Agricultural Production and Technological Change

Advanced Producer Theory and Analysis: Stochastic Frontier Analysis

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AREC 705: Efficiency and Productivity Analysis

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Stochastic frontier analysis (SFA): A parametric approach that asserts a particular functional form for $f(x)$ and uses the data to estimate the parameters.

- SFA is based on theory, need to select a production function and a form of
- SFA has distribution-free and distribution approaches. These relate to the assumptions on the structure of the error term.
- SFA distribution approaches allow you to separate random noise or disturbances from your estimates of efficiency (but this requires assumptions on the distribution of the error term)

Statistics from Production Functions

Economic effect	Formula
Output level	$y = f(\mathbf{x})$
Elasticity of output	$\varepsilon_i = \frac{\partial f}{\partial x_i} \cdot \frac{x_i}{f}$
Returns to scale (RTS)	$\mu = \left(\sum_{i=1}^{n} x_i f_i\right) / f$
Elasticity of substitution	$\sigma_{ij} = \frac{-f_{ii}/f_i^2 + 2(f_{ii}/f_i f_j) - f_{ii}/f_j^2}{1/x_i f_i + 1/x_j f_j}$
Rate of technical change	$TC = \frac{\partial \ln y}{\partial t}$
Speed of technical change	$\frac{\partial TC}{\partial t}$

Where $f_i = \frac{\partial f}{\partial x_i}$ and $f_{ij} = \frac{\partial f_i}{\partial x_j}$

Some Common Production Functions for SFA

Cobb-Douglas (without technical change and inefficiency):

$$y = f(x) = A \prod_{j=1}^{J} x_j^{\beta_j}$$

$$\Rightarrow \ln y = \beta_0 + \sum_j \beta_j \ln x_j$$

Where $\beta_0 = \ln A$

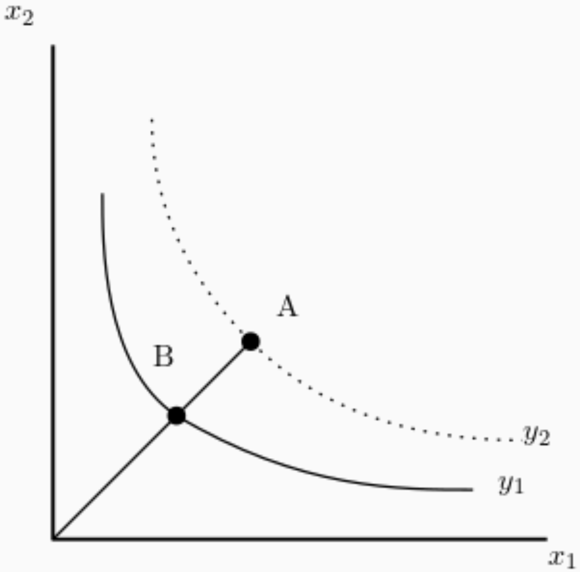
Cobb-Douglas (with technical change, without inefficiency):

$$\ln y = \beta_0 + \sum_j \beta_j \ln x_j + \beta_t t$$

Statistics from Cobb-Douglas

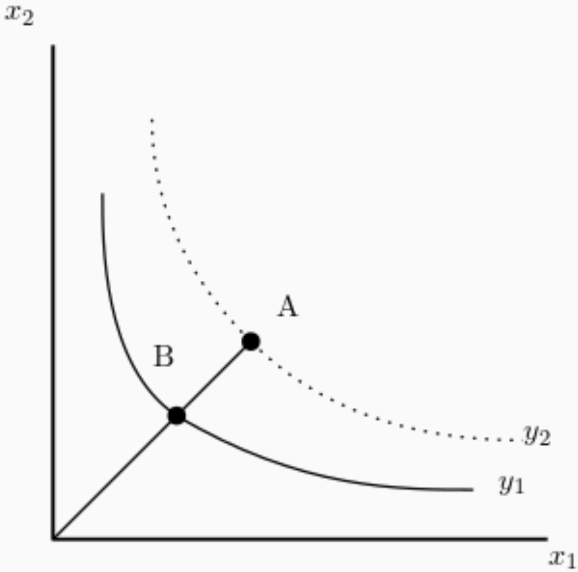
Economic effect	C-D formula
Output level	$\ln y = \beta_0 + \sum_j \beta_j \ln x_j + \beta_t t$
Elasticity of output	$\varepsilon_i = \frac{\partial \ln y}{\partial \ln x_i} = \beta_j$
Returns to scale (RTS)	$\sum_{\forall j} \beta_j$
Elasticity of substitution	1
Rate of technical change	$TC = \beta_t$
Speed of technical change	$\frac{\partial TC}{\partial t} = 0$

