Accelerating Block-Structured Adaptive Multiresolution Schemes: Applications to Reactive Flows

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Introduction

Many engineering applications depend on numerically solving systems of conservation laws of the form

$$\begin{cases} u_t + f(u)_x = s(u) \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where $x \in \Omega$ and $t \in [t_0, t_f]$. Volume averages are defined for each cell $I_i = \left[x_{i-1/2}, x_{i+1/2}\right]$,

$$\overline{u}_i(t) = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} u(\xi, t) d\xi, \tag{2}$$

Discretization

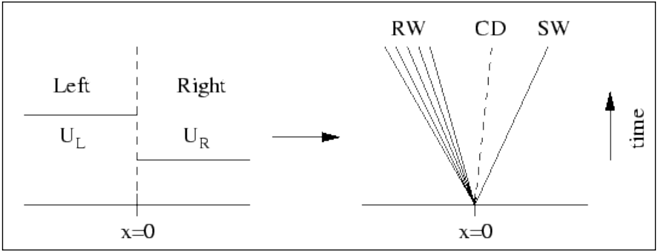
The governing equations are cast into the semi-discrete conservative form,

$$\frac{d\overline{u}_i(t)}{dt} = L(\overline{\boldsymbol{u}}) = -\frac{1}{\Delta x}\left(\hat{f}_{i+1/2} - \hat{f}_{i-1/2}\right) + \overline{s}_i, \tag{3}$$

where $\overline{s} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} s(\xi)d\xi$, and the numerical flux is evaluated as

$$\hat{f}_{i\pm1/2} = \hat{f}(\overline{u}^-_{i\pm1/2}, \overline{u}^+_{i\pm1/2}). \tag{4}$$

Riemann problem



Time integration

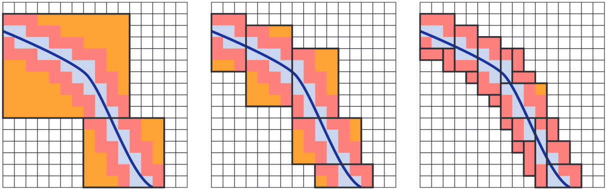
This system of ODEs is evolved forward in time

$$\overline{\boldsymbol{u}}^{n+1} = \overline{\boldsymbol{u}}^{n} + \Delta t \sum_{j=1}^{s} b_j \boldsymbol{k}_j, \tag{5}$$

where the stages k_j are solved either implicitly or explicitly.

Adaptive mesh refinement

These equations are typically solved on non-uniform mesh spacing:



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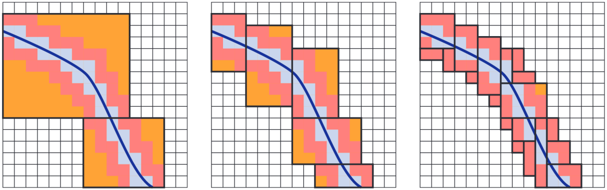

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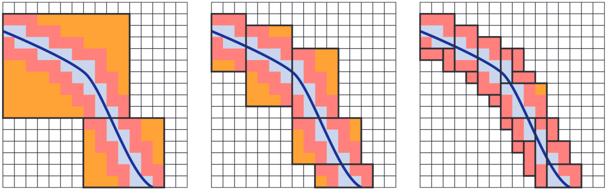

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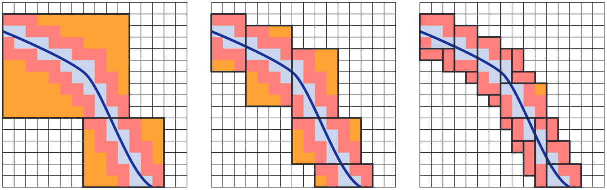

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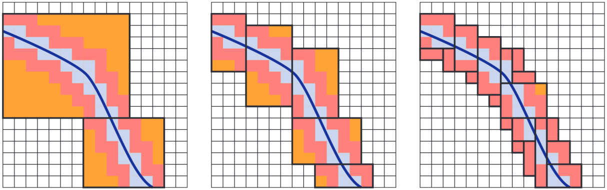

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- some type of estimator of the local error is needed
- typically a collection of cells (a block) is refined for efficiency
- blocks introduce inherent "overresolution" in some regions of the mesh

Mesh hierarchy

Define multiple levels of representation of the discrete data

$$\mathcal{G}_l = \left\{ x_i^l \right\}_{i=0}^{N_l}, \quad x_i^l = i \cdot \Delta x_l, \ \ \Delta x_l = 2^{L-l} \cdot \Delta x_L, \ \ N_l = N_L / 2^{L-l},$$

where Δx_l and N_l denote the cell width and number of cells, respectively, on level l. The index space of cells on each level of the hierarchy is denoted by $\mathcal{I}^l = \{1, \ldots, N_l\}$.

