Article Review - Multi-resolution dynamic mode decomposition for foreground/background separation and object tracking Week 9, Spring 2019

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Introduction

Dynamic Mode Decomposition provides spatio-temporal modes that correlate data across spatial features (like PCA), but also pins the spatially correlated data to unique temporal Fourier modes.

Overview

The task is to approximate nonlinear system

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}, t)\,. \tag{1}$$

with

$$\frac{d\tilde{\mathbf{x}}}{dt} = \mathbf{A}\tilde{\mathbf{x}} \tag{2}$$

with the solution being

$$\tilde{\mathbf{x}}(t) = \sum_{k=1}^{K} b_k \psi_k \exp(\omega_k t) \tag{3}$$

This is done in a least-square sense. The result is a low-rank structure, allowing for dimensionality reduction of the original system.

Method

The DMD is a data-driven method, taking M snapshots with N spatial points saved per snapshot. These are stored in large data matrices

$$\mathbf{X} = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_M \\ | & | & & | \end{bmatrix}, \quad \mathbf{X}' = \begin{bmatrix} | & | & & | \\ \mathbf{x}'_1 & \mathbf{x}'_2 & \cdots & \mathbf{x}'_M \\ | & | & & | \end{bmatrix} \quad (6)$$

where X is the original data, and X' is the data after some evolution of the nonlinear system. One can think of the DMD modes as the eigenvectors of $A = X'X^{\dagger}$. Here A is the Koopman operator, central to the DMD, but too 'mathy' for right now...

Algorithm

First, an SVD of the original data set is computed

$$\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*, \tag{9}$$

The SVD is truncated by retaining largest singular values and corresponding modes. An approximation $\widetilde{A}$ of the Koopman operator A is given by projecting A onto the low-rank modes of U:

$$\begin{aligned} \mathbf{A} &= \mathbf{X}'\mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^* \\ \Longrightarrow \quad \tilde{\mathbf{A}} &= \mathbf{U}^*\mathbf{A}\mathbf{U} = \mathbf{U}^*\mathbf{X}'\mathbf{V}\mathbf{\Sigma}^{-1}. \end{aligned} \tag{10}$$

Algorithm

Then an eigendecomposition of $\widetilde{A}$ is done

$$\tilde{\mathbf{A}}\mathbf{W} = \mathbf{W}\mathbf{\Lambda}, \qquad (11)$$

and the ultimate goal is to reconstruct the eigendecomposition of A from these eigenvectors and eigenvalues. Thus the eigenvectors of A (DMD modes) are given by the columns of

$$\mathbf{\Psi} = \mathbf{X}'\mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{W}. \qquad (12)$$

Using these modes as a basis, a projected evolution of the system in time is given by

$$\tilde{\mathbf{x}}(t) = \sum_{k=1}^{K} b_k(0)\psi_k(\boldsymbol{\xi})\exp(\omega_k t) = \mathbf{\Psi}\mathrm{diag}(\exp(\boldsymbol{\omega} t))\mathbf{b} \quad (13)$$

Multiresolution

Multiresolution adds another flavor to DMD. Whereas in the standard method the spatially correlated data is represented in terms of Fourier modes, the multiresolution approach separates the DMD analysis into varying levels of time resolution. This has the benefit not allowing a single set of modes that dominate the SVD to marginalize features at other time scales.

Multiresolution