Belief Change and Non-Monotonic Reasoning sans Compactness

Jandson S. Ribeiro
Macquarie University, Australia
University of São Paulo, Brazil
jandson.santos-ribeiro-sant@hdr.mq.edu.au
jandson@ime.usp.br

Abhaya Nayak
Macquarie University, Australia
abhaya.nayak@mq.edu.au

Renata Wassermann
University of São Paulo, Brazil
renata@ime.usp.br

Abstract

Belief change and non-monotonic reasoning are arguably different perspectives on the same phenomenon, namely, jettisoning of currently held beliefs in response to some incompatible evidence. Investigations in this area typically assume, among other things, that the underlying (background) logic is compact, that is, whatever can be inferred from a set of sentences X can be inferred from a finite subset of X. Recent research in the field shows that this compactness assumption can be dispensed without inflicting much damage on the AGM paradigm of belief change. In this paper we investigate the impact of such relaxation on non-monotonic logics instead. In particular, we show that, when compactness is not guaranteed, while the bridge from the AGM paradigm of belief change to expectation logics remains unaffected, the "return trip" from expectation logics to AGM paradigm is no longer guaranteed. We finally explore the conditions under which such guarantee can be given.

1 Introduction

Classical logics are monotonic: accumulation of new information (in form of additional premises) does not invalidate old conclusions. For instance, since from *All birds fly* and *Tweety is a bird* we can infer *Tweety flies*, acquiring another piece of information, *Tweety is a penguin* is not going to invalidate that inference – we will, classically, still be able to infer that *Tweety flies*. However, commonsense dictates that in light the new piece of information, that *Tweety is a penguin*, we should no longer be able to infer that *Tweety flies* since we "know" that penguins, though birds, cannot fly. *Commonsense reasoning is* **non-monotonic**: although some set of premises Δ entails some conclusion x, the inference of x from Δ' may not be guaranteed even if $\Delta \subseteq \Delta'$. This realisation has led to the development of a number of approaches to non-monotonic logics, including, *default logics* (Besnard 2013), *defeasible logics* (Nute 1994) and *expectation logics* (Gärdenfors and Makinson 1994).

One may attribute the non-monotonic behaviour of the commonsense inference to our tendency to "deactivate" some premises in light of some conflicting information – it is as if the new information, *Tweety is a penguin*, triggers deactivation of the premise, *All birds fly*. When viewed from this angle, non-monotonic reasoning appears to be closely connected to accounts of rational belief change, such as the classic AGM account, (Alchourrón, Gärdenfors, and Makinson 1985), further developed in many works such as (Gärdenfors 1988; Hansson 1999; Rott 2001). Indeed it has been argued that belief dynamics and non-monotonic reasoning are different perspectives on the same phenomenon (Makinson and Gärdenfors 1991): *I am licensed to (commonsensically/non-monotonically) infer sentence y from sentence x if it is rational on my part to believe y after accepting evidence x*. This idea is concisely captured in the standard notation:

BRNM: $x \mid\!\sim_K y$ if and only if $y \in K * x$

where K represents a contextually fixed background knowledge, $\mid\!\sim_K$ is the non-monotonic inference operation that employs K in the background, and $*$ is a belief *revision* operation that yields the new "belief set" $K * x$ from the old belief set K in light of evidential input x. Oftentimes the subscript $_K$ from the relation $\mid\!\sim_K$ is dropped for notational convenience when the intention is clear from the context.

The connection between belief revision and non-monotonic reasoning captured by BRNM comes out handy in going back and forth between these two systems. Consider for instance a constraint on belief revision, that revision of a set of beliefs K by evidence x should not license one to believe in something that does not follow from K and x together, that is,

Inclusion: $K * x \subseteq Cn(K \cup \{x\})$

where the consequence operation Cn represents the background (classical) logic. Assuming Cn satisfies *Deduction*, this constraint on belief revision can be rewritten as:

If $y \in K * x,$ then $x \rightarrow y \in K.$

Now, since a tautology $\top$ brings in no new information, we can take $K * \top$ to be equivalent to K, which allows us to rewrite *Inclusion* as:

If $y \in K * x,$ then $x \rightarrow y \in K * \top.$

Since this version of *Inclusion* is of the same syntactic form as BRNM, we get

Weak Conditionalisation: If $x \mid\!\sim y,$ then $\top \mid\!\sim x \rightarrow y$

which says that if y is a non-monotonic consequence of x, then the material conditional $x \rightarrow y$ should be a theorem of that non-monotonic system. Principles of non-monotonic logic can similarly be translated to and viewed as principles of belief revision. The translation process is not always as straightforward as in this example; a rigorous recipe for this translation procedure as well as a comprehensive list of such translations can be found in in the seminal work (Makinson and Gärdenfors 1991). This work was later generalised to non-monotonic reasoning based on *expectation orderings*, ordering over sentences reflecting the extent they are expected to hold true (Gärdenfors and Makinson 1994). As noted in (Dix and Makinson 1992), non-monotonic reasoning based on expectation orderings is very similar to the classic approach to non-monotonic reasoning based on preference structures, often called the KLM-system, as propounded in (Kraus, Lehmann, and Magidor 1990).

