Families of Shape Functions, Numerical Integration, Physical and Master Element, Concept and Mapping, Element Stiffness and Load Vector Calculations

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in Engineering Application

Knowledge Incubation for TEQIP

Two Dimensional Problems

Two Dimensional Problems:

- Single variable or multivariable problems
- The phenomenon is represented through partial differential equations
- Finite element formulation involves same steps as one dimensional case
- Boundary Γ of a two dimensional domain Ω, in general, is a curve
- Two dimensional shapes like triangle and rectangle/quadrilaterals are used to approximate the geometry.

Sample Boundary Value Problems:

- Poisson equation

$$-\nabla\cdot\left(k\nabla u\right)= f \qquad in\ \Omega$$

gradient operator $\nabla = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y}$

These equation represent

- Heat conduction
- Electrostatics
- Stream function
- Magnetic statics
- Torsion of non-circular sections, Transverse deflection of elastic membranes

Sample Boundary Value Problems: Heat Conduction

$$-\left[\frac{\partial}{\partial x}\left(k_{11}\frac{\partial u}{\partial x}+k_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(k_{21}\frac{\partial u}{\partial x}+k_{22}\frac{\partial u}{\partial x}\right)\right]=f\left(x,y\right)\quad \text{in } \Omega$$

with $k_{12}=k_{21}=0$ and $k_{11}=k_{22}=k$ it becomes Poisson's equation.

Finite element discretization:

- The number, shape and type (linear, quadratic,……….) of elements should be such that the geometry of the domain is accurately represented.

- Density of elements should be such that the regions of large gradients of the solution are adequately modeled.

Heat Conduction Equation: Weak Formulation

Weighted residual form:

$$\int_{\Omega} -\left[\frac{\partial}{\partial x}\left(k_{11} \frac{\partial u}{\partial x} + k_{12} \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y}\left(k_{22} \frac{\partial u}{\partial x} + k_{22} \frac{\partial u}{\partial y} \right) \right] w \, dA = \int_{\Omega} w \, dA$$

We know that,

$$\frac{\partial}{\partial x}(q_x w) = \frac{\partial q_x}{\partial x} w + q_x \frac{\partial w}{\partial x}$$

Therefore,

$$\left(\frac{\partial}{\partial x} q_x \right) w = \frac{\partial}{\partial x}(q_x w) - q_x \frac{\partial w}{\partial x}$$

Similarly,

$$\left(\frac{\partial}{\partial y} q_y \right) w = \frac{\partial}{\partial y}(q_y w) - q_y \frac{\partial w}{\partial y}$$

Heat Conduction Equation: Weak Formulation

Now using the Divergence Theorem,

$$\int_{\Omega} \frac{\partial}{\partial x}(q_x w)dA = \int_{\Gamma} q_x w n_x ds \quad \text{and} \quad \int_{\Omega} \frac{\partial}{\partial y}(q_y w)dA = \int_{\Gamma} q_y w n_y ds$$

where n_x and n_y are the direction cosines or the components of the unit normal vector $\boldsymbol{n}$

$$\mathbf{n} = n_x \hat{i} + n_y \hat{j} = \cos\alpha \hat{i} + \sin\alpha \hat{j}$$

Heat Conduction Equation: Weak Formulation

writing the weak form over an element

$$0=\int_{\Omega^e}\left[\frac{\partial w}{\partial x}\left(k_{11}\frac{\partial u}{\partial x}+k_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial w}{\partial y}\left(k_{21}\frac{\partial u}{\partial x}+k_{22}\frac{\partial u}{\partial y}\right)-wf\right]dA$$

$$-\oint_{\Gamma^e} w\left[n_x\left(k_{11}\frac{\partial u}{\partial x}+k_{22}\frac{\partial u}{\partial y}\right)+n_y\left(k_{21}\frac{\partial u}{\partial x}+k_{22}\frac{\partial u}{\partial y}\right)\right]ds$$

Looking at boundary term:

- $w \Rightarrow u \rightarrow$ Primary variable – essential BC and
- coefficient of weight function in boundary term form the natural BC.

Heat Conduction Equation: Weak Formulation

$$q_n = n_x\left(k_{11}\frac{\partial u}{\partial x} + k_{12}\frac{\partial u}{\partial y}\right) + n_y\left(k_{21}\frac{\partial u}{\partial x} + k_{22}\frac{\partial u}{\partial y}\right) \qquad \text{on } \Gamma^e$$

q_n - is the secondary variable.

q_n - projection of the vector $\mathbf{k}\cdot\nabla\mathbf{u}$ along the unit normal ***n***

- positive outward from the surface as the as we more counter-clock wise along the boundary

Thus,

$$0 = \int_{\Omega^e}\left[\frac{\partial w}{\partial x}\left(k_{11}\frac{\partial u}{\partial x} + k_{12}\frac{\partial u}{\partial y}\right) + \frac{\partial w}{\partial y}\left(k_{21}\frac{\partial u}{\partial x} + k_{22}\frac{\partial u}{\partial y}\right) - wf\right]dA - \oint_{\Gamma^e} w q_n \, ds$$

Heat Conduction Equation: Weak Formulation

Or $$B(w,u) = F(w)$$

where,

$$B(w,u) = \int_{\Omega^e} \left[\frac{\partial w}{\partial x}\left(k_{11}\frac{\partial u}{\partial x} + k_{12}\frac{\partial u}{\partial y} \right) + \frac{\partial w}{\partial y}\left(k_{21}\frac{\partial u}{\partial x} + k_{22}\frac{\partial u}{\partial y} \right) \right] dA$$

$$F(w) = \oint_{\Omega^e} w\, f\, dA + \oint_{\Gamma^e} w q_n\, ds$$

$B(w,u)$ - bilinear from, is not symmetric. It is symmetric only when $k_{12} = k_{21}$

$F(w)$ - linear from

Quadratic Potential: $$I(u) = B(u,u) - F(u)$$

Heat Conduction Equation: Finite Element Formulation

Approximation of u over Ω^e as

$$u(x,y) \approx u_h(x,y) = \sum_{j=1}^{NODE} u_j^e \psi(x,y)_j^e$$

u_j^e - are the nodal values of at node.

ψ_j^e - are the Lagrange interpolation functions with the property $\psi_i^e(x_j, y_j) = \delta_{ij}$

Substitute $u_h(x,y) = \sum_{j=1}^{NODE} u_j^e \psi_j^e(x,y)$ in the weak form and to get equation replace *w* by ψ_i^e to give

$$0 = \sum_{j=1}^{NODE} \left\{ \int_{\Omega^e} \left[\frac{\partial \psi_i}{\partial x}\left(k_{11}\frac{\partial \psi_j}{\partial x} + k_{12}\frac{\partial \psi_j}{\partial y} \right) + \frac{\partial \psi_i}{\partial y}\left(k_{21}\frac{\partial \psi_j}{\partial x} + k_{22}\frac{\partial \psi_j}{\partial y} \right) \right] dA \right.$$

$$- \int_{\Omega^e} f\psi_i \, dA - \oint_{\Gamma^e} \psi_i \, q_n \, ds \qquad i, j = 1, 2, \cdots, NODE$$

Heat Conduction Equation: Finite Element Formulation

This leads to:

$$\sum_{j=1}^{NODE} k_{ij}^e u_j^e = f_i^e + Q_i^e$$

where,

$$k_{ij}^e = \int_{\Omega^e} \left[\frac{\partial \psi_i}{\partial x} \left(k_{11} \frac{\partial \psi_j}{\partial x} + k_{12} \frac{\partial \psi_j}{\partial y} \right) + \frac{\partial \psi_i}{\partial y} \left(k_{21} \frac{\partial \psi_i}{\partial x} + k_{22} \frac{\partial \psi_j}{\partial y} \right) \right] dA$$

and

$$f_i^e = \int_{\Omega^e} f \, \psi_i^e \, dA, \quad Q_i^e = \oint_{\Gamma^e} q_n \, \psi_i^e \, ds$$

In matrix notation, $\left[k^e\right]\left\{u^e\right\} = \left\{f^e\right\} + \left\{Q^e\right\}$

$\left[k^e\right]$ is symmetric only when $k_{12} = k_{21}$

Interpolation Functions in Two Dimensions

Interpolation Functions: Requirements

The approximate solution $u_h(x, y)$ over an element, for the assurance of convergence to the actual solution as the number of elements is increased and their size is decreased, must satisfy:

1) It must be continuous over the element and differentiable as required by the weak form.

   - It ensures a nonzero coefficient matrix

2) It must not allow a strain to appear if the nodal displacements are compatible with a rigid-body displacements.

Interpolation Functions: Requirements

3) If the nodal displacements are compatible with a uniform strain in the elements, then the interpolation functions must yield this strain for the nodal displacements.

In other fields:

If the nodal variables are compatible with uniform states of the variable ϕ and any of its derivatives up to the highest in the appropriate quadratic functional $I\left(\phi\right)$, then these uniform states must be preserved in the element as it shrinks to zero size. This condition is called completeness.

- This requirement is necessary in order to capture all possible states of the actual solution

Interpolation Functions: Requirements

Example:

If a linear polynomial without the constant term is used to represent the temperature distribution in a one-dimensional system, the approximate solution can never be able to represent a uniform state of temperature in the element.

Interpolation Functions: Requirements

4) The interpolation functions should be chosen so that the strain remains finite at the boundary. Note, however, that the strain can be indeterminate at a boundary.

For example, if the strains involve only the first derivatives, then the displacement field must be continuous across a boundary. (strain energy must be finite)

In other fields:

The dependent variable and all its derivatives up to a ***one less*** than the highest order derivative in the appropriate quadratic functional should be continuous at the element interfaces. (The quadratic potential must be finite)

This requirement is called compatibility condition.
Elements satisfying these requirements are called conforming elements.

Interpolation Functions: Requirements

Degree of Compatibility achieved by interpolation functions at the element:

- C^0 continuity is achieved when the field variable only and none of its derivatives maintain continuity at an interelement interface.

- C^1 continuity is achieved when the field variable and its first derivatives maintain continuity at an interelement interface.

- C^2 continuity is achieved when the field variable, its first derivatives and second derivatives all maintain continuity at an interelement interface.

Interpolation Functions: Requirements

Continuity Requirement for completeness and compatibility:

For assurance of finite element convergence for a functional having p as the highest order of a derivative –

- C^p continuity – requirement for completeness
- $C^{(p-1)}$ continuity – requirement for compatibility

Interpolation Functions: Types of Families

Lagrange Family:

- Interpolation functions derived using the dependent unknown only and not its derivatives, at nodes.

- C^0 - continuity

Hermite Family:

- Interpolation functions derived using the dependent unknown and its derivatives as well, at nodes.

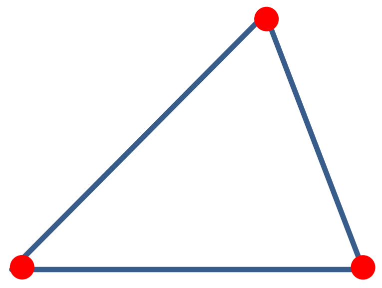
- C^1 continuity – continuity of the first derivative of the dependent unknown. Cubic interpolation functions
- C^2 continuity – continuity of the first and second derivatives of the dependent unknown. Quintic interpolation functions

Linear Interpolation Functions:

ψ_i^e must be at least linear in *x* and *y.*

Let $$u_h(x, y) = c_1 + c_2 x + c_3 y$$

- Three linearly independent terms, linear in both *x* and *y*.
- c_i to be expressed terms of nodal values of u_h. Hence, we must have three nodes.
- This is possible for a triangle with its vertices as nodes.



Linear Interpolation Functions:

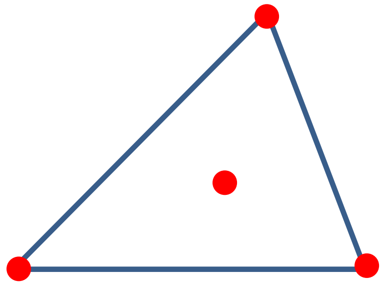
- Consider a polynomial

$$u_h(x, y) = c_1 + c_2 x + c_3 y + c_4 xy$$

- Four linear independent terms, linear in *x, y* with a bilinear term in *x* and *y.*
- This requires an element with four nodes.
- Two possible geometries:

- A triangle with the three nodes at its vertices and the fourth node at its centre or centroid.

- A rectangle with the nodes at the vertices



Linear Interpolation Functions:

- A triangle with the fourth node at its centre does not provide a single valued variation of primary variable at interelement boundaries.

- This results in incompatible variations of primary variable there. Therefore, this is not admissible.

- Thus, the only possible element with assumed polynomial approximation is rectangle with the nodes at the four vertices

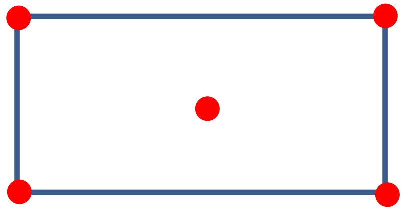
Quadratic Interpolation Functions:

- Consider a polynomial with five constants

$$u_h(x,y) = c_1 + c_2 x + c_3 y + c_4 xy + c_5 (x^2 + y^2)$$

- This is an incomplete quadratic polynomial
- Note that the terms x^2 and y^2 can not be varied independently.

- The polynomial requires an element with five nodes.
- The possible geometry is a rectangle with a node at each vertex and the fifth node at its centre.



- This again does not give single valued variation of primary variable.

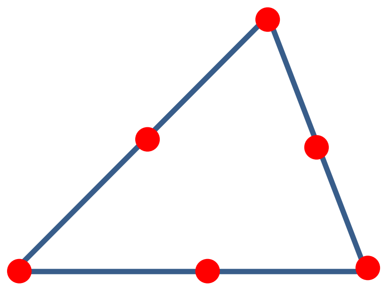
Quadratic Interpolation Functions:

- Consider a polynomial with six constants

$$u_h(x,y) = c_1 + c_2 x + c_3 y + c_4 xy + c_5 x^2 + c_6 y^2$$

- This is a complete quadratic polynomial
- The polynomial requires an element with six nodes.

- The possible geometry is a triangle with a node at each vertex and a node at the midpoint of each side.

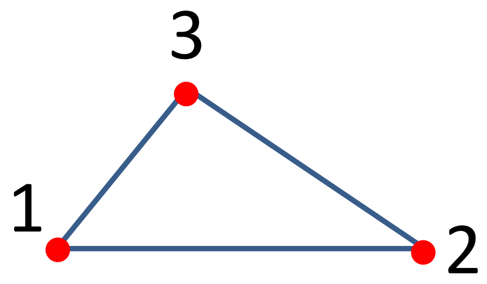


Lagrange Interpolation Functions in Two Dimensions

Linear Interpolation Functions in Two Dimensions

Interpolation Functions: Linear Triangular Element

A triangular element with three nodes at vertices



Approximate solution over an element is of the form

$$u_h(x, y) = c_1 + c_2 x + c_3 y$$

We need three conditions to find the unknowns C_i ‘s.

Using nodal values of solution we get three conditions

Interpolation Functions: Linear Triangular Element

Thus, we have

$$u_1 = u_h(x_1, y_1) = c_1 + c_2 x_1 + c_3 y_1$$
$$u_2 = u_h(x_2, y_2) = c_1 + c_2 x_2 + c_3 y_2$$
$$u_3 = u_h(x_3, y_3) = c_1 + c_2 x_3 + c_3 y_3$$

Or in a matrix form

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = [A] \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

unknowns C_i 's are found as

$$[A]^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

where,

$$[A]^{-1} = \frac{1}{2A_e} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

$$2A_e = \alpha_1 + \alpha_2 + \alpha_3$$

Interpolation Functions: Linear Triangular Element

Solving for C_i 's:

$$c_1 = \frac{1}{2A_e}\left(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3\right)$$

$$c_2 = \frac{1}{2A_e}\left(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3\right)$$

$$c_3 = \frac{1}{2A_e}\left(\gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3\right)$$

$2A_e$ - determinant of $[A]$

A_e - area of the triangle

where,

$$\left.\begin{aligned} \alpha_i &= x_j y_k - x_k y_j \\ \beta_i &= y_j - y_k \\ \gamma_i &= -\left(x_j - x_k\right) \end{aligned}\right\} \quad i \neq j \neq k \text{ and } i,\, j, k \text{ permute in a natural order}$$

Interpolation Functions: Linear Rectangular Element

Substituting values of C_i ‘s:

$$u_h(x,y)=\frac{1}{2A_e}\left[(\alpha_1u_1+\alpha_2u_2+\alpha_3u_3)+(\beta_1u_1+\beta_2u_2+\beta_3u_3)x+(\gamma_1u_1+\gamma_2u_2+\gamma_3u_3)y\right]$$

$$=\sum_{i=1}^{3}u_i^e\psi_i^e(x,y)$$

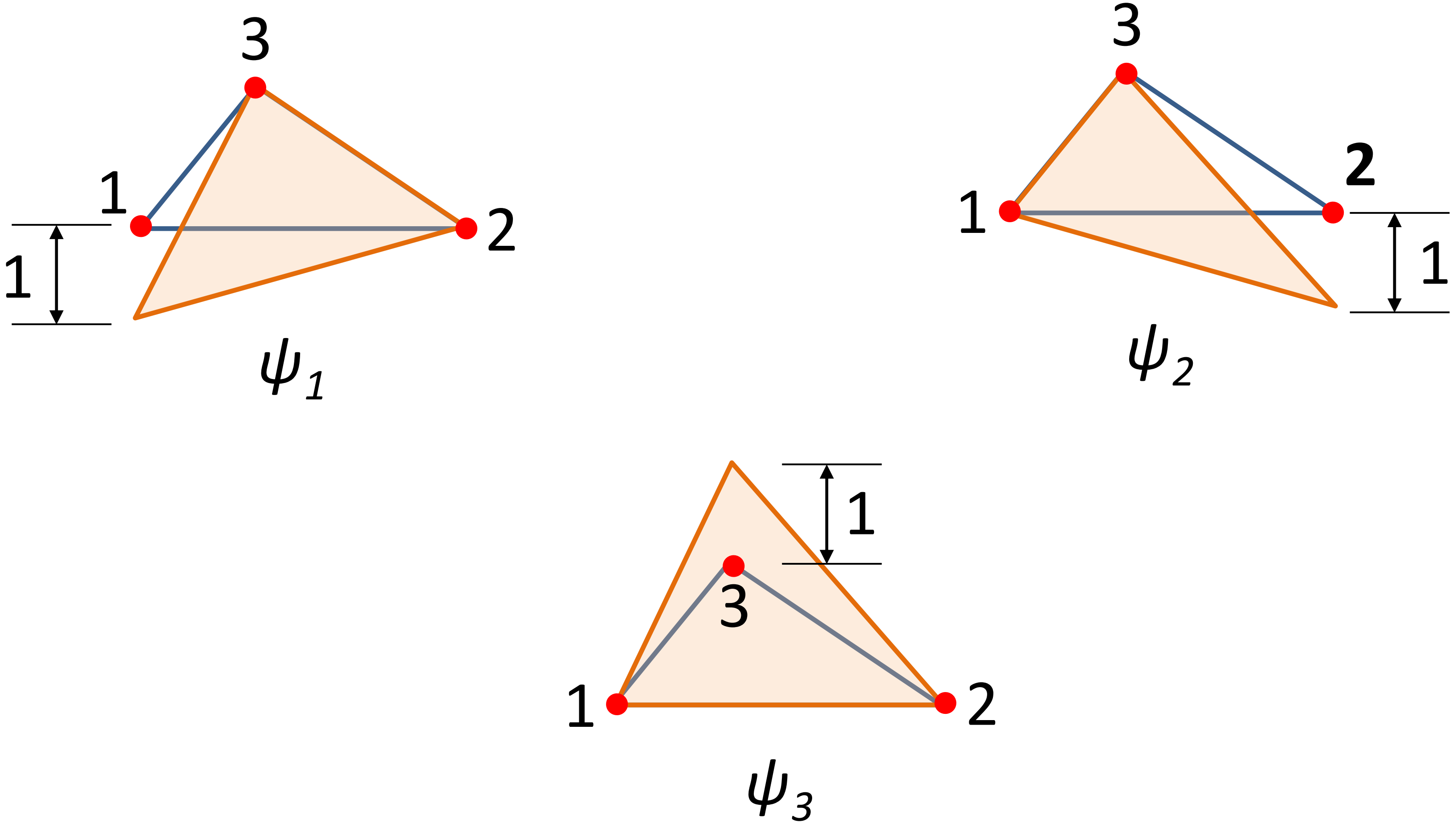
where, $$\psi_i^e(x,y)=\frac{1}{2A_e}\left\{\alpha_i^e+\beta_i^e+\gamma_i^e\right\}\quad i=1,2,3$$

With the properties: $$\psi_i^e\left(x_j^e,y_j^e\right)=\delta_{ij},\qquad \sum_{i=1}^{3}\psi_i^e=1,$$

$$\sum_{i=1}^{3}\frac{\partial\psi_i^e}{\partial x}=0,\qquad \sum_{i=0}^{3}\frac{\partial\psi_i^e}{\partial y}=0$$

Lagrange Interpolation Functions: Linear Triangular Element

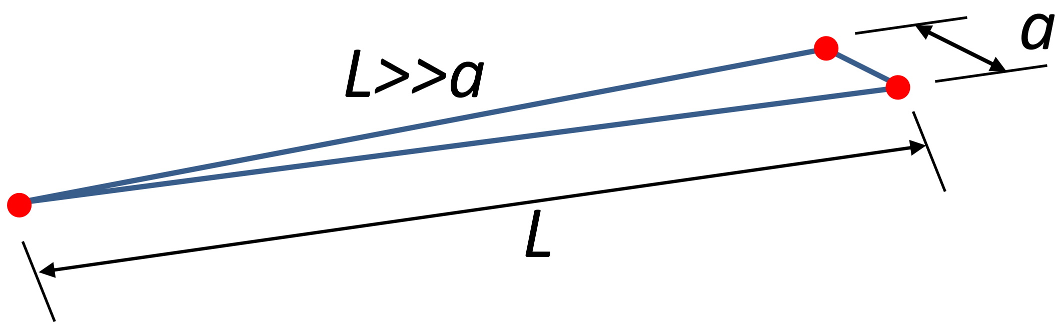
Linear interpolation functions for the three-noded triangular element

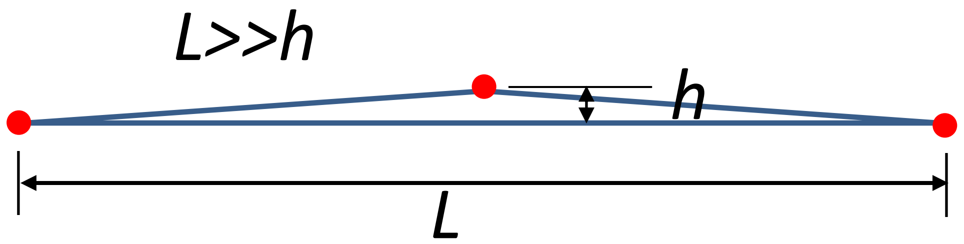


Lagrange Interpolation Functions: Triangular Geometries to Avoid

- The nodes are almost in-line.

