

Belief Change without Compactness - The Case of Knowledge Update

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Abstract

The main paradigms of belief change impose the background logic to be, among others, Tarskian and compact. However, logics such as temporal logics that are interesting from the vantage point of AI do not satisfy the compactness property. In this work, we extend the KM approach to belief update through relaxing the requirement for the background logic to be compact. We show that, in absence of compactness, the classical construction for KM update, which relies on preorders, does not capture all the rationality postulates for KM belief update. Indeed, in absence of compactness such construction, being fully characterised by a subset of the KM update postulates, is weaker. We explore the reason behind this, and subsequently provide an alternative and new constructive accounts of belief update which is characterised by the full set of KM postulates.

1 Introduction

Belief change studies the dynamics of a body of knowledge employed in order to accommodate a new piece of information. The two most influential paradigms of belief change are the AGM paradigm [Alchourrón *et al.*, 1985] and the KM paradigm [Katsuno and Mendelzon, 2003]. Broadly speaking, the former concerns the process of changing beliefs in the context of *static* worlds, whereas the latter consider knowledge change in a *dynamic* worlds.

In both approaches, belief change may be represented as a function f that takes an epistemic state into another one, given a new piece of information. These functions are governed by rationality postulates whose purpose is to prescribe good set of behaviours for belief change.

In both these paradigms, the epistemic state of an agent is represented as a formula, or a set of formulae, in a propositional language, and the background logic is assumed to be, among others, Tarskian and compact. Compactness states that any formula entailed by a set of formulae X should be entailed by a finite subset X' of X . The purpose of this work is to examine belief update when the background logic is not assumed to be compact.

Efforts to apply AGM paradigm to non-classical logics were studied in [Restall and Slaney, 1995] and [Wassermann, 2011]. In the context of the AGM contraction (a form of AGM belief change in which an agent relinquish a piece of information) it was shown in [Ribeiro *et al.*, 2013] that it is not possible to define contraction functions satisfying the AGM contraction postulates in intuitionistic logics. The reason for this is that these logics are not closed under classical negation.

Recently, complete accounts of AGM contraction for non-compact logics has been provided in [Ribeiro *et al.*, 2018], subsequently extended in [Ribeiro *et al.*, 2019]. In this work we study KM update while dispensing with the compactness requirement. A number of interesting logics, both from both perspective of AI and computing science are not compact. Temporal logics, such as CTL and LTL [Gabbay *et al.*, 1994], which are used in planning and formal methods. Work such as [Zhang and Ding, 2008] propose to use KM update to repair CTL models that do not comply with the temporal requirements in the context of formal specification and verification of systems. One of the limitation of this work is that it considers only knowledge expressed via a single CTL model.

In this work, we take a step further on belief update in the context of non-compact logics. We will assume very little about the background logic, and then provide a complete account of KM update for these logics. Our main contributions are:

- (1) a new class of KM update functions (fully) characterised by the KM postulates, in logics without compactness;
- (2) we keep the classical KM update functions, and identify the set of rationality postulate that characterise them.

In Section 2 we briefly outline the notation employed in this paper, and introduce some useful concepts that we will need later. In Section 3, we briefly review the KM rationality postulates and the classical KM update functions. We also show the consequences that dispensing compactness has over these functions. In Section 4, we devise a new class of functions characterised by the set of basic KM postulates. In particular, we extend these constructions to capture the supplementary postulates and provide a fully KM-rational account. Then in Section 5 we show that without compactness the classical KM update functions become weaker in the sense that they are characterised by a proper subset of the KM postulates. In the subsequent section we consider the issue of

total relation in the KM update relational function, and show how to dispense with them. Finally, in Section 7, we recap the main results obtained and discuss some future directions for our work.