The belief *revision* operation $*$ alluded to above is one of three forms of belief change operations dealing with belief dynamics arguably in a static environment (Katsuno and Mendelzon 1992):[1]

(a) one for adding the new information possibly inviting inconsistency (*expansion*);

(b) one for removing an existing belief (*contraction*); and

(c) one for incorporating the new information without courting inconsistency (*revision*).

The desired behaviour of these operations are captured by corresponding *rationality postulates* that are motivated by the principle that the change involved should be minimised, and different constructions of such operations are well known. Furthermore, belief revision and belief contraction are inter-definable via well known identities. It is typical to assume in these approaches that the background logic that drives this process is, among other things, *compact*, that is, if this background logic allows inference of a sentence x from a set of premises Δ, then it allows the inference of x from a finite subset Δ' of Δ. In a recent work we have shown that this compactness assumption can be relaxed and yet rational belief contraction operations can be constructed given innocuous conditions on the background language and logic (Ribeiro, Nayak, and Wassermann 2018):

Theorem 1. *(Ribeiro, Nayak, and Wassermann 2018) AGM (full) rational contraction functions can be defined in every non-compact logic as long as it is Tarskian and closed under classical negation and disjunction.*

Given the inter-definability between contraction and revision via the Levi Identity and the Harper Identity, it is easily shown that the compactness assumption can be similarly relaxed in case of belief revision. Indeed, one can check step-by-step the relevant proofs in (Gärdenfors 1988) and verify that compactness plays no special role in them, and hence can be dispensed with without any damage.

That many logics such as Temporal Logics (Gabbay, Rodrigues, and Russo 2008) that play important roles in both Artificial Intelligence and Computing Science do not satisfy compactness lends practical significance to such relaxation. For instance, temporal logics such as Computation Tree Logic and Linear Temporal Logic, though not compact, are widely used in both Computing Science (e.g. Formal specification and verification of systems) and Artificial Intelligence (in planning, for instance). Arguably, such extension of belief revision to the realm of non-compact logics will be of particular relevance to research on AI agents dealing with (semi)-automatic repair of formal system specifications when the desired formal requirements (specified in temporal logics) are not complied with. Belief revision can be used in this scenario to recommend rational modification of a system specification so that the required properties are satisfied. Some initial efforts towards this end can be found in (Guerra and Wassermann 2010; Ribeiro and Andrade 2015) and in (Guerra, Andrade, and Wassermann 2013).

Since non-monotonic reasoning and belief revision are strongly connected, the nature of non-monotonic reasoning is worth enquiring when the background logic of the corresponding belief revision is not assumed to be compact. That is precisely the problem we address in this paper. In particular, we study the connection between the AGM paradigm of belief revision and the non-monotonic logic based on expectation orderings, and show that when compactness is dropped, it is still possible to walk from belief revision to non-monotonic reasoning. However, the walking back from non-monotonic reasoning to belief revision cannot be assured without further appropriate measures, and we identify such measures.

In the rest of this section we will briefly outline the notation we employ in this paper and the formal background. We review in the next section, Section 2, the AGM account of belief revision (Alchourrón, Gärdenfors, and Makinson 1985) as well as the non-monotonic inference system based on Expectation orderings (Gärdenfors and Makinson 1994), and indicate the significance of the compactness assumption. This is followed by a demonstration in Section 3 that relaxing the compactness assumption does not affect the transition from belief revision to non-monotonic reasoning. We show in the next section, Section 4, and explain why, the trip back from non-monotonic systems to belief revision cannot be completed in absence of the compactness assumption. The subsequent Section 5 is devoted to identify the required conditions under which the desired connection between the belief revision and non-monotonic reasoning can be established in the absence of compactness and provide the expected representation results. Finally, in section 6, we conclude with a brief discussion and considerations of future work.

We sketch proof of selected results in this paper; others will be provided in a planned future publication.

Notation and Technical Preliminaries

Given a set A, the power set of A will be denoted as 2^A. We use the terms *formula* and *sentence* interchangeably. We will use upper case Roman letters (A, B, X, Y ...) to denote sets, and lower case Greek letter $(\varphi, \psi, \alpha, \beta, \dots)$ will

[1] The corresponding operations of contraction and revision in a dynamic environment are respectively known as *erasure* and *update*.

be used to denote formulas. We will reserve the upper case letter K for a sepcial kind of sets called belief sets, and the Greek lower case letter γ to denote a kind of function called *selection function*. The letter Γ is reserved to denote a collection of sets. Propositional symbols will be denoted by lower case Roman letters (p, q, r, ...), while $\top$ and $\perp$ will be used to denote `truth` and `falsum`. The letter M will denote a model. Moreover we will use the symbol $\subseteq$ for subset, whereas $\subset$ will denote proper subset.