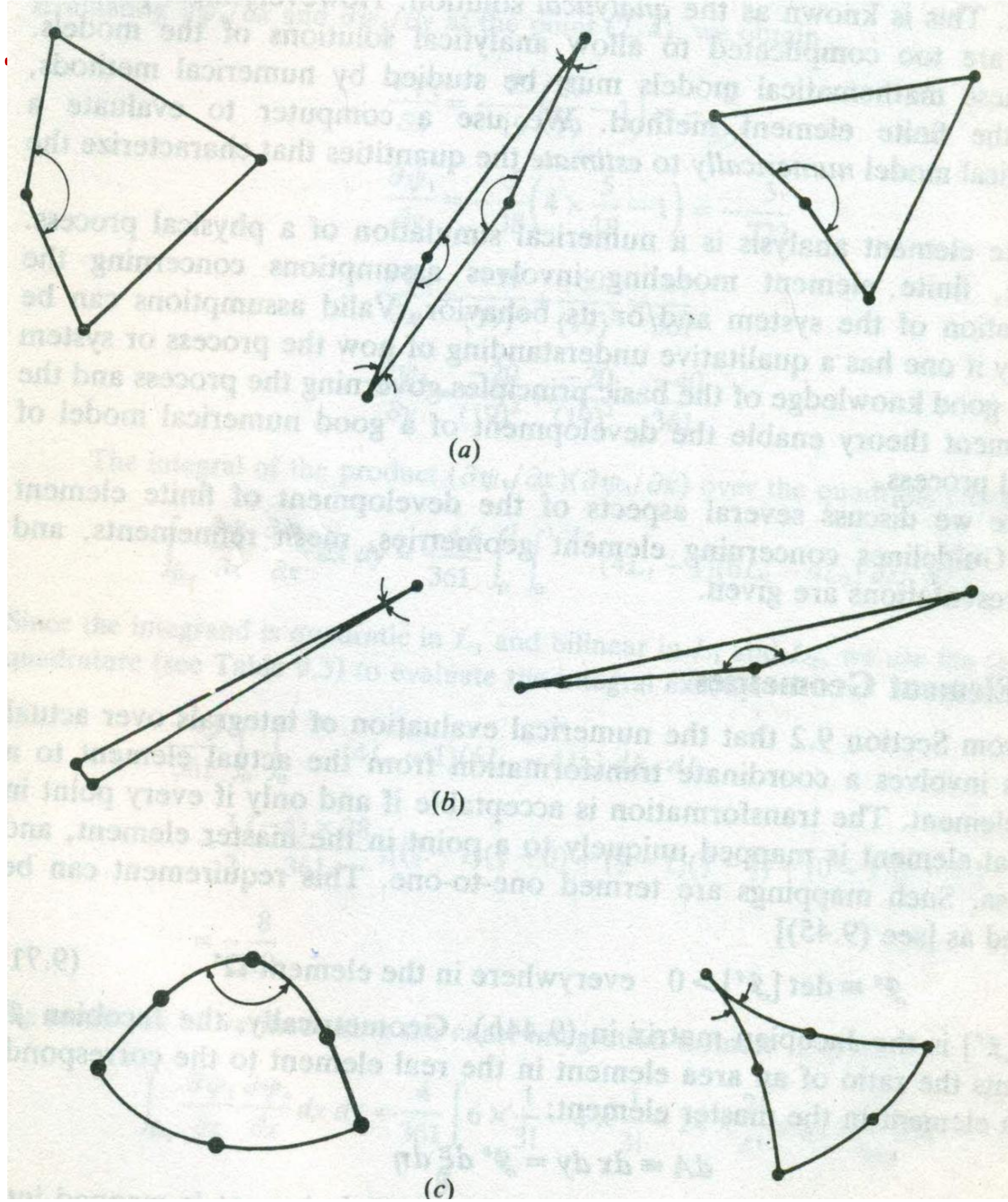
- The resulting coefficient matrix is nearly singular or non-invertible.





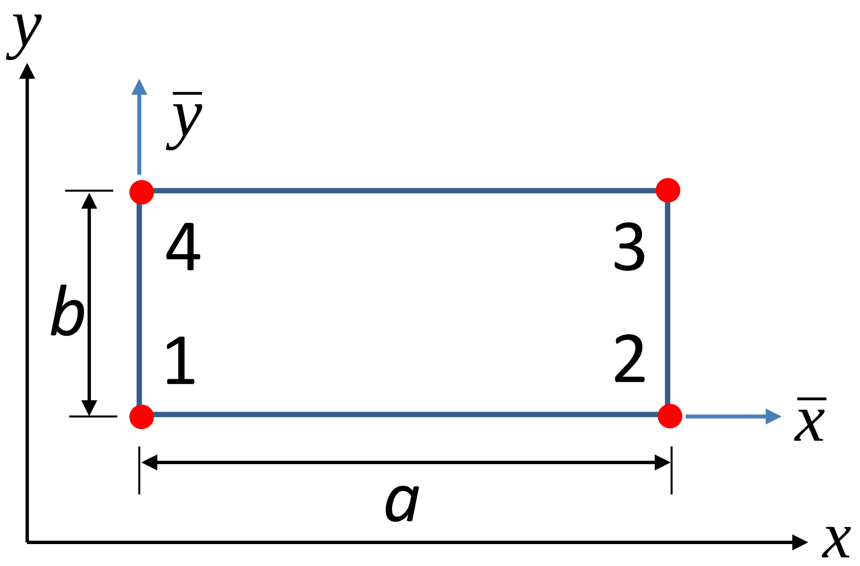
Lagrange Interpolation Functions: Linear Triangular Element

Shapes that should be avoided in the meshing.



Interpolation Functions: Linear Rectangular Element

A rectangular element with four nodes at the corner and of sides *a* and *b*.



Approximate solution over an element is of the form

$$u_h(\bar{x},\bar{y}) = c_1 + c_2\bar{x} + c_3\bar{y} + c_4\bar{x}\bar{y}$$

We need four conditions to find the unknowns C_i 's.

Interpolation Functions: Linear Rectangular Element

Using the nodal solution values, we get four conditions:

$$u_h(0,0) = u_1 = c_1 \qquad u_h(a,0) = u_2 = c_1 + a\,c_2$$

$$u_h(a,b) = u_3 = c_1 + c_2\,a + c_3\,b + c_4\,ab \qquad u_h(0,b) = u_4 = c_1 + c_3\,b$$

Gives the values of constants as

$$c_1 = u_1,\ c_2 = \frac{u_2 - u_1}{a},\ c_3 = \frac{u_4 - u_1}{b},\ c_4 = \frac{u_3 - u_4 + u_1 - u_2}{ab}$$

Putting these back and re-arranging,

$$u_h(\bar{x},\bar{y}) = u_1\left(1 - \frac{\bar{x}}{a} - \frac{\bar{y}}{b} + \frac{\bar{x}\bar{y}}{ab}\right) + u_2\left(\frac{\bar{x}}{a} - \frac{\bar{x}}{a}\frac{\bar{y}}{b}\right) + u_3\left(\frac{\bar{x}}{a}\frac{\bar{y}}{b}\right) + u_4\left(\frac{\bar{y}}{b} - \frac{\bar{x}}{a}\frac{\bar{y}}{b}\right)$$

Interpolation Functions: Linear Rectangular Element

$$u_h(\bar{x},\bar{y}) = u_1\psi_2 + u_2\psi_2 + u_3\psi_3 + u_4\psi_4$$

where,

$$\psi_1^e = \left(1-\frac{\bar{x}}{a}\right)\left(1-\frac{\bar{y}}{b}\right),\ \psi_2^e = \frac{\bar{x}}{a}\left(1-\frac{\bar{x}}{b}\right)$$

$$\psi_3^e = \frac{\bar{x}}{a}\frac{\bar{y}}{b},\ \psi_4^e = \left(1-\frac{\bar{x}}{a}\right)\frac{\bar{y}}{b}$$

And can be given as

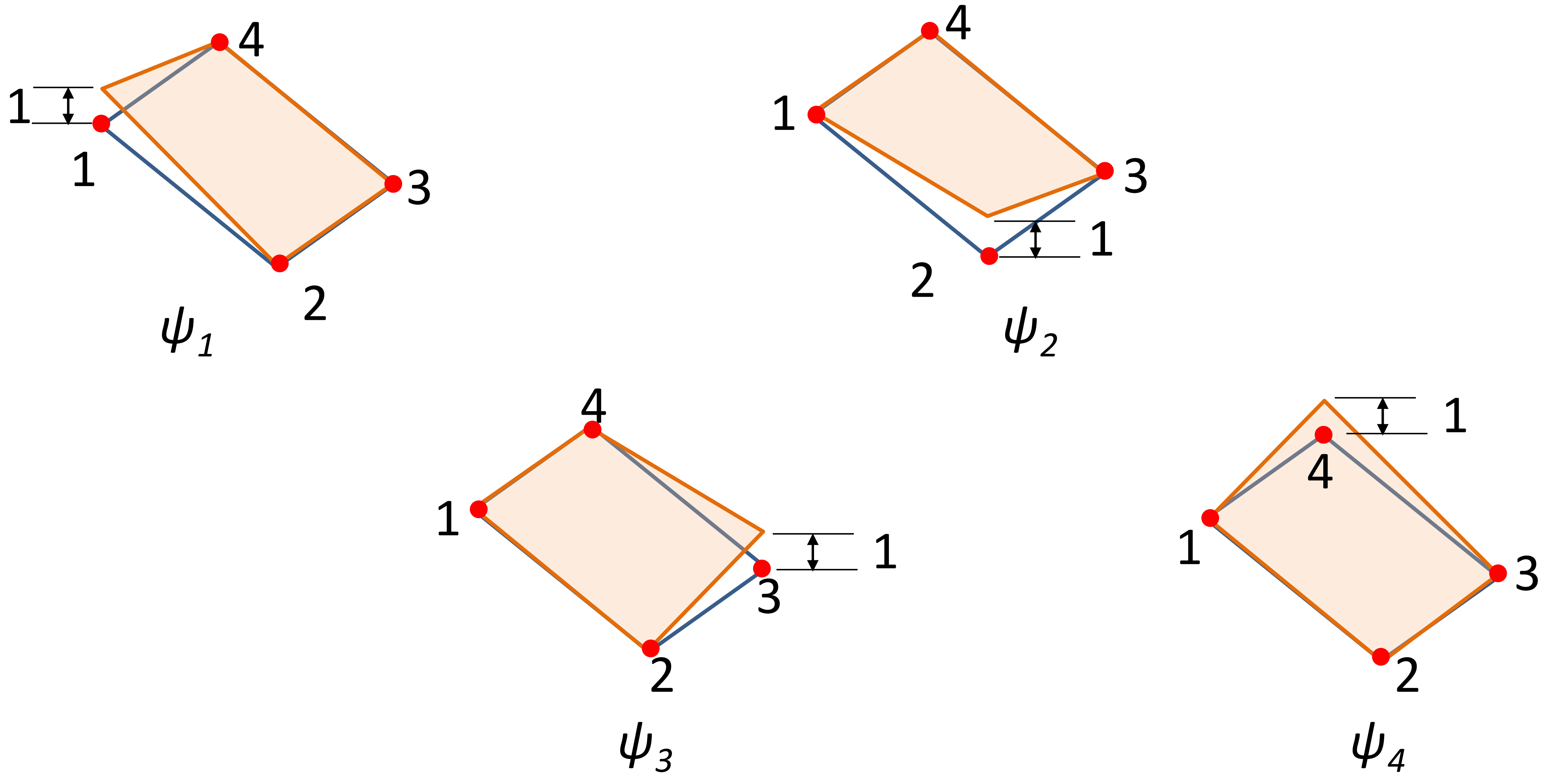
$$\psi_{\mathrm{i}}^{\mathrm{e}}(\bar{x},\bar{y}) = (-1)^{i+1}\left(1-\frac{\bar{x}+\bar{x}_i}{a}\right)\left(1-\frac{\bar{y}+\bar{y}_i}{b}\right)$$

With the properties:

$$\psi_{\mathrm{i}}^{\mathrm{e}}\left(\bar{x}_j,\bar{y}_j\right) = \delta_{ij}, \qquad \sum_{i=1}^{4}\psi_i^e = 1$$

Lagrange Interpolation Functions: Linear Rectangular Element

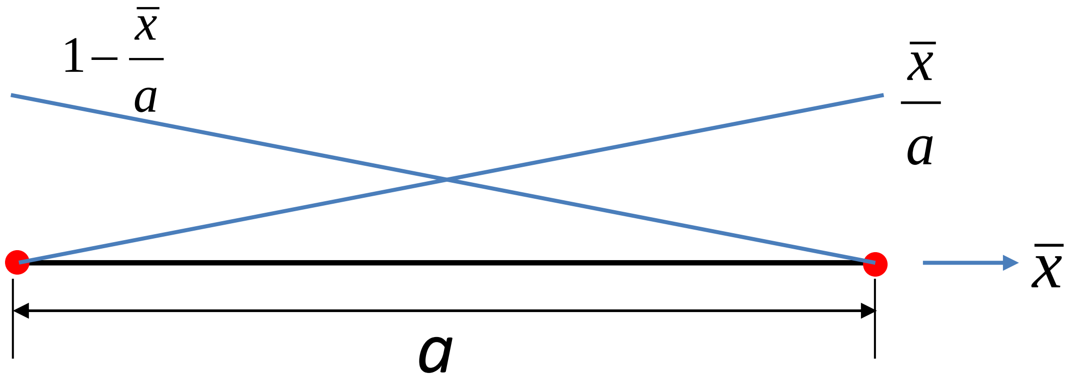
Linear interpolation functions for the four-noded rectangular element

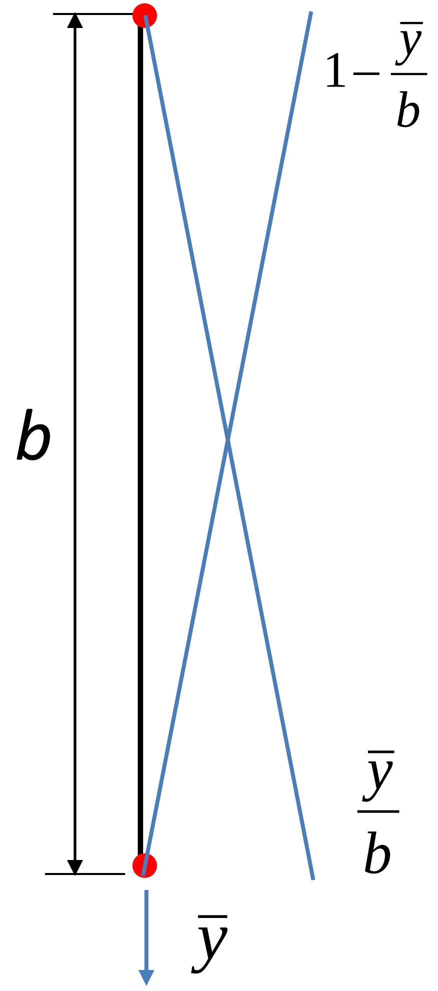


Interpolation Functions: Linear Rectangular Element

- Can also be obtained from tensor product of *1D* linear interpolation functions associated with sides 1-2 and 2-3

$$\psi_1^e = \left(1-\frac{\bar{x}}{a}\right)\left(1-\frac{\bar{y}}{b}\right),\ \psi_2^e = \frac{\bar{x}}{a}\left(1-\frac{\bar{y}}{b}\right),\ \psi_3^e = \frac{\bar{x}}{a}\frac{\bar{y}}{b},\ \psi_4^e = \left(1-\frac{\bar{x}}{a}\right)\frac{\bar{y}}{b}$$





$$\begin{Bmatrix} 1-\frac{\bar{x}}{a} \\ \frac{\bar{x}}{a} \end{Bmatrix} \begin{Bmatrix} 1-\frac{\bar{y}}{b} & \frac{\bar{y}}{b} \end{Bmatrix} = \begin{bmatrix} \psi_1 & \psi_4 \\ \psi_2 & \psi_3 \end{bmatrix}$$

Interpolation Functions: Linear Rectangular Element

Another approach:

Derivation of $\psi_1^e(\bar{x}, \bar{y})$

We know that $\psi_1^e(x_1, y_1) = 1$ and $\psi_1^e(\bar{x}_i, \bar{y}_i) = 0 \;\; \text{for}\, i = 2,3,4$

And zero on the lines $\bar{x} = a$ and $\bar{y} = b$

Therefore, ψ_1^e is of the form: $\psi_1^e(\bar{x}, \bar{y}) = c_1(a - \bar{x})(b - \bar{y})$

Putting first condition that $\psi_1^e(\bar{x}, \bar{y}) = 1$ at $\bar{x} = 0, \bar{y} = 0$ $\Rightarrow c_1 = \dfrac{1}{ab}$

Hence, ψ_1^e is

$$\psi_1^e(\bar{x}, \bar{y}) = \frac{1}{ab}(a - \bar{x})(b - \bar{y}) = \left(1 - \frac{\bar{x}}{a}\right)\left(1 - \frac{\bar{y}}{b}\right)$$

In a similar way, the other interpolation functions can be derived.

Higher Order Interpolation Functions
For Triangular Element

Interpolation Functions: Higher Order Element

- The interpolation functions of higher degree can be systematically developed with the help of Pascal's Triangle.
- It contain the terms of polynomials of various degrees in the two coordinates *x* and *y*.
- *x* and *y* denote the local coordinate (and not the global coordinates of the problem).
- Shape of the triangle need not be equilateral triangle.
- p^{th} order triangular element has *n* nodes given by the relation

$$n = \frac{1}{2}(p+1)(p+2)$$

- A complete polynomial of p^{th} degree is given by

$$u(x,y) = \sum_{i=1}^{n} a_i x^r y^s = \sum_{j=1}^{n} u_j \psi_j \qquad \text{with} \quad r+s \le p$$

Interpolation Functions: Pascal's Triangle

$$1$$

$$x \quad y$$

$$x^2 \quad xy \quad y^2$$

$$x^3 \quad x^2y \quad xy^2 \quad y^3$$

$$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$$

$$x^5 \quad x^4y \quad x^3y^2 \quad x^2y^3 \quad xy^4 \quad y^5$$

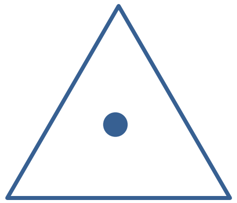
$$x^6 \quad x^5y \quad x^4y^2 \quad x^3y^3 \quad x^2y^4 \quad xy^5 \quad y^6$$

$$x^7 \quad x^6y \quad x^5y^2 \quad x^4y^3 \quad x^3y^4 \quad x^2y^5 \quad xy^6 \quad y^7$$

Interpolation Functions: Pascal's Triangle

We choose the terms in the polynomial

$$
\begin{array}{c}
1 \\
x \quad y \\
x^2 \quad xy \quad y^2 \\
x^3 \quad x^2y \quad xy^2 \quad y^3 \\
x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4 \\
x^5 \quad x^4y \quad x^3y^2 \quad x^2y^3 \quad xy^4 \quad y^5 \\
x^6 \quad x^5y \quad x^4y^2 \quad x^3y^3 \quad x^2y^4 \quad xy^5 \quad y^6 \\
x^7 \quad x^6y \quad x^5y^2 \quad x^4y^3 \quad x^3y^4 \quad x^2y^5 \quad xy^6 \quad y^7
\end{array}
$$

Polynomial Degree	Element	
0		(1)

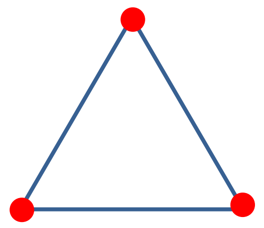
$$u_h(x, y) = c_1$$

Interpolation Functions: Pascal's Triangle

Polynomial Degree | Element

$$1$$
$$x \quad y$$
$$x^2 \quad xy \quad y^2$$
$$x^3 \quad x^2y \quad xy^2 \quad y^3$$
$$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$$
$$x^5 \quad x^4y \quad x^3y^2 \quad x^2y^3 \quad xy^4 \quad y^5$$
$$x^6 \quad x^5y \quad x^4y^2 \quad x^3y^3 \quad x^2y^4 \quad xy^5 \quad y^6$$
$$x^7 \quad x^6y \quad x^5y^2 \quad x^4y^3 \quad x^3y^4 \quad x^2y^5 \quad xy^6 \quad y^7$$