Decomposition

Project: The cells at level $l+1$ are projected by means of averaging, onto the coarser grid level l. The projection is defined by a linear operator which performs the mapping $\boldsymbol{P}^{l}_{l+1} : \overline{\boldsymbol{u}}^{l+1} \mapsto \overline{\boldsymbol{u}}^{l}$.

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Predict: Cell averages at level $l+1$ are predicted by an average-interpolating polynomial constructed of cells on level l. The prediction operator performs the mapping $\boldsymbol{P}_{l}^{l+1} : \overline{\boldsymbol{u}}^{l} \mapsto \tilde{\boldsymbol{u}}^{l+1}$.

Project

Coarsening/projection is done via

$$\overline{u}_i^l = \left(\boldsymbol{P}_{l+1}^l \overline{\boldsymbol{u}}^{l+1} \right)_i = \frac{1}{2}(\overline{u}_{2i-1}^{l+1} + \overline{u}_{2i}^{l+1}), \quad \forall i \in \mathcal{I}^l. \tag{6}$$

Predict

Prediction is done using average-interpolating polynomials, with $m \in \{0, 1\}$,

$$\tilde{u}_{2i-m}^{l+1} = \overline{u}_i^l - (-1)^m \sum_{p=1}^{s} \gamma_p \left(\overline{u}_{i+p}^l - \overline{u}_{i-p}^l \right), \quad \forall i \in \mathcal{I}^l.$$

Difference information and thresholding

We compute a difference between the true value at the higher resolution $l+1$, and its prediction,

$$d_i^l = \overline{u}_{2i}^{l+1} - \tilde{u}_{2i}^{l+1}, \quad \forall i \in \boldsymbol{\mathcal{I}}^l. \tag{7}$$

Then we can eliminate small detail coefficients (where solution is smooth enough),

$$\tilde{d}_i^l = \begin{cases} d_i^l, \ \text{if } |d_i^l| > \varepsilon \\ 0, \ \text{if } |d_i^l| \leq \varepsilon. \end{cases} \tag{8}$$

Then reconstructing field is done by

$$\overline{u}_{2i}^{l+1} = \tilde{u}_{2i}^{l+1} + \tilde{d}_i^l.$$

Grid adaptation

(MOVIE)

Adaptive flux and source computations on blocks

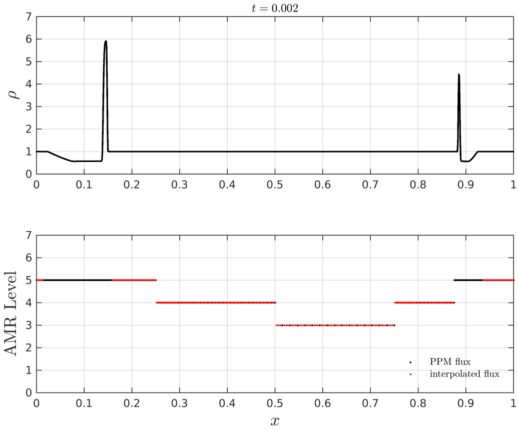
Where the mask is not active (solution is smooth enough), we can also replace redundant flux calculations with interpolation,

$$\hat{f}^{l+1}_{2i+1} \approx \sum_{p=1}^{s+1} \beta_p \left(\hat{f}^{l}_{i-p+1} + \hat{f}^{l}_{i+p} \right). \tag{9}$$

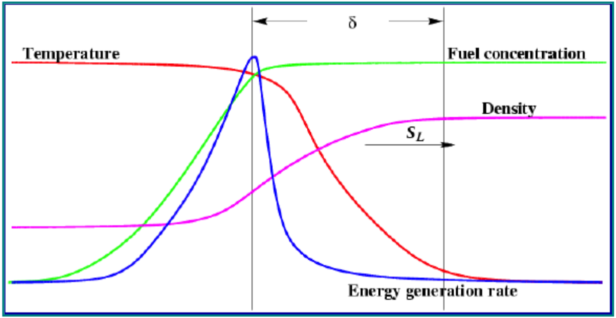
We can do the same for the source terms,

$$\overline{s}^{l+1}_{2i-m} \approx \overline{s}^{l}_{i} - (-1)^m \sum_{p=1}^{k} \gamma_p \left(\overline{s}^{l}_{i+p} - \overline{s}^{l}_{i-p} \right). \tag{10}$$

Two interacting blast waves



Laminar flame



Nuclear burning