We consider a logic as a pair $\langle L, Cn \rangle$, where L is a language and Cn: $2^L \rightarrow 2^L$ is a logical consequence operator that maps a set of formulas to the set of all formulas that can be inferred from it. For readability, for any formula φ, the set $Cn(\{\varphi\})$ will often simply be written as $Cn(\varphi)$. We will often pretend that the consequence operation Cn itself represents a logic when no confusion is imminent. We limit ourselves to logics that are Tarskian, that is, logics whose consequence operator satisfies the following three properties:

1. **(Monotonicity)**: $A \subseteq B$ iff $Cn(A) \subseteq Cn(B)$;
2. **(Idempotence)**: $Cn(A) = Cn(Cn(A))$;
3. **(Inclusion)**: $A \subseteq Cn(A)$;

Apart from being Tarskian, the consequence operation is often taken to satisfy some other properties in the AGM belief change literature:

(deduction): $\varphi \in Cn(A \cup \{\psi\})$ iff $\psi \rightarrow \varphi \in Cn(A)$;

(supraclassicality): if φ is a logical consequence of A in classical propositional logic, then $\varphi \in Cn(A)$;

(compactness): if $\varphi \in Cn(A)$ then there is a finite subset A' of A such that $\varphi \in Cn(A')$.

We will say that a logic $\langle L, Cn \rangle$ is closed under classical negation iff the language L is closed under the negation operator $\neg$ such that, for each formula $\varphi \in L$, $Cn(\varphi) \cap Cn(\neg\varphi) = Cn(\emptyset)$, and $Cn(\{\varphi, \neg\varphi\}) = L$. In other words, the negation is interpreted classically. Analogously, a logic is closed under the disjunction if the associated language is closed under such a connective (classically interpreted, that is, if $\varphi \in Cn(X)$ then $\varphi \vee \psi \in Cn(X)$, for every $\psi \in L$ and $X \subseteq L$).

2 Belief Revision and Non-monotonic Reasoning

Earlier in the introductory section we outlined in an informal way accounts of belief revision and non-monotonic reasoning, and the interconnection between them. We now elaborate it a bit in a more formal setting, and discuss the role that compactness plays in them.

AGM Belief Revision

All the beliefs of an agent as a whole is represented as a set of sentences K, called a *belief set* (or *theory*), that is assumed to be closed under logical consequence: $K = Cn(K)$. For notational convenience we take $K + \varphi$ to mean $Cn(K \cup \{\varphi\})$, for any belief set K and sentence φ.[2] In the AGM paradigm of belief revision, as well as other forms of AGM belief change (Alchourrón, Gärdenfors, and Makinson 1985), the background logic Cn is assumed to satisfy a set of properties called *AGM assumptions*, namely, it is Tarskian, supra-classical, compact, closed under all boolean connectives (conjunction, disjunction and negation), and satisfies deduction. Let $\mathbf{K}$ be the collection of all belief sets. Then any function $f : \mathbf{K} \times L \rightarrow \mathbf{K}$ is a belief change operation. The full set of AGM rationality postulates is listed below.[3] For any theory K, sentences φ and ψ, and belief revision function $*$:

K1 $K * \varphi = Cn(K)$ *(Closure)*
K2 $\varphi \in K * \varphi$ *(Success)*
K3 $K * \varphi \subseteq K + \varphi$ *(Inclusion)*
K4 if $\neg\varphi \notin K$, then $K + \varphi \subseteq K * \varphi$ *(Preservation)*
K5 $K * \varphi = Cn(\perp)$ iff $\neg\varphi \in Cn(\emptyset)$ *(Consistency)*
K6 if $Cn(\varphi) = Cn(\psi)$, then $K * \varphi = K * \psi$ *(Extensionality)*
K7 $K * (\varphi \wedge \psi) \subseteq (K * \varphi) + \psi$ *(Conjunctive Inclusion)*
K8 if $\neg\psi \notin K * \varphi$, then $(K * \varphi) + \psi \subseteq K * (\varphi \wedge \psi)$ *(Conjunctive Preservation)*.

Postulates (K1) to (K6) are the basic AGM postulates for revision, and the last two constitute the supplementary postulates. Discussion of and rationale behind these postulates can be found in (Gärdenfors 1988), among others. We will call any belief change operation that satisfies postulates (K1) to (K6) an *AGM rational belief revision* operation. Any AGM rational belief revision operation that also satisfies the supplementary postulates (K7) and (K8) will be said to be *fully AGM rational*.

The postulates (K1) to (K8) prescribe a good set of behaviours for a revision function, but do not tell us where to find such a function. There are different available constructions of (fully) AGM-rational belief revision functions. These constructions employ certain extra-logical mechanisms in the form of preference relations over beliefs/sentences reflecting how *epistemically entrenched* each belief is (how hard it is to give up a belief) or over possible worlds captured by Grove's *System of Spheres*(Grove 1988) reflecting their *plausibility*. Expectation orderings studied in (Gärdenfors and Makinson 1994) are generalisations of the epistemic entrenchment relation, and preference structures propounded in the KLM-system (Kraus, Lehmann, and Magidor 1990) are generalisation of such plausibility orderings.

Non-monotonic reasoning

Unlike in the AGM approach to belief revision, in the non-monotonic reasoning system based on expectation order-

[2]This is actually called the *belief expansion* operator that is used to add beliefs without consideration of whether or not the result is consistent.

[3]In (Alchourrón, Gärdenfors, and Makinson 1985), the rationality postulates for belief revision effectively incorporated the Harper Identity which defines contraction in terms of revision as one of the postulates. Here we have followed the presentation in (Gärdenfors 1988), which has become the *de facto* standard in the literature.

ings there are seven basic axioms and two supplementary axioms. This separation into two groups, first proposed in (Gärdenfors and Makinson 1994), makes the alignment of these axioms with the postulates for AGM belief revision explicit.

