Data Structures and Algorithms

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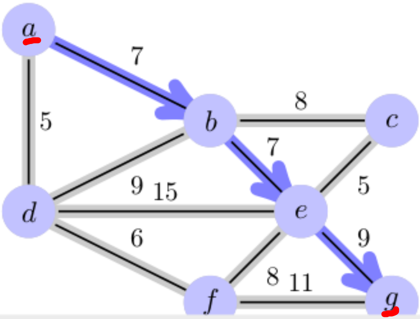
Session: Shortest Path Algorithm (Dijkstra's Algorithm)

Finding Shortest Paths in Graphs

1. Shortest Path in Weighted Graphs
2. Shortest Path Properties in Weighted Graphs
3. Dijkstra's algorithm
4. Edge Relaxaton

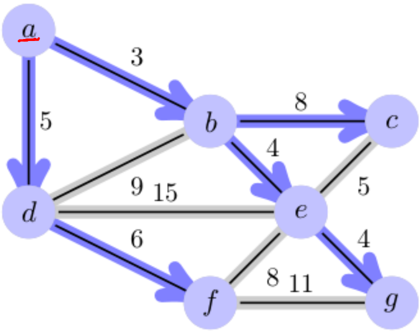
Shortest Path in Weighted Graphs

1. Given two vertices a and g in the weighted graph below, find a path of minimum total weight between them.
2. Length of path = sum of weight of constituent edges
3. Applications: Routing of vehicles, flights, internet packets



Shortest Path Properties in Weighted Graphs

1. **Optimal Substructure:** A subpath of a shortest path is also a shortest path
2. **Tree of shortest paths** exists[1] from a start vertex s to every other vertex



[1] Illustrated below is the tree of shortest paths from a

Dijkstra's Algorithm and Shortest Path Properties

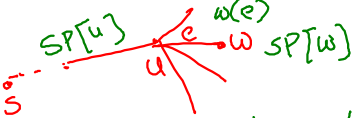
1. **Optimal Substructure:** A subpath of a shortest path is also a shortest path
2. **Tree of shortest paths** exist from a start vertex s to every other vertex
3. Dijkstra's Algorithm combines the two:
   - Store for each v a label d(v) = the distance of v from s
   - Successively computes $d(v)$, starting from the neighbors of s, assuming
     - graph is connected → H/W problem: To deal with graph containing several connected components
     - edges are undirected → H/W: deal with mixture of directed & undirected edges
     - edge weights are nonnegative

Dijkstra's Shortest Path Algorithm

```
Algorithm ComputeDijkstraSPs(G, s)
Output: Array SP[.] with length of shortest path from s to v stored in SP[v].
P ← new Min-heap
for v ∈ G.getVertices() do
    if v = s then
        SP[v] = 0
    else
        SP[v] = ∞
    end if
    P.insert(v, SP[v])
end for
while P.isNotEmpty() do
    u ← P.getMin()
    P.removeMin() //Remove u
    for e ∈ G.incidentEdges(u) do
        w ← G.otherVertex(e, u)
        r ← SP[u] + weight(e) //Relax e
        if r < SP[w] then
            SP[w] = r
            P.update(w, r)//Update the value for key w
        end if
    end for
end while
```

P = interim stat (distance) of every vertex from s that has been found to be shortest so far

u is the closest node in P from s



Loop invariant: For each $u \notin P$, shortest path to u is SP[u]

Figure: Computing Shortest Paths using Dijkstras Algorithm

Shortest Path (from d)

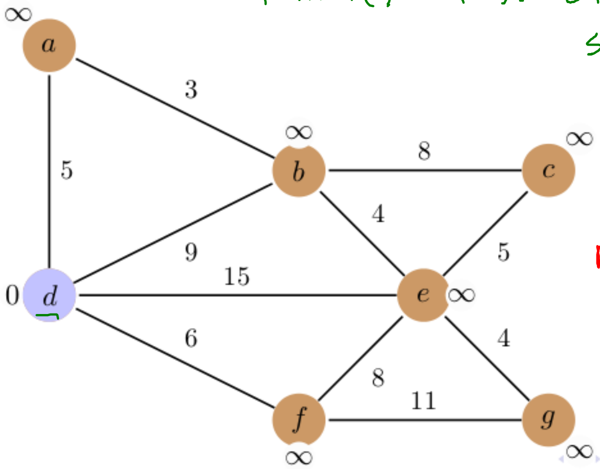
$SP[d] = 0 \quad SP[a, b \ldots] = \infty$

