Cryptography

Corso di Laurea Magistrale in Informatica

Public-Key Encryption

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Ciphers in an Asymmetrical Framework

- In asymmetric cryptography, anyone who wants to *receive* messages generates not a key but a *pair* of keys (pk, sk) where:
  - pk is a *public key*, used by the sender when encoding messages and must reach as many users as possible (through authenticated channels, even if not private).
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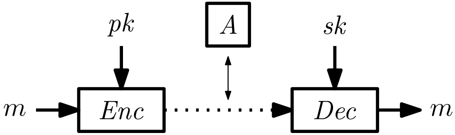
  - sk is a *private key*.
- The framework then becomes the following one:



Symmetric Key vs. Asymmetric Key

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  - Only *one part* of the key is kept secret, while the other is made public.
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- **Advantages** of the Asymmetric Key:
  - It is no longer necessary to distribute keys on *private* channels.
  - Each user must manage the secrecy of *only one* key.
- **Disadvantages** of the Asymmetric Key:
  - The performance of asymmetric-key schemes is usually orders of magnitude lower than that of symmetric-key ones.
  - Public keys must be distributed over *authenticated* channels, without which a very simple attack is possible.

Public-Key Encryption Scheme

- The definition of the encryption scheme $\Pi = (Gen, Enc, Dec)$ needs to be suitably modified:
  - Gen takes a string in the form 1^n as input and outputs a pair of keys (pk, sk), such that that $|pk|, |sk| \geq n$ and such that n can be inferred by pk or sk.
  - The Enc algorithm takes as input a message m and a public key pk and outputs a ciphertext.
  - The algorithm Dec can be probabilistic, it takes as input a ciphertext c and a secret key sk and outputs either a message or a special symbol $\perp$.
- Let us assume that the scheme is **correct**, this time in the *probabilistic* sense: there must exist a negligible function ε such that for every pair (pk, sk) produced by $Gen(1^n)$ and for every n,

$$Pr(Dec_{sk}(Enc_{pk}(m)) \neq m) \leq \varepsilon(n)$$

- Often, Enc_k is defined only for messages of length equal to n, or over the whole space $\{0, 1\}^*$.

Security of a Public-Key Encryption Scheme

- The notion of experiment should be modified:

$\mathsf{PubK}^{eav}_{A,\Pi}(n)$:
$(pk, sk) \leftarrow Gen(1^n)$;
$(m_0, m_1) \leftarrow A(1^n, pk)$;
if $|m_0| \neq |m_1|$ **then**
  Result: 0
$b \leftarrow \{0,1\}$; $c \leftarrow Enc(k, m_b)$;
$b^* \leftarrow A(c)$;
Result: $\neg(b \oplus b^*)$

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Definition

A public key encryption scheme Π is said to be *secure against passive attacks* iff for every adversary PPT A there exists a function $\varepsilon \in \mathcal{NGL}$ such that

$$Pr(\mathsf{PubK}^{eav}_{\Pi,A}(n) = 1) = \frac{1}{2} + \varepsilon(n)$$

Comments on the Definition

- The definition of security we have just given is imperceptibly different from that seen in a symmetrical context: A obviously has also access to pk.
- This small difference has *important* consequences:
  1. The fact that A has access to pk implies that A can encrypt any message, even without access to oracles.
  2. Given pk and $c = Enc_{pk}(m)$, it is always possible to reconstruct m having arbitrary time available.

Theorem

If Π is secure against passive attacks, then it is CPA-secure.

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There are no asymmetric ciphers that are secure in a perfect sense.

Insecurity of Deterministic Encryption

- We know that every *passive* adversary, having access to pk, is actually also *active*.
  - Therefore, many properties that we have seen for the symmetrical case and for CPA attacks hold also in this case.

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Theorem

No public key scheme in which Enc is deterministic can be secure with respect to PubK^{eav}.