We will assume that a non-monotonic inference relation, denoted $\mid\sim$, builds up upon to an underlying logic Cn. We will assume very little about Cn, except that it is required to be Tarskian. The basic postulates (or axioms) for non-monotonic inference relation $\mid\sim$ are:

N1 If $\psi \in Cn(\varphi)$, then $\varphi \mid\sim \psi$ *(Supraclassicality)*

N2 If $Cn(\varphi) = Cn(\psi)$ and $\varphi \mid\sim \alpha$ then $\psi \mid\sim \alpha$ *(Left Logical Equivalence)*

N3 If $\varphi \mid\sim \psi$ and $\alpha \in Cn(\psi)$ then $\varphi \mid\sim \alpha$ *(Right Weakening)*

N4 If $\varphi \mid\sim \psi$ and $\varphi \mid\sim \alpha$ then $\varphi \mid\sim \psi \wedge \alpha$ *(And)*

N5 If $\varphi \mid\sim \psi$ then $\mid\sim \varphi \to \psi$ *(Weak Conditionalization)*

N6 If $\not\mid\sim \neg\varphi$ and $\mid\sim \varphi \to \psi$, then $\varphi \mid\sim \psi$ *(Weak Rational Monotony)*

N7 If $\varphi \mid\sim \perp$ then $Cn(\varphi) = Cn(\perp)$ *(Consistency Preservation).*

For simplicity, we will call any inference relation that satisfies the above seven (basic) postulates of non-monotonicity a *non-monotonic inference relation*.

Besides these basic postulates, two supplementary postulates were also introduced for non-monotonic reasoning:

N8 If $\varphi \wedge \psi \mid\sim \phi$ then $\varphi \mid\sim \psi \to \phi$ *(Conditionalization)*

N9 If $\varphi \not\mid\sim \neg\psi$ and $\varphi \mid\sim \phi$ then $\varphi \wedge \psi \mid\sim \phi$ *(Rational Monotony).*

Let us, for the time being, assume that Cn satisfies Compactness. Then it is easily verified that postulates N3 and N4 jointly imply the following closure axiom:

Closure$_{\text{NM}}$ If $\varphi \mid\sim \psi$ for all $\psi \in A$, then $\varphi \mid\sim \alpha$ for all $\alpha \in Cn(A)$.

named after the closure postulate **K1** of belief revision that it corresponds to (Gärdenfors and Makinson 1994). On the other hand, postulates **N5** through **N7** respectively correspond to the AGM revision postulates **K3** through **K5**. Furthermore, postulate **N2** corresponds to **K6** (extensionality). Similarly, postulates **N8** and **N9** correspond respectively to the revision postulates **K7** and **K8**.

We will see in Section 4 that this nice correspondence between the revision operation $*$ and the inference relation $\mid\sim$ breaks down at many points when the compactness assumption is dropped. Here we give an indication of how compactness can have consequences for non-monotonic reasoning systems. Let us consider **Closure**$_{\text{NM}}$ which follows from axioms **N3** and **N4** when Cn satisfies compactness. However, in absence of compactness, **Closure**$_{\text{NM}}$ no longer follows from **N3** and **N4**.

Theorem 2. *If the underlying logic Cn is not compact, then **N3** and **N4** do not imply **Closure**$_{\text{NM}}$.*

Proof Sketch. Assume that Cn is not compact. Also suppose that $\mid\sim$ is a nonmonotonic inference relation that satisfies **N3** and **N4**. Let φ be such that $\varphi \mid\sim \beta$ for all $\beta \in A$ for some (infinite) set of formulae A, $\alpha \in Cn(A)$, but $\alpha \notin Cn(A')$ for any finite subset A' of A. Satisfaction of **Closure**$_{\text{NM}}$ would require that $\varphi \mid\sim \alpha$. However, **N3** and **N4** would jointly allow non-monotonic derivation of α only if $\alpha \in Cn(A')$ for some finite subset A' of A contradicting our assumption. $\square$

3 From * to $\mid\sim$ without Compactness

We recall from the introductory section the inter-translatability between revision operation * and non-monotonic inference relation $\mid\sim$ captured by **BRNM**:

$\psi \in K * \varphi$ if and only if $\varphi \mid\sim \psi$.

This allows us to navigate between a belief revision operator $*$ and an inference relation $\mid\sim$. For convenience, when an inference relation $\mid\sim$ is taken to have been obtained from a revision operation *, we say that * induces $\mid\sim$. Conversely, when navigating on the opposite direction, we will say that $\mid\sim$ induces $*$. We will **not** assume that the background logic Cn is compact.

Let us at this point make the following observation, that if we start with a revision operator $*$, obtain the inference $\mid\sim$ induced by it, and then again obtain the revision operator induced by this inference relation in turn, we will get back the revision operator $*$ we started with. Similarly, if we started with an inference operation $\mid\sim$ and obtain another inference via a revision operation induced by it, we will go back to our original inference operation $\mid\sim$. This simple observation will give us much flexibility with writing the proofs, and we will implicitly assume it throughout.

