_z}$$

$$\tilde{H}_x^{N_z} \mid_{t+\frac{\Delta t}{2}} = \tilde{H}_x^{N_z} \mid_{t-\frac{\Delta t}{2}} + m_{H_x}^{N_z} \left(\frac{e_3 - E_y^{N_z} \mid_t}{\Delta z} \right)$$

Sources

There are two ways to incorporate a source into our models, Hard Source and Soft Source. The hard source is the simplest to implement. After we have performed our update equations over the entire grid we overwrite some fields with our source. This injects the source energy into our model. This does cause a problem because the point where the source was inject behaves like a perfect electric or magnetic conductor to scattered fields.

$$\tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} = g_H \mid_k \quad E_y^k \mid_{t+\Delta t} = g_E \mid_k$$

The soft source on the otherhand is the best way to inject energy into our model. After performing the udpate equations across the entire grid, the source is added to the current value of our source injection points. This injects energy into the model, but allows scattered waves to pass through the source region like it was transparent.

$$\tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} = \tilde{H}_x^k \mid_{t+\frac{\Delta t}{2}} + g_H \mid_k \quad E_y^k \mid_{t+\Delta t} = E_y^k \mid_{t+\Delta t} + g_E \mid_k$$

Model Parameters

Grid Resolution

There are multiple considerations that need to be addressed in order to properly calculate the grid resolution.

Wavelength

The grid resolution must be sufficient to resolve the shortest wavelength. First we must determine the smallest wavelength then we must devide this by a value that can be found in the chart. These values are used as a guideline depending on the material with the highest refractive index.

$$\lambda_{min} = \frac{c_0}{f_{max} nmax}$$

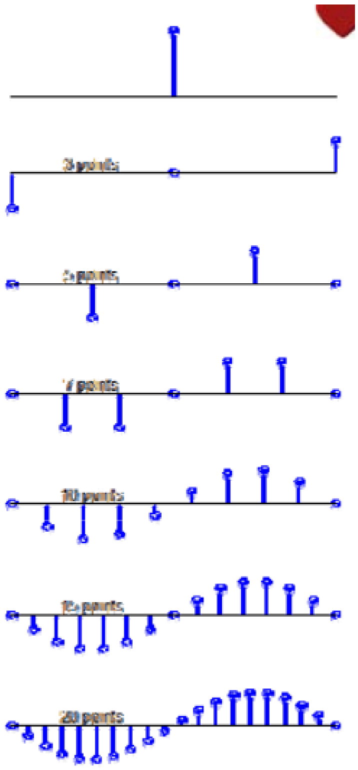


Figure 3: Building up wavelength resolution

$$\Delta_\lambda \approx \frac{\lambda_{min}}{N_\lambda} \quad N_\lambda \geq 10$$

Approximate N_λ Values		
N_λ	Refractive Index	Comment
10 to 20	$n = 1 - 10$	Low contrast dielectrics
10 to 30	$n = 10 - 30$	High contrast dielectrics
40 to 60	$n = 40 - 60$	Most metallic structures
100 to 200	$n \geq 60$	Plasmonic devices

Mechanical Features

The grid resolution mst be sufficient to resolve the smallest physical feature of the device. First, we must determine the smallest feature 7.1 d_{min}. then we must device this by 1 to 4 to accurately manage the width of our smallest feature.

$$\Delta_d \approx \frac{d_{min}}{N_d} \quad N_d \geq 1$$

Calculate Grid Resolution

We will first calculate an initial grid resolution by taking the minimum between the model feature Δ_d and wavelength Δ_λ.