$P.min() = d \quad \ldots \quad SP[a] = 5$

$SP[b] = 9$

$SP[f] = 6$

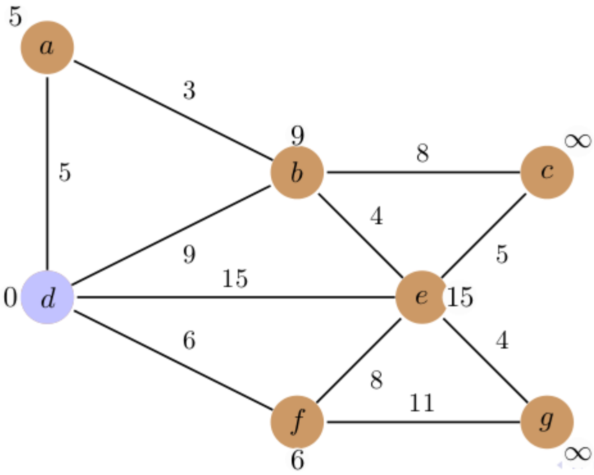
$SP[e] = 15$



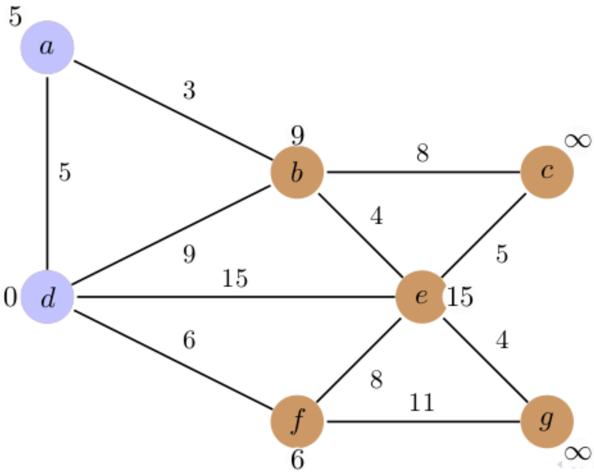
Next

$P.min() = (a, 5)$

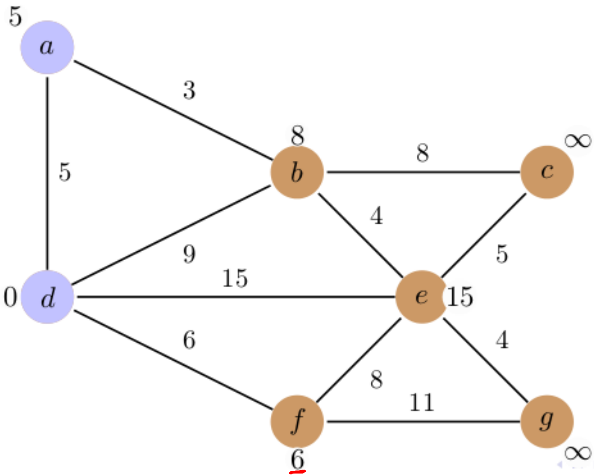
Shortest Path (from d) contd.



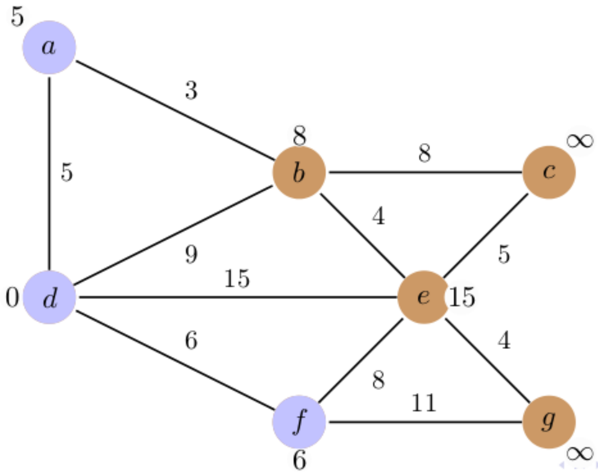
Shortest Path (from d) contd.