Observation 1. *Let $*$ be a belief change operator and $\mid\sim$ an inference relation. Further more, with a bit of notational overloading, let us denote by $\mid\sim (*)$ the inference relation induced from $*$, and similarly, by $* (\mid\sim)$ the belief revision operator induced by $\mid\sim$. Then,*

(a) $* (\mid\sim (*)) = *$, and

(b) $\mid\sim (* (\mid\sim)) = \mid\sim$.

Proof. Let K be fixed. We sketch the proof of (a) here, and the proof of (b) is analogous. We need to show that

$$\alpha \in K * \varphi \text{ iff } \alpha \in K * (\mid\sim *)\ \varphi,$$

($\Rightarrow$.) Suppose $\alpha \in K*\varphi$. Thus, $\varphi \mid\sim (*)\ \alpha$, which implies that $\alpha \in K * (\mid\sim (*))\varphi$.

($\Leftarrow$.) Suppose $\alpha \in K * (\mid\sim (*))$. Then, $\varphi \mid\sim (*)\ \alpha$ which implies that $\alpha \in K * \varphi$. $\square$

First we show that as long as $*$ satisfies the six basic AGM revision postulates, the induced $\mid\sim$ relation satisfies the seven basic axioms of non-monotonicity (Theorem 3). Towards this end we first make the following useful observation which trivially follows from Deduction.

Observation 2. *If $\varphi \in K + \psi$ then $\psi \to \varphi \in K$, for every theory K.*

Theorem 3. *If a belief change operator $*$ is (basic) AGM rational then its induced inference relation $\mid\!\sim$ is a non-monotonic inference relation.*

As the basic revision postulates translates to the basic non-monotonic axioms, the next step is to investigate if the supplementary postulates also imply the supplementary non-monotonic axioms. In Corollary 1 below we show that a fully AGM rational belief revision operator induces a non-monotonic inference relation that satisfies the supplementary axioms of non-monotonicity. Theorem 4 below shows a strong alignment, respectively between the revision postulates **K7** and **K8**, and the axioms of non-monotonicity **N8** and **N9**.

Theorem 4. *Let $*$ be a belief revision operator that satisfies **K1**, and $\mid\!\sim$ be its induced inference relation. Then,*

*(i) if $*$ satisfies **K7**, then $\mid\!\sim$ satisfies **N8**;*

*(ii) if $*$ satisfies **K8**, then $\mid\!\sim$ satisfies **N9**.*

Putting Theorems 3 and 4 together tells us that a fully AGM rational belief revision operator induces a non-monotonic inference relation that satisfies the supplementary axioms of non-monotonicity:

Corollary 1. *Let $*$ be a fully AGM rational revision operation. The non-monotonic inference relation $\mid\!\sim$ induced by it satisfies the two supplementary axioms of non-monotonicity.*

What remains to be shown is that the supplementary axioms **N8** and **N9** induce belief change operators that satisfy respectively **K7** and **K8**. However, we postpone this discussion to Section 5, since the transition from the basic non-monotonic inference axioms to the basic AGM postulates proves problematic, as we show in the next section.

4 From $\mid\!\sim$ to * without Compactness

We have shown that a rational belief revision function behaves as a non-monotonic inference relation. Moreover, we also saw that the supplementary revision postulates imply the supplementary non-monotonic axioms, where: **N8** is obtained from **K7**, and **N9** from **K8**. In this section, we examine the converse problem, that is, if every non-monotonic inference induces a rational AGM revision operator.

As we will see, the basic non-monotonic axioms are too general to capture some of the basic AGM postulates in absence of compactness. We subsequently identify the conditions under which a non-monotonic inference relation behaves in accordance with the belief revision postulates.

Let us first show that the basic non-monotonic axioms do not correspond to the basic AGM revision postulates. For this purpose we need construct a non-monotonic inference relation $\mid\!\sim$ such that the revision function $*$ induced by it violates some of the basic AGM postulates. We will conveniently take a revision function $*$ which violates one of the AGM postulates (namely, **K3**), and then we show that the non-monotonic inference relation $\mid\!\sim$ induced by it satisfies *all* the basic non-monotonic axioms. In light of Observation 1, this will suffice our purpose. We construct such a belief revision operator in Example 1.

Example 1. *Let $\otimes$ be a fully AGM rational revision function, and p an arbitrary formula. The belief revision operation $*$ is constructed as follows:*

$$K * \varphi = \begin{cases} (K+\varphi)+p & \text{if } \varphi \to \neg p \notin K \\ K \otimes \varphi & \text{otherwise.} \end{cases}$$

The revision operation from Example 1 behaves in a simple way. It adds both p and a formula φ to K, if both φ and p are jointly consistent with K. On the other hand, if K is inconsistent with φ or p, it resorts to the rational AGM function $\otimes$. It is trivial to show that although it violates postulate **K3**, the inference relation $\mid\!\sim$ induced by it satisfies **N1** to **N7**. This example helps us to prove the following interesting result:

Theorem 5. *Let $\mid\!\sim$ be the inference relation induced from the belief change function $*$. It is not necessary for * to satisfy K1-K6 in order that $\mid\!\sim$ satisfies all the basic axioms of non-monotonicity.*

Which trivially yields the following as a corollary:

Corollary 2. *There is a non-monotonic inference relation $\mid\!\sim$ whose induced belief change operator $*$ violates some AGM revision postulates.*

One might wonder that although the basic non-monotonic axioms do not lead to the six basic AGM postulates, perhaps the supplementary axioms of non-monotonicity can play a compensating role. However, the inference relation $\mid\!\sim$ induced from the belief change function of Example 1 already satisfies these supplementary axioms. This means that even in the presence of the supplementary axioms it is not possible to navigate from a non-monotonic inference relation to the AGM postulates.