Incorporating Inefficiency in Production Functions



If production is technically efficient, inputs at point A should produce output $y_2 > y_1$; with technical inefficiency inputs at A only produce y_1.

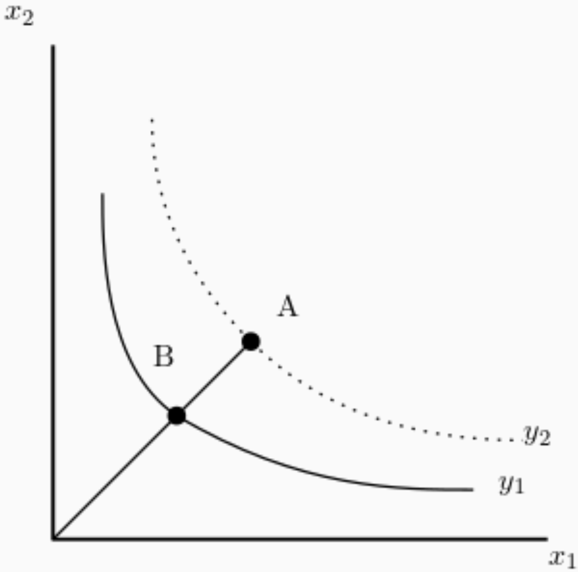
Incorporating Inefficiency in Production Functions



IO technical inefficiency can be measured by moving radially downward from point A to point B. Output y_1 can be produced using less of both inputs $\Rightarrow$ inputs can be reduced by $\overline{AB}/\overline{OA}$.

A production plan with IO technical inefficiency can be written as: $y = f(\mathbf{x} \cdot exp(-\eta)),\ \eta \geq 0$

Incorporating Inefficiency in Production Functions



OO technical inefficiency: the input quantities at point A can be used to produce $y_2 > y_1$. Output can be increased by $\frac{y_2 - y_1}{y_2}$

A production plan with OO technical inefficiency can be written as: $y = f(\mathbf{x}) \cdot exp(-u), \ u \geq 0$

Cobb-Douglas with IO:

$$y = f(xe^{-\eta})$$

$$\ln y = \beta_0 + \sum_j \beta_j \ln x_j - \left(\sum_j \beta_j\right) \eta$$

Cobb-Douglas with OO:

$$y = f(x)e^{-u}$$

$$\ln y = \ln f(x) - u = \beta_0 + \sum_j \beta_j \ln x_j - u$$

Some Common Production Functions for SFA

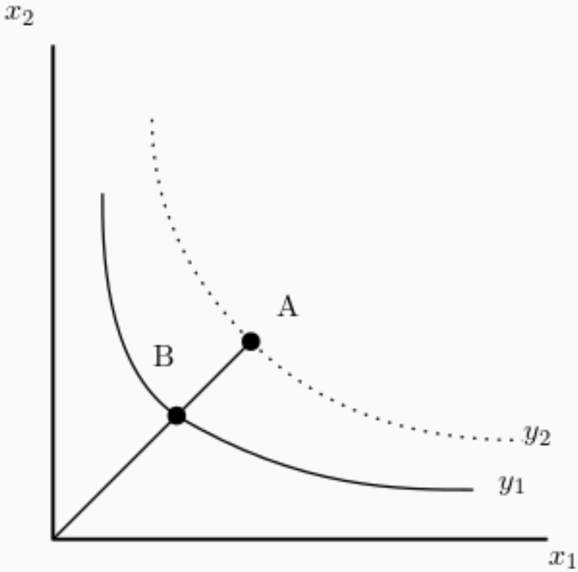
Because Cobb-Douglas is homogeneous of degree 1, there is (essentially) no difference between IO and OO technical inefficiency. This implies that none of the economic measures we care about are affected by the presence of inefficiency:

Elasticity of output: $\frac{\partial \ln y}{\partial \ln x_j} = \beta_j$

Returns to scale: $\sum_j \frac{\partial \ln y}{\partial \ln x_j} = \sum_j \beta_j$

Elasticity of substitution: $\sigma_{ij} = 1$

SFA: Deterministic vs. Stochastic



A production plan with OO technical inefficiency can be written as: $y = f(\mathbf{x}) \cdot exp(-u), \; u \geq 0$.

