Data Structures and Algorithms

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Session: Shortest Path Algorithm
(All Pair Shortest Path)

All Pair Shortest Path Problem [Floyd-Warshall Algorithm]

Instance of an algorithmic paradigm called dynamic programming

1. Given: A directed weighted graph $G(V, E)$, for each edge $(v_1, v_2) \in E$ and an associated weight of an edge $w(v_1, v_2)$
2. Find: A shortest path from v_1 to v_2 for every pair of vertices v_1 and v_2 in V

If $d^{(k-1)}[v_1, v_k]$ is shortest path from v_1 to v_k using nodes $(1 \cdots k-1)$

$d^{(k-1)}[v_k, v_2]$ " v_k to v_2 " " $(1 \cdots k-1)$

$$d^{(k)}[v_1, v_2] = \min\left(d^{(k-1)}[v_1, v_k] + d^{(k-1)}[v_k, v_2],\ w(v_1, v_2)\right)$$

Floyd-Warshall Algorithm: Setting up Notation

1. Given a directed weighted graph $G(V, E)$, the weight of each edge $(v_1, v_2) \in E$ can be defined using an an adjacency-matrix representation W
2. If W is an n x n matrix,

$$w_{v_i v_j} = \begin{cases} 0 & \text{if i = j,} \\ \text{weight of directed edge } (v_i, v_j) & \text{if i} \neq \text{j and (i, j)} \in \text{E,} \\ \infty & \text{if i} \neq \text{j OR (i, j)} \notin \text{E.} \end{cases}$$

w[i,j] =

3. Distance matrix $D^0 = W$ → shortest path matrix for all pairs with path through $\emptyset$ (empty set)
4. Predecessor matrix L can be defined as

Book keeping for shortest path

$$L^0(i, j) = \begin{cases} \text{i} & \text{if (i, j)} \in \text{E} \\ \text{NULL} & \text{if i = j and (i, j)} \notin \text{E.} \end{cases}$$

Floyd-Warshall Algorithm

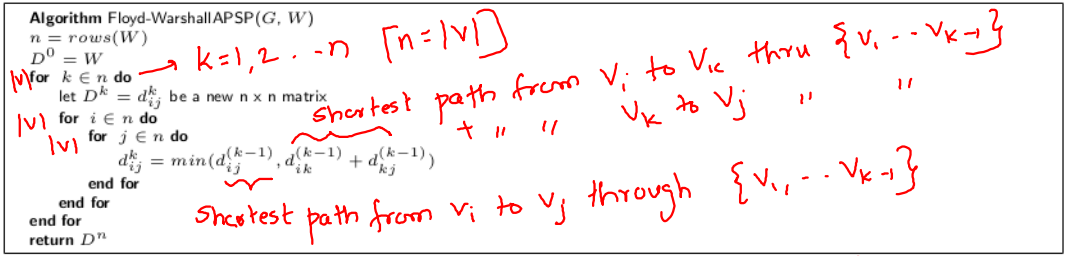
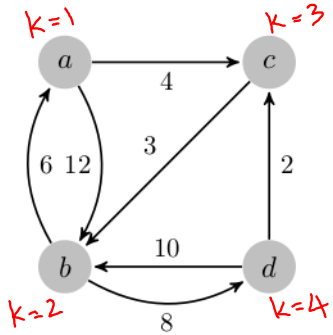


Figure: Floyd-Warshall Algorithm : $O(|V|^3)$

Example

Graph



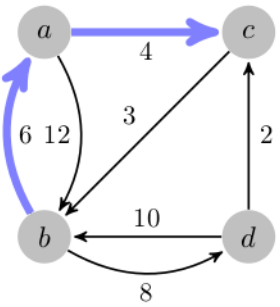
k=0

$D^0 = W$

	a	b	c	d
a	0	12	4	∞
b	6	0	∞	8
c	∞	3	0	∞
d	∞	10	2	0

Example

Graph



k = 1 (v_i = b)

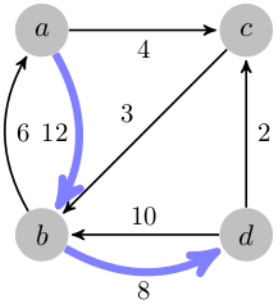
a

	a	b	c	d
a	0	12	4	∞
b	6	0	10	8
c	∞	3	0	∞
d	∞	10	2	0

j

Example

Graph



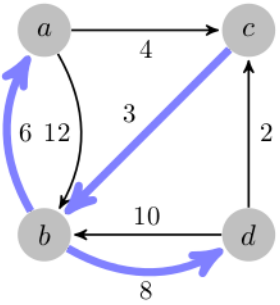
k = 2 (v_i =a)

	a	b	c	d
a	0	12	4	20
b	6	0	10	8
c	∞	3	0	∞
d	∞	10	2	0

j

Example

Graph



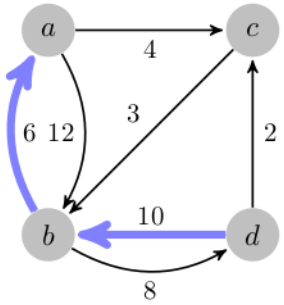
k = 2 (v_i =c)

	a	b	c	d
a	0	12	4	20
b	6	0	10	8
c	9	3	0	11
d	∞	10	2	0

→ j

Example

Graph



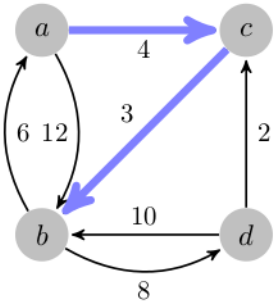
k = 2 (v_i =b)

	a	b	c	d
a	0	12	4	20
b	6	0	10	8
c	9	3	0	11
d	16	10	2	0

j

Example

Graph

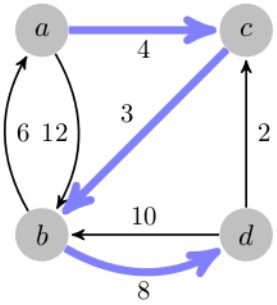


k = 3 (v_i =a, v_j =b)

	a	b	c	d
a	0	7	4	20
b	6	0	10	8
c	9	3	0	11
d	16	10	2	0

Example

Graph



k = 3 (v_i =a, v_j =d)

	a	b	c	d
a	0	7	4	15
b	6	0	10	8
c	9	3	0	11
d	16	10	2	0

Correctness : Loop invariant

$d^{(k-1)}(v_i, v_j)$ = Shortest path from v_i to v_j using $\{1, \ldots k-1\}$

Thank you