Observation 3. *The belief change operator $*$ used in Example 1 satisfies the supplementary AGM revision postulates.*

Theorem 6. *There is a non-monotonic inference relation that satisfies the supplementary non-monotonic axioms, but violate some of the basic AGM revision postulates.*

Theorem 6 shows us that even in the presence of the supplementary axioms, it is not possible to establish the desired connection between AGM revision postulates and non-monotonic inference axioms without the compactness assumption. This naturally leads to the question whether there are conditions under which the revision operation induced by a rational inference relation $\mid\!\sim$ is AGM-rational in absence of compactness. This is the issue we examine in the next section.

5 Bridging the gap

We have seen that the basic axioms of non-monotonicity alone are not strong enough to capture all the basic AGM revision postulates. We also saw that resorting to the supplementary axioms does not help in capturing the basic AGM revision postulates either, let alone the supplementary AGM revision postulates.

In this section, we will show that it is possible to connect AGM revision and non-monotonic inference. We will also show what conditions are needed to establish such a connection. Towards this end let us first identify the largest set of AGM postulates that are induced by the basic axioms of non-monotonicity.

Let us start with the easy ones. Axiom **N2** corresponds to **K6** (extensionality), and says that if two formulae are logically equivalent modulo Cn, then they entail the same set of formulae. The success postulate **K2** is captured by the axiom **N1**. The following theorem shows that postulate **K5** is jointly captured by axioms **N7** and **N1**.

Theorem 7. *Let $\mid\!\sim$ be an inference relation and $*$ its induced belief change operator. Then:*

*(i) if $\mid\!\sim$ satisfies **N1**, then $*$ satisfies **K2** (success);*

*(ii) if $\mid\!\sim$ satisfies **N2**, then $*$ satisfies **K6** (extensionality);*

*(iii) if $\mid\!\sim$ satisfies both **N7** and **N1** then $*$ satisfies **K5** (consistency).*

So far, of the basic AGM postulates, the ones that are missing are **K1**, **K3** and **K4**. As we will see, the basic axioms of non-monotonicity are not strong enough to capture these three AGM postulates. We start with **K1** (the closure).

As we discussed in Section 2, when the underlying logic Cn is compact, axioms **N3** and **N4** trivially imply **Closure**$_{\text{NM}}$. However, when Cn is not assumed to be compact, **N3** and **N4** may no longer guarantee **Closure**$_{\text{NM}}$. Consequently, a non-monotonic inference may fail to induce a belief change operator $*$ that satisfies **K1**. Therefore, satisfaction of **K1** necessitates the presence of the **Closure**$_{\text{NM}}$ axiom.

Proposition 1. *An inference relation $\mid\!\sim$ satisfies **Closure**$_{\text{NM}}$ if and only if the belief change operator $*$ induced by it satisfies **K1**.*

One question that immediately arises is: Do we still need **N3** and **N4** in presence of **Closure**$_{\text{NM}}$? It turns out that **Closure**$_{\text{NM}}$ implies both **N3** and **N4**.

Proposition 2. ***Closure**$_{\text{NM}}$ implies **N3** and **N4**.*

Now we analyse the other two missing postulates: **K3** and **K4**. In Section 4, we showed in Example 1 that non-monotonic inference relations may fail to induce postulate **K3**. In that example, reasoning from a tautological evidence allowed the inference of new information not previously believed: recall that $Cn(\top) * \top$ entailed the propositional symbol p which is clearly not in $Cn(\top)$.

This means that, the axioms do not forbid a non-monotonic inference to augment its body of knowledge in the light of a tautology. In other words, if a formula φ does not belong to a theory K, it is possible that $\top \mid\!\sim \varphi$. So, our first task is to prevent such inappropriate acquisition of information. Hence we propose the following condition:

(keeper) if $\mid\!\sim \varphi$ then $\varphi \in K$.

This allows us to capture postulate **K3**.

Proposition 3. *Let $\mid\!\sim$ be an inference relation. If it satisfies both **N5** and the **keeper**, then the belief change operator $*$ induced by it satisfies **K3**.*

Proof. Let $\mid\!\sim$ be an inference relation that satisfies both **N5** and **keeper**. We will show that its induced belief change operator $*$ satisfies **K3**, that is, $K * \varphi \subseteq K + \varphi$. Let α be a formula in $K * \varphi$. It follows then that $\varphi \mid\!\sim \alpha$. Then it follows from **N5** that $\mid\!\sim \varphi \to \alpha$. Thus, from the **keeper**, we get that $\varphi \to \alpha \in K$. Therefore, $\alpha \in K + \varphi$. $\square$

Though **keeper** is strong enough to capture **K3**, it fails to capture postulate **K4** even in the presence of the seven basic axioms of non-monotonicity. To notice that, let us take the Example 2 below.