1

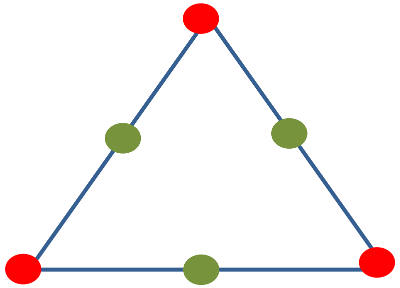


(3)

$$u_h(x, y) = c_1 + c_2 x + c_3 y$$

Interpolation Functions: Pascal's Triangle

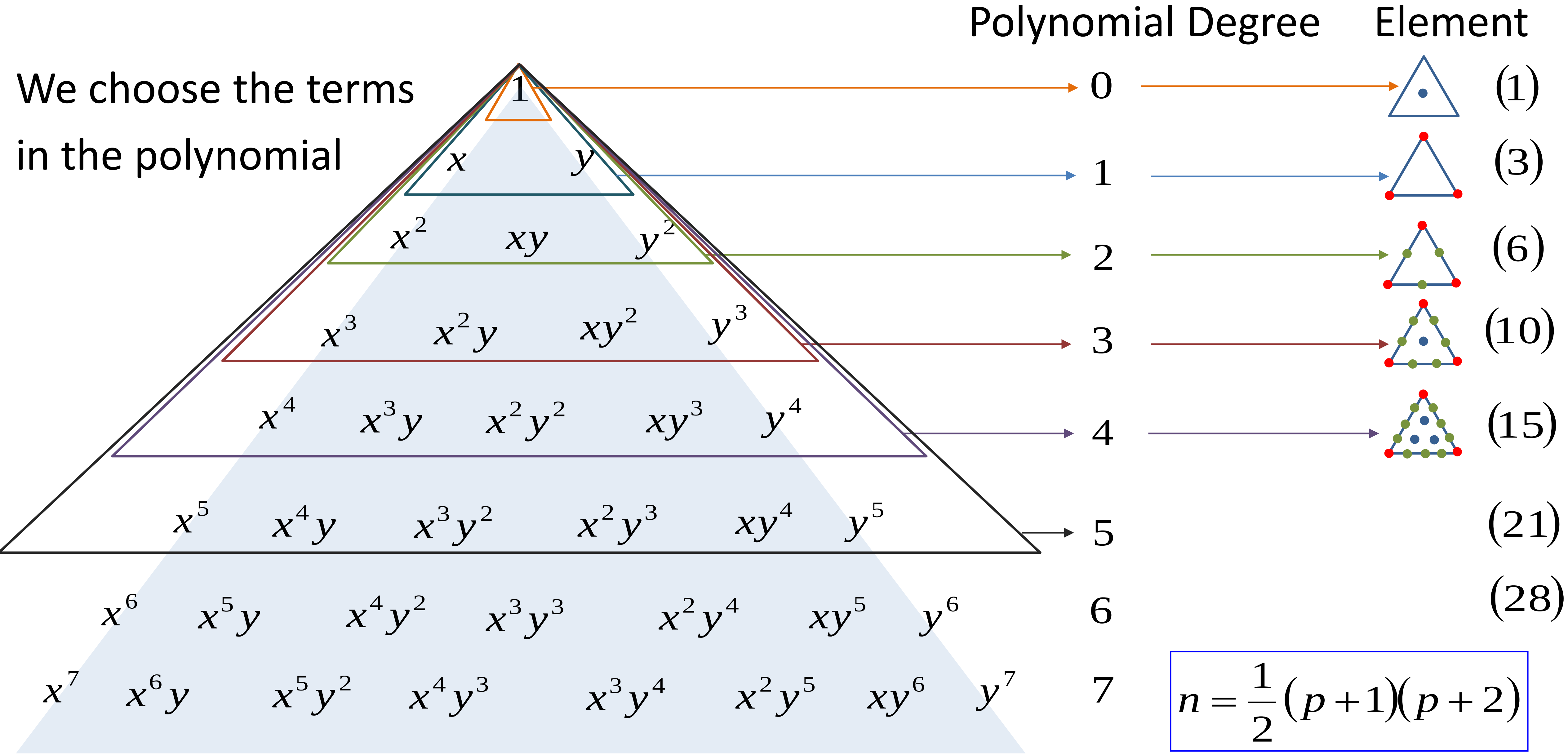
$$1$$
$$x \quad y$$
$$x^2 \quad xy \quad y^2$$
$$x^3 \quad x^2y \quad xy^2 \quad y^3$$
$$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$$
$$x^5 \quad x^4y \quad x^3y^2 \quad x^2y^3 \quad xy^4 \quad y^5$$
$$x^6 \quad x^5y \quad x^4y^2 \quad x^3y^3 \quad x^2y^4 \quad xy^5 \quad y^6$$
$$x^7 \quad x^6y \quad x^5y^2 \quad x^4y^3 \quad x^3y^4 \quad x^2y^5 \quad xy^6 \quad y^7$$

Polynomial Degree	Element
2	 (6)

$$u_h(x, y) = c_1 + c_2 x + c_3 y + c_4 xy + c_5 x^2 + c_6 y^2$$

Interpolation Functions: Pascal's Triangle

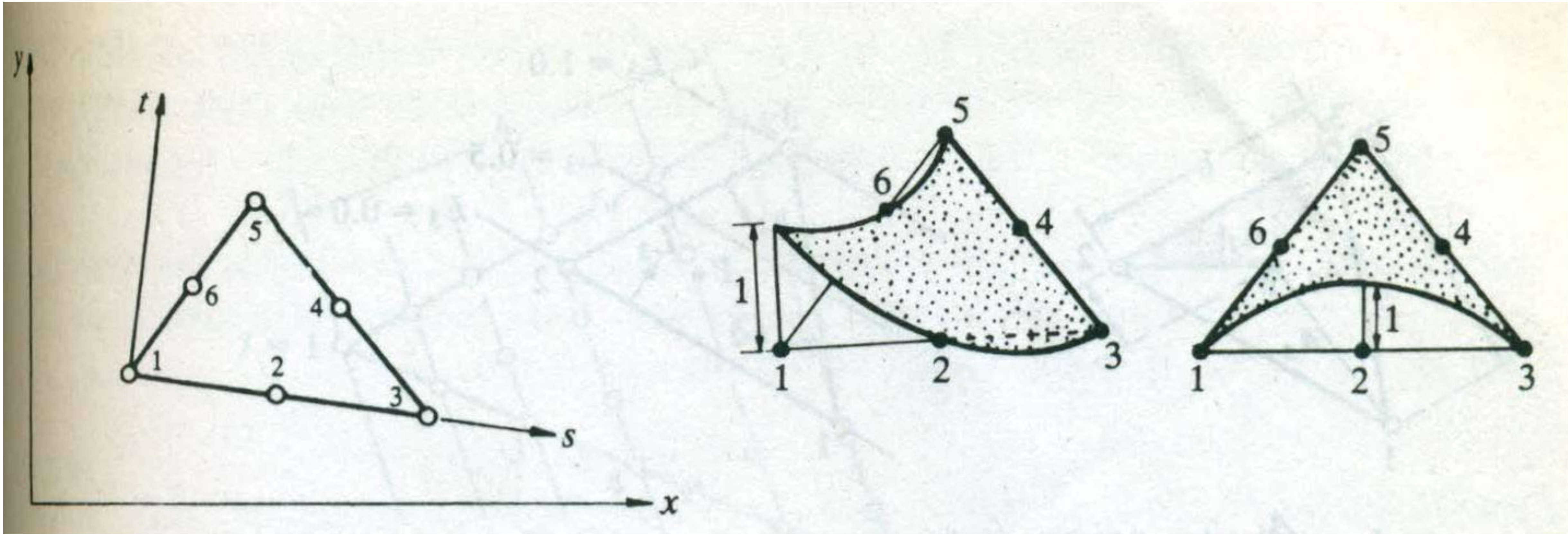
We choose the terms in the polynomial



$$\boxed{n = \frac{1}{2}(p+1)(p+2)}$$

Lagrange Interpolation Functions: Quadratic Triangular Element

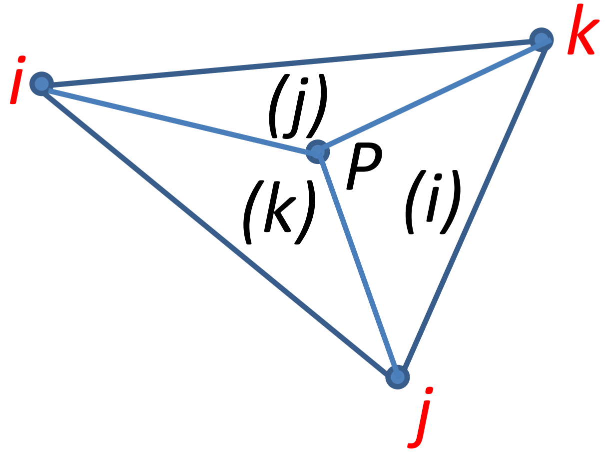
Geometric variation of Lagrangian interpolation functions



Interpolation Functions For Triangular Element using Area Coordinates

Interpolation Functions: Using Area Co-ordinates

P is an arbitrary point inside a linear triangular element



Let A_i, A_j and A_k be the area of triangles formed by point P with an edge of the triangle.

For example, A_k is the area formed by edge *i-j* and point P.

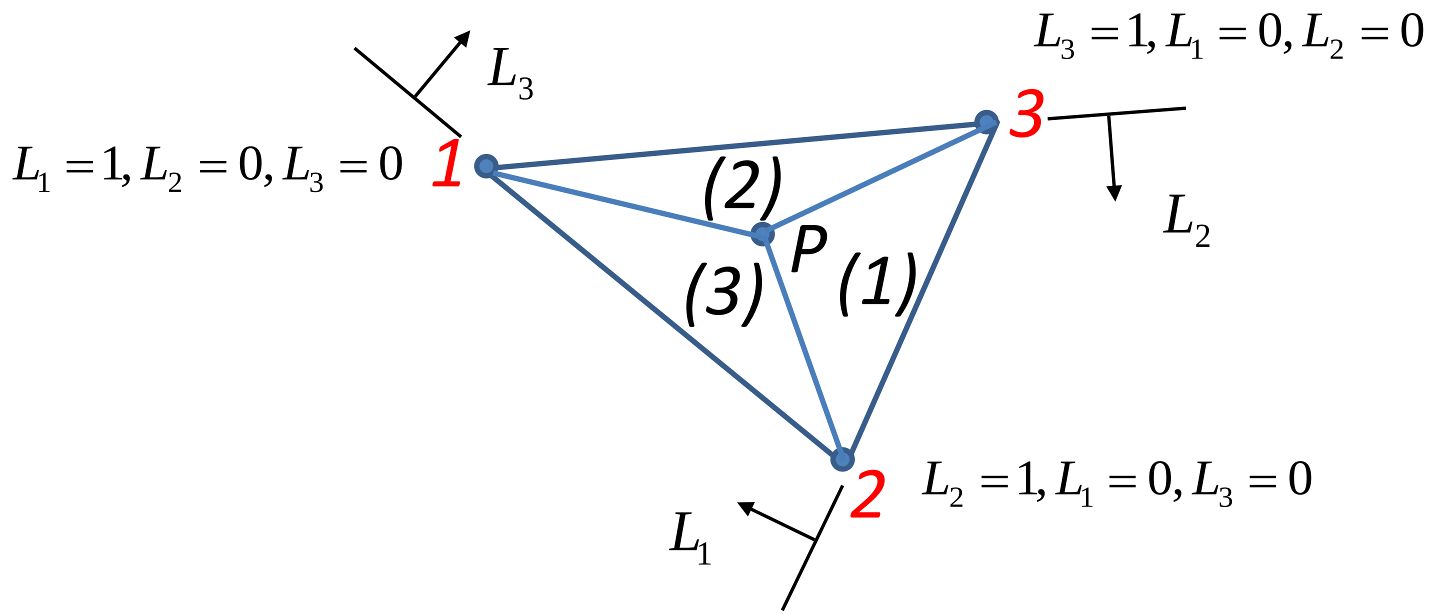
Define: Natural /triangular/ barycentric coordinate L_i

$$L_i = \frac{A_i}{A}, L_j = \frac{A_j}{A}, L_k = \frac{A_k}{A} \quad \text{and} \quad A = A_i + A_j + A_k$$

Interpolation Functions: Using Area Co-ordinates

- For P at node i: $L_i = 1, L_j = 0, L_k = 0$
- For P at node j: $L_i = 0, L_j = 1, L_k = 0$
- For P at node k: $L_i = 0, L_j = 0, L_k = 1$

Connection with triangular element:

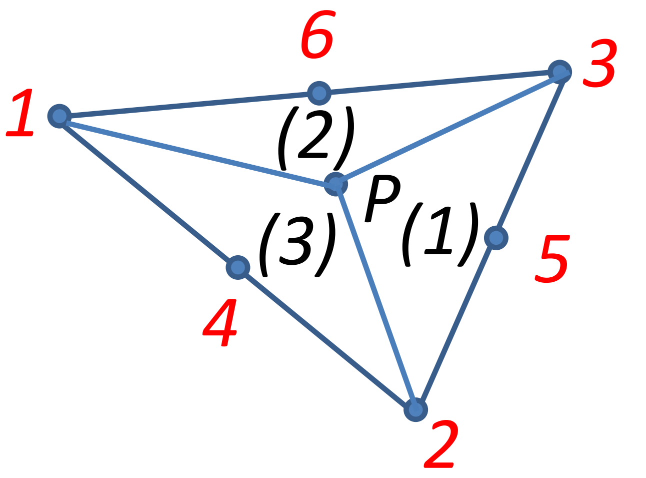


Interpolation Functions: Using Area Co-ordinates

- We needed to invert coefficient matrix when we derived interpolation functions.

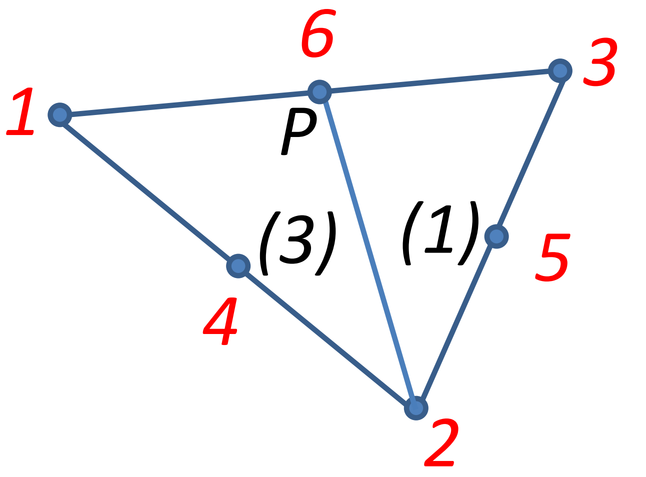
This can be avoided in this approach.

Consider a quadratic triangular element



We express natural co-ordinate L_i at any point by using superscript to identify the nodes.

Example: Point P moves to node 6.

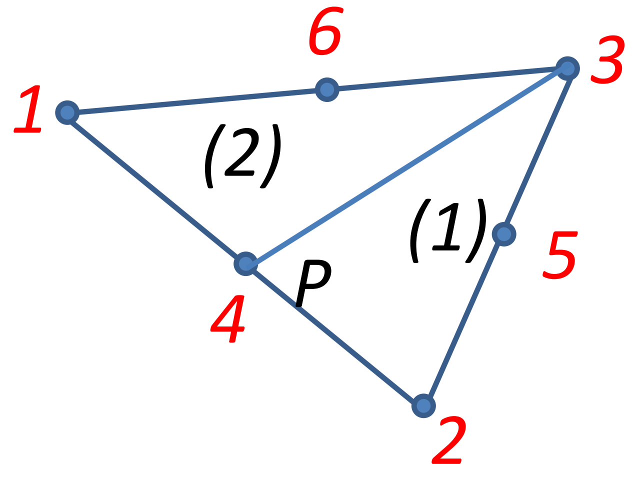


$$L_1^{(6)} = \frac{A_1}{A} = \frac{1}{2}, \qquad L_2^{(6)} = \frac{A_2}{A} = \frac{0}{A} = 0, \qquad L_3^{(6)} = \frac{A_3}{A} = \frac{1}{2}$$

Interpolation Functions: Using Area Co-ordinates

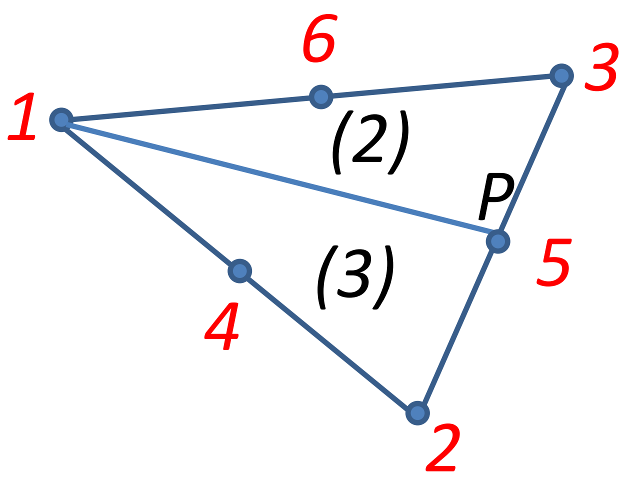
- Point P moves to node at 4.

$$L_1^{(4)} = \frac{A_1}{A} = \frac{1}{2}, \qquad L_2^{(4)} = \frac{A_2}{A} = \frac{1}{2}, \qquad L_3^{(4)} = \frac{A_3}{A} = \frac{0}{A} = 0$$



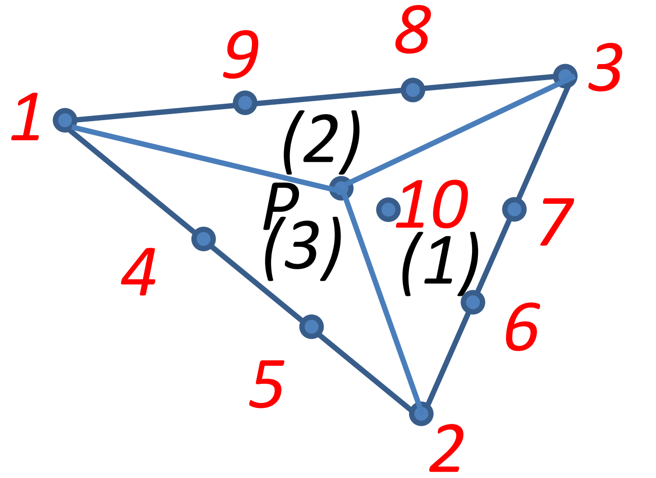
- Point P moves to node at 5.

$$L_1^{(5)} = \frac{A_1}{A} = \frac{0}{A} = 0, \qquad L_2^{(5)} = \frac{A_2}{A} = \frac{1}{2}, \qquad L_3^{(5)} = \frac{A_3}{A} = \frac{1}{2}$$



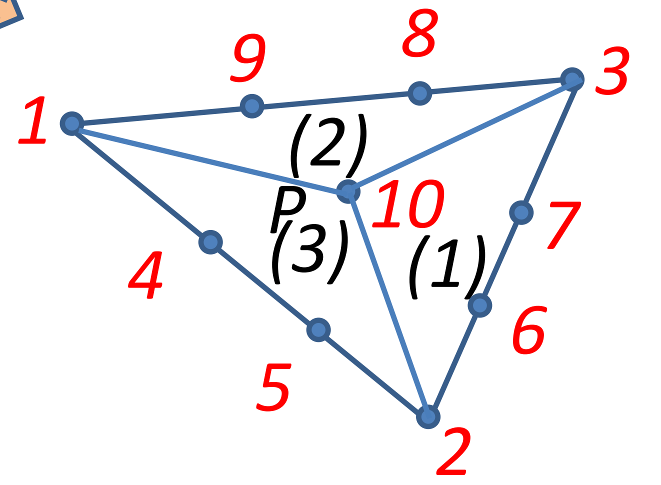
Interpolation Functions: Using Area Co-ordinates

- For a cubic triangular element

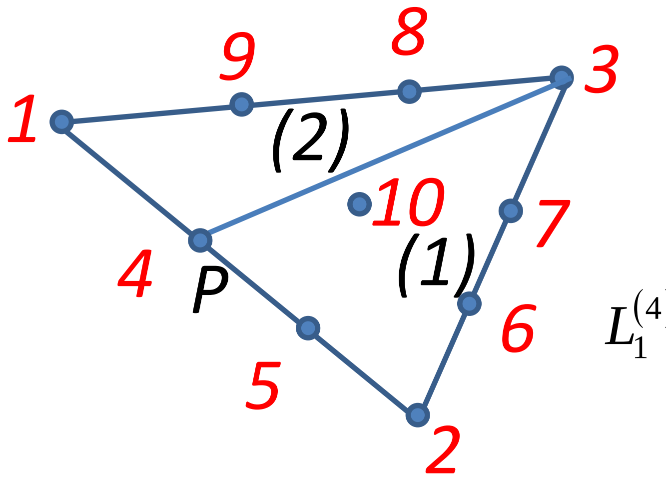


Example: Point P moves to node 10.

$$L_1^{(10)} = \frac{A_1}{A} = \frac{1}{3}, \qquad L_2^{(10)} = \frac{A_2}{A} = \frac{1}{3}, \qquad L_3^{(10)} = \frac{A_3}{A} = \frac{1}{3}$$



Example: Point P moves to node 4.



$$L_1^{(4)} = \frac{A_1}{A} = \frac{2}{3}, \qquad L_2^{(4)} = \frac{A_2}{A} = \frac{1}{3}, \qquad L_3^{(4)} = \frac{A_3}{A} = \frac{0}{A} = 0$$

Interpolation Functions: Using Area Co-ordinates

To get the interpolation functions, we first transform the Lagrangian interpolation formula from Cartesian to natural coordinates.

$$\left[L\left(L_j\right)\right]_r = \prod_{i=1}^{nL_j^{(r)}} \left(\frac{nL_j - i + 1}{i}\right) \quad \text{for } nL_j^{(r)} \geq 1$$

$$\left[L\left(L_j\right)\right]_r = 1 \quad \text{for } nL_j^{(r)} = 0$$

where j = one of the three natural coordinates
n $(=p)$ = order of interpolation polynomial
r = free index from 1 up to the total number of nodes