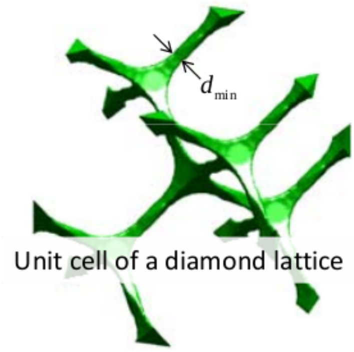


Figure 4:

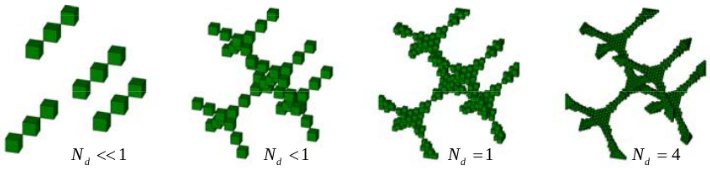


Figure 5:

$$\Delta_x = \Delta_y = min[\Delta_\lambda, \Delta_d]$$

Finally we need to "Snap" grid to critical dimensions. In 7.1 we have modified delta to not fall halfway through a grid cell. We can't fill half a cell in a discrete environment. First we Decide what dimensions along each axis are critical. d_x and d_y. We then compute how many grid cells comprise d_c and round up (13). Finally adjust grid resolution to fit this dimension in grid exactly (14).

$$\begin{aligned} M_x &= ceil(\frac{d_x}{\Delta_x}) \\ M_y &= ceil(\frac{d_y}{\Delta_y}) \end{aligned} \tag{13}$$

$$\begin{aligned} \Delta_x &= \frac{d_x}{M_x} \\ \Delta_y &= \frac{d_y}{M_y} \end{aligned} \tag{14}$$

Courant Stability Condition

For each iteration of our update equations a currently existing electric field will not immediatly affect adjacent fields. A disturbance in the electric field at one point will only be felt by the immediately adjacent magnetic fields. For an adjancent electric field to feel the disturbance it takes at least two time steps.

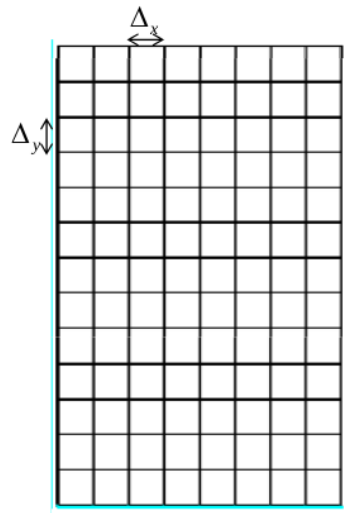


Figure 6: Grid X/Y

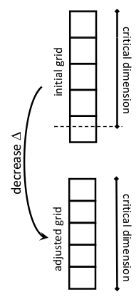


Figure 7: Critical Dimension Expansion

Keeping in mind that Electromagnetic waves propagate at the speed of light in a vacuum. Inside a material, a wave propagates at a reduce speed based on the refractive index (15)

$$v = \frac{c_0}{n} \tag{15}$$

Because we are dealing with a numerical algorithm it is not possible for a disturbance to travel farther than one unit cell in a single time step. So we need to force a condition that would force a wave not to propage farther than a single unit cell in one time step.

$$\frac{c_0 \Delta t}{n} < \Delta z$$

So now we can place an upper limit on a time steps

$$\Delta t < \frac{n \Delta z}{c_0}$$

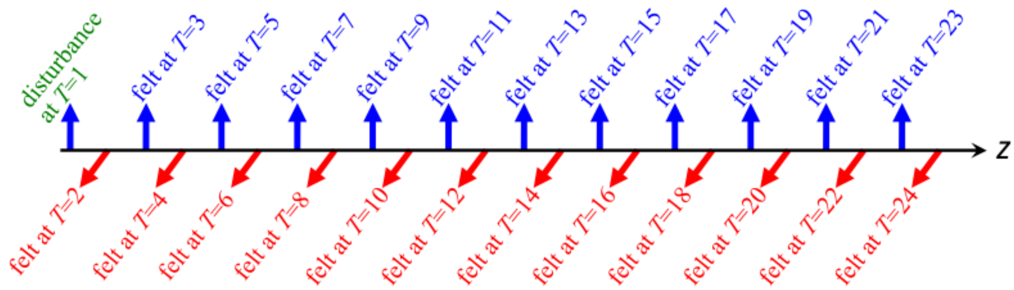


Figure 8: Timeline of electric field propation

To force our wave to travel the distance of one grid cell in exactly two time steps we finish up with (16)