Example 2. *Let $\otimes$ be a fully AGM rational revision function. We construct the following belief change operation:*

$$K * \varphi = \begin{cases} Cn(\varphi) & \text{if } \neg\varphi \notin K \\ K \otimes \varphi & otherwise. \end{cases}$$

Example 2 is similar to Example 1 except that when the input evidence is consistent with the current knowledge, the agent believes the information contained in that evidence alone (and forgoes all the old information). Clearly * so defined violates **K4**. We will show that $*$ induces an inference relation $\mid\!\sim$ that satisfies all basic non-monotonic inference axioms and the ***keeper***.

Proposition 4. *Satisfaction of K4 by a revision operation * is not necessary to induce an inference relation $\mid\!\sim$ which satisfies all basic axioms of non-monotonicity.*

The next step deals with keeper.

Proposition 5. *The belief change operation used in Example 2 induces an inference relation $\mid\!\sim$ that satisfies the keeper.*

Proof. Let us suppose that $\mid\!\sim \varphi$, we need to show that $\varphi \in K$. As $\mid\!\sim$ is induced from $*$, we have that $\varphi \in K * \top$. We have two cases to consider $K * \top = K \otimes \top$ or $K * \top = Cn(\top)$.

For $K * \top = K \otimes \top$, as $\otimes$ satisfies all the basic AGM revision postulates, we have from **K3** that $K * \top \subseteq K$, thus $\varphi \in K$.

For $K * \top = Cn(\top)$ we have that $\varphi \in Cn(\top)$, which implies that $\varphi \in K$, as K is a theory. $\square$

Example 2 helps to show that adding the **keeper** is not enough to capture the **K4**. In that example, the main reason that **K4** failed is that we are allowing loss of information. This suggests that we should avoid losing formulae when the input formula is consistent with a theory K. Hence we introduce a further condition:

(rooting) if $\varphi \in K$ then $\mid\!\sim_K \varphi$.

The **rooting** axiom is the converse of the **keeper** and enforces that the formulae present in a theory K should remain in the non-monotonic inferences by a tautology. The *rooting* is the last piece of the puzzle to our first side of the representation theorem. The *rooting* together with the *keeper* and **N6** captures **K4**.

Proposition 6. *If an inference relation $\mid\!\sim$ satisfies keeper, rooting and **N6**, then its induced belief change operator $*$ satisfies **K4**.*

Proof. Let $\mid\!\sim$ be an inference relation that satisfies *keeper*, *rooting* and **N6**, and $*$ its induced belief change operator. We will show that $*$ satisfies **K4**. To show this, let us suppose that $\neg\varphi \notin K$, we need to show that $K + \varphi \subseteq K * \varphi$. In other words, for every formula $\alpha \in K + \varphi$ we need to show that $\alpha \in K * \varphi$. By hypothesis, $\neg\varphi \notin K$ which from the contrapositive of the **keeper** implies that $\not\mid\!\sim \neg\varphi$.

As $\alpha \in K + \varphi$, we have that $\varphi \to \alpha \in K$ from Observation 1. Thus, from *rooting* $\mid\!\sim \varphi \to \alpha$. Thus we have that both $\not\mid\!\sim \neg\varphi$ and $\mid\!\sim \varphi \to \alpha$. Thus, from **N6** we have that $\varphi \mid\!\sim \alpha$ which implies that $\alpha \in K * \varphi$. □

Now we have all the pieces of the puzzle to show the first result of the representation theorem, that is, the basic non-monotonic axioms augmented with the keeper and rooting induces AGM rational revision operators.

Theorem 8. *Let $\mid\!\sim$ be a non-monotonic inference relation, if it satisfies the keeper, rooting and Closure$_{\text{NM}}$ then its induced belief revision operator $*$ is AGM rational.*

Proof. From Propositions 3 and 6 we have that $*$ satisfies **K3** and **K4**. Moreover, from Theorem 7 we have **K2**, **K5** and **K6**. Finally, from Proposition 1 we have that Closure$_{\text{NM}}$ implies **K1**. This finishes the proof.

□

So far, we were able to show one direction of the representation theorem. However, as we have included two new axioms to the systems, we need to show that rational AGM revision operators induces non-monotonic inferences that satisfy, besides the basic axioms, the **keeper**, **rooting** and the **Closure**$_{\text{NM}}$.

Theorem 9. *An AGM rational belief revision operator induces a non-monotonic inference that satisfies rooting, keeper and Closure$_{\text{NM}}$.*

Proof. Let $*$ be an AGM rational operator, that is, it satisfies the six basic AGM revision postulates. We have from Theorem 3 that it induces a non-monotonic inference relation $\mid\!\sim$. We will show that it also satisfies *rooting*, *keeper* and *Closure*$_{\text{NM}}$. From **K3** and **K4**, we have that

$$K * \top = K. \tag{1}$$

As a formula $\varphi \in K * \psi$ iff $\psi \mid\!\sim \varphi$, we have then from (1) that $\varphi \in K * \top$ iff $\mid\!\sim \varphi$. This means that $\mid\!\sim$ satisfies both **keeper** and **rooting**. Furthermore, since * satisfies **K1**, it follows from the contrapositive of Proposition 2 that $\mid\!\sim$ satisfies the **Closure**$_{\text{NM}}$. □

We reached our first representation theorem, in the form of Corollary 3 below, which comprises Theorems 8 and 9. This results is due to that adding **keeper**, **rooting** and **Closure**$_{\text{NM}}$ axioms to the basic set of non-monotonic inference axioms stablish a bridge between AGM belief revision operators and non-monotonic logics.