2 Notation and Technical Background

We consider a logic as a pair $\langle L, Cn \rangle$, where L is a language and $Cn: 2^L \rightarrow 2^L$ is a logical consequence operator that maps a set of formulas to the set of all formulas that can be inferred from it. For readability, for any formula φ , the set $Cn(\{\varphi\})$ will often simply be written as $Cn(\varphi)$. We limit ourselves to logics that are Tarskian, that is, logics whose consequence operator satisfies the following three properties:

1. **(Monotonicity)**: if $A \subseteq B$ then $Cn(A) \subseteq Cn(B)$;
2. **(Idempotence)**: $Cn(A) = Cn(Cn(A))$;
3. **(Inclusion)**: $A \subseteq Cn(A)$;

The KM paradigm was initially proposed for Proposition Logic, whereby the consequence operator Cn satisfies the following properties:

- **(deduction)**: $\varphi \in Cn(A \cup \{\psi\})$ iff $\psi \rightarrow \varphi \in Cn(A)$;
- **(supraclassicality)**: if φ is a logical consequence of A in classical propositional logic, then $\varphi \in Cn(A)$;
- **(compactness)**: if $\varphi \in Cn(A)$ then there is a finite subset A' of A such that $\varphi \in Cn(A')$.

We will say that a logic $\langle L, Cn \rangle$ is closed under classical negation iff the language L is closed under the negation operator \neg such that, for each formula $\varphi \in L$, $Cn(\varphi) \cap Cn(\neg\varphi) = Cn(\emptyset)$, and $Cn(\{\varphi, \neg\varphi\}) = L$. In other words, the negation is interpreted classically. Analogously, a logic is closed under the disjunction if the associated language is closed under such a connective (classically interpreted, that is, if $\varphi \in Cn(X)$ then $\varphi \vee \psi \in Cn(X)$, for every $\psi \in L$ and $X \subseteq L$).

A pre-order is a binary relation \leq that satisfies reflexivity and transitivity:

(reflexivity) $X \leq X$;

(transitivity) if $X \leq Y$ and $Y \leq Z$ then $X \leq Z$.

The maximal elements of a set A with respect to a binary relation \leq is given by $\max_{\leq}(A) = \{x \in A \mid \nexists y \in A, x \leq y\}$. Analogously, the minimal elements are given by $\min_{\leq}(A) = \{x \in A \mid \nexists y \in A, y \leq x\}$. A belief set, or simply a theory, K is a set of formulae logically closed under a consequence relation Cn , that is, $K = Cn(K)$. Given a belief set K , the collection of all the interpretations that satisfy K is given by $[K]$. For convenience, we write $[\varphi]$ to denote the interpretations that satisfy the theory $Cn(\varphi)$. A theory K is complete if and only if, for every sentence φ , either $\varphi \in K$ or $\neg\varphi \in K$, and we use T_L to denote all such complete theories that are consistent. We say that $\omega(K)$ is the decomposition of K in terms of its consistent complete theories. Conveniently, we simply write $\omega(\varphi)$ to refer to $\omega(Cn(\varphi))$. A function that maps each pair of theory and formula (K, φ) to another theory K' is a belief change function.

3 Belief Update

In the classic KM approach to belief update, epistemic states are represented as a formula from a propositional language. When dealing with non-compact logics, however, representing epistemic states in this way may be inconvenient, mainly because without compactness not every belief set can be represented via a single formula or a finite set of formulae. For this reason, we will translate the original KM update functions and rationality postulates in terms of epistemic states as belief sets. We start with the Classical Update (CUP) functions¹:

$$\text{CUP: } K \diamond \varphi = Th\left(\bigcup_{M \in [K]} \text{Max}_{\leq_M}([\varphi])\right).$$

Given a theory K and a formula φ , a CUP operator \diamond behaves as follows. For each model M of K , the operator \diamond chooses from the models of φ those closest to M . The notion of distance between models is usually given by a pre-order \leq_M . A model M_1 is closer to M than a model M_2 , if $M_2 \leq_M M_1$. These selected models are then assembled, and the theory of these models corresponds to $K \diamond \varphi$.

In [Katsuno and Mendelzon, 2003] it is shown that CUP functions are characterised precisely by a set of eight postulates U1 to U8. We translate this postulates in terms of epistemic states as belief sets which correspond respectively to the postulates (K \diamond 2) to (K \diamond 9) below.