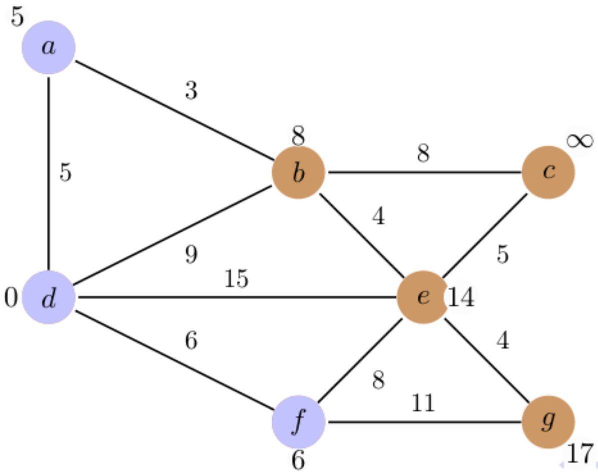
Shortest Path (from d) contd.



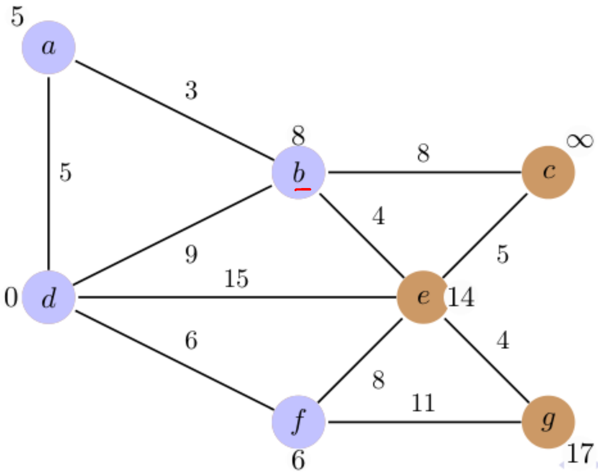
Shortest Path (from d) contd.



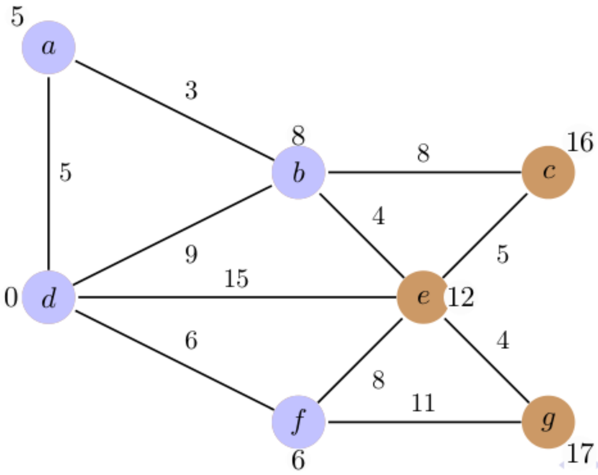
Shortest Path (from d) contd.



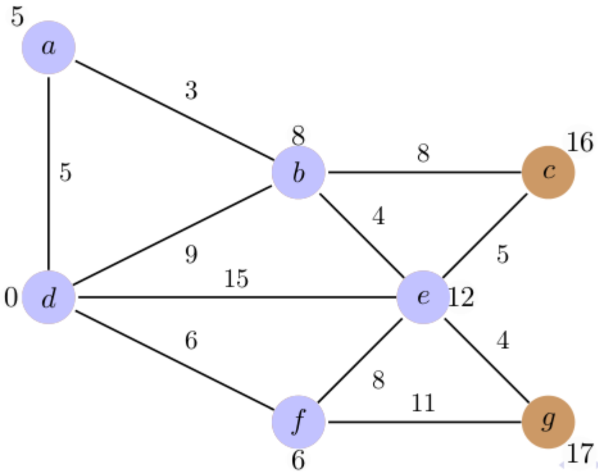
Shortest Path (from d) contd.