$$\Delta t = \frac{n\Delta z}{2c_0} \tag{16}$$

Learning from our Model

Efficent Fourier Transform

In order to be able to learn from our model we must be able to somehow collect the data and put it in a form that is useful. In our case we can use the Fourier Transform to calculate the transmitted and reflected energy. In order to save CPU and Memory consumption insteam of using the traditional method of recording the fields as functions of time. Then perform the FFT on these functions. We are instead going to perform a numerical approximation of the Fourier Transform as the model is running. In each iteration of the model we can sum up parts of the energy that exist as time goes by.

The standard Fourier transform is defined in (17)

$$F(f) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft} \tag{17}$$

Since we only know the f(t) at descrete points, the Fourier transform is approximated numerically in (18)

$$F(f) \cong \sum_{m=1}^{M} \Delta t f(m\Delta t)e^{-j2\pi fm\Delta t} \tag{18}$$

We can write this in a form that is beneficial for numerical modeling (19)

$$F(f) \cong \Delta t \sum_{m=1}^{M} \left(e^{-j2\pi f\Delta t}\right)^m f(m\Delta t) \tag{19}$$

Finally we can break this down into parts

$$F(f) \cong \Delta t \sum_{m=1}^{M} \left(e^{-j2\pi f\Delta t}\right)^m f(m\Delta t) \quad (2$$

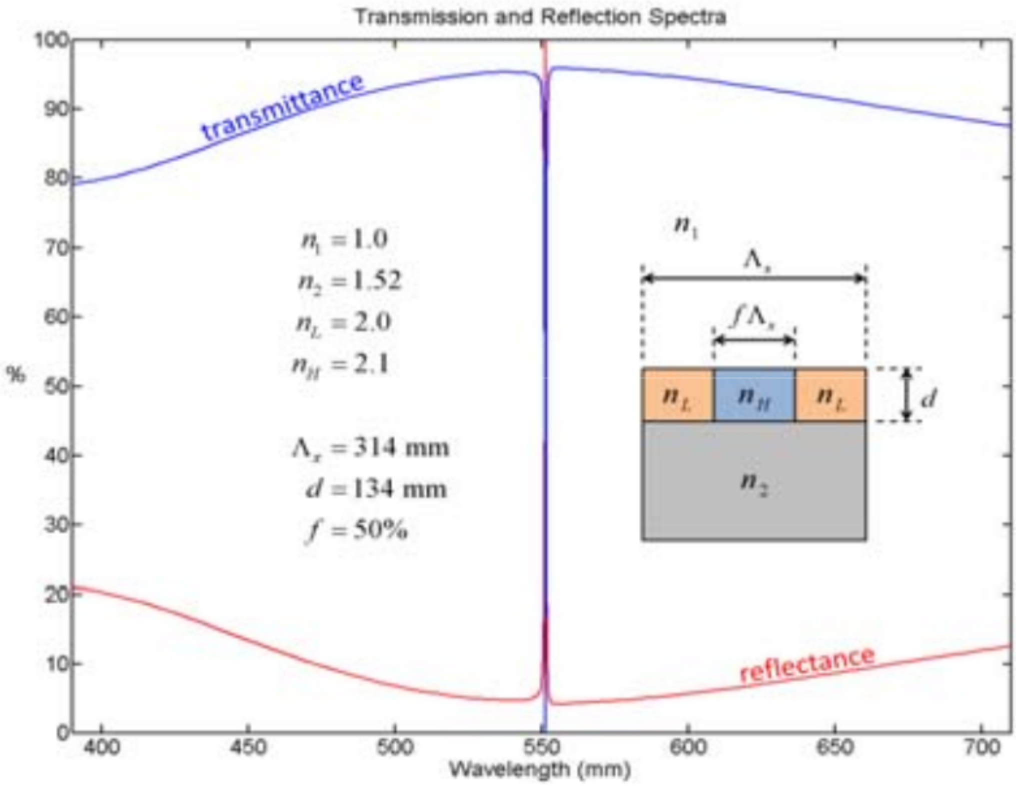
Δt Multiplication can be done after the main FDTD loop in a post-processing step (2