- Historically, a large number of public-key encryption schemes are such that *Enc* is deterministic.
  - This had (and still has) disastrous consequences.

On Multiple Encryptions

- Similarly to what we have seen in the symmetrical case, we can talk about security for *multiple encryptions*.
  - We just define a new experiment PubK^{mult} in which the adversary outputs not a pair of messages (m_0, m_1) but a pair of tuple of messages $(\mathbf{m}_0, \mathbf{m}_1)$ where $\mathbf{m}_0 = (m_0^1, \ldots, m_0^t)$, $\mathbf{m}_1 = (m_1^1, \ldots, m_1^t)$, and $|m_0^j| = |m_1^j|$.

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- As usual, a public-key encryption scheme Π is said to be secure with respect to multiple encodings iff for every PPT A there exists ε with

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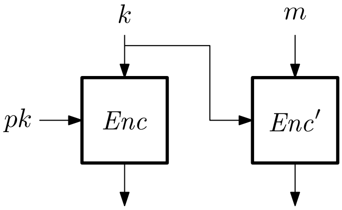
Theorem

If an encryption scheme Π is secure with respect to PubK^{eav}, then it is secure with respect to PubK^{mult}.

Hybrid Encryption

- We have already mentioned that public-key encryption schemes are *less performing* than private-key ones.
- With hybrid encryption we simply try to put together *the positive aspects* of public-key and private-key encryptions.
- Given $\Pi = (Gen, Enc, Dec)$ with a public key and $\Pi' = (Gen', Enc', Dec')$ with a private key, we can construct Π^{Hy} in which the encryption is more or less as follows:



Hybrid Encryption

- When defining the hybrid encryption, we will make the assumption that Gen' returns a random string in $\{0,1\}^n$ and Π includes $\{0,1\}^n$ in the message space.
- Formally, the scheme Π^{Hy} is defined from Π and Π', as follows :

$Gen^{Hy}(1^n)$:
Result: $Gen(1^n)$

$Enc^{Hy}(pk, m)$:
$k \leftarrow \{0,1\}^n$;
$c \leftarrow Enc_{pk}(k)$;
$d \leftarrow Enc_k(m)$;
Result: (c, d)

$Dec^{Hy}(sk, (c, d))$:
$k \leftarrow Dec_{sk}(c)$;
$m \leftarrow Dec_k(d)$;
Result: m

Theorem

If Π is CPA-secure and Π' has indistinguishable encryptions, then Π^{Hy} is secure.

Hybrid Encryption: Why?

1. **Encryption Time.**
   - Suppose that the encryption of the key takes time α and that the encryption of the message takes time β for each bit.
   - Therefore, the average time taken by Enc^{Hy} for each bit will be , for messages t long, equal to $TIME(t) = (\alpha + \beta t)/t$.
   - Note that
   $$\lim_{t \to \infty} \frac{\alpha + \beta t}{t} = \beta$$

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2. **Ciphertexts' Length**
   - A very similar reasoning to that made for the encryption time can be made for the length of the ciphertexts.
   - As $|m|$ increases, the quantity $|c|$ stays constant, while there are private-key encryption schemes such that $|d| = |m| + n$.
   - Therefore, as $|m|$ increases, the length of (c, d) is linear.

The RSA Encryption Scheme

- We have considered the security of public-key encryption schemes, giving interesting results.
- However, we have not dealt with any concrete encryption scheme.
  - Hybrid Encryption cannot be used in this sense, as it requires the existence of a public-key encryption scheme to start from.
- We will first present a scheme call **Textbook RSA**:

$Gen(1^n)$:
$(N, e, d) \leftarrow \mathsf{GenRSA}(1^n)$;
Result: $((N, e), (N, d))$

$Enc(((N, e), m)$:
$c \leftarrow m^e \mod N$;
Result: c

$Dec((N, d), c)$:
$m \leftarrow c^d \mod N$;
Result: m

- The correctness of the scheme follows from the fact that if the pair $((N, e), (N, d))$ is obtained from Gen, then f_d is the inverse of f_e.