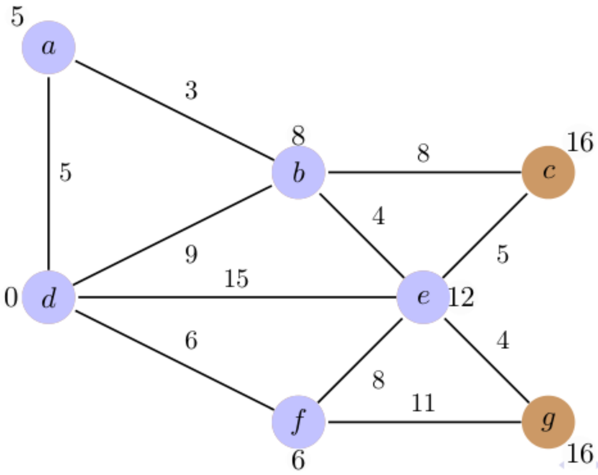
Shortest Path (from d) contd.



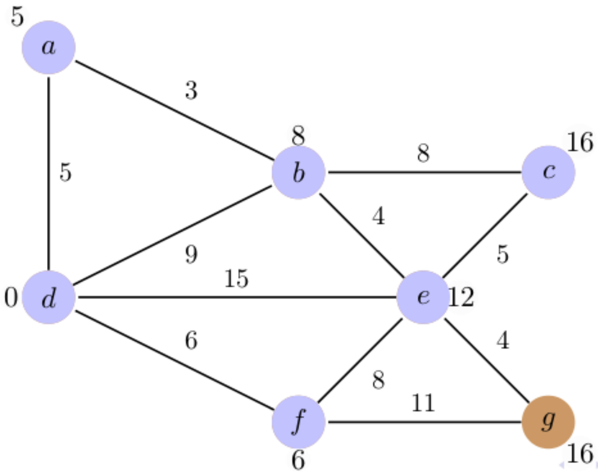
Shortest Path (from d) contd.



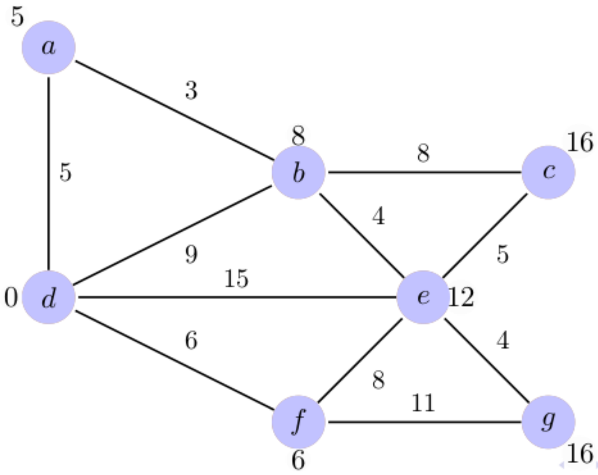
Shortest Path (from d) contd.



Shortest Path (from d) contd.



Shortest Path (from d) contd.



Analysis of Dijkstra's Algorithm

Algorithm ComputeDijkstraSPs(G, s)
Output: Array $SP[.]$ with length of shortest path from s to v stored in $SP[v]$.
$P \leftarrow$ new Min-heap
for $v \in G.getVertices()$ **do**
  if $v = s$ **then**
    $SP[v] = 0$
  else
    $SP[v] = \infty$
  end if
  $P.insert(v, SP[v])$
end for $\Longrightarrow \quad c_1 \quad \times$ $|V|$ times
while $P.isNotEmpty()$ **do**
  $u \leftarrow P.getMin() \quad \Longrightarrow \quad c_2 \quad \times$ $\log|V|$ times
  $P.removeMin()$ //Remove u
  for $e \in G.incidentEdges(u)$ **do**
    $w \leftarrow G.otherVertex(e, u)$
    $r \leftarrow SP[u] + weight(e)$ //Relax e
    if $r < SP[w]$ **then**
      $SP[w] = r$
      $P.update(w, r)$//Update the value for key $w \quad \Longrightarrow \quad c_3 \quad \times$ $\log|V|$ times
    end if
  end for $\Longrightarrow \quad c_4 \quad \times$ $deg(v)$ times
end while $\Longrightarrow \quad c_5 \quad \times$ $|V|$ times

$$T(n) = c_3 c_4 \log|V| \sum_{v \in V} deg(v) + c_2 c_5 |V|(\log|V|) + c_1|V| = O(|E| + |V|)\log|V|$$

Thank you