- (K \diamond 1) $K \diamond \varphi$ is a theory.
- (K \diamond 2) $\varphi \in K \diamond \varphi$.
- (K \diamond 3) If $\varphi \in K$, then $K \diamond \varphi = K$.
- (K \diamond 4) $K \diamond \varphi$ is consistent, if both K and φ are consistent.
- (K \diamond 5) If $Cn(\varphi) = Cn(\psi)$, then $K \diamond \varphi = K \diamond \psi$.
- (K \diamond 6) $K \diamond (\varphi \wedge \psi) \subseteq K \diamond \varphi + \psi$.
- (K \diamond 7) If $\psi \in K \diamond \varphi$ and $\varphi \in K \diamond \psi$, then $K \diamond \varphi = K \diamond \psi$.
- (K \diamond 8) If K is complete, then $K \diamond \varphi \vee \psi \subseteq (K \diamond \varphi) \cup (K \diamond \psi)$.
- (K \diamond 9) $(K \cap K') \diamond \varphi = (K \diamond \varphi) \cap (K' \diamond \varphi)$.

As we require an epistemic state to be a belief set, the postulate (K \diamond 1) is added here to ensure that every update function goes from a belief set to another one. Discussion of the rationality behind these postulates can be found in [Katsuno and Mendelzon, 2003].

Theorem 1. [Katsuno and Mendelzon, 2003] *A belief change function satisfies the KM update postulates iff it is a CUP function.*

We stress that this representation between KM postulates and CUP functions was shown for propositional logic. This connection, however, fails for non-compact logics. For instance, without compactness CUP functions do not satisfy postulate (K \diamond 7), as illustrated in Example 1.

¹Originally, a CUP function is defined over minimal elements, here for convenience we will resort to maximal elements, as we will need to deal with infinite chains, and maximal elements makes the reasoning task easier.

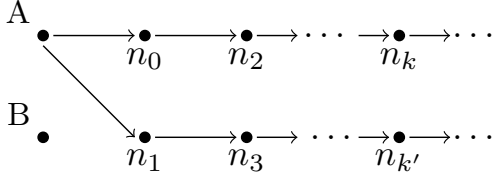


Figure 1: A preorder for the function of Example 1.

Example 1. Let \diamond be a CUP function over the pre-order \leq depicted in Figure 1. An edge $x \rightarrow y$ means that $x \leq y$. Reflexive and transitive edges are omitted in the figure for convenience. Moreover, let K be a complete consistent theory, and φ and ψ be two formulae such that $[\varphi] = \{A, B, n_0, n_2, \dots\}$, $[\psi] = \{A, B, n_1, n_3, \dots\}$, and $[\varphi \wedge \psi] = \{A, B\}$. According to the relation \leq , $\text{Max}_{\leq}(\varphi) = \{B\}$, $\text{Max}_{\leq}(\psi) = \{B\}$ and $\text{Max}_{\leq}(\varphi \wedge \psi) = \{A, B\}$. Therefore,

$$\begin{aligned} K \diamond \varphi &= \text{Th}(\text{Max}_{\leq}([\varphi])) &= \text{Th}(\{B\}) \\ K \diamond \psi &= \text{Th}(\text{Max}_{\leq}([\psi])) &= \text{Th}(\{B\}) \\ K \diamond \varphi \wedge \psi &= \text{Th}(\text{Max}_{\leq}([\varphi \wedge \psi])) &= \text{Th}(\{A, B\}). \end{aligned}$$

Though \leq is a pre-order, the update function \diamond does not satisfy (K \diamond 7). The reason for this is that according to (K \diamond 7) it is required that $K \diamond \varphi = K \diamond \varphi \wedge \psi$, however this does not occur in this example. To see this, note that $\varphi \wedge \psi \in K \diamond \varphi$, as B is model of both φ and ψ . Clearly, $\varphi \in K \diamond \varphi \wedge \psi$. However, $K \diamond \varphi \neq K \diamond \varphi \wedge \psi$ which violates (K \diamond 7).

The reason why the function \diamond of Example 1 does not satisfy (K \diamond 7) is related to the characteristic of non-compact logics having infinity models. In this case, a pre-order may induce an infinite chain in which it is not possible to determine a minimal (resp. maximal) element. To see this, let us consider what happens to the \diamond function of Example 1 if there were no infinity chain. For this, we can easily slightly change that example to consider a finite subset of elements: n_0 to n_3 besides A and B . This gives us that $[\varphi] = \{A, B, n_0, n_2\}$ and $[\psi] = \{A, B, n_1, n_3\}$. Thus, $\text{Max}_{\leq}(\varphi) = \{B, n_2\}$ and $\text{Max}_{\leq}(\psi) = \{B, n_3\}$. As n_2 is a model of φ but not a model of ψ , we do not have anymore that $\varphi \wedge \psi \in K \diamond \varphi$. Therefore, in this case, (K \diamond 7) is trivially satisfied.