Thus, for each node: $$N_r = \left[L\left(L_1\right)\right]_r \left[L\left(L_2\right)\right]_r \left[L\left(L_3\right)\right]_r$$

Interpolation Functions: Using Area Co-ordinates

Example: For quadratic triangular element ($n=p=2$)

For N_1, put r=1 and considering $j=1$ first

$$nL_j^{(r)} = 2L_1^{(1)} = (2)(1) = 2$$

Thus,

$$\left[L(L_1)\right]_1 = \prod_{i=1}^{2L_1^{(1)}}\left(\frac{2L_1 - i + 1}{i}\right) = \prod_{i=1}^{2}\left(\frac{2L_1 - i + 1}{i}\right)$$

$$= \left(\frac{2L_1 - 1 + 1}{1}\right)\left(\frac{2L_1 - 2 + 1}{2}\right) = L_1(2L_1 - 1)$$

and $\left[L(L_2)\right]_1 = 0, \left[L(L_3)\right]_1 = 0$. Therefore, $N_1 = L_1(2L_1 - 1)$

Interpolation Functions: Using Area Co-ordinates

Example: For quadratic triangular element   ($n=p=2$)

For N_4, put r=4 and considering $j=1$ first

$$nL_j^{(r)} = 2L_1^{(4)} = (2)\left(\frac{1}{2}\right) = 1$$

Thus,

$$\left[L(L_1)\right]_4 = \prod_{i=1}^{2L_1^{(4)}}\left(\frac{2L_1 - i + 1}{i}\right) = \prod_{i=1}^{1}\left(\frac{2L_1 - i + 1}{i}\right) = \left(\frac{2L_1 - 1 + 1}{1}\right) = 2L_1$$

Now, considering $j=2$

$$nL_j^{(r)} = 2L_2^{(4)} = (2)\left(\frac{1}{2}\right) = 1$$

$$\left[L(L_2)\right]_4 = 2L_2$$

Interpolation Functions: Using Area Co-ordinates

Example: For quadratic triangular element (*n=p=2*)

For N_4, considering *j=3*

$$nL_j^{(r)} = 2L_3^{(4)} = (2)(0) = 0 \qquad \left[L(L_3)\right]_4 = 1$$

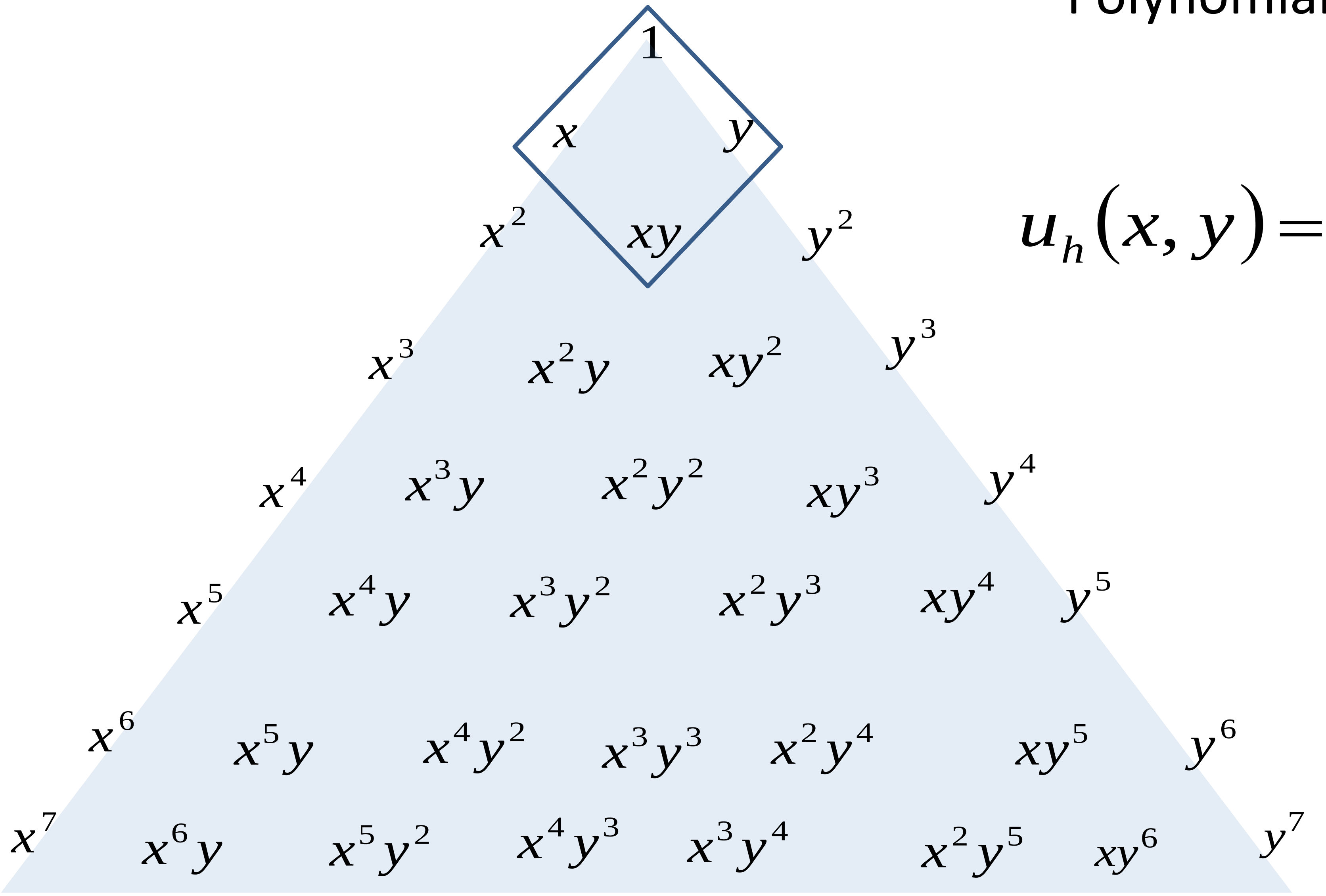
Thus, $N_4 = 4L_1L_2$

For remaining nodes,

$$N_2 = L_2(2L_2 - 1), \quad N_3 = L_3(2L_3 - 1),$$
$$N_5 = 4L_2L_3, \qquad N_6 = 4L_1L_3$$

Higher Order Interpolation Functions
For Rectangular Element

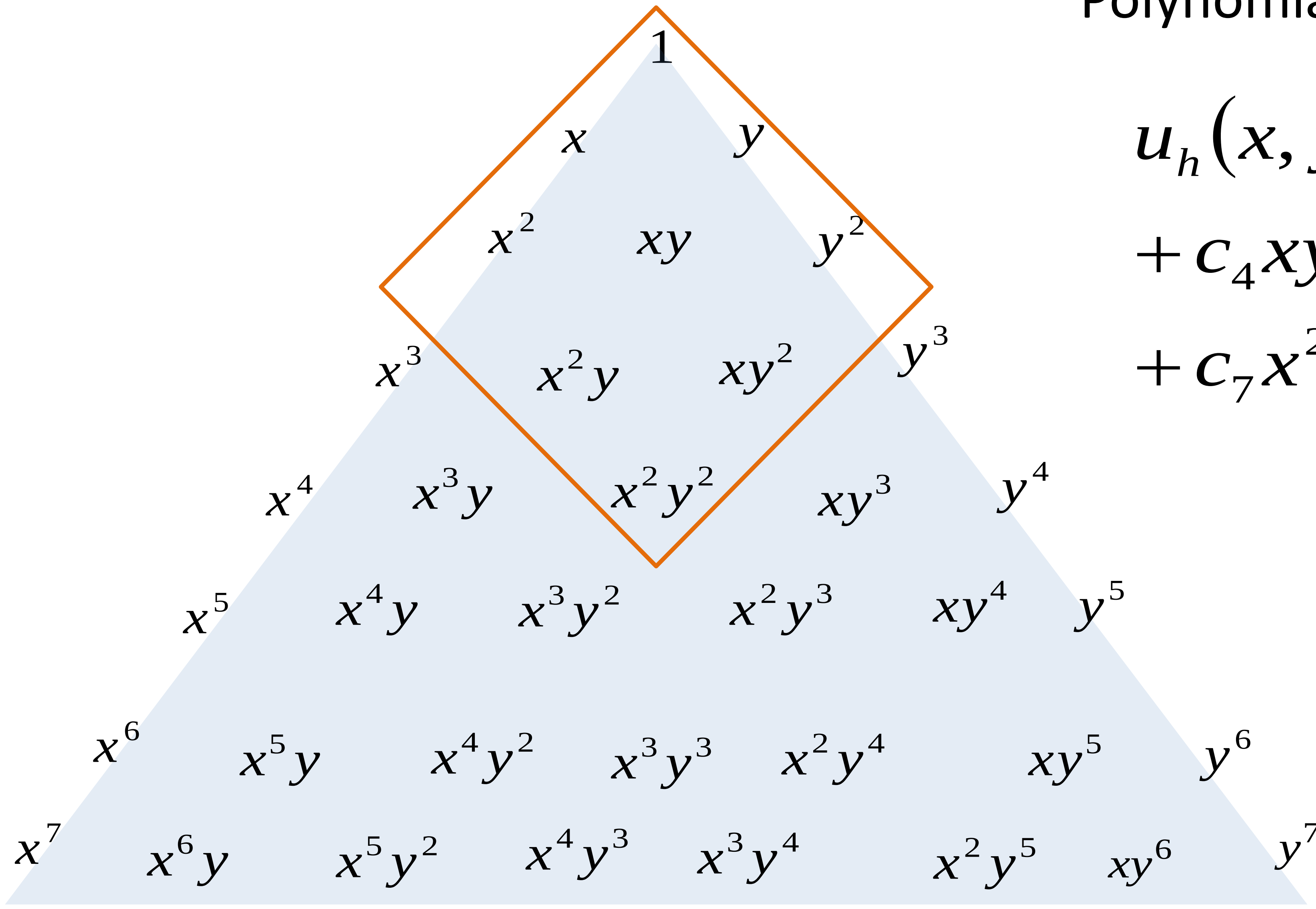
Interpolation Functions: Pascal's Triangle



Polynomial Degree = 1

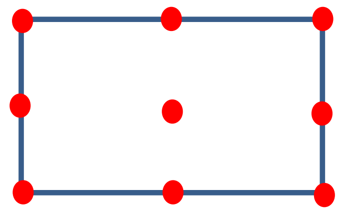
$$u_h(x, y) = c_1 + c_2 x + c_3 y + c_4 xy$$

Interpolation Functions: Pascal's Triangle



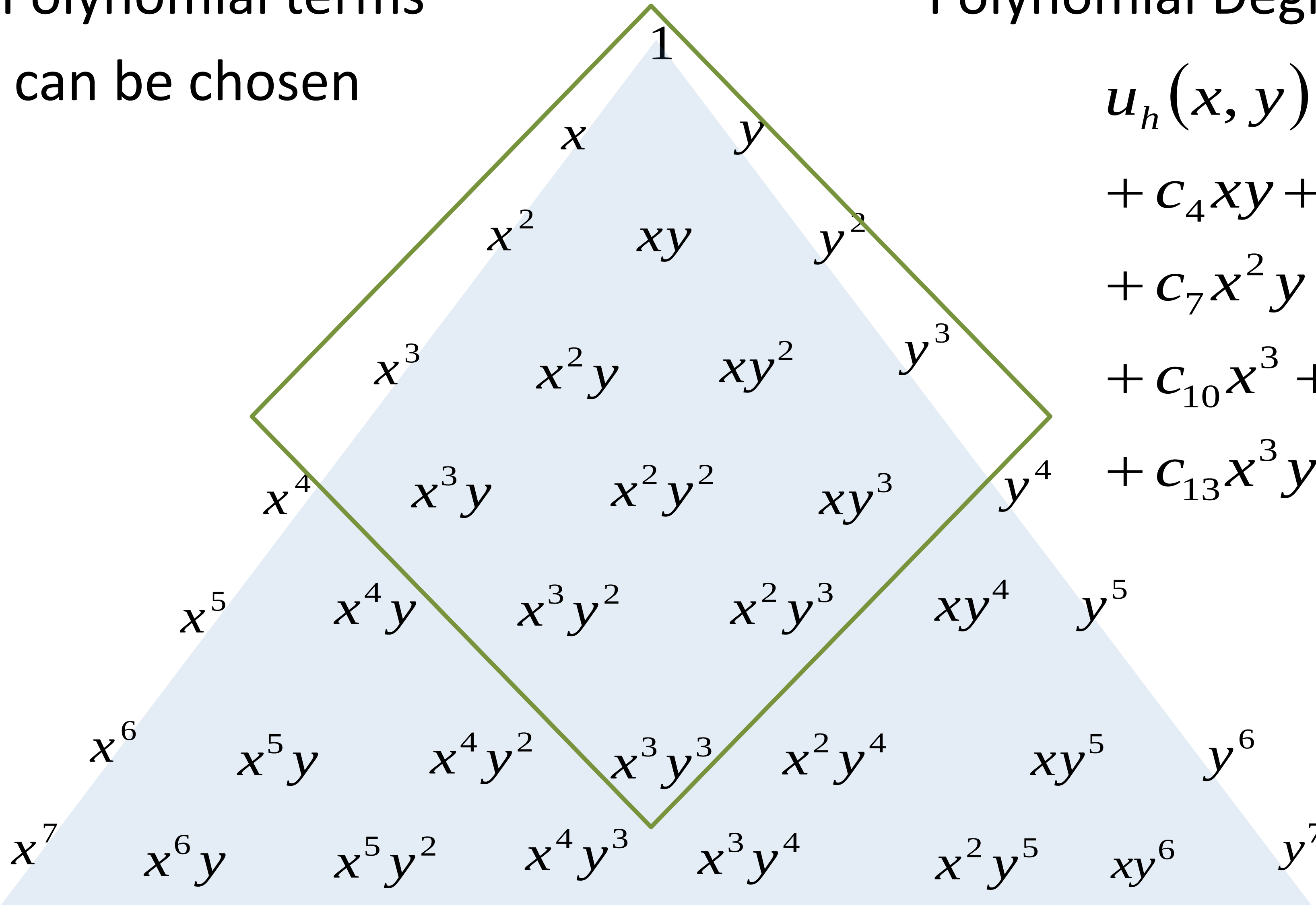
Polynomial Degree = 2

$$u_h(x,y) = c_1 + c_2 x + c_3 y + c_4 xy + c_5 x^2 + c_6 y^2 + c_7 x^2 y + c_8 xy^2 + c_9 x^2 y^2$$



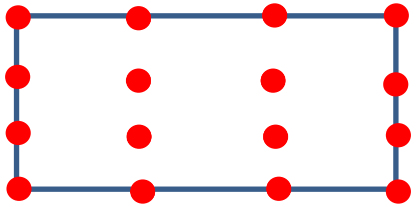
Interpolation Functions: Pascal's Triangle

Polynomial terms can be chosen



Polynomial Degree = 3

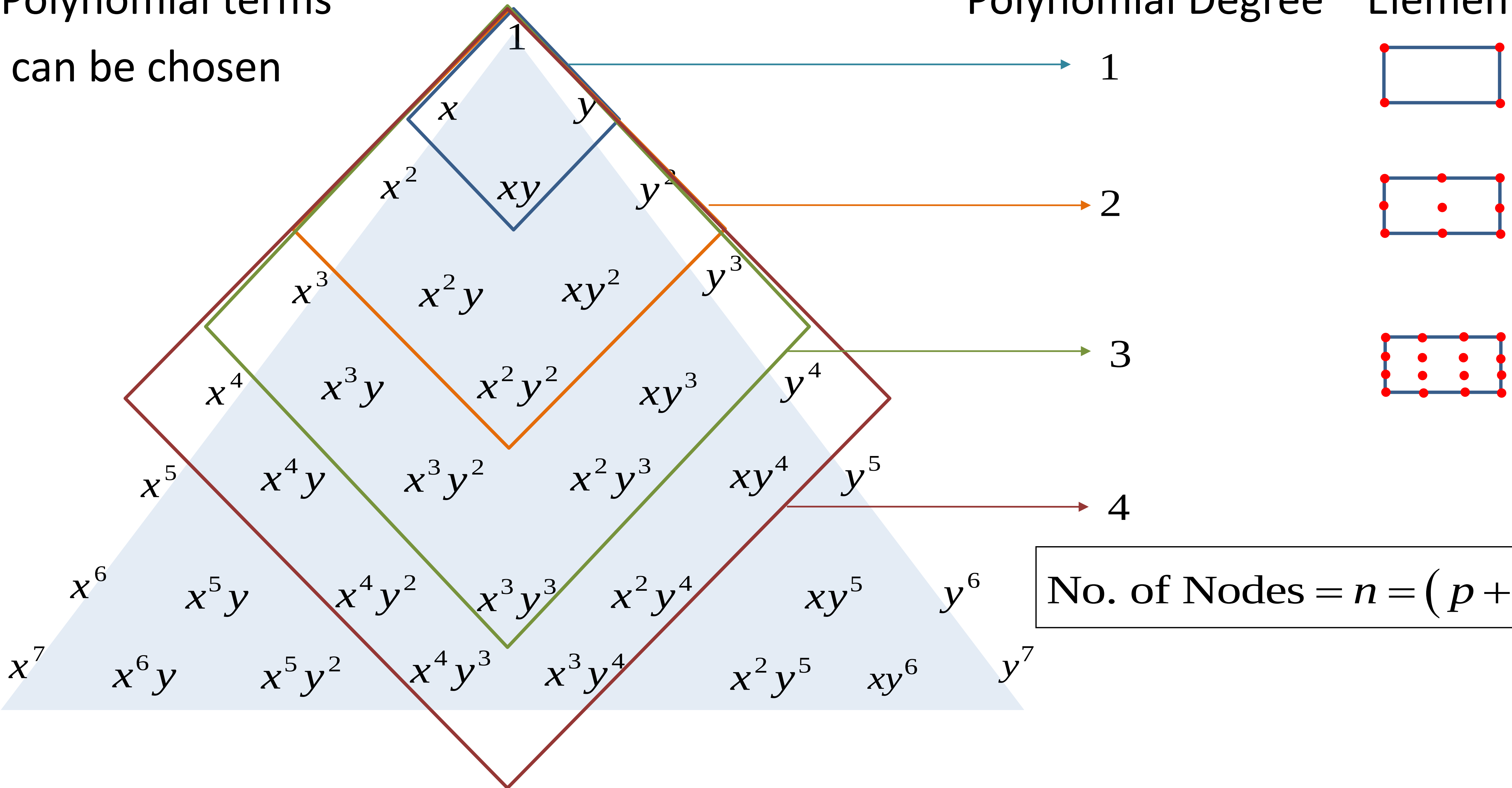
$$
\begin{aligned}
u_h(x,y) = {} & c_1 + c_2 x + c_3 y \\
& + c_4 xy + c_5 x^2 + c_6 y^2 \\
& + c_7 x^2 y + c_8 xy^2 + c_9 x^2 y^2 \\
& + c_{10} x^3 + c_{11} x^3 y + c_{12} x^3 y^2 \\
& + c_{13} x^3 y^3 + c_{14} y^3 + c_{15} xy^3 + c_{16} x^2 y^3
\end{aligned}
$$



Interpolation Functions: Pascal's Triangle

Polynomial terms can be chosen

Polynomial Degree Element

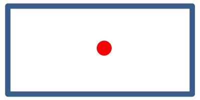
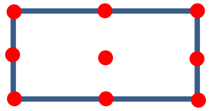
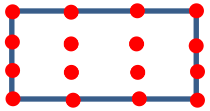


$$\text{No. of Nodes} = n = (p+1)^2$$

Interpolation Functions: Pascal's Triangle in Rectangular Array Form

Rectangular array

$$\begin{array}{ccccc} 1 & x & x^2 & x^3 & \cdots \\ y & xy & x^2y & x^3y & \cdots \\ y^2 & xy^2 & x^2y^2 & x^3y^2 & \cdots \\ y^3 & xy^3 & x^2y^3 & x^3y^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{array}$$

Polynomial Degree	Element
0	
1	
2	
3	

$$\text{No. of Nodes} = n = (p+1)^2$$

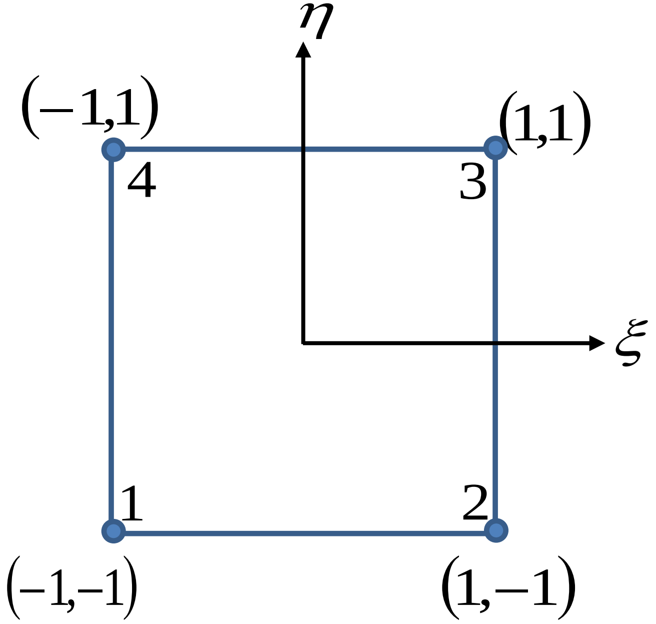
Interpolation Functions: Higher Order Rectangular Elements

The Lagrange interpolation functions associated with rectangular elements can be obtained from corresponding one-dimensional Lagrange interpolation functions by taking tensor product.

