CV LAB
Computer Vision Laboratory

Image Processing and Computer Vision

Image Formation and Acquisition – The Fourier Transform for Digital Images

Continuous 1D Signals

$$i(x) \rightarrow \Im\left[i(x)\right] = \int_{-\infty}^{+\infty} i(x)e^{-j2\pi fx}dx = I(f)$$

$\left|I(f)\right| \rightarrow$ **Amplitude Spectrum** $\quad \tan^{-1}\left[\frac{I\left(I(f)\right)}{R\left(I(f)\right)}\right] \rightarrow$ **Phase Spectrum**

$$i(x) = \Im^{-1}\left[I(f)\right] = \int_{-\infty}^{+\infty} I(f)e^{j2\pi fx}df$$

The signal can be represented as an infinite weighted sum of harmonic functions at different frequencies:

$$e^{j2\pi fx} = \cos(2\pi fx) + j\sin(2\pi fx)$$

Continuous 2D Signals

$$i(x,y) \rightarrow \Im\left[i(x,y)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} i(x,y)e^{-j2\pi(f_x x + f_y y)}dxdy = I(f_x, f_y)$$

$$i(x,y) = \Im^{-1}\left[I(f_x, f_y)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I(f_x, f_y)e^{j2\pi(f_x x + f_y y)}df_x df_y$$

The signal can be represented as an infinite weighted sum of 2D harmonic functions at different horizontal-vertical frequency pairs:

$$e^{j2\pi\left(f_x x + f_y y\right)} = \cos\left(2\pi\left(f_x x + f_y y\right)\right) + j\sin\left(2\pi\left(f_x x + f_y y\right)\right)$$

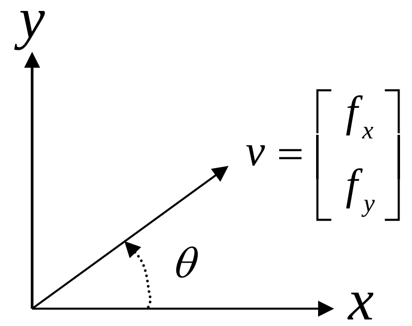
2D Harmonic Functions (1)

$e^{j2\pi(f_x x + f_y y)} \rightarrow$ **Periodic function varying along a direction of the 2D plane *(x,y)***

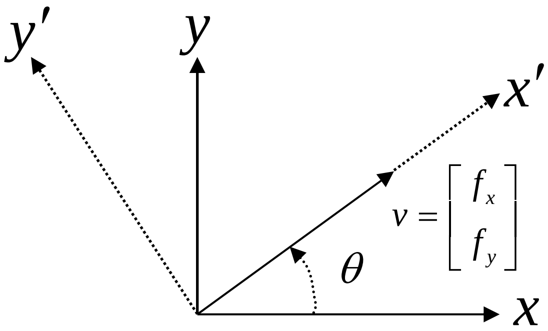
The function varies with frequency: $\tilde{f} = \sqrt{f_x^{\,2} + f_y^{\,2}}$

along direction:

$$\theta = \tan^{-1}\left(\frac{f_y}{f_x}\right)$$



2D Harmonic Functions (2)



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\cos\theta = \frac{f_x}{\sqrt{f_x^2 + f_y^2}}$$

$$\sin\theta = \frac{f_y}{\sqrt{f_x^2 + f_y^2}}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{f_x^2 + f_y^2}} \begin{pmatrix} f_x & -f_y \\ f_y & f_x \end{pmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

2D Harmonic Functions (3)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{f_x^2 + f_y^2}} \begin{pmatrix} f_x & -f_y \\ f_y & f_x \end{pmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \Rightarrow \begin{cases} x = \dfrac{1}{\sqrt{f_x^2 + f_y^2}} \left(f_x x' - f_y y' \right) \\ y = \dfrac{1}{\sqrt{f_x^2 + f_y^2}} \left(f_y x' + f_x y' \right) \end{cases}$$

$$e^{j2\pi\left(f_x x + f_y y \right)} = e^{j2\pi \frac{1}{\sqrt{f_x^2 + f_y^2}} \left(f_x \left(f_x x' - f_y y' \right) + f_y \left(f_y x' + f_x y' \right) \right)}$$

$$e^{j2\pi \frac{1}{\sqrt{f_x^2 + f_y^2}} \left(f_x^2 + f_y^2 \right) x'} = e^{j2\pi \left(\sqrt{f_x^2 + f_y^2} \right) x'} \Rightarrow$$

Harmonic varying along direction θ with frequency $\tilde{f}$

Fourier Transform and Convolution

$$i(x,y) * h(x,y) = \Im^{-1}\left[I(f_x, f_y) \cdot H(f_x, f_y) \right]$$

Where $H(f_x, f_y)$ **denotes the Fourier Transform of the impulse response of the LSI operator.**