Example 1 shows us that the connection between the postulates for KM update and CUP functions breaks down in the absence of compactness. It is natural, then, to pose the following question: how can one perform belief update in non-compact logics? Two natural answers are:

1. one can devise a new class of functions characterised by the KM update postulates in non-compact logics;
2. one can identify which set of postulates follow from CUP functions in non-compact logics.

We pursue both these paths. We will propose a new class of functions that satisfy all the KM update postulates, and the results we obtain will prove useful to show that CUP functions, in non-compact logics, are characterised by a proper subset of the KM update postulates.

4 Reconstructing Rational Update Functions

In this section, we provide complete new accounts of rational KM update functions. For convenience, we split the KM postulates in two categories: basic postulates which comprises postulates (K \diamond 1) to (K \diamond 5) and (K \diamond 9); and supplementary postulates which consists of postulates (K \diamond 6) to (K \diamond 8). We will call a belief change function that satisfies the basic KM postulates a belief update function.

4.1 KM Basic Rationality

The functions we will devise here will operate over complete consistent theories rather than models, following the approach in [Ribeiro *et al.*, 2018]. This will help up avoid technicalities issues. For instance, in modal logics, unlike in Propositional logic, two different models (but not consistent complete theories) may satisfy exactly the same set of formulae.

Without compactness pre-orders are not strong enough to properly indicate which are the best elements to be used in order to update a theory K by a formula φ , as illustrated in Example 1. This means we will need another way to define the ‘closeness’ criterion between consistent complete theories. We will assume that, instead of a pre-order, an agent has an *appointee* function that judges which complete theories of a formula φ are closest to a complete theory K . This appointee function is used to perform the update of K by φ . An appointee considers only consistent complete theories. We recall from Section 2 that the set of all complete theories comprises T_L .

Definition 1. An *appointee* is a function $\delta : T_L \times L \rightarrow 2^{T_L}$ that maps each consistent complete theory and formula φ to a set of consistent complete theories subject to the following conditions:

- (D1) $\delta(S, \varphi) \neq \emptyset$, if S and φ are consistent;
- (D2) $\delta(S, \varphi) \subseteq \omega(\varphi)$;
- (D3) $\delta(S, \varphi) = \{S\}$, if $\varphi \in S$;
- (D4) $\delta(S, \varphi) = \delta(S, \psi)$, if $\text{Cn}(\varphi) = \text{Cn}(\psi)$.

The purpose of an appointee is to select the best complete consistent theories that satisfy a formula φ (condition D2). This criterion is given locally, and it depends on the fixed complete theory S as background. An appointee is compelled to choose at least one consistent complete theory that contains φ , as long as φ is consistent (condition D1). If a formula φ belongs to a complete theory S , then there is no need to go after another theory, that is, the best theory for φ is S itself (condition D3). Finally, condition (D4) simply determines that an appointee is syntax independent.

The main purpose of defining an appointee function is that it captures the concept of distance between complete consistent theories in a general way. With an appointee in hand, it will become easier to explore which conditions we need to capture all the KM update postulates. We start by defining a new class of update functions, the *splinter functions* which are similar in spirit to CUP functions:

Definition 2. Given an appointee δ , a *splinter* is a function \diamond_δ such that

1. if K and φ are consistent, then

$$K \diamond_{\delta} \varphi = \bigcap_{S \in \omega(K)} (\bigcap \delta(S, \varphi));$$

2. otherwise, $K \diamond_{\delta} \varphi = Cn(\perp)$.

We recall from Section 2 that $\omega(K)$ corresponds to the set of all complete consistent theories that contains a theory K .

To understand how a splinter works, we will first need a property that follows from the converse of postulate (K \diamond 4). To see this, first notice that φ cannot be inconsistent since is required to be in $K \diamond \varphi$. On the other hand, if K were inconsistent, then we would have $\varphi \in K$, rendering $K \diamond \varphi$ inconsistent. So the only options remaining is that both K and φ are consistent. Conversely, if either K or φ are inconsistent, we get that $K \diamond \varphi$ is inconsistent. This case is captured by the second condition of splinter function definition. For the other cases, a theory K is split into its decomposition of complete consistent theory. An appointee then is used to select for each of these consistent theories the best complete theories from φ . Thereafter they are all intersected and it results in $K \diamond \varphi$.