$$
\begin{bmatrix} \psi_1 & \psi_4 & \psi_7 \\ \psi_2 & \psi_5 & \psi_8 \\ \psi_3 & \psi_6 & \psi_9 \end{bmatrix} = \left\{ \begin{array}{c} \dfrac{\left(x - \dfrac{a}{2}\right)(x - a)}{\left(-\dfrac{a}{2}\right)(-a)} \\ \dfrac{x(x-a)}{\frac{1}{2}a\left(\frac{1}{2}a - a\right)} \\ \dfrac{x\left(x - \frac{a}{2}\right)}{a\left(\frac{1}{2}a\right)} \end{array} \right\} \left\{ \begin{array}{c} \dfrac{\left(y - \frac{b}{2}\right)(y-a)}{\frac{b^2}{2}} \\ \dfrac{y(y-b)}{-\frac{b^2}{4}} \\ \dfrac{y\left(y - \frac{b}{2}\right)}{\frac{b^2}{2}} \end{array} \right\}^T
$$

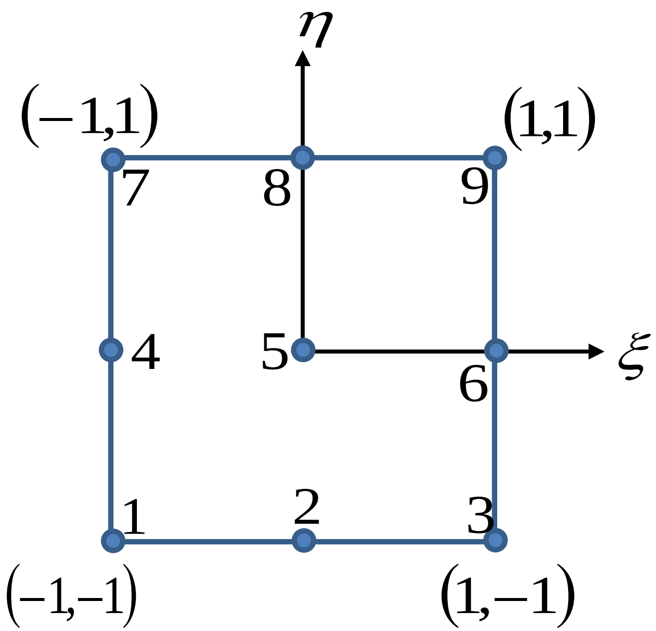
Interpolation Functions: Master Rectangular Elements

Tensor product of 1D functions:



$$\psi_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad \psi_2 = \frac{1}{4}(1+\xi)(1-\eta),$$

$$\psi_1 = \frac{1}{4}(1-\xi)(1+\eta), \quad \psi_4 = \frac{1}{4}(1+\xi)(1+\eta)$$



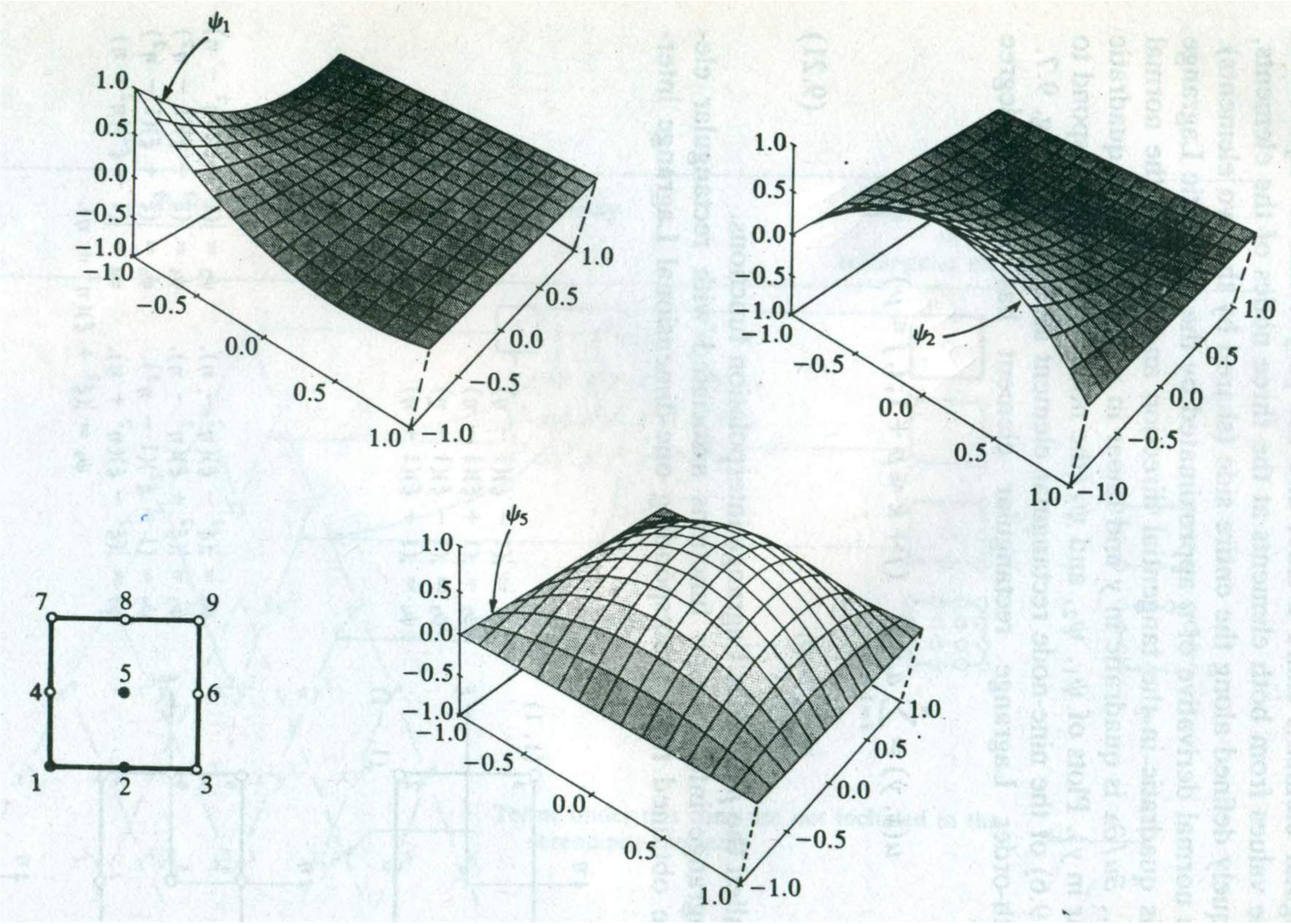
$$\psi_1 = \frac{1}{4}\left(\xi^2-\xi\right)\left(\eta^2-\eta\right), \quad \psi_2 = \frac{1}{2}\left(1-\xi^2\right)\left(\eta^2-\eta\right), \quad \psi_3 = \frac{1}{4}\left(\xi^2+\xi\right)\left(\eta^2-\eta\right),$$

$$\psi_4 = \frac{1}{2}\left(\xi^2-\xi\right)\left(1-\eta^2\right), \quad \psi_5 = \frac{1}{4}\left(1-\xi^2\right)\left(1-\eta^2\right), \quad \psi_6 = \frac{1}{2}\left(\xi^2+\xi\right)\left(1-\eta^2\right),$$

$$\psi_7 = \frac{1}{4}\left(\xi^2-\xi\right)\left(\eta^2+\eta\right), \quad \psi_8 = \frac{1}{2}\left(1-\xi^2\right)\left(\eta^2+\eta\right), \quad \psi_9 = \frac{1}{4}\left(\xi^2+\xi\right)\left(\eta^2+\eta\right)$$

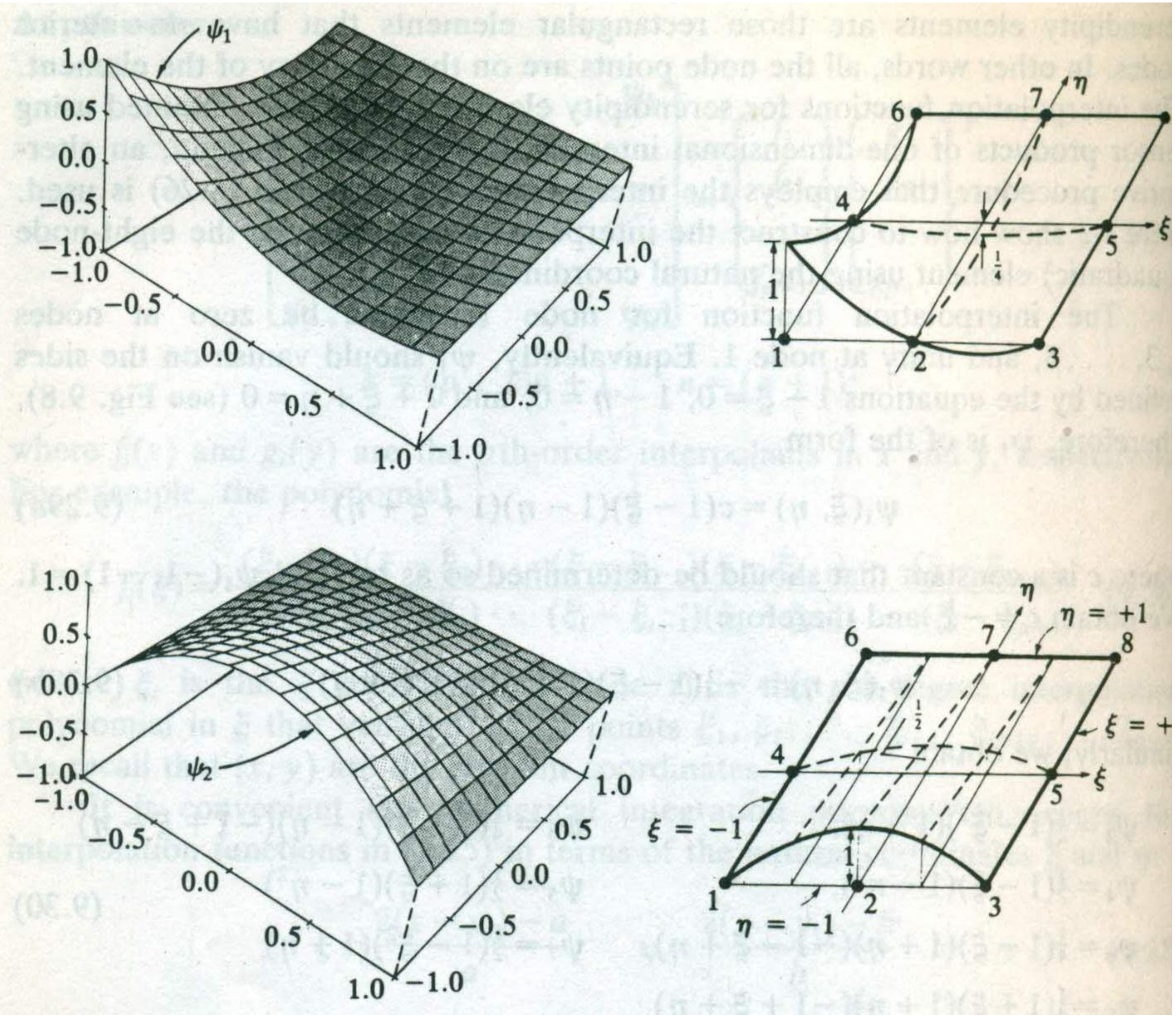
Lagrange Interpolation Functions: Quadratic Rectangular Element

Geometric variation of Lagrangian interpolation functions



Lagrange Interpolation Functions: Quadratic Rectangular Element

Geometric variation of Lagrange interpolation functions:



Hermite Interpolation Functions: Master Rectangular Element

- The shape function presented earlier are for the interpolation of primary variables.

- The Hermite cubic interpolation functions based on the interpolation of can be generated.

$$u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x \partial y}$$

Interpolation for

Variable *u*

$$\frac{1}{16}(\xi+\xi_i)^2(\xi\xi_i-2)(\eta+\eta_i)^2(\eta\eta_i-2)$$

derivative $\frac{\partial u}{\partial \xi}$

$$-\frac{1}{16}\xi_i(\xi+\xi_i)^2(\xi\xi_i-1)(\eta+\eta_i)^2(\eta\eta_i-2)$$

Hermite Interpolation Functions: Master Rectangular Element

Derivative $\dfrac{\partial u}{\partial \eta}$ $$-\frac{1}{16}\left(\xi+\xi_i\right)^2\left(\xi\xi_i-2\right)\eta_i\left(\eta+\eta_i\right)^2\left(\eta\eta_i-1\right)$$

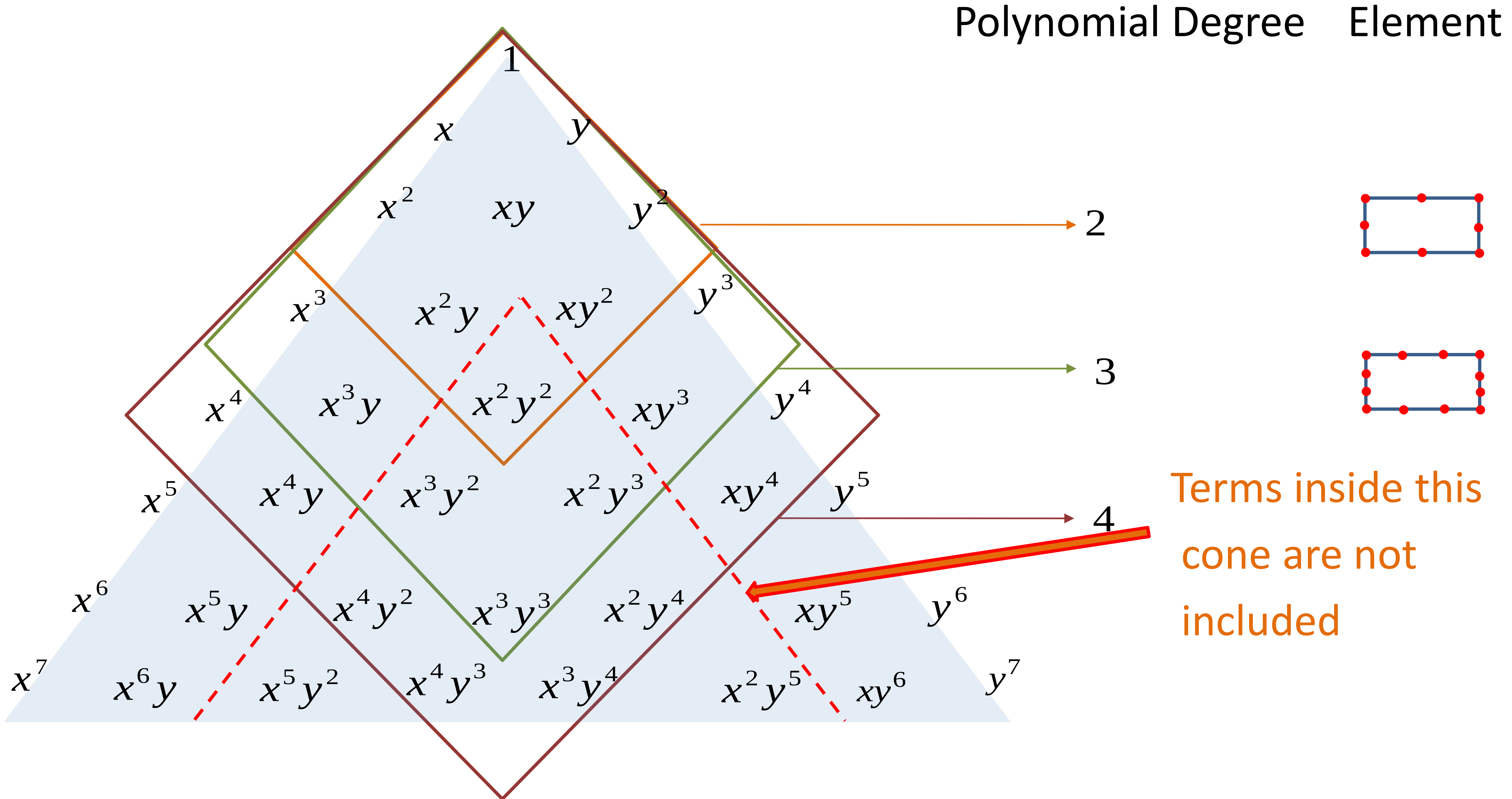
Derivative $\dfrac{\partial^2 u}{\partial \xi \partial \eta}$ $$\frac{1}{16}\xi_i\left(\xi+\xi_i\right)^2\left(\xi\xi_1-1\right)\eta_i\left(\eta+\eta_i\right)^2\left(\eta\eta_i-1\right)$$

where, *i=1,….4* are the nodes of the element.

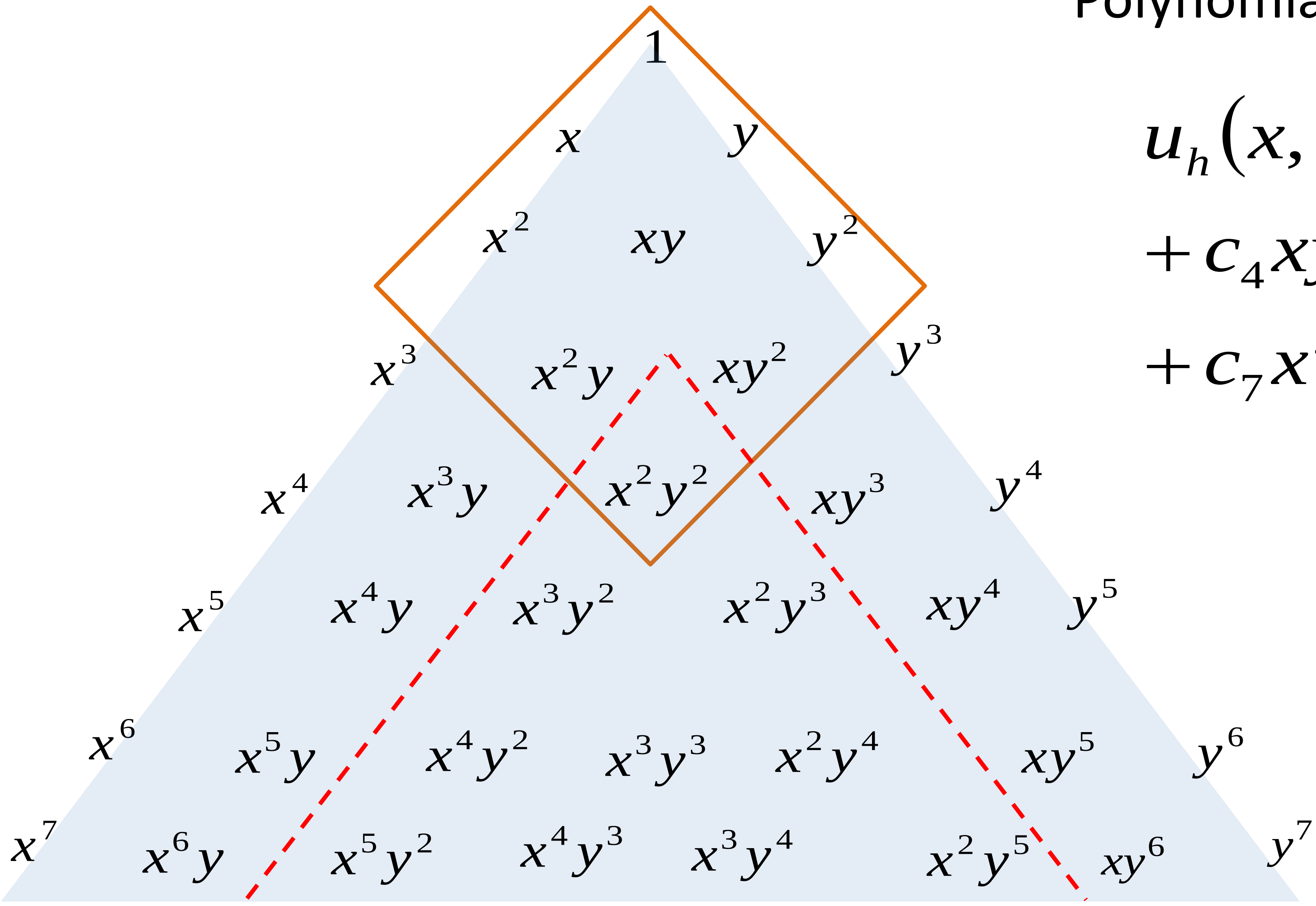
Serendipity Family of Interpolation Functions