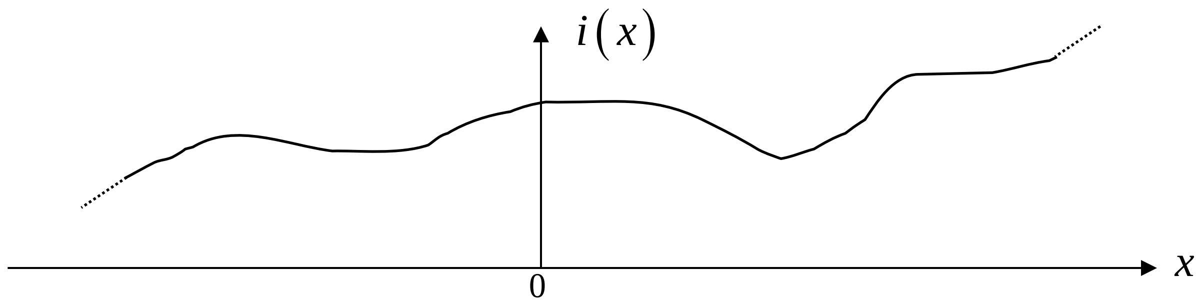
As for *correlation* :

$$h(x,y) \circ i(x,y) = i(x,y) * h(-x,-y)$$

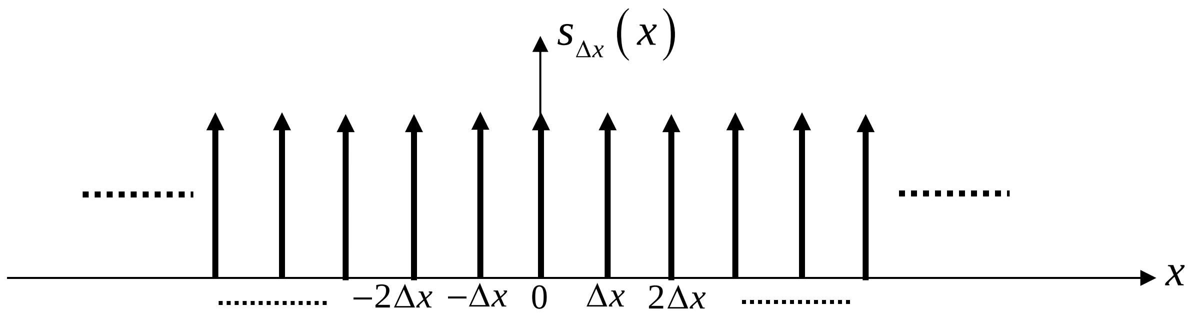
If $h(x,y)$ **is real-valued** $\Rightarrow$ $\Im\left[h(-x,-y)\right] = H^*(f_x, f_y)$

$$h(x,y) \circ i(x,y) = i(x,y) * h(-x,-y) = \Im^{-1}\left[I(f_x, f_y) \cdot H^*(f_x, f_y) \right]$$

Discrete 1D Signals (1)