This assumes the production frontier is **"deterministic"**, that is, it can be precisely estimated using observed data.

However, it is more likely that the true frontier is **stochastic**, that is, it cannot be estimated precisely.

SFA: Deterministic vs. Stochastic

A deterministic production plan with OO technical inefficiency can be written as:

$y = f(\mathbf{x}) \cdot exp(-u),\ \ u \geq 0 \Rightarrow \ln y_i = \ln f(\mathbf{x_i}; \beta) - u_i$

A stochastic production plan with OO technical inefficiency can be written as: $\ln y = \ln f(\mathbf{x_i}; \beta) + v_i - u_i$

where v_i is a zero-mean random error.

To estimate inefficiency relative to the stochastic frontier, we need to assume a distribution on $\mathbf{v_i}$. This is sometimes called a "parametric approach" vs. the distribution-free approach.

Extra: Some Common Production Functions for SFA

Translog (without technical inefficiency):

$$\ln y = \beta_0 \sum_j \beta_j \ln x_j + \frac{1}{2} \sum_j \sum_k \beta_{jk} \ln x_j \ln x_k, \;\; \beta_{jk} = \beta_{kj}$$

$$\Rightarrow \ln y = \beta_0 + \sum_j \beta_j \ln x_j$$

With technical change we can write:

$$\ln y = \beta_0 \sum_j \beta_j \ln x_j + \frac{1}{2} \sum_j \sum_k \beta_{jk} \ln x_j \ln x_k + \beta_t t \frac{1}{2} \beta_{tt} t^2 + \sum_j \beta_j t \ln x_j t$$

Where the rate of technical change is: $\beta_t \beta_{tt} t + \sum_j \beta_{jt} ln x_j$

Extra: Some Common Production Functions for SFA

Translog with OO:

$$\ln y = \beta_0 \sum_j \beta_j \ln x_j + \frac{1}{2} \sum_j \sum_k \beta_{jk} \ln x_j \ln x_k - u, \;\; \beta_{jk} = \beta_{kj}$$

Translog with IO:

$$\ln y = \beta_0 \sum_j \beta_j (\ln x_j - \eta) + \frac{1}{2} \sum_j \sum_k \beta_{jk} (\ln x_j - \eta)(\ln x_k - \eta), \;\; \beta_{jk} = \beta_{kj}$$

$$\Rightarrow \ln y = \beta_0 \sum_j \beta_j \ln x_j + \frac{1}{2} \sum_j \sum_k \beta_{jk} \ln x_j \ln x_k - \eta \left[\sum_j \beta_j + \sum_j \left(\sum_k \beta_{jk} \ln x_k \right) \right] + \frac{1}{2} \eta^2 \sum_j \sum_k \beta_{jk}$$

Extra: Some Common Production Functions for SFA

Assuming the Translog production function leads to a few differences in economic measures we might be interested in when using IO and OO TE.

Translog with OO technical inefficiency:

Elasticity of output: $\frac{\partial \ln y}{\partial \ln x_j} = \beta_j + \sum_k \beta_{jk} \ln x_k$

Returns to scale: $\sum_j \frac{\partial \ln y}{\partial \ln x_j} = \sum_j (\beta_j + \sum_k \beta_{jk} \ln x_k)$

With IO technical inefficiency:

Elasticity of output: $\frac{\partial \ln y}{\partial \ln x_j} = \beta_j + \sum_k \beta_{jk} \ln x_k + (\sum_k \beta_{jk})\, \eta$

Returns to scale: $\sum_j \frac{\partial \ln y}{\partial \ln x_j} = \sum_j (\beta_j + \sum_k \beta_{jk} \ln x_k + (\sum_k \beta_{jk})\, \eta)$