For Rectangular Element

Interpolation Functions: Pascal's Triangle

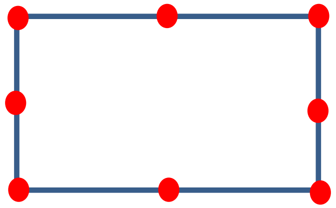


Interpolation Functions: Pascal's Triangle

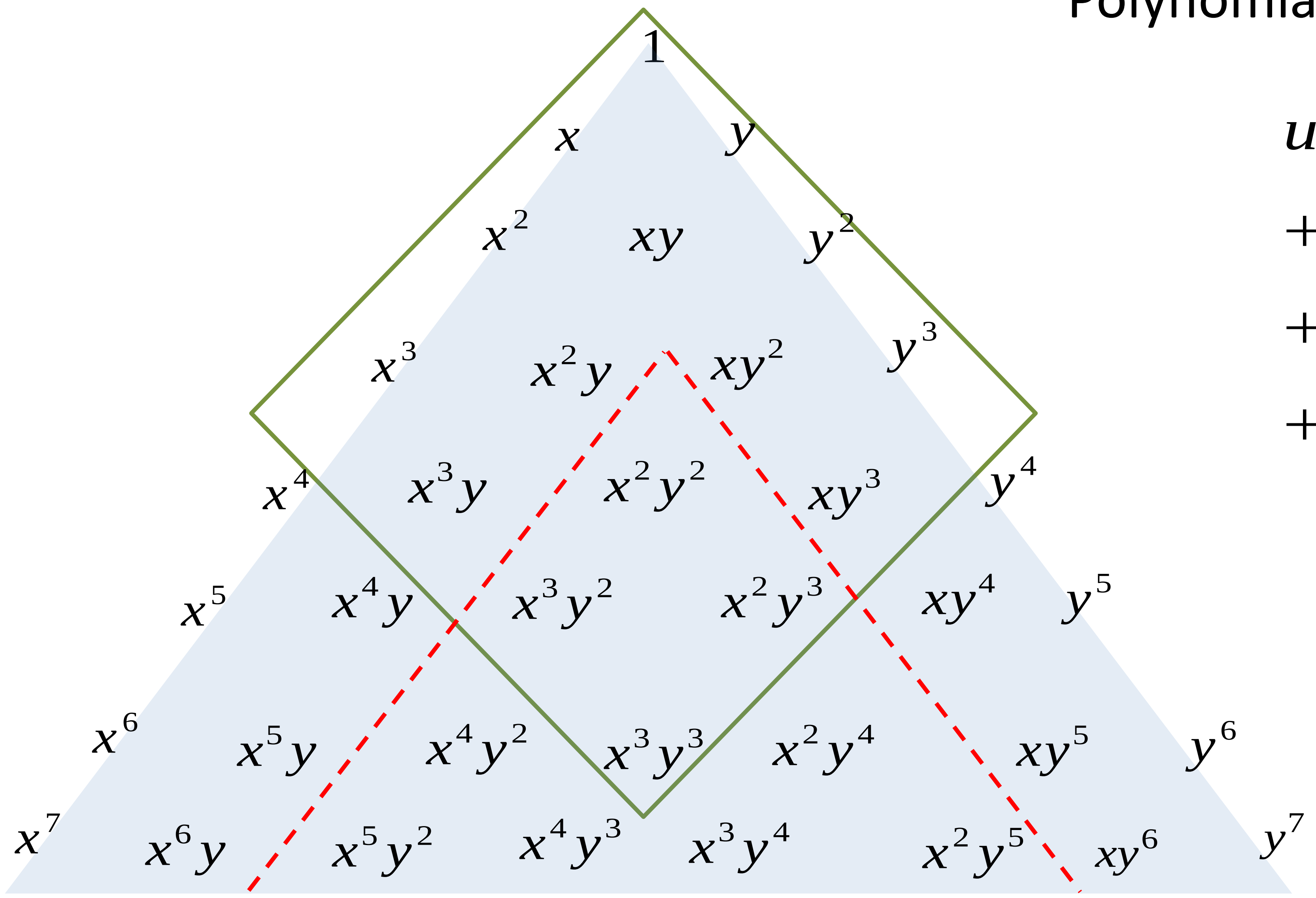


Polynomial Degree = 2

$$u_h(x, y) = c_1 + c_2 x + c_3 y + c_4 xy + c_5 x^2 + c_6 y^2 + c_7 x^2 y + c_8 xy^2$$

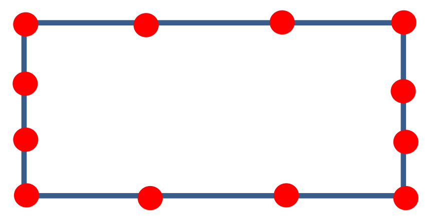


Interpolation Functions: Pascal's Triangle



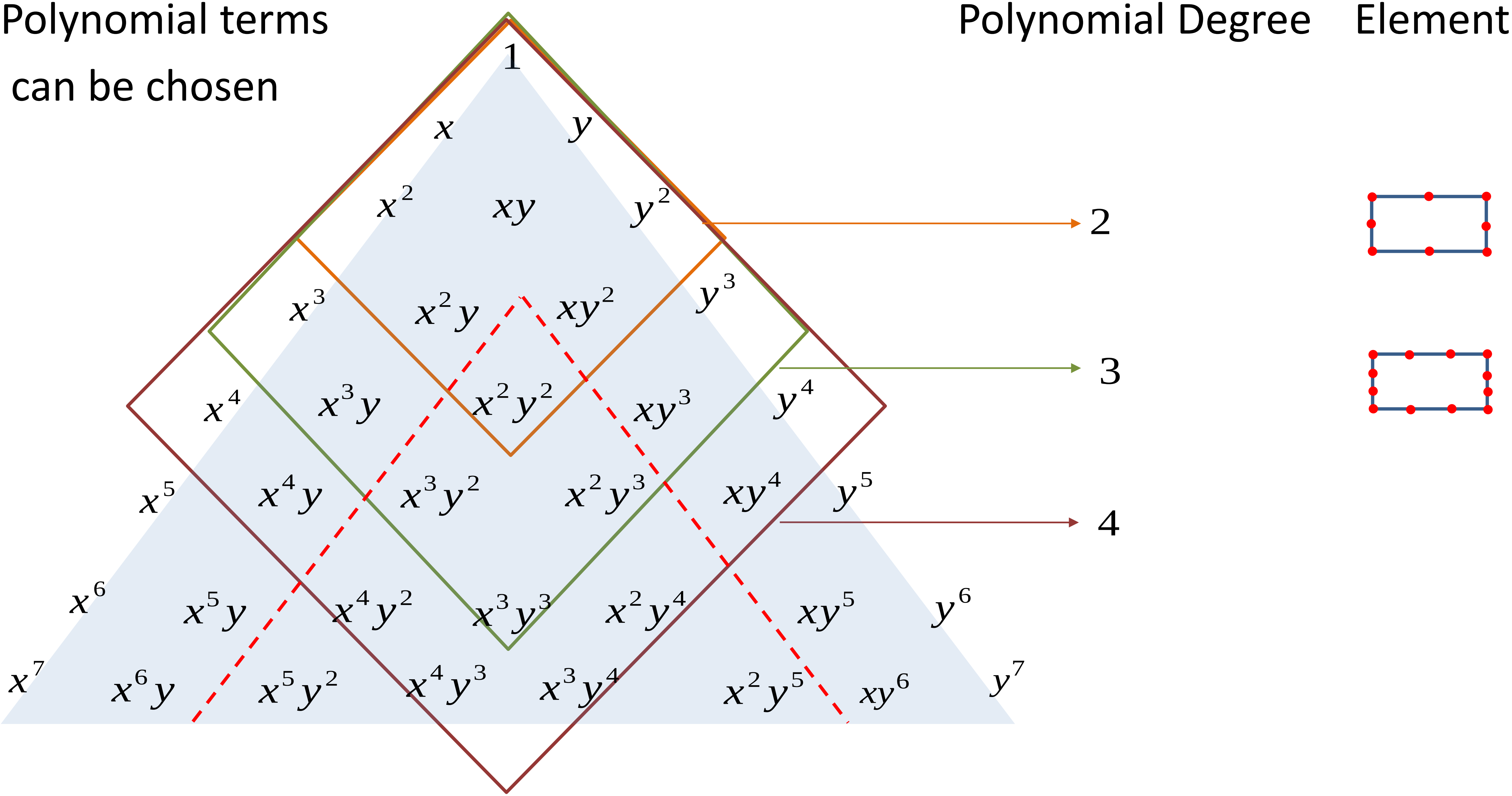
Polynomial Degree = 3

$$
\begin{aligned}
u_h(x, y) &= c_1 + c_2 x + c_3 y \\
&+ c_4 xy + c_5 x^2 + c_6 y^2 \\
&+ c_7 x^2 y + c_8 xy^2 + c_9 x^3 \\
&+ c_{10} x^3 y + c_{11} y^3 + c_{12} xy^3
\end{aligned}
$$



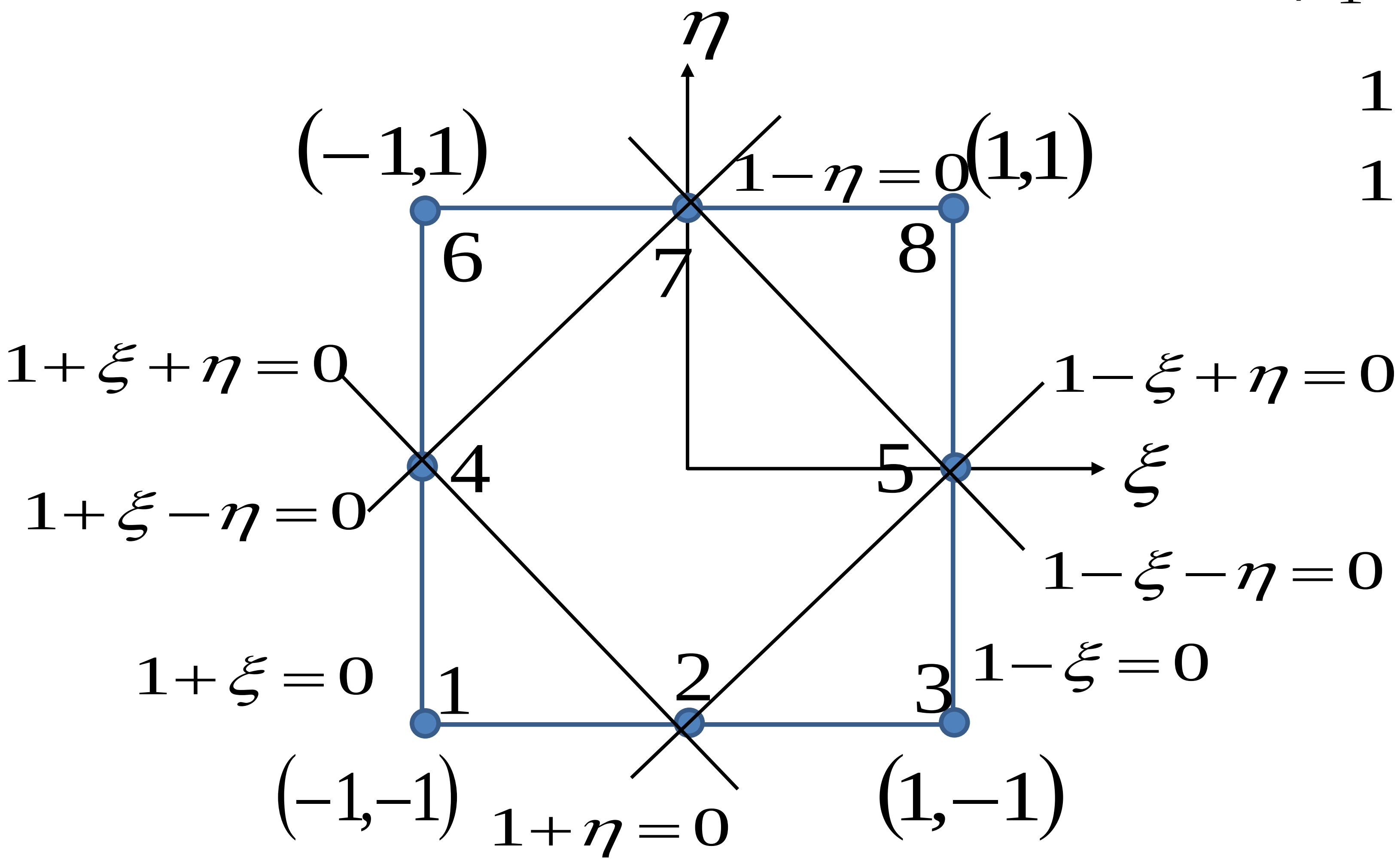
Interpolation Functions: Pascal's Triangle

Polynomial terms can be chosen



Interpolation Functions: Derivation of Serendipity Functions

Derivation of ψ_1:



ψ_1 vanishes on the lines

$$1-\eta=0,\ 1-\xi=0,$$
$$1+\xi+\eta=0$$

Interpolation Functions: Derivation of Serendipity Functions

Therefore, ψ_1 is of the from

$$\psi_1(\xi,\eta) = C(1-\xi)(1-\eta)(1+\xi+\eta)$$

It has a value of 1 at $(-1,-1)$ $\Rightarrow c = -1/4$

$$\psi_1 = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

ψ_2 vanishes along the lines $1+\xi = 0, 1-\xi = 0, 1-\eta = 0$

Let $\quad \psi_2 = c(1+\xi)(1-\xi)(1-\eta)$

$\psi_2 = 1$ at $(0,\text{-}1) \Rightarrow c = 1/2$

$$\psi_2 = \frac{1}{2}\left(1-\xi^2\right)(1-\eta)$$

Interpolation Functions: Derivation of Serendipity Functions

Similarly,

$$\psi_3 = \tfrac{1}{4}(1+\xi)(1-\eta)(-1+\xi-\eta)$$

$$\psi_4 = \tfrac{1}{4}(1-\xi)\left(1-\eta^2\right)$$

$$\psi_5 = \tfrac{1}{2}(1+\xi)\left(1-\eta^2\right)$$

$$\psi_6 = \tfrac{1}{4}(1-\xi)(1+\eta)(-1-\xi+\eta)$$

$$\psi_7 = \tfrac{1}{2}\left(1-\xi^2\right)(1+\eta)$$

$$\psi_8 = \tfrac{1}{4}(1+\xi)(1+\eta)(-1+\xi+\eta)$$

Note: The serendipity interpolation functions are different than Lagrange interpolation functions as internal nodes are not present.

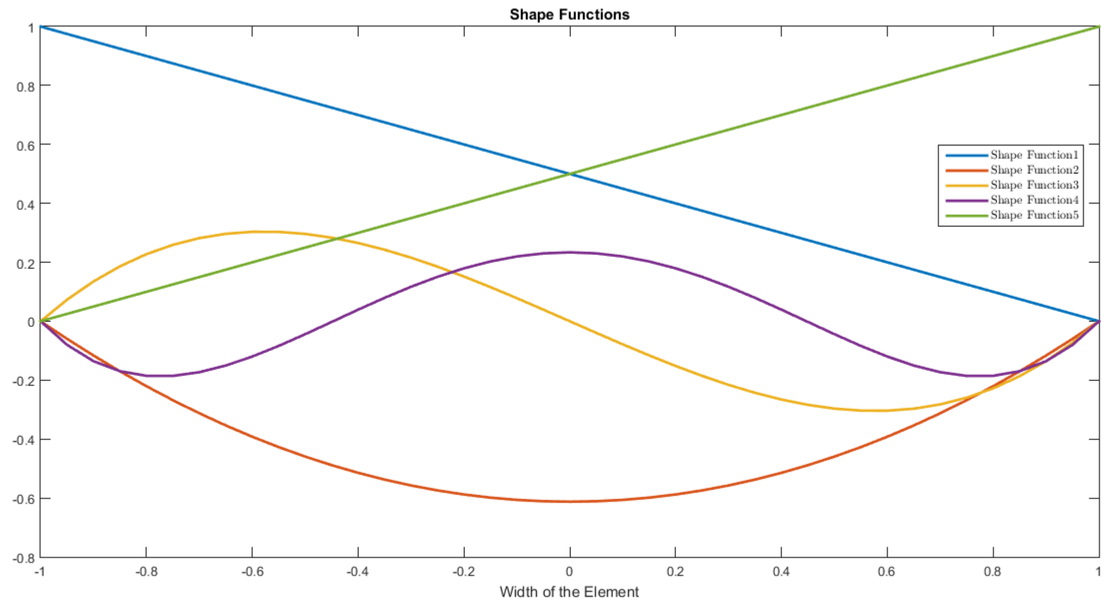
Hierarchic Interpolation Functions
For Rectangular Element

Interpolation Functions: Hierarchic Functions

- The set of interpolation functions of polynomial degree *p* should be in the set of interpolation functions of polynomial degree (*p+1*).

- The number of interpolation functions which do not vanish at vertices and sides should be smallest possible.

- Example: 1D Hierarchic interpolation functions



Hierarchic Shape Functions: Rectangular Elements

- Nodal Interpolation functions:

  - Four nodal interpolation functions
  - Exactly same as the functions for four noded quadrilateral element

$$\psi_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad \psi_2 = \frac{1}{4}(1+\xi)(1-\eta),$$
$$\psi_1 = \frac{1}{4}(1-\xi)(1+\eta), \quad \psi_4 = \frac{1}{4}(1+\xi)(1+\eta)$$

Hierarchic Shape Functions: Rectangular Elements

- Side Nodes Interpolation functions:

  - There are $4(p-1)$ interpolation functions associated with the sides of finite elements with the order of approximation $p \geq 2$

  - For Side 1 $\quad \psi_i^{(1)} = \frac{1}{2}(1-\eta)\phi_i(\xi) \quad i = 2, \cdots, p$

  - For Side 2 $\quad \psi_i^{(2)} = \frac{1}{2}(1+\xi)\phi_i(\eta) \quad i = 2, \cdots, p$

  - For Side 3 $\quad \psi_i^{(3)} = \frac{1}{2}(1+\eta)\phi_i(\xi) \quad i = 2, \cdots, p$

  - For Side 4 $\quad \psi_i^{(4)} = \frac{1}{2}(1-\xi)\phi_i(\eta) \quad i = 2, \cdots, p$

Hierarchic Shape Functions: Rectangular Elements

- Internal Nodes Interpolation functions:

- There are $(p-2)(p-3)/2$ interpolation functions associated with the internal nodes of finite elements with the order of approximation $p \geq 4$

$$\psi_1^{(0)} = \phi_2(\xi)\phi_2(\eta), \quad \psi_2^{(0)} = \phi_3(\xi)\phi_2(\eta),$$
$$\psi_3^{(0)} = \phi_2(\xi)\phi_3(\eta), \quad \psi_4^{(0)} = \phi_4(\xi)\phi_2(\eta),$$
$$\psi_5^{(0)} = \phi_3(\xi)\phi_3(\eta), \quad \psi_6^{(0)} = \phi_2(\xi)\phi_4(\eta),$$

and so on.

Hierarchic Shape Functions: Rectangular Elements

where, polynomial of degree *j* is:

$$\phi_j(\xi) = \sqrt{\frac{2j-1}{2}} \int_{-1}^{\xi} P_{j-1}(t)\,dt, \quad j = 2, 3, \cdots$$

P_j - is Legendre polynomial.

Using the properties of Legendre polynomials

$$\phi_j(\xi) = \frac{1}{\sqrt{2(2j-1)}} \left(P_j(\xi) - P_{j-2}(\xi)\right)$$

Hierarchic Shape Functions: Rectangular Elements

Legendre polynomial is given by Bonnet Recursion formula:

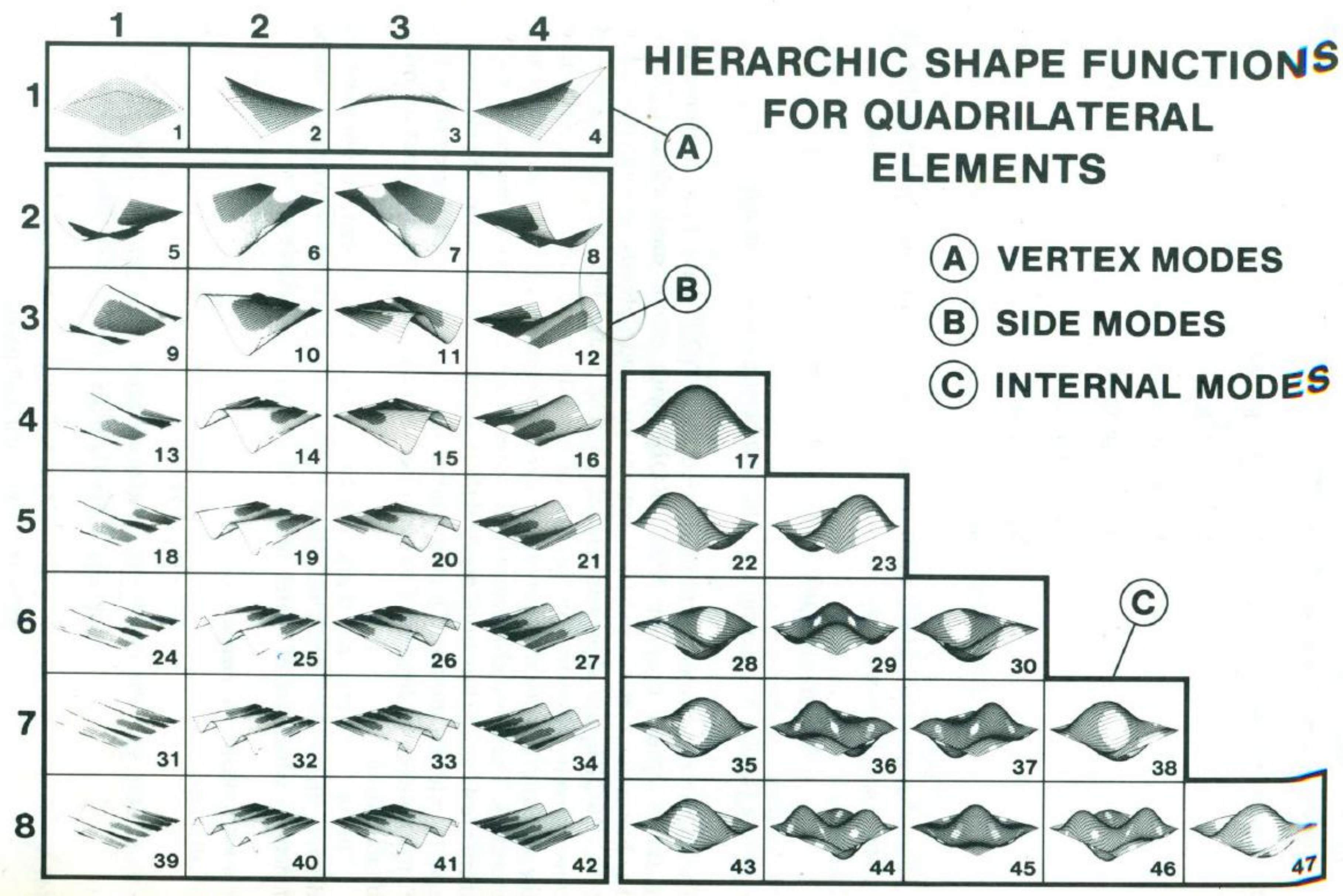
$$(n+1)P_{n+1}(\xi)=(2n+1)\xi P_n(\xi)-nP_{n-1}(\xi), \qquad n=1,2,\cdots$$

with $P_0(\xi)=1, \quad P_1(\xi)=\xi$

and

$$\int_{-1}^{+1} P_i(\xi)P_j(\xi)\,d\xi=\begin{cases}\dfrac{2}{2i+1} & \text{for } i=j\\ 0 & \text{for } i\neq j\end{cases}$$

Hierarchic Shape Functions: Rectangular Elements



Hierarchic Interpolation Functions

For Triangular Element

Hierarchic Shape Functions: Rectangular Elements

- Nodal Interpolation functions:

  - Three nodal interpolation functions - L_1, L_2, L_3

  - Exactly same as the functions for three noded triangular element

- Side Modes:

  - There are $3(p-1)$ side modes and vanish on the other two sides

Sides 1: $\psi_i^{(1)} = L_2 L_1 \varphi_i (L_2 - L_1), \quad i = 2, \cdots, p$

where, $\phi_j(\xi) = \frac{1}{4}(1-\xi^2)\varphi_j(\xi), \quad j = 2, \cdots, p$

Hierarchic Shape Functions: Rectangular Elements

For example:

$$\varphi_2(\xi) = -\sqrt{6}, \varphi_3(\xi) = -\sqrt{10}\xi, \varphi_4(\xi) = -\sqrt{\frac{7}{8}}\left(5\xi^2 - 1\right)$$

- Internal Modes:

  - There are $(p-1)(p-2)/2$ internal modes

Mode 1: $\psi_1^{(0)} = L_1 L_2 L_3$

Other modes:

$$\psi_2^{(0)} = L_1 L_2 L_3 P_1(L_2 - L_1), \quad \psi_3^{(0)} = L_1 L_2 L_3 P_1(2L_3 - 1)$$

and so on.