$e^{-j2\pi f\Delta t}$ This "kernel" can be computed prior to the main FDTD loop for each frequency of interest(2

$f(m\Delta t)$ This is simply the field value of interst at the current time step. (2

Reflectance and Transmittance

When simulating passive device we generally want to plot an ouput of reflectance and transmittance as a function of frequency. Reflectance is the fraction of power reflected from a device. Transmittance is the fraction of power transmitted through the device.



We typically start the computation of power by Fourier transforming the reflected and transmitted field using our approximated fourier transform (19). Our results will typically look like figure 8.2

Finally we must normalize the spectra to calculate transmittancee and reflectance. We do this by dividing the reflection and transmission by the source spectrum. We check for energy conversvation by adding the reflectance and transmittance and ensuring the sum equals 100%. This is assuming the device itself as no loss or gain.

$$R(f) = \left(\frac{FFT[E_{ref}(t)]}{FFT[E_{src}(t)]}\right)^2 \quad (24)$$

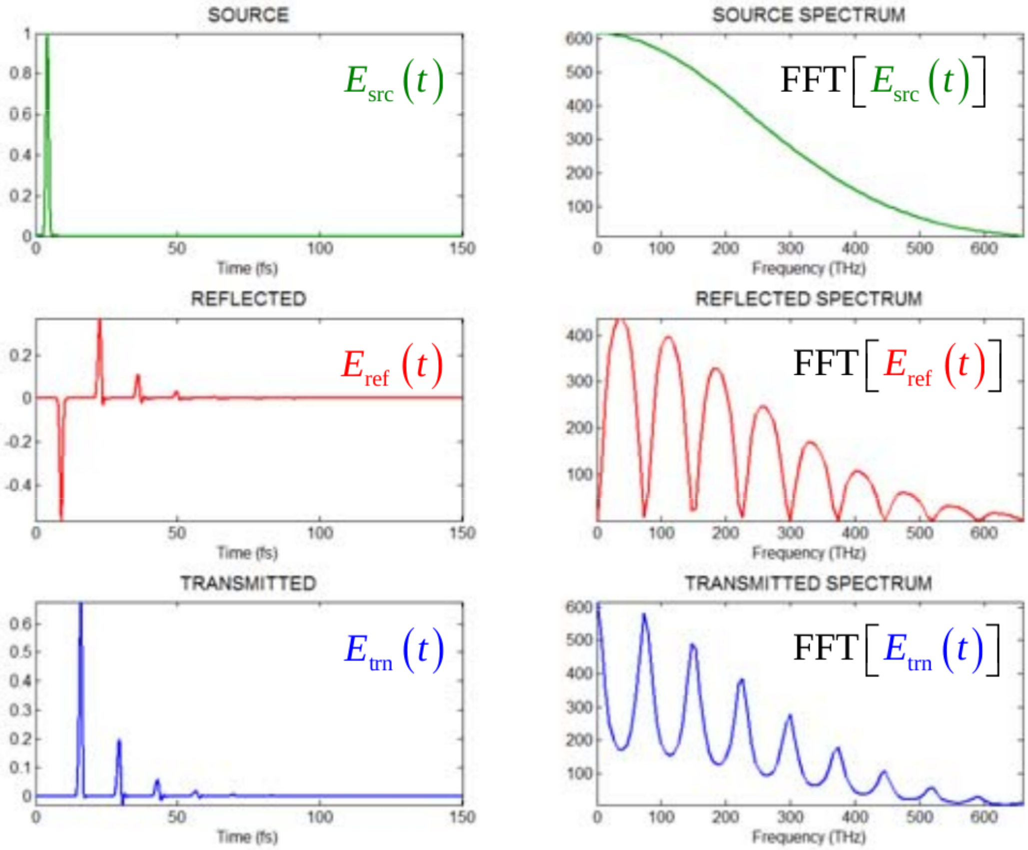


Figure 9: FDTD typical Reflected and Transmitted Power results