Observation 1. Let \diamond be a belief change function that satisfies (K \diamond 1), (K \diamond 3) and (K \diamond 4). Then, $K \diamond \varphi = Cn(\perp)$ iff either φ or K is inconsistent.

Example 2. Let $K = Cn(p)$ be a theory such that its decomposition has only two complete consistent theories: K_1 and K_2 . Let an appointee δ be as follows: $\delta(K_1, \neg p) = \{Cn(\neg p, q)\}$ and $\delta(K_2, \neg p) = \{Cn(\neg p, \neg q)\}$. Thus,

$$\begin{aligned} K \diamond_{\delta} \neg p &= \delta(K_1, \neg p) \cap \delta(K_2, \neg p) \\ &= Cn(\neg p, q) \cap Cn(\neg p, \neg q) \\ &= Cn(\neg p). \end{aligned}$$

A splinter function behaves similar to a CUP function, with the difference of resorting to an appointee instead of pre-orders to choose elements for the update. It is easy to check that every splinter function satisfies the basic KM postulates.

Theorem 2. Every splinter function \diamond_{δ} satisfies (K \diamond 1) to (K \diamond 5) and (K \diamond 9).

Not surprisingly, splinter functions not only satisfy the basic KM update postulates, but every update function is a splinter function.

Another interesting property of update functions is that they are monotonic, that is, if $K \subseteq K'$ then $K \diamond \varphi \subseteq K' \diamond \varphi$. Update functions are monotonic due to postulate (K \diamond 9).

Proposition 1. Every belief change operation \diamond that satisfies (K \diamond 9) is monotonic.

The last piece of the puzzle towards our representation result requires us to look at splinter functions from another perspective. We begin with a special case: given complete theory K and a consistent formula φ , how does $K \diamond_{\delta} \varphi$ look like? As K is a complete consistent theory, the only complete theory that contains K is K itself. Therefore, $\omega(K) = \{K\}$, which gives us a simpler version of splinter:

$$K \diamond \varphi = \bigcap \delta(K, \varphi).$$

Now let us proceed for a more general case. Imagine that K is a consistent theory, but not complete. Let S be one of the complete consistent theories from the decomposition of K , then have that $S \diamond \varphi = \bigcap \delta(S, \varphi)$. Thus, we can easily see that the update of K by φ consists in taking each $S \diamond \varphi$, for $S \in \omega(K)$, and intersecting all of them. This leads us to the following observation.

Observation 2. Given a splinter \diamond_{δ} , a theory K and a formula φ , if both K and φ are consistent then

$$K \diamond_{\delta} \varphi = \bigcap_{S \in \omega(K)} S \diamond_{\delta} \varphi.$$

Observation 2 will be useful to prove the Theorem 3 below. For this, we will also need the following two lemmas from [Ribeiro et al., 2018].

Lemma 1. $[Cn(K \cup K')] = [Cn(K)] \cap [Cn(K')]$, for all theories K and K' .

Lemma 2. $Cn(K \cap K') = Th([K] \cup [K'])$, for all theories K and K' .

Theorem 3. Every update function is a splinter.

Our first representation result follows from Theorems 2 and 3 which jointly show that splinter functions and the basic KM postulates are equivalent.

4.2 Reviewing the Supplementary Postulates

Although splinter functions satisfy the basic KM postulates, they are not strong enough to capture the supplementary postulates. In this section we introduce a new class of update functions that satisfies the supplementary postulates. The main idea is to make the appointees to resort to binary relations in order to pick the consistent complete theories.

An appointee δ will assign, for each consistent complete theory K , a binary relation \leq_K . The appointee chooses the maximal elements within $\omega(\varphi)$ modulo \leq_K . These relations will need to satisfy some requirements in order to yield splinter functions that satisfies the supplementary postulates. The first one is

Maximal Cut: for every consistent formula φ , $\omega(\varphi)$ has a maximal element w.r.t \leq .