Physical to Master Element Mapping

Mapping:

- Non rectangular domains cannot be represented using rectangular elements. However, it can be represented more accurately by quadrilateral elements.

- Interpolation functions are easily available for a rectangular element
- And it is easy to evaluate integral over rectangular geometric

- Therefore, we transform the finite element integral statement over quadrilaterals to a rectangular.
- Similarly, the finite element integral statement over a triangular element is transformed to an isosceles triangle.
- Caution: Transformation is for numerical integration purpose only ! The actual domain is not mapped to another domain.

Mapping:

- Then numerical integration schemes like Gauss-Legendre are used to evaluate the integrals.
- These schemes require that the integral be evaluated on a specific domain or with respect to a specific coordinate system.
- In general it is a coordinate system (ξ,η) such that $-1 \leq \xi,\eta \leq 1$
- Transformation between physical element Ω^e and master element $\hat{\Omega}$

$$x = \sum_{j=1}^{m} x_j^e \hat{\psi}_j^e(\xi,\eta), \quad y = \sum_{j=1}^{m} y_j^e \hat{\psi}_j^e(\xi,\eta)$$

where, $\hat{\psi}_j^e$ interpolation functions over master element $\hat{\Omega}$

Mapping:

In general, the dependent variables(s) are approximated by expressions of the form

$$u(x,y)=\sum_{j=1}^{n}u_j^e\psi_i^e(x,y)$$

The degree of approximations used for the geometry and the dependent variables(s) are different.

Depending upon it, the finite element formulations are classified as:

- Sub-parametric
- Iso-parametric
- Super parametric

Mapping:

a) super- parametric $\left(m > n\right)$: The approximation used for the geometry is of higher order than used for the dependent variable (s).

b) iso-parametric $\left(m = n\right)$: Equal degree of approximation for geometry and dependent variable(s).

- This is extensively used in h- version of finite element programs.

Mapping:

c) Sub-parametric $(m < n)$: Approximation order used for geometry is lower than that used for primary variable (s).

- Coordinate transformations is only for the purpose of numerical integration.

- When an element is transformed to its master element for the purpose of numerical integration, the integrand must also be expressed in terms of the coordinates (ξ, η) of the master element.

Mapping :

For example, consider the stiffness coefficient

$$k_{ij}^e = \int_{\Omega^e} \left[a(x,y)\frac{\partial \psi_i^e}{\partial x}\frac{\partial \psi_j^e}{\partial x} + b(x,y)\frac{\partial \psi_i^e}{\partial y}\frac{\partial \psi_j^e}{\partial y} + c(x,y)\psi_i^e \psi_j^e \right] dx\,dy$$

Here, we need to relate $\frac{\partial \psi_i^e}{\partial x}, \frac{\partial \psi_i^e}{\partial y}$ with $\frac{\partial \psi_i^e}{\partial \xi}$ and $\frac{\partial \psi_i^e}{\partial \eta}$ using the transformation.

- $\psi_i^e(x,y)$ can be expressed in terms of local coordinate ξ and η. Hence, the chain rule for partial differentiation gives

$$\frac{\partial \psi_i^e}{\partial \xi} = \frac{\partial \psi_i^e}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \psi_i^e}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial \psi_i^e}{\partial \eta} = \frac{\partial \psi_i^e}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \psi_i^e}{\partial y}\frac{\partial y}{\partial \eta}$$

In matrix form,

$$\begin{Bmatrix} \frac{\partial \psi_i^e}{\partial \xi} \\ \frac{\partial \psi_i^e}{\partial \eta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{\text{Jacobian Matrix}} \begin{Bmatrix} \frac{\partial \psi_i^e}{\partial x} \\ \frac{\partial \psi_i^e}{\partial \eta} \end{Bmatrix}$$

Mapping :

In matrix form,

$$\begin{Bmatrix} \dfrac{\partial \psi_i^e}{\partial \xi} \\ \dfrac{\partial \psi_i^e}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \psi_i^e}{\partial x} \\ \dfrac{\partial \psi_i^e}{\partial \eta} \end{Bmatrix}$$

where, the Jacobian matrix is given as

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}^e$$

Mapping :

We need to relate $\dfrac{\partial \psi_i^e}{\partial x}, \dfrac{\partial \psi_i^e}{\partial y}$ with $\dfrac{\partial \hat{\psi}_i^e}{\partial \xi}$ and $\dfrac{\partial \hat{\psi}_i^e}{\partial \eta}$

Hence, we need the inverse of the Jacobian matrix.

$$\begin{Bmatrix} \dfrac{\partial \psi_i^e}{\partial x} \\ \dfrac{\partial \psi_i^e}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial \hat{\psi}_i^e}{\partial \xi} \\ \dfrac{\partial \hat{\psi}_i^e}{\partial \eta} \end{Bmatrix}$$

$[J]$ must be non singular.

Now, using the transformation, we can write

$$\frac{\partial x}{\partial \xi} = \sum_{j=1}^{m} x_j \frac{\partial \hat{\psi}_j^e}{\partial \xi}, \qquad \frac{\partial y}{\partial \xi} = \sum_{j=1}^{m} y_j \frac{\partial \hat{\psi}_i^e}{\partial \xi},$$

$$\frac{\partial x}{\partial \eta} = \sum_{j=1}^{m} x_j \frac{\partial \hat{\psi}_j^e}{\partial \eta}, \qquad \frac{\partial y}{\partial \eta} = \sum_{j=1}^{m} y_j \frac{\partial \hat{\psi}_i^e}{\partial \eta}$$

Mapping :

$$[J] = \begin{bmatrix} \sum_{j=1}^{m} x_j \frac{\partial \hat{\psi}_i^e}{\partial \xi} & \sum_{j=1}^{m} y_j \frac{\partial \hat{\psi}_i^e}{\partial \xi} \\ \sum_{j=1}^{m} x_j \frac{\partial \hat{\psi}_i^e}{\partial \eta} & \sum_{j=1}^{m} y_j \frac{\partial \hat{\psi}_i^e}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{\psi}_1}{\partial \xi} & \frac{\partial \hat{\psi}_2}{\partial \xi} & \cdots & \frac{\partial \hat{\psi}_m}{\partial \xi} \\ \frac{\partial \hat{\psi}_1}{\partial \eta} & \frac{\partial \hat{\psi}_2}{\partial \eta} & \cdots & \frac{\partial \hat{\psi}_m}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_m & y_m \end{bmatrix}$$

A necessary and sufficient condition for $[J]^{-1}$ to exist is that the determinant $\boldsymbol{J}$, called the Jacobian, be non negative at every point (ξ,η) in $\hat{\Omega}$.

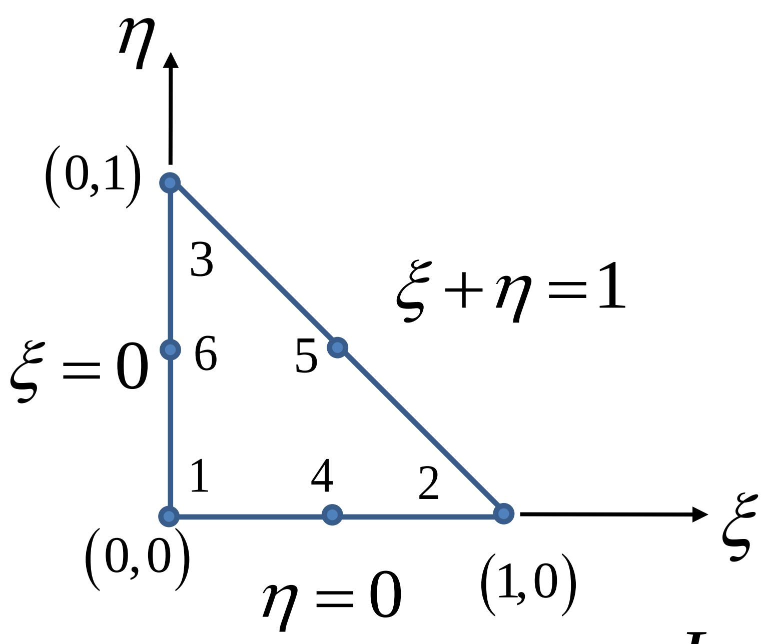
$$J = \det[J] = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} > 0$$

Thus, the function $\xi = \xi(x,y), \eta = \eta(x,y)$ must be continuous, differentiable and invertible.

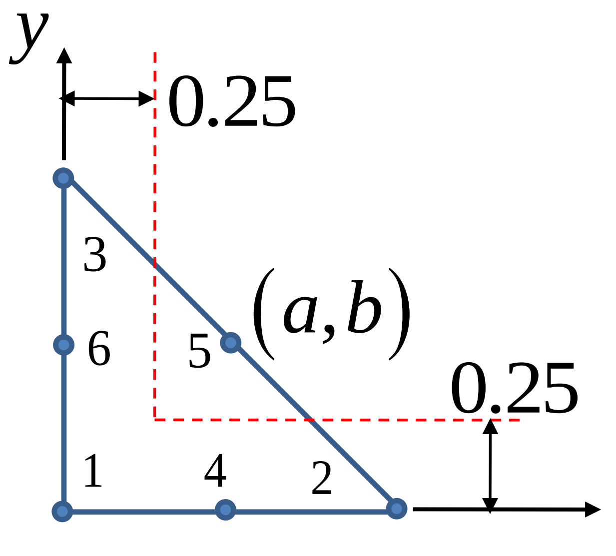
Mapping :

- In case of higher order triangular and rectangular elements the placement of edge and interior nodes is restricted.

- It can be shown that the side nodes should be placed at a distance greater than a quarter of the length of the side form either corner node.



$$x=\xi+4\xi\eta(a-1/2)$$
$$y=\eta+4\xi\eta(b-1/2)$$



$$J=1+2(2b-1)\xi+2(2a-1)\eta>0$$

Mapping :

Mapping the elemental area *dA* from *x,y* to $d\xi\, d\eta$

We have

$$dA = dx \cdot dy = \left[\vec{dx} \times \vec{dy} \right] \cdot \hat{k}$$

$$= \left\{ \left[\frac{\partial x}{\partial \xi} d\xi\, \hat{i} + \frac{\partial x}{\partial \eta} d\xi\, \hat{j} \right] \times \left[\frac{\partial y}{\partial \xi} d\xi\, \hat{i} + \frac{\partial y}{\partial \eta} d\eta\, \hat{j} \right] \right\} \cdot \hat{k}$$

$$= \left[\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \eta} \frac{\partial y}{\partial \xi} \right] d\xi\, d\eta$$

$$= J\, d\xi\, d\eta$$

Now, the derivatives of the shape functions with respect to *x,y* can be given as

$$\begin{Bmatrix} \dfrac{\partial \psi_i^e}{\partial x} \\ \dfrac{\partial \psi_i^e}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial \hat{\psi}_i^e}{\partial \xi} \\ \dfrac{\partial \hat{\psi}_i^e}{\partial \eta} \end{Bmatrix} = \left[J^* \right] \begin{Bmatrix} \dfrac{\partial \hat{\psi}_i^e}{\partial \xi} \\ \dfrac{\partial \hat{\psi}_i^e}{\partial \eta} \end{Bmatrix}$$

Mapping :

Now, using this we can write the element calculations over master element as

$$k_{ij}^{e}=\int_{\Omega^{e}}\left[a\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}+b\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y}+c\psi_i\psi_j\right]dxdy$$

$$=\int_{\hat{\Omega}}\left[\hat{a}\left(J_{11}^{*}\frac{\partial\hat{\psi}_i}{\partial\xi}+J_{12}^{*}\frac{\partial\hat{\psi}_i}{\partial\eta}\right)\left(J_{11}^{*}\frac{\partial\hat{\psi}_j}{\partial\xi}+J_{12}^{*}\frac{\partial\hat{\psi}_j}{\partial\eta}\right)\right.$$

$$\left.+\hat{b}\left(J_{21}^{*}\frac{\partial\psi_i}{\partial\xi}+J_{22}^{*}\frac{\partial\psi_i}{\partial\eta}\right)\left(J_{21}^{*}\frac{\partial\psi_i}{\partial\xi}+J_{22}^{*}\frac{\partial\psi_i}{\partial\eta}\right)+\hat{c}\psi_i\psi_j\right]J\,d\xi\,d\eta$$

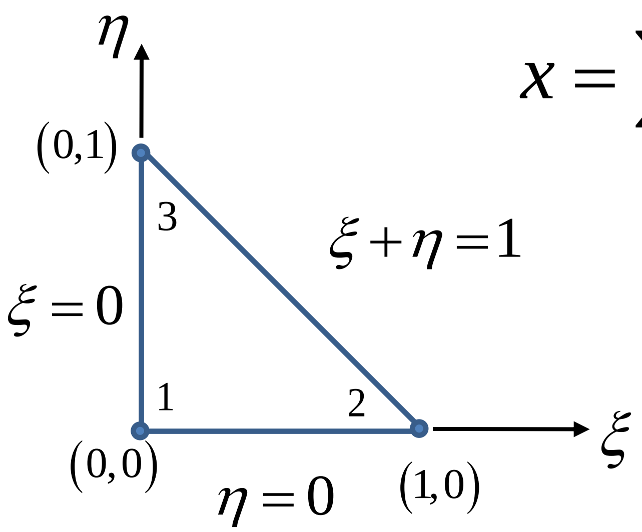
or

$$k_{ij}^{e}=\int_{\hat{\Omega}}F(\xi,\eta)\,d\xi\,d\eta$$

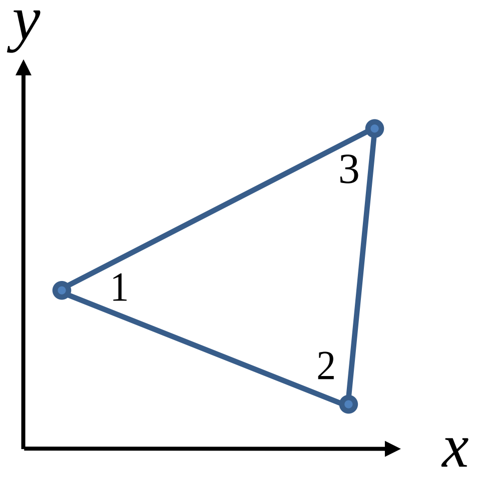
Mapping of Physical Elements to Master Elements

Affine Mapping

Physical to Master Element Mapping: Linear Triangular Element



$$x=\sum_{j=1}^{3} x_j\hat{\psi}_j^e,\quad y=\sum_{j=1}^{3} y_j\hat{\psi}_j^e$$



Linear Master Element

Linear Physical Element

- The affine mapping essentially stretches, translates, and rotates the triangle.
- Straight or planar faces of the reference cell are therefore mapped onto straight or planar faces in the physical coordinate system.
- Preferred when all the edges of an element are straight.

Numerical Integration: Triangular Element

For three noded triangular element the transformation of co-ordinates is given by

$$x = \sum_{i=1}^{3} x_i \hat{\psi}_i(\xi,\eta), \quad y = \sum_{i=1}^{3} y_i \hat{\psi}_i(\xi,\eta)$$

- where, $\hat{\psi}_i$ are the interpolation functions over the master three noded triangular element.

$$\hat{\psi}_1 = 1 - \xi - \eta$$
$$\hat{\psi}_2 = \xi$$
$$\hat{\psi}_3 = \eta$$

- Master triangle is an isosceles triangle with unit length and right angle

Mapping: Triangular Element

The Jacobian matrix, [J] for the linear triangular element is

$$[J] = \begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{bmatrix}$$

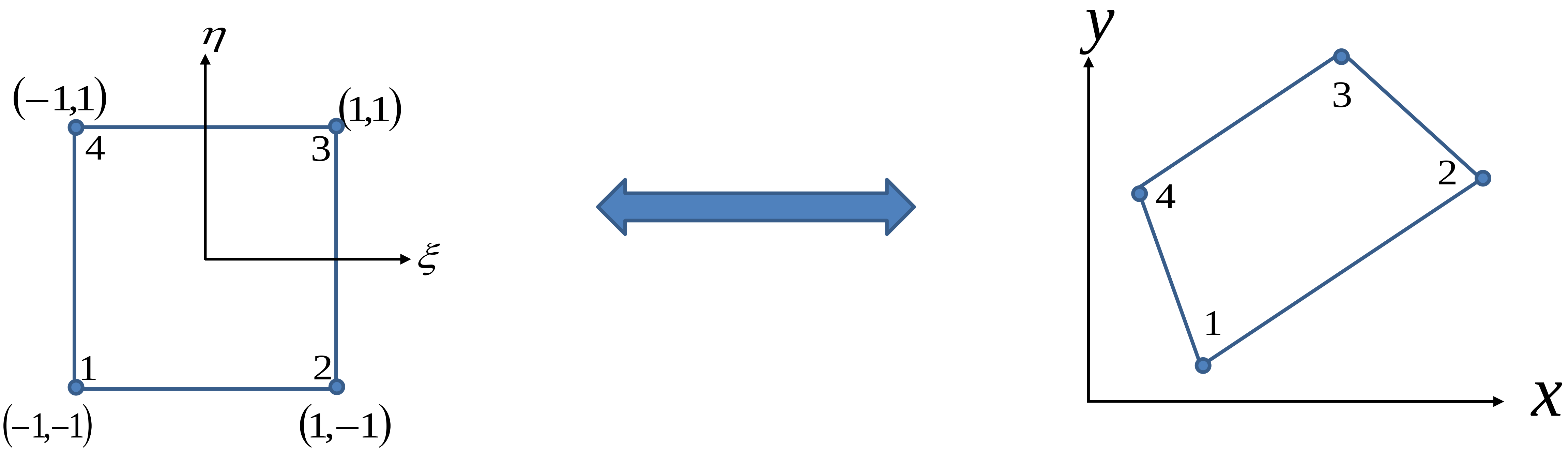
Inverse transformation from the element Ω^e to $\hat{\Omega}$ is:

$$\xi = \frac{1}{2A}\left[(x - x_1)(y_3 - y_1) - (y - y_1)(x_3 - x_1)\right]$$

$$\eta = \frac{1}{2A}\left[(x - x_1)(y_1 - y_2) + (y - y_1)(x_2 - x_1)\right]$$

where, *A* the area of the element Ω^e

Physical to Master Element Mapping: Linear Rectangular Element



Linear Master Element

Linear Physical Element

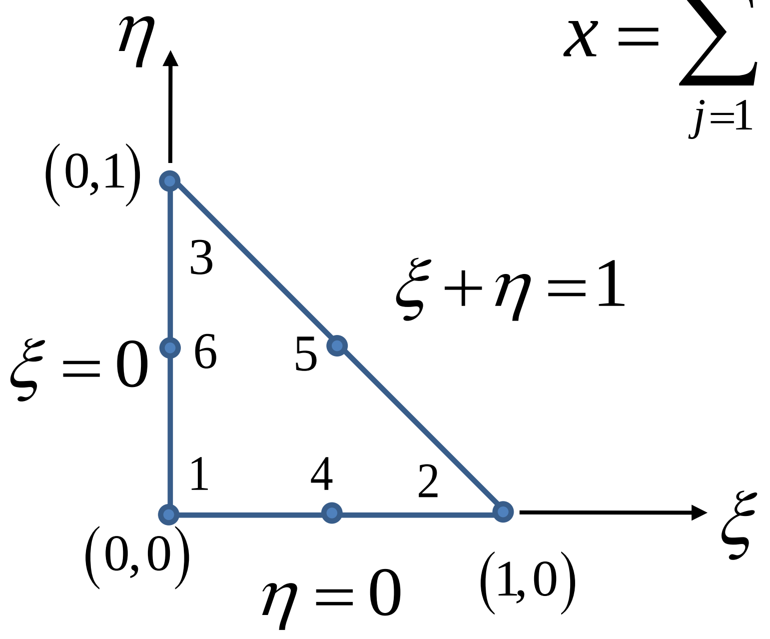
$$x = \sum_{j=1}^{4} x_j \hat{\psi}_j^e, \quad y = \sum_{j=1}^{4} y_j \hat{\psi}_j^e$$