$$T(f) = \left(\frac{FFT[E_{trn}(t)]}{FFT[E_{src}(t)]}\right)^2 \tag{25}$$

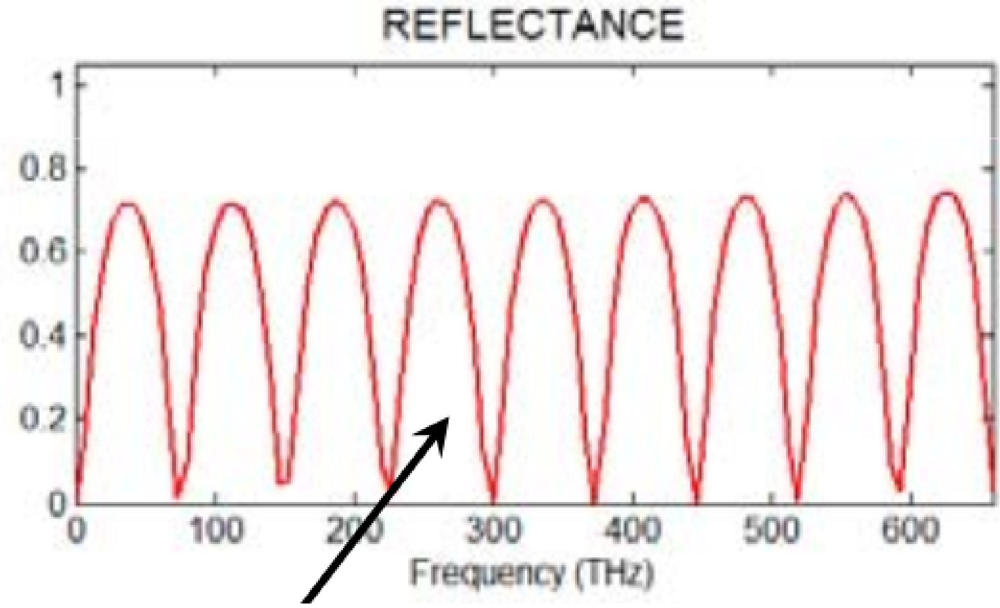
$$C(f) = R(f) + T(f) \tag{26}$$

Total-Field / Scattered-Field

The total-field/scattered-field (TF/SF) is a technique to inject a “one-way” source. This eliminates backward propagating energy, ensures waves at the boundaries are only traveling outward, and 100% of energy inejected by the source is incident on the device being simulated. In figure 8.3 shows the typical grid layout with the TF/SF interface included.

On the scattered field side of the TF/SF interface, the finite-difference equation contains a term from the total-field side. This is due to the staggered nature of the yee grid, this only occurs in the udpate equation for a magnetic field.

$$\tilde{H}_x \mid_{t+\frac{\Delta t}{2}}^{k_{src}-1} = \tilde{H}_x \mid_{t-\frac{\Delta t}{2}}^{k_{src}-1} + m_{H_x}^{k_{src}-1}\left(\frac{E_y \mid_t^{k_{src}} - E_y \mid_t^{k_{src}-1}}{\Delta z}\right)$$



This equation that is trying to calculate a Total-Field value contains a Scattered-Field value $E_y^{k_{src}}$. To fix this issue we must now subtract out the source from this Scattered-Field value.