*Maximal Cut*² was introduced in [Ribeiro et al., 2018] and its purpose is to guarantee that for every formula φ , as long as it is consistent, the epistemic agent chooses at least one consistent complete theory of φ . This ensures that φ will be successfully incorporated through the update process. A binary relation that satisfies *maximal-cut* is called a *contra-headed* relation³.

Appointee functions that rely strictly on *contra-headed* relations will be called relational appointees.

Definition 3. An appointee μ is relational iff for every consistent complete theory K , there is a contra-headed relation \leq_K such that $\mu(K, \varphi) = \max_{\leq_K}(\omega(\varphi))$, for every consistent formula φ .

²In [Ribeiro et al., 2018], maximal cut is defined over the complements of a formula φ , since the focus is contraction. Here, we conveniently translate maximal cut in terms of the complete consistent theories of φ .

³This terminology is borrowed from [Ribeiro et al., 2018].

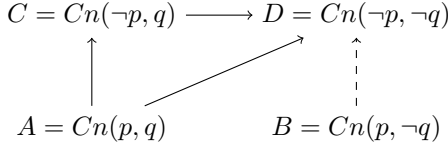


Figure 2: A contra-headed relation.

The new update functions we introduce are the *royal splinter functions* which consist of splinters functions founded on relational appointees.

Definition 4. A splinter \diamond_μ is royal iff its appointee μ is relational.

Example 3. Consider the complete theories depicted in Figure 2 and the respective contra-headed relation \leq represented by solid arrows. Let μ be a relational appointee such that for the theory A it considers the contra-headed \leq . Thus, $\mu(A, \neg q) = \{B, D\}$ which implies that

$$A \diamond_\mu \neg q = B \cap D = Cn(\neg q).$$

At this point we investigate what we gain by restricting ourselves to royal splinters. The first benefit is that the *contra-headed* relations guarantee satisfaction of postulates (K \diamond 6) and (K \diamond 8) which is shown on Theorem 4 below. Lemma 3 will be of immediate assistance thought this section.

Lemma 3. Let \diamond_δ be a fully KM rational splinter, and K a complete consistent theory,

- (i) $\delta(K, \varphi) \cap \omega(\psi) \subseteq \delta(K, \varphi \wedge \psi)$;
- (ii) $\delta(K, \varphi) \cap \delta(K, \psi) \subseteq \delta(K, \varphi \vee \psi)$.

The conditions (i) and (ii) from Lemma 3 concern properties that follow from fully rational splinters, precisely the appointees of these functions, and are related respectively to postulates (K \diamond 6) and (K \diamond 8).

Theorem 4. Every royal splinter satisfies (K \diamond 6) and (K \diamond 8).

In order to capture postulate (K \diamond 7), we will need to impose one more restriction over the relations considered by a relational appointee:

Quasi-reflection: if $A \not\leq B$ and $B \not\leq A$ but $A \leq C$ then

- (i) either $B \leq C$, or
- (ii) if $C \leq C'$, then $B \leq C'$.

To explain quasi-reflection, we will consider conditions (i) and (ii) separately. The main intuition behind (i) is that if an agent has no preference between two elements A and B , then all elements preferable to A should be preferred to B and vice versa. The second condition elevates this mimicking process by one step. For instance, consider the relation \leq depicted in Figure 2 by both solid and dashed arrows. In that relation, there is no preference between A and B . Note that $A \leq C$ and $C \leq D$. However, C is not preferable to B , as condition (i) demands. In this case, instead of forcing $B \leq C$, condition (ii) pushes B to mimic one of the elements immediately above A , in this case mimic the preferences over C . Thus, it makes $B \leq D$ (depicted as a dashed arrow).

It turns out that quasi-reflection is all we need to capture (K \diamond 7). A relational appointee that considers only *contra-headed* relations that satisfy quasi-reflection will be called a *quasi-reflected appointee*.

To see how *quasi-reflection* captures (K \diamond 7), we begin by looking at a special case of (K \diamond 7) over complete consistent theories.

Lemma 4. Let \diamond_μ be a royal splinter, such that μ is quasi-reflected. Given a complete consistent theory K , if $\varphi \in K \diamond_\mu \psi$ and $\psi \in K \diamond_\mu \varphi$, then $K \diamond_\mu \varphi = K \diamond_\mu \psi$.