Mapping of Physical Elements to Master Elements

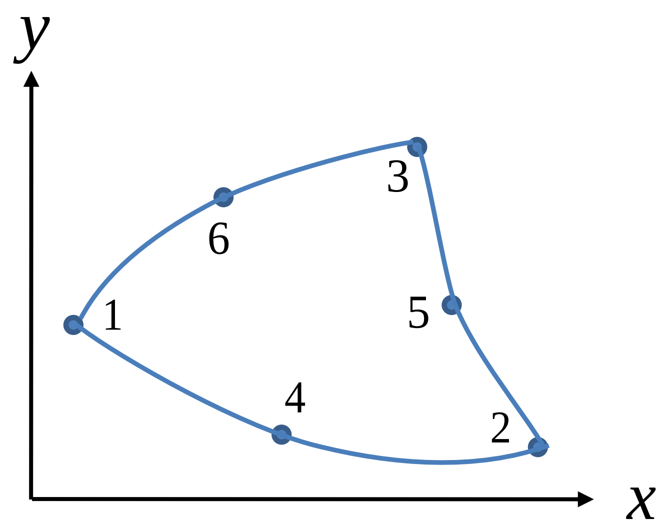
Iso-parametric Mapping

Physical to Master Element Mapping: Quadratic Triangular Element

$$x = \sum_{j=1}^{6} x_j \hat{\psi}_j^e, \;\; y = \sum_{j=1}^{6} y_j \hat{\psi}_j^e$$



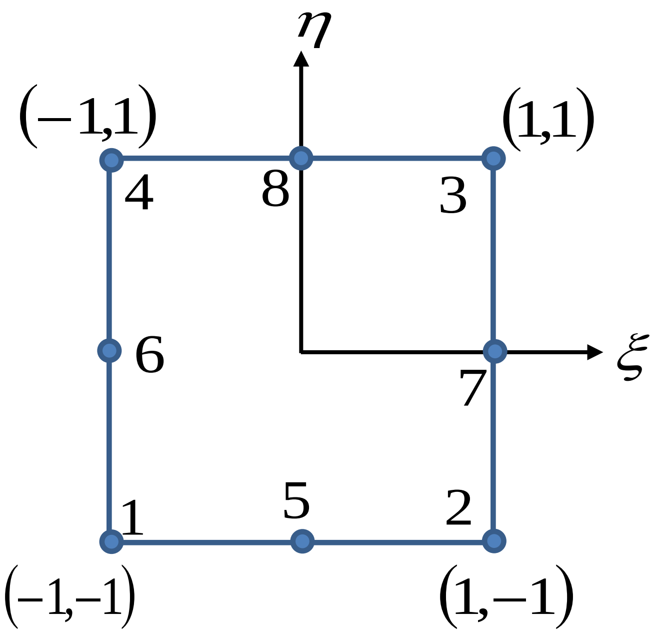
Quadratic Master Element

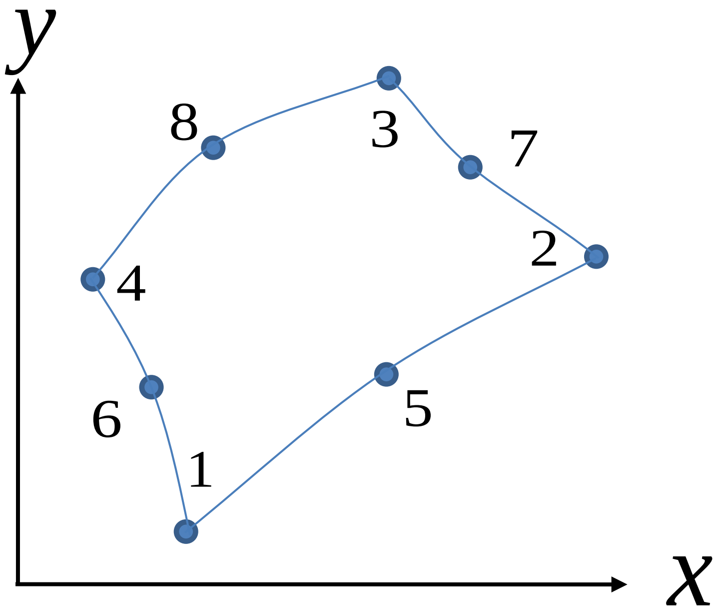


Quadratic Physical Element

- The straight faces of the reference triangle are mapped onto curved faces of parabolic shape in the physical coordinate system.

Physical to Master Element Mapping: Quadratic Rectangular Element





Quadratic Master Element

Quadratic Physical Element

$$x = \sum_{j=1}^{8} x_j \hat{\psi}_j^e, \quad y = \sum_{j=1}^{8} y_j \hat{\psi}_j^e$$

Mapping of Physical Elements to Master Elements

Blending Function Mapping

Physical to Master Element Mapping: Quadratic Rectangular Element

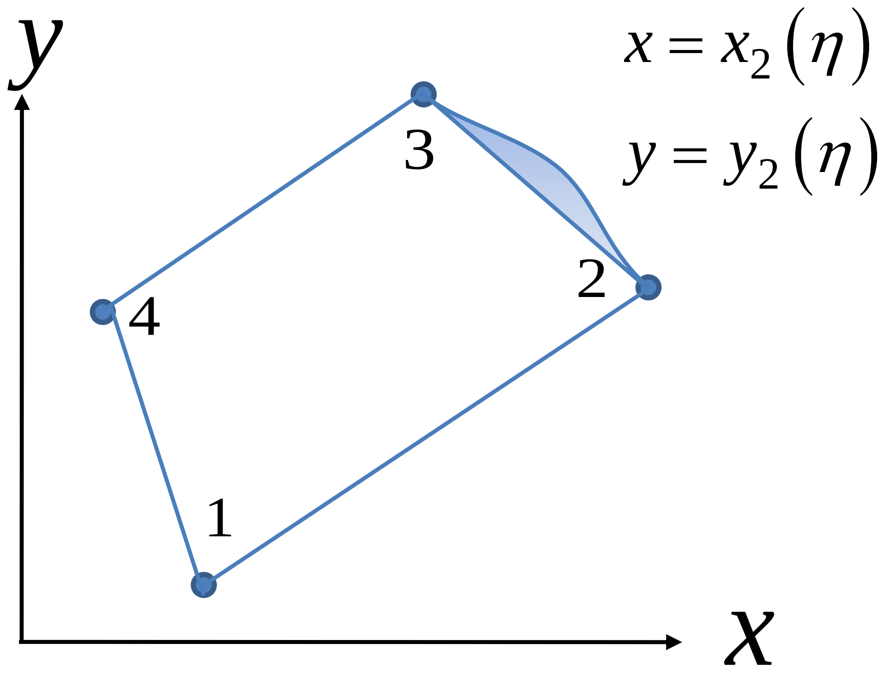
Curved Quadratic Physical Element with

Side 2 is the only curved side.

Curve $x = x_2(\eta), y = y_2(\eta)$ given in parametric form so that

$$x_2(-1) = X_2, y_2(-1) = Y_2, x_2(1) = X_3, y_2(1) = Y_3$$

(X_i, Y_i) are the coordinates of node *i*



Physical to Master Element Mapping: Quadratic Rectangular Element

We have

$$x=\frac{1}{4}(1-\xi)(1-\eta)X_1+\frac{1}{4}(1+\xi)(1-\eta)X_2+\frac{1}{4}(1+\xi)(1+\eta)X_3+\frac{1}{4}(1-\xi)(1+\eta)X_4$$
$$+\left(x_2(\eta)-\frac{1-\eta}{2}X_2-\frac{1+\eta}{2}X_3\right)\frac{1+\xi}{2}$$

The first four terms are the linear mapping terms.

The fifth term is the product of two functions:

The bracketed term represents the difference between $x_2(\eta)$

and the *x*-coordinates of the chord that connects (X_2,Y_2) and (X_3,Y_3)

The other is the linear blending function $(1+\xi)/2$, which is unity along side 2 and zero along side 4.

Physical to Master Element Mapping: Quadratic Rectangular Element

Therefore, we simplify as

$$x=\frac{1}{4}(1-\xi)(1-\eta)X_1+\frac{1}{4}(1-\xi)(1+\eta)X_4+x_2(\eta)\frac{1+\xi}{2}$$

Similarly,

$$y=\frac{1}{4}(1-\xi)(1-\eta)Y_1+\frac{1}{4}(1-\xi)(1+\eta)Y_4+y_2(\eta)\frac{1+\xi}{2}$$

In the general case all sides may be curved. We write the curved sides in the parametric form:

$$x=x_i(\xi),y=y_i(\xi),\quad -1\le\xi\le1\quad (i=1,2,3,4)$$

Mapping functions are:

$$x=-\frac{1}{4}(1-\xi)(1-\eta)X_1-\frac{1}{4}(1+\xi)(1-\eta)X_2-\frac{1}{4}(1+\xi)(1+\eta)X_3-\frac{1}{4}(1-\xi)(1+\eta)X_4$$

$$+\frac{1-\eta}{2}x_1(\xi)+\frac{1+\xi}{2}x_2(\eta)+\frac{1+\eta}{2}x_3(\xi)+\frac{1-\xi}{2}x_4(\eta)$$

Physical to Master Element Mapping: Quadratic Rectangular Element

Mapping functions for *y* coordinate are:

$$y = -\frac{1}{4}(1-\xi)(1-\eta)Y_1 - \frac{1}{4}(1+\xi)(1-\eta)Y_2 - \frac{1}{4}(1+\xi)(1+\eta)Y_3 - \frac{1}{4}(1-\xi)(1+\eta)Y_4$$
$$+\frac{1-\eta}{2}y_1(\xi) + \frac{1+\xi}{2}y_2(\eta) + \frac{1+\eta}{2}y_3(\xi) + \frac{1-\xi}{2}y_4(\eta)$$

Inverse mapping cannot be given explicitly.

Newton-Raphson method or similar procedure is used.

One can see the quadratic iso-parametric mapping as a special application of the blending function method.

Numerical Integration over a Master Rectangular Element

Numerical Integration:

$$k_{ij}^{e} = \int_{\hat{\Omega}} F(\xi,\eta)\,d\xi\,d\eta$$

$$\int_{\hat{\Omega}} F(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{1}\left[\int_{-1}^{1} F(\xi,\eta)\,d\eta\right]d\xi \approx \int_{-1}^{1}\left[\sum_{j=1}^{M}(\xi,\eta_j)w_j\right]d\xi$$

$$\approx \sum_{i=1}^{M}\sum_{j=1}^{N} F(\xi_i,\eta_j)w_i w_j$$

M,N - Number of quaddrature points in ξ and η directions.

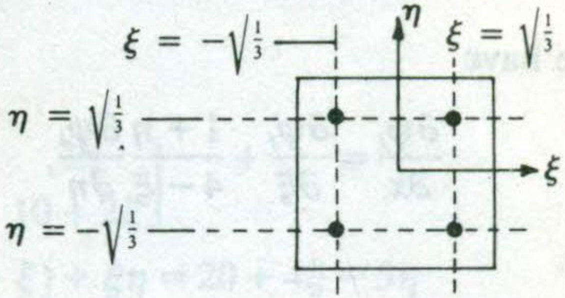
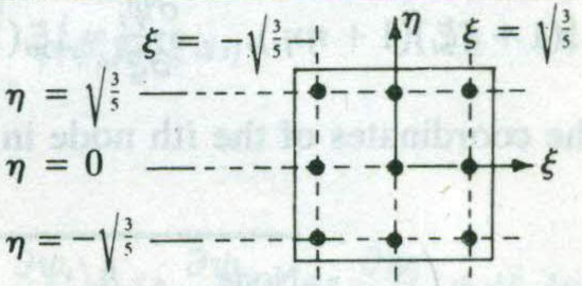
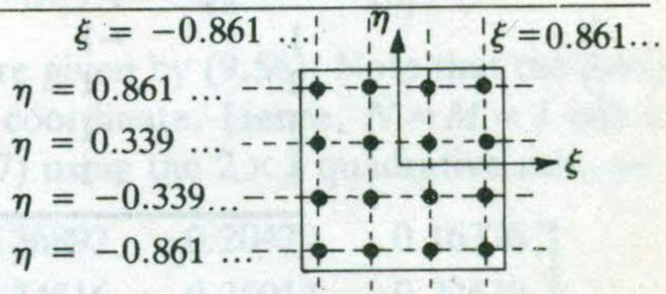
(ξ_i,η_i) - Gauss points,

w_i, w_j - corresponding weights.

The $M \times N$ Gauss point locations are given by the tensor product of one dimensional Gauss points.

Numerical Integration: Rectangular Element

Selection of integration order and location of the Gauss points:

Element type	Maximum polynomial degree	Order of integration $(r \times r)$	Order of the residual	Location of integration points* in master element
Linear $(r = 2)$	2	2×2	$O(h^4)$	
Quadratic $(r = 3)$	4	3×3	$O(h^6)$	
Cubic $(r = 4)$	6	4×4	$O(h^8)$	

Numerical Integration over a Master Triangular Element

Numerical Integration:

$$k_{ij}^{e} = \int_{\hat{\Omega}} F(\xi,\eta)\, d\xi\, d\eta$$

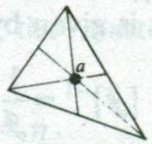
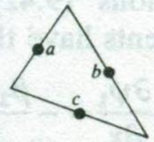
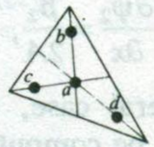
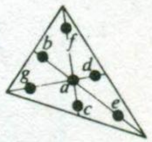
$$\int_{\hat{\Omega}} F(\xi,\eta)\, d\xi\, d\eta = \int_0^1\int_0^1 F(\xi,\eta)\, d\eta\, d\xi \approx \left[\sum_{j=1}^{NINT} \left(\xi_j,\eta_j\right) w_j\right]$$

NINT – Number of quadrature points

$\left(\xi_i,\eta_i\right)$ - Gauss points,

w_i - corresponding weight.

Numerical Integration: Triangular Element

Number of integration points	Degree of polynomial and order of the residual	L_1	L_2	L_3	W	Geometric locations	
1	1 $O(h^2)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	a	
3	2 $O(h^3)$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{3}$	a	
		$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{3}$	b	
		0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	c	
4	3 $O(h^4)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{27}{48}$	a	
		0.6	0.2	0.2	$\frac{25}{48}$	b	
		0.2	0.6	0.2	$\frac{25}{48}$	c	
		0.2	0.2	0.6	$\frac{25}{48}$		
7	5 $O(h^6)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0.225	a	
		α_1	β_1	β_1	w_2	b	
		β_1	α_1	β_1	w_2	c	
		β_1	β_1	α_1	w_2	d	
		α_2	β_2	β_2	w_3	e	
		β_2	α_2	β_2	w_3	f	
		β_2	β_2	α_2	w_3	g	

$\alpha_1 = 0.797\ 426\ 985\ 353$

$\beta_1 = 0.101\ 286\ 507\ 323$

$\alpha_2 = 0.059\ 715\ 871\ 789$

$\beta_2 = 0.470\ 142\ 064\ 105$

$W_2 = 0.125\ 939\ 180\ 544$

$W_3 = 0.132\ 394\ 152\ 788$

Numerical Integration: Using Area Co-ordinates

Line integration:

$$\int_a^b (L_1)^m (L_2)^k \, ds = \frac{m!k!}{(m+k+1)!}(b-a)$$

Area integration:

$$\iint_A (L_1)^m (L_2)^k (L_2)^l \, dA = \frac{m!k!l!}{(m+k+l+2)!} 2A$$

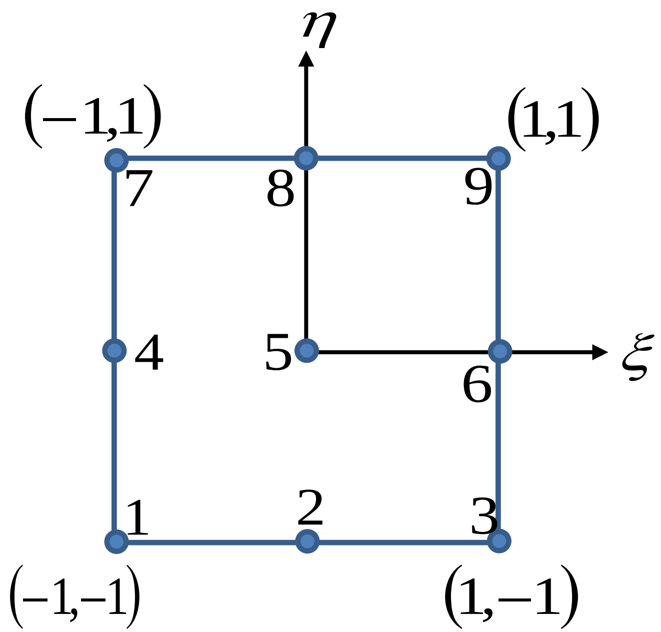
Transformation of the co-ordinates to natural co-ordinates:

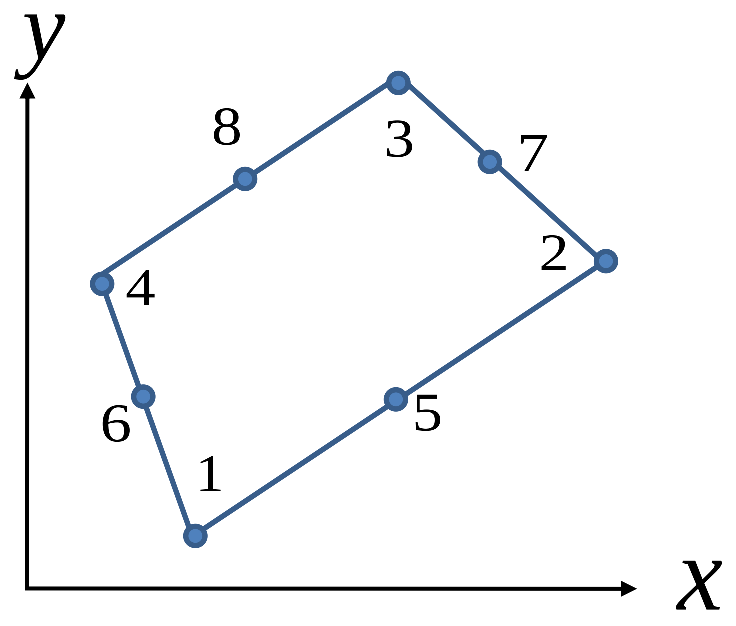
$$x = \sum_{i=1}^{n} x_i L_i, \quad y = \sum_{i=1}^{n} y_i L_i$$

where, (x_i, y_i) are the global coordinates of i^{th} node.

Thank you.

Physical to Master Element Mapping: Quadratic Rectangular Element





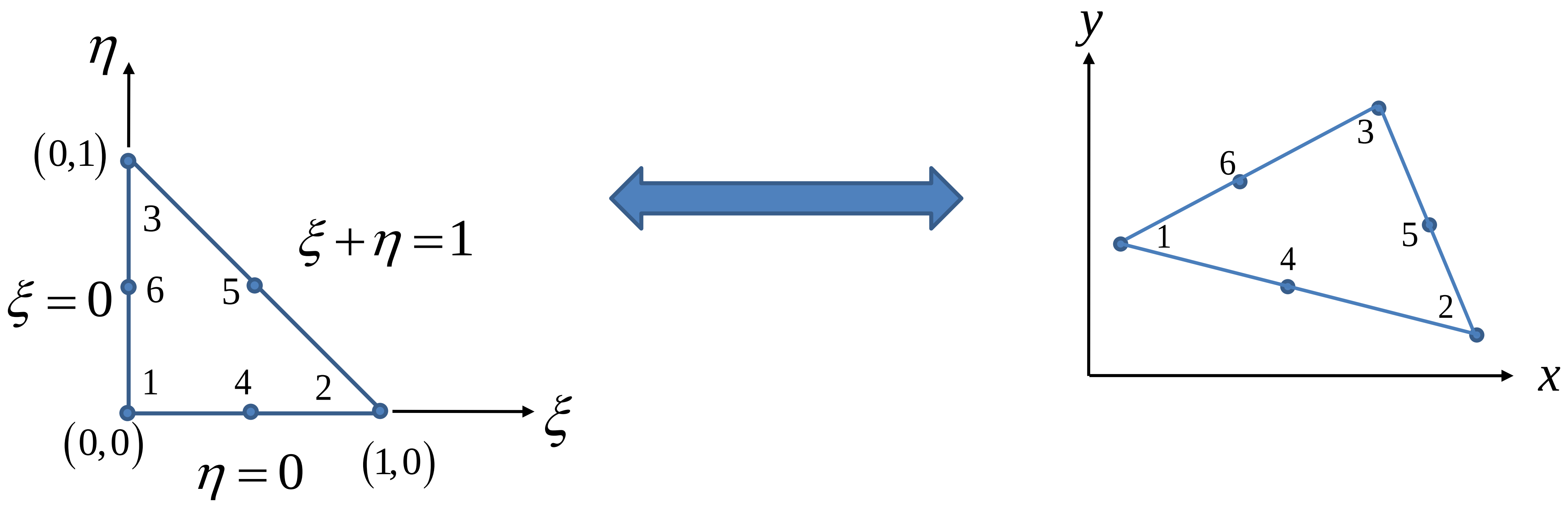
Quadratic Master Element

Quadratic Physical Element

$$x = \sum_{j=1}^{8} x_j \hat{\psi}_j^e, \quad y = \sum_{j=1}^{8} y_j \hat{\psi}_j^e$$

- This is also an Iso-parametric mapping of quadratic element.

Physical to Master Element Mapping: Quadratic Triangular Element



Quadratic Master Element

Quadratic Physical Element

- This is also an Iso-parametric mapping of quadratic element.

$$x = \sum_{j=1}^{6} x_j \hat{\psi}_j^e, \;\; y = \sum_{j=1}^{6} y_j \hat{\psi}_j^e$$