$$\tilde{H}_x \mid_{t+\frac{\Delta t}{2}}^{k_{src}-1} = \tilde{H}_x \mid_{t-\frac{\Delta t}{2}}^{k_{src}-1} + m_{H_x}^{k_{src}-1} \left(\frac{\left(E_y \mid_t^{k_{src}} - E_y^{src} \mid_t^{k_{src}} \right) - E_y \mid_t^{k_{src}-1}}{\Delta z} \right)$$

Which can be simplified to

$$\tilde{H}_x \mid_{t+\frac{\Delta t}{2}}^{k_{src}-1} = \tilde{H}_x \mid_{t-\frac{\Delta t}{2}}^{k_{src}-1} + m_{H_x}^{k_{src}-1} \left(\frac{E_y \mid_t^{k_{src}} - E_y \mid_t^{k_{src}-1}}{\Delta z} \right) - \frac{m_{H_x}^{k_{src}-1}}{\Delta z} E_y^{src} \mid_t^{k_{src}} \tag{27}$$

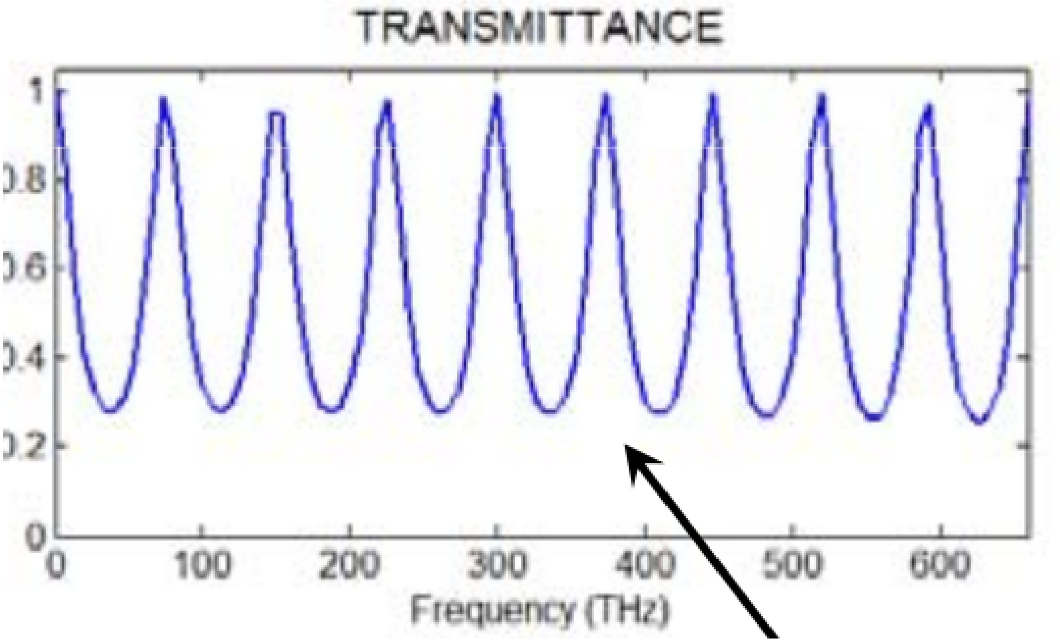
This has to be accomplished for electric field update equation as well.

$$E_y \mid_{t+\Delta t}^{k_{src}} = E_y \mid_t^{k_{src}} + m_{E_y}^{k_{src}} \left(\frac{\tilde{H}_x^{k_{src}} \mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x \mid_{t+\frac{\Delta t}{2}}^{k_{src}-1}}{\Delta z} \right) \tag{28}$$

$$E_y \mid_{t+\Delta t}^{k_{src}} = E_y \mid_t^{k_{src}} + m_{E_y}^{k_{src}} \left(\frac{\tilde{H}_x^{k_{src}} \mid_{t+\frac{\Delta t}{2}} - \left(\tilde{H}_x \mid_{t+\frac{\Delta t}{2}}^{k_{src}-1} + \tilde{H}_x^{src} \mid_{t+\frac{\Delta t}{2}}^{k_{src}-1} \right)}{\Delta z} \right) \tag{29}$$