The tricky part is to show that quasi-reflection not only captures (K \diamond 7) in complete theories, but also in every theory. We make use of the monotonicity property of update functions to show, via lemma above, that (K \diamond 7) spreads from complete consistent theories to all theories.

Theorem 5. Given a royal splinter \diamond_μ , if μ is quasi-reflected then \diamond_μ satisfies (K \diamond 7).

We get the following corollary from Theorems 4 and 5.

Corollary 1. A royal splinter whose appointee is quasi-reflected is fully KM rational.

Corollary 1 provides the first piece of our representation result. The next step is to prove that every fully KM update function is a royal splinter function whose appointee is *quasi-reflected*, in short, a royal reflected splinter.

We begin by showing that fully KM update functions are royal. From Theorem 2 we have that every KM update function is a splinter, which means that all of them have an appointee. All we need to show now is that appointees from fully KM splinters are relational.

This is not a trivial task. We will start with a special case and then extend it to a more general case. This special case regards consistent complete theories. Let K be a complete consistent theory, and \diamond_δ a splinter fully KM rational. We want to identify a contra-headed relation \leq_K , such that $\delta(K, \varphi) = \text{Max}_{\leq_K}(\omega(\varphi))$, for every consistent formula φ . For now, we will say that \leq_K is the silhouette of δ within K .

To achieve this, let us assign to each formula φ a relation \leq_φ such that the maximal elements of $\omega(\varphi)$ modulo \leq_φ match $\delta(K, \varphi)$. This can be easily achieved by making $\leq_\varphi = \{(A, B) \in \omega(\varphi) \times \omega(\varphi) \mid A \notin \delta(K, \varphi) \text{ and } B \in \delta(K, \varphi)\}$. Then, we can assemble all \leq_φ together to reach the general relation \leq_K . Although this seems a good idea, the relation \leq_K constructed in this way is not strong enough to satisfy *quasi-reflection*.

In order to capture *quasi-reflection*, we adopt a variation of the triangulation method employed in [Ribeiro et al., 2018].

Definition 5. Given a complete consistent theory K , an appointee δ and two consistent formulae φ and ψ , the δ -triangulation of φ and ψ modulo K is the relation $\nabla(\varphi, \psi) = \{(A, B) \in \omega(\varphi) \times \omega(\psi) \mid A \in \omega(\varphi) \setminus (\delta(K, \varphi) \cup \omega(\psi)) \text{ and } B \in \delta(K, \psi)\}$.

Triangulation was introduced in [Ribeiro et al., 2018] in order to construct relations that satisfy quasi-reflected relations constrained by condition (i) - dubbed *mirroring*. The δ -triangulation we employ here is a weaker variation, though the intuition is quite similar. Given a theory K , and a appointee δ , a triangulation connects the complete consistent

theories of φ that were not chosen by δ to the theories of ψ that were chosen by $\delta(K, \psi)$. This makes all non-chosen elements of φ immediately less preferable to all elements chosen by ψ . The main difference between the triangulation version we use here from the original one is that we are indifferent about the complete theories that entails both φ and ψ . For instance, if a complete theory A entails both φ and ψ then A will not participate in the triangulation $\nabla(\varphi, \psi)$.

Now we go back to our problem of finding the silhouette of a complete consistent theory K within an appointee δ . This can be achieved by combining triangulation and our first naive idea of assembling relations \leq_φ .

Definition 6. *Given a complete theory K and an appointee δ , the projection of δ modulo K is the relation \leq_K such that $(A, B) \in \leq_K$ iff for some formulae φ and ψ*

$$\text{either } (A, B) \in \leq_\varphi \quad \text{or} \quad (A, B) \in \nabla(\varphi, \psi).$$

A projection concerns only the relation for a single complete consistent theory. To construct a relational appointee, we take the collection of all the projections of the consistent complete theories modulo δ . We say that this is the shadow of δ .

It turns out that the projection of δ modulo K is also the silhouette of δ within K , when the splinter function \diamond_δ is fully KM rational.

Proposition 2. *Consider a splinter \diamond_δ , and a consistent complete theory K . If \diamond_δ is fully KM rational, then the projection of δ modulo K is the silhouette of δ within K .*

An immediate consequence of this is that appointees of fully KM rational splinters are relational. To see this, notice that from the proposition above $\delta(K, \varphi) = \max_{\leq}(\omega(\varphi))$, for each consistent complete theory K , when \diamond_δ is fully KM rational.

