Average-Interpolating Wavelet Scheme

Brandon Gusto

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Multiresolution Analysis Procedure

We are interested in obtaining the difference between approximation spaces at varying levels of resolution. We are given cell-averaged values as input data to our wavelet transform. This data is fed to the scheme at some arbitrary maximum resolution level J, and the wavelet transform produces details coefficients at each lower level until the coarsest level, $j = 0$, is reached. The coefficients in this case are interchangeable with the cell-averages and are denoted by c_k^j, where the level of resolution is denoted by j, and the spatial index is denoted by k. We consider an interpolating polynomial $p(x)$ such that

$$c_{k-1}^j = \int_{x_{k-1}^j}^{x_k^j} p(x)dx \tag{1}$$

$$c_k^j = \int_{x_k^j}^{x_{k+1}^j} p(x)dx \tag{2}$$

$$c_{k+1}^j = \int_{x_{k+1}^j}^{x_{k+2}^j} p(x)dx. \tag{3}$$

The polynomial $p(x)$ should then predict the finer cell-averages of cell c_k^j as

$$\hat{c}_{2k}^{j+1} = 2\int_{x_k^j}^{x_{k+1/2}^j} p(x)dx \tag{4}$$

$$\hat{c}_{2k+1}^{j+1} = 2\int_{x_{k+1/2}^j}^{x_{k+1}^j} p(x)dx \tag{5}$$

At present, it may not be clear how to implement such a scheme on a computer. However this interpolation procedure can be cast in a more suitable form by introducing another polynomial, the integral of $p(x)$:

$$P(x) = \int_0^x p(y)dy. \tag{6}$$

Now the problem is to interpolate the following data

$$0 = P(x^j_{k-1}) \tag{7}$$

$$c^j_{k-1} = P(x^j_k) \tag{8}$$

$$c^j_{k-1} + c^j_k = P(x^j_{k+1}) \tag{9}$$

$$c^j_{k-1} + c^j_k + c^j_{k+1} = P(x^j_{k+2}). \tag{10}$$

This can easily be done using Lagrange polynomials. Then the predictions are given in terms of $P(x)$ by

$$\hat{c}^{j+1}_{2k} = 2\left(P(x^j_{k+1/2}) - P(x^j_k)\right) \tag{11}$$

$$\hat{c}^{j+1}_{2k+1} = 2\left(P(x^j_{k+1}) - P(x^j_{k+1/2})\right). \tag{12}$$

This interpolating polynomial is cast in the Lagrange form,

$$P(x) = \sum_{i=0}^{n} y_i l_i(x), \tag{13}$$

where y_i are the functional data, and $l_i(x)$ are the Lagrange polynomials. For $n = 3$ these are given by

$$l_0(x) = \frac{x - x_1}{x_0 - x_1}\frac{x - x_2}{x_0 - x_2}\frac{x - x_3}{x_0 - x_3} \tag{14}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0}\frac{x - x_2}{x_1 - x_2}\frac{x - x_3}{x_1 - x_3} \tag{15}$$

$$l_2(x) = \frac{x - x_0}{x_1 - x_0}\frac{x - x_1}{x_2 - x_1}\frac{x - x_3}{x_2 - x_3} \tag{16}$$

$$l_3(x) = \frac{x - x_0}{x_3 - x_0}\frac{x - x_1}{x_3 - x_1}\frac{x - x_2}{x_3 - x_2}, \tag{17}$$

and the final interpolating polynomial is

$$P(x) = (0)l_0(x) + (c^j_{k-1})l_1(x) + (c^j_{k-1} + c^j_k)l_2(x) + (c^j_{k-1} + c^j_k + c^j_{k+1})l_3(x). \tag{18}$$

Several evaluations are necessary in order to obtain the predictions. Using intervals of equal length, these values are

$$P(x^j_k) = c^j_{k-1} \tag{19}$$

$$P(x^j_{k+1/2}) = \frac{17}{16}c^j_{k-1} + \frac{1}{2}c^j_k - \frac{1}{16}c^j_{k+1} \tag{20}$$

$$P(x^j_{k+1}) = c^j_{k-1} + c^j_k. \tag{21}$$

Then the predictions of the cell-averages at the higher level of resolution are finally given by

$$\hat{c}^{j+1}_{2k} = c^j_k + \frac{1}{8}\left(c^j_{k-1} - c^j_{k+1}\right) \tag{22}$$

$$\hat{c}^{j+1}_{2k+1} = c^j_k - \frac{1}{8}\left(c^j_{k-1} - c^j_{k+1}\right). \tag{23}$$

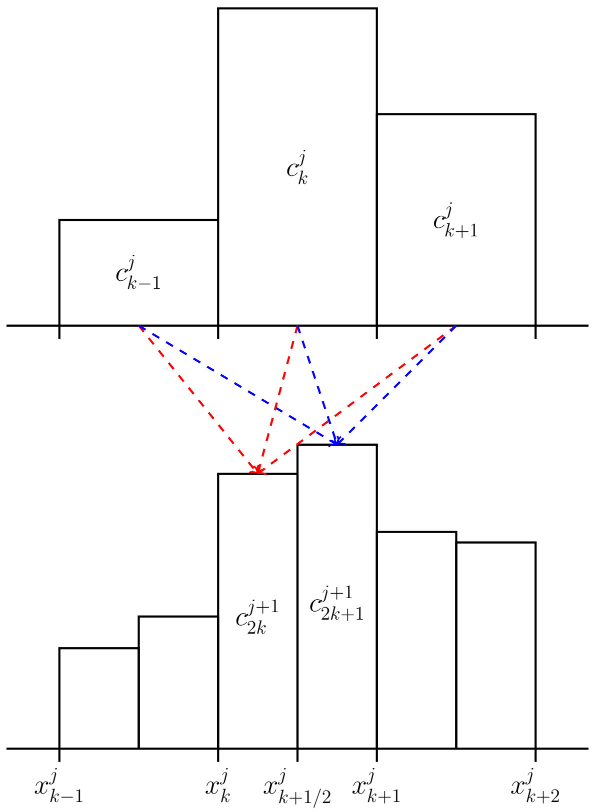

Figure 1: Quadratic prediction operator from coarse-scale j to fine-scale $j+1$, given cell-averaged data $\mathbf{c}^j$. Red and Blue arrows indicate interpolation dependency for each of the new cells, c_{2k}^j and c_{2k+1}^j, respectively.

This procedure could easily be extended to non-uniformly spaced intervals, giving different weights. The detail, or wavelet coefficient on level j is then given by the difference between known value and prediction,

$$d_k^j = c_{2k+1}^{j+1} - \hat{c}_{2k+1}^{j+1}. \tag{24}$$

Note that only the odd indices are counted because in the multiresolution scheme the data is initially split into even and odd signals. All data at level j are just considered to be a copy of the even-index data at level $j+1$, whereas the odd-indexed data at level $j+1$ is what is predicted by even-indexed data at level $j+1$.