$$E_y \mid_{t+\Delta t}^{k_{src}} = E_y \mid_t^{k_{src}} + m_{E_y}^{k_{src}} \left(\frac{\tilde{H}_x^{k_{src}} \mid_{t+\frac{\Delta t}{2}} - \tilde{H}_x \mid_{t+\frac{\Delta t}{2}}^{k_{src}-1}}{\Delta z} \right) - \frac{m_{E_y}^{k_{src}}}{\Delta z} \tilde{H}_x^{src} \mid_{t+\frac{\Delta t}{2}}^{k_{src}-1} \tag{30}$$

Source Terms



Now we know we need to calculate two source functions $\tilde{H}_x^{src}\,|_{t+\frac{\Delta t}{2}}^{k_{src}-1}$ and $E_y^{src}\,|_t^{k_{src}}$ We need to make a few observations that must be accounted for before we can calcuate these source functions correctly.

- The amplitude of these functions can be different as E and H are related through the material impedance.
- These functions are a half grid cell apart and have a small time delay between them.
- These functions exist at different time steps.

Assuming the source is injected inside a homogeneious material, the fields must be plane waves.

$$E_y(t) = \sin(\omega t - \beta z) \tag{31}$$

We now must find the amplitude A and phase ϕ of the magnetic field related to the elctric field

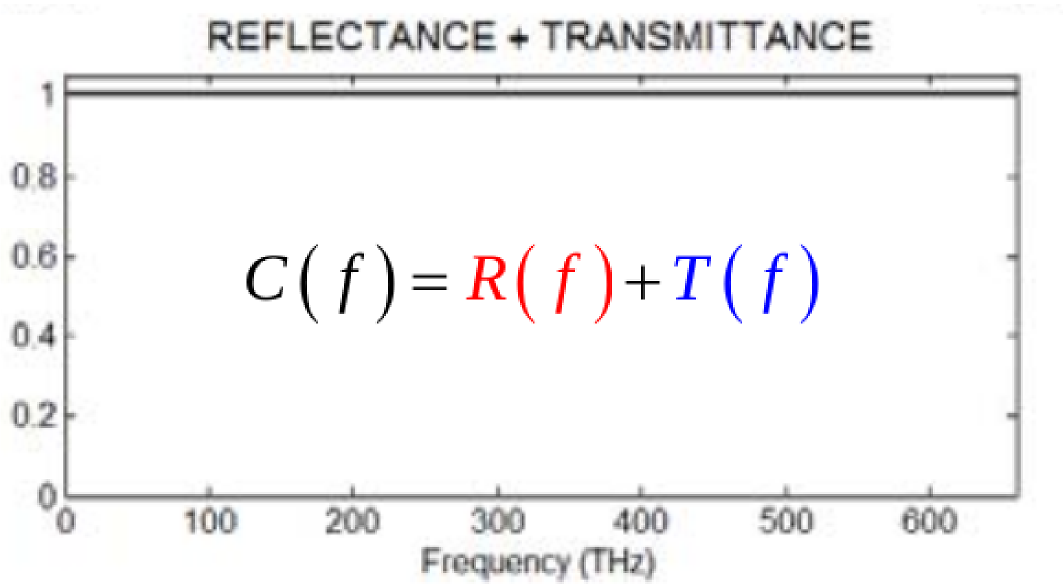
$$\tilde{H}_x(t) = A\sin(\omega t - \beta z - \phi) \tag{32}$$

The Amplitude and phase depend on the mode

$$A_{E_y H_x} = -\sqrt{\frac{\epsilon_r^{src}}{\mu_r^{src}}} \quad \phi_{E_y H_x} = 0 \tag{33}$$

$$A_{E_x H_y} = \sqrt{\frac{\epsilon_r^{src}}{\mu_r^{src}}} \quad \phi_{E_x H_y} = 0 \tag{34}$$