Theorem 6. *The appointee of a fully KM rational function \diamond_δ is relational.*

Hence we get that fully KM rational splinters are indeed royal.

Corollary 2. *Fully KM rational splinters are royal splinters.*

To show that fully KM rational functions are royal reflected splinters, we need only to show that fully KM rational splinters are reflected. For this, we need to show that the appointee of fully rational splinters are *quasi-reflected*. The triangulation used to construct projection gives us this property in an easy way.

Theorem 7. *Let μ be the appointee from a fully KM rational splinter. Given a complete consistent theory K , the projection \leq_K of δ modulo K is quasi-reflected.*

Therefore, we have that every fully KM rational belief change function is a royal reflected splinter. This together with Corollary 1 constitute the representation result between KM postulates and royal reflected splinters.

Corollary 3. *An update function is fully KM rational iff it is a royal reflected splinter.*

5 Looking back at CUP functions

Relying solely on preorders do not guarantee the postulate (K \diamond 2) of success, since it does not have the *maximal-cut* condition to guarantee the existence of maximal elements to be picked. We will assume a stronger version of the CUP functions: that their preorder are *contra-headed*. Let us call it CUP⁺ functions. It is easy to see that CUP⁺ are a special case of royal splinters. Thus,

Theorem 8. *Every CUP⁺ function satisfies (K \diamond 1) to (K \diamond 6) and (K \diamond 8) and (K \diamond 9).*

It turns out that transitivity and reflexivity add no substantial change in the royal functions.

Proposition 3. *Let \leq be a contra-headed relation and \leq_+ its reflexive and transitive version. Then, $\text{Max}_{\leq}(A) = \text{Max}_{\leq_+}(A)$.*

Proposition 4. *Every royal splinter function that satisfies (K \diamond 6) and (K \diamond 8) is a CUP⁺ function.*

6 Are total relations really needed?

Besides the three KM supplementary postulates, the following postulate was proposed to be used in place of postulates (K \diamond 7) and (K \diamond 8):

(K \diamond 10) if K is complete and $\neg\psi \notin K \diamond \varphi$ then $K \diamond \varphi + \psi \subseteq K \diamond \varphi \wedge \psi$.

Postulate (K \diamond 10) is intimately connected with total preorders as shown in [Katsuno and Mendelzon, 2003], which is taken to be a limitation [Lindström and Rabinowicz, 1991; Rott, 1992]. We show a new class of royal splinter functions characterised by the postulates (K \diamond 1) to (K \diamond 6) and (K \diamond 9) and (K \diamond 10) that do not depend on total preorders.

As the basic KM postulates and (K \diamond 6) are satisfied by royal splinter functions, we will still maintain the restriction of appointees to be relational. To capture, (K \diamond 10), we will replace the quasi-reflection condition by the mirroring condition:

(Mirroring) if $S_1 \not\leq S_2$ and $S_2 \not\leq S_1$; then for any $S' \in T_L$, if $S_1 \leq S'$ then $S_2 \leq S'$.

proposed in [Ribeiro *et al.*, 2018]. This satisfies postulate (K \diamond 10). It is worth highlighting that the *mirroring* corresponds to *quasi-reflection* constrained to its condition (i). Thus, every mirrored relation is a quasi-reflected relation.

Theorem 9. *Every mirrored royal splinter satisfies (K \diamond 10).*

Theorem 10. *Every update function that satisfies (K \diamond 6) and (K \diamond 10) is a mirrored royal splinter.*

7 Conclusion and Future Works

In this work, we have addressed the issue of KM update in the context of non-compact logics. We have found both negative and positive results. On the negative side, the CUP functions, (fully) characterised by the KM update postulates in Propositional Logic, cannot satisfy all the postulates without compactness. We showed that without compactness CUP functions, and a slightly stronger version of them (the CUP⁺) are characterised by a proper subset of the KM rationality postulates. Assuming the KM update postulate as a good guide for

belief update, we have devised a new class of update functions, the royal splinter functions, which are (fully) characterised by the KM postulates.

We intend to explore, in the future, another forms of belief change in the context of non-compact logics: the erasure functions. Our intention is to provide complete accounts of KM belief change for non-compact logics.

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