Textbook RSA: Problems

- First of all, it should be noted that Textbook RSA is **insecure** with respect to our definition.
  - To realise this, it is sufficient to observe that Enc is deterministic!
  - However, a very weak security notion holds: given the public key (N, e) and $c = m^e \mod N$, it is not possible to determine the message m in its entirety, at least when the RSA Assumption holds.

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- From a theoretical point of view, it would be necessary to guarantee that $m \in \mathbb{Z}_N^*$. Also when $m \in \mathbb{Z}_N$, encryption and decryption work.
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  - It can also be shown that $\phi(N)/N$, considered as a function of n, is in the form $1 - \varepsilon(n)$.
- In the literature, there are many examples of attacks against Textbook RSA.
  - If, as is often the case, e is chosen as a *fixed* and very *small* value (e.g. 3), then m is the cube root of m (modulo N), which can be easily computed.
  - The complexity of the brute force attack can be reduced from N to $\sqrt{N}$.

Padded RSA

- Is there any way to make RSA secure?

Padded RSA

- Is there any way to make RSA secure?
- The answer is yes. Consider the following diagram, called **Padded RSA**:

Gen(1^n):
$(N, e, d) \leftarrow$ $\mathsf{GenRSA}(1^n)$;
Result: $((N, e), (N, d))$

Enc$(((N, e), m)$:
$r \leftarrow \{0,1\}^{|N|-\ell(n)-1}$;
$c \leftarrow (r||m)^e \bmod N$;
Result: c

Dec$((N, d), c)$:
$m \leftarrow LSB(c^d \bmod N, \ell(n))$;
Result: m

where ℓ is a function such that $|m| \leq \ell(n) \leq 2n - 2$ and LSB returns the least significant bits.

- It is necessary to choose $\ell(n)$ sufficiently small, less than linear.

Theorem

If the RSA Assumption holds with respect to GenRSA *and if* $\ell(n) = O(\lg n)$*, then Padded RSA is secure with respect to passive attacks.*

The Elgamal Encryption Scheme

- In addition to RSA, there is another secure encryption scheme based on the assumptions we talked about few lessons ago.
- In particular, there is one encryption scheme, due to Elgamal, which can be proved secure from the DDH Assumption.
- The observation to start from is that, when fixed two elements $m, c \in \mathbb{G}$ of a finite group, the probability that a random element $k \in \mathbb{G}$ is such that $m \cdot k = g$ is equal to $\frac{1}{|\mathbb{G}|}$.
  - All this can be easily proved by observing that

$$Pr(m \cdot k = c) = Pr(k = m^{-1} \cdot c) = \frac{1}{|\mathbb{G}|}$$

- In other words, we are in a situation similar to the one we saw in OTP.

The Elgamal Encryption Scheme

- Formally, the Elgamal scheme is defined as follows:

$Gen(1^n)$:
$(\mathbb{G}, q, g) \leftarrow \mathsf{GenCG}(1^n)$;
$x \leftarrow \mathbb{Z}_q$;
$sk \leftarrow (\mathbb{G}, q, g, x)$;
$pk \leftarrow (\mathbb{G}, q, g, g^x)$;
Result: (sk, pk)

$Enc((\mathbb{G}, q, g, h), m)$:
$y \leftarrow \mathbb{Z}_q$;
Result: $(g^y, h^y \cdot m)$

$Dec((\mathbb{G}, q, g, x), (c, d))$:
Result: d/c_1^x

- The correctness of the scheme is easy to prove:

$$\frac{d}{c_1^x} = \frac{h^y \cdot m}{g^{yx}} = \frac{(g^x)^y \cdot m}{g^{xy}} = m$$

Theorem

If Assumption DDH holds with respect to GenCG, *then the Elgamal scheme is secure.*