Therefore Ey/Hx mode



$$E_y^{src}\,|_t^{k_{src}} = g(t) \tag{35}$$

$$\tilde{H}_x^{src}\,|_{t+\frac{\Delta t}{2}}^{k_{src}-1} = -\sqrt{\frac{\epsilon_r^{src}}{\mu_r^{src}}}\, g\left(t + \frac{n\Delta Z}{2C_0} + \frac{\Delta t}{2}\right) \tag{36}$$

FDTD Algorithm

We first must do some initialization work

- Compute Grid Resolution
- Compute time step
- Compute Source
- Compute Update Coefficeints
- Intialize Fields

The meat of the algorithm is the loop

- Update H from E
- Update E from H
- Inject Source
- Visualize Fields

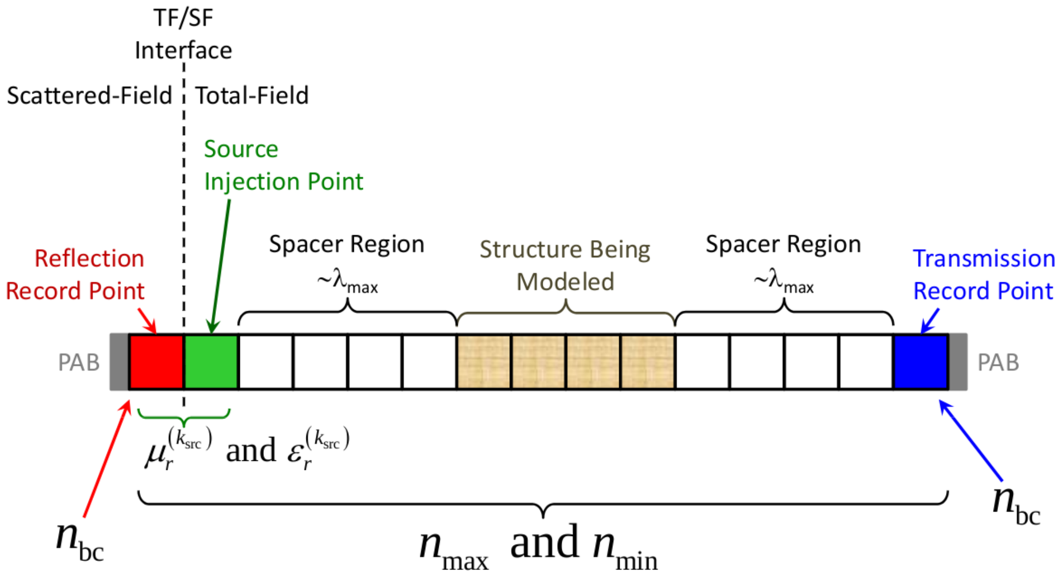


Figure 10: 1D Grid Layout including TF/SF

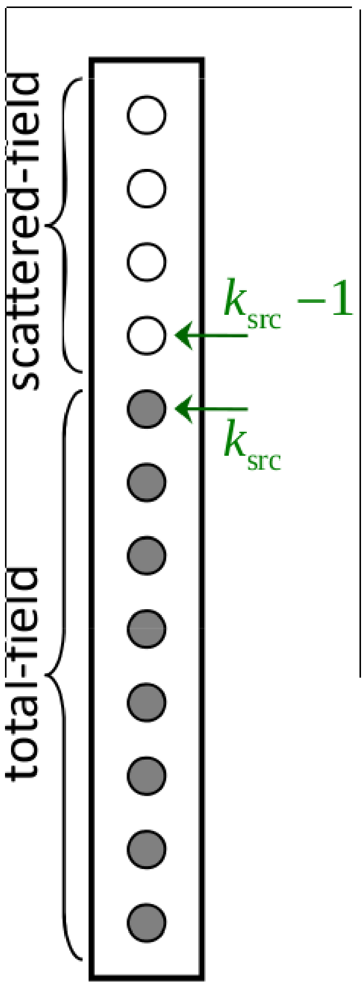


Figure 11: 1D Grid Source Boundary