Corollary 3. *A belief change operator $*$ is AGM rational iff its induced inference relation $\mid\!\sim$ is a non-monotonic inference relation that satisfies **Closure**$_{\text{NM}}$, **keeper** and **rooting**.*

Proof. It is straightforward from Theorems 8 and 9. □

Capturing Supplementary postulates

Our first representation result comprises Corollary 3, which says that a non-monotonic inference relations augmented with the keeper and rooting induce an AGM rational belief revision operators, and AGM rational belief revision operators also induce non-monotonic inference relations that satisfy, besides the basic axioms, the keeper and rooting.

The next representation theorem concerns the supplementary postulates and the supplementary axioms. We start showing that axioms **N8** and **N9** induce belief change operators that satisfy **K7** and **K8** respectively.

Theorem 10. *If a non-monotonic inference relation $\mid\!\sim$ satisfies **N8** then the belief change function $*$ induced by it satisfies **K7**.*

Proof. Given a formula $\alpha \in K * \varphi \wedge \psi$, we need to show that $\alpha \in K * \varphi + \psi$. As $\alpha \in K * \varphi \wedge \psi$, we have that $\varphi \wedge \psi \mid\!\sim \alpha$ which from **N8** implies that $\varphi \mid\!\sim \psi \to \alpha$. This means that $\psi \to \alpha \in K * \varphi$. Thus, $\alpha \in K * \varphi + \psi$. □

Theorem 11. *If an inference relation $\mid\!\sim$ satisfies both **N9** and **Closure**$_{\text{NM}}$, then its induced belief change operation $*$ satisfies **K8**.*

Proof. Let us suppose that $\neg\psi \notin K * \varphi$, the we need to show that $K * \varphi + \psi \subseteq K * \varphi \wedge \psi$. In other words, given a formula $\alpha \in K * \varphi + \psi$, we need to show that $\alpha \in K * \varphi \wedge \psi$. From $\neg\psi \notin K * \varphi$, we have that $\varphi \not\mid\!\sim \neg\psi$. Moreover, as $\mid\!\sim$ satisfies Closure$_{\text{NM}}$, we have that both $K * \varphi$ and $K * \varphi \wedge \psi$ are theories. Therefore, $\alpha \in (K * \varphi) + \psi$ implies Observation 1 that $\psi \to \alpha \in K * \varphi$, which means that $\varphi \mid\!\sim \psi \to \alpha$. This together with $\varphi \not\mid\!\sim \neg\psi$ implies from **N9** that $\varphi \wedge \psi \mid\!\sim \psi \to \alpha$. This means that $\psi \to \alpha \in K * \varphi \wedge \psi$. Thus, as $\psi \in K * \varphi \wedge \psi$, we have that $\alpha \in K * \varphi \wedge \psi$. □

This second representation result, an immediate consequence from Corollary 3, together with Theorems 10 and 11, shows that **N8** and **N9** translates respectively into **K7** and **K8**, and vice-versa.

Corollary 4. *A belief change operator $*$ is fully AGM rational iff its induced inference relation $\mid\!\sim$ satisfies all the basic, and supplementary axioms of non-monotonicity together with keeper and rooting.*

We know from Proposition 2 that **N3** and **N4** are superfluous in presence of **Closure**$_{\text{NM}}$. Therefore, we may assume a shorter set of non-monotonic axioms: **N1**, **N2**, **N5** to **N7** and the **keeper**, **rooting** and **Closure**$_{\text{NM}}$. Hence the Corollary 3 can be stated more simply as:

Corollary 5. *A belief change operator $*$ is AGM rational iff its induced inference relation $\mid\!\sim$ satisfies **N1**, **N2**, **(N5 - N7)**, **keeper**, **rooting** and **Closure**$_{\text{NM}}$.*

6 Conclusion and Future Works

In this work we have addressed the connection between AGM revision operators and non-monotonic logics. Unlike the previous works, we no longer assume compactness. We showed that dropping compactness has some immediate consequences on the non-monotonic system. For instance,

Closure$_{\text{NM}}$ needs to be assumed as it no longer immediately follows from **N3** and **N4**. Moreover, in presence of **Closure**$_{\text{NM}}$, **N3** and **N4** become redundant. Another two consequences is that, though the direction from AGM revision postulates to the non-monotonic system we addressed remains untouched, the converse is not true. We showed that the postulates **K3** and **K4** are violated in this side of the trip. To solve this problem, we studied which axioms remained as a translation of the AGM postulates, and then we proposed two new axioms, the **keeper** and **rooting**, in order to strength the non-monotonic system to be able to reconnect with AGM paradigm.

At the end, we were able to show a connection between the enhanced non-monotonic system and the AGM belief postulates: both for the basic postulates as well as the supplementary postulates. One more difference between our work and the classical one is that whereas the study between AGM revision and non-monotonic inference rides on specific belief revision operators such as partial meet or other constructions, our proofs navigate directly from postulates to axioms and vice-versa without the need to assume any specific construction. This makes the study more general, as we assume very little about the underlying logics, namely, to be only Tarskian and closed under classical negation and disjunction.

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