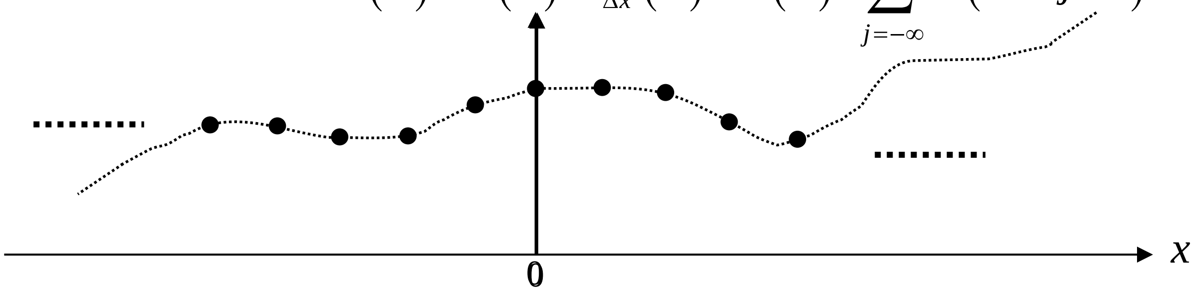


Continuous Signal



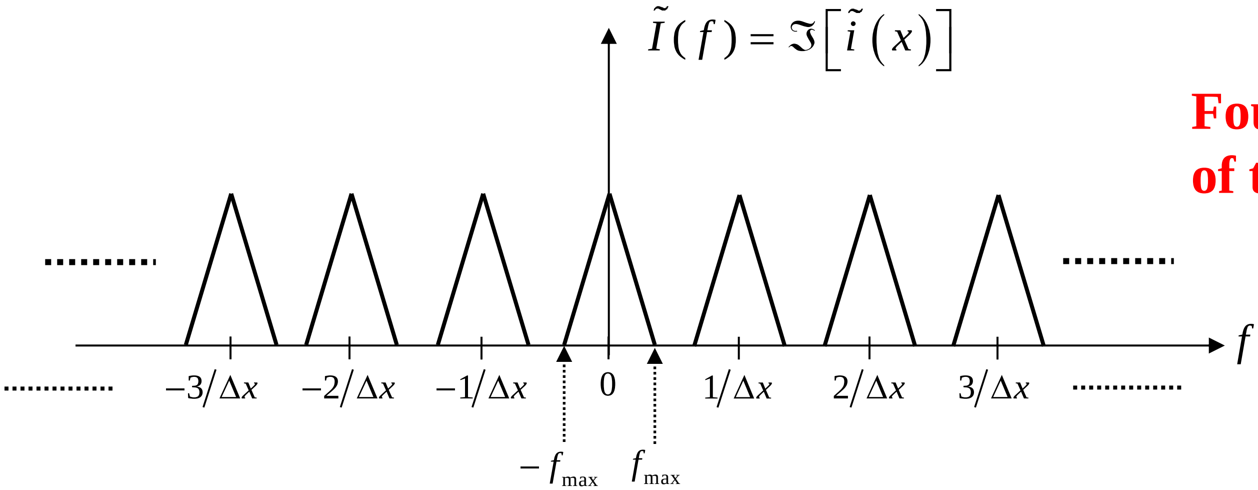
Sampling Impulses

$$\tilde{i}(x) = i(x) \cdot s_{\Delta x}(x) = i(x) \cdot \sum_{j=-\infty}^{\infty} \delta(x - j\Delta x)$$



Sampled Signal

Discrete 1D Signals (2)



Fourier Transform of the sampled signal

$$\tilde{I}(f) = \sum_{j=-\infty}^{+\infty} I\left(f - \frac{j}{\Delta x}\right)$$

$$\frac{1}{\Delta x} \geq 2 f_{max}$$

the original signal can be reconstructed from its samples without any loss of information (*Sampling Theorem*).

DFT (1)

The formula depicted in the previous slide does not allow computing the transform of the sampled signal as a function of its samples.

The DFT (Discrete Fourier Transform) allows computing *M* samples of the Fourier Transform of the sampled signal from M signal samples:

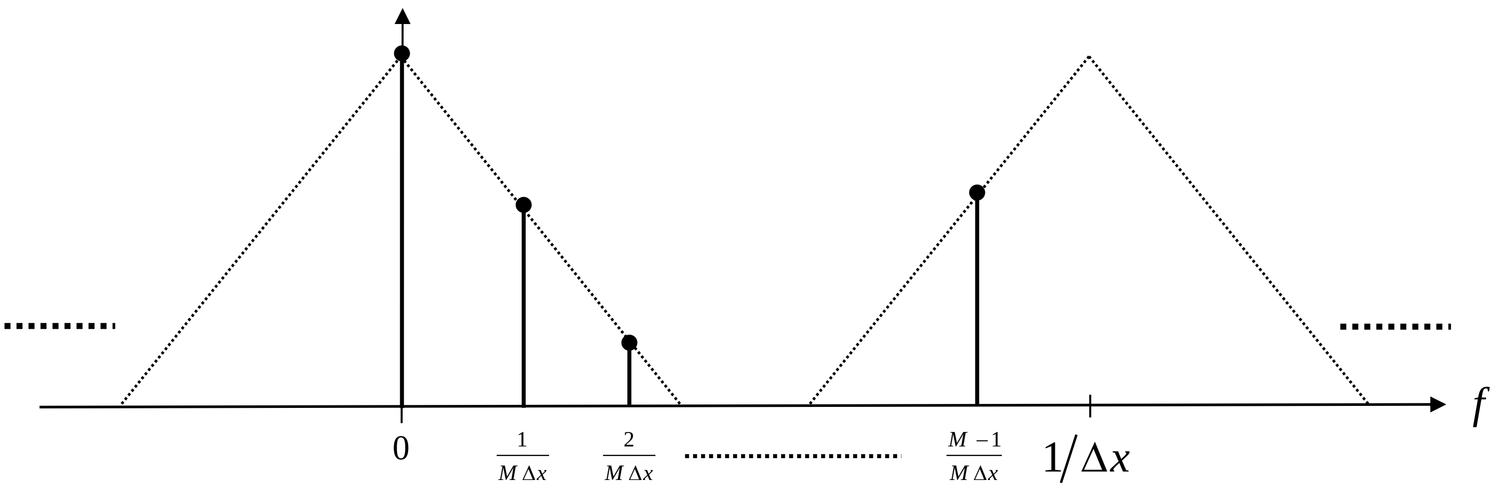
$$\tilde{I}(u) = \sum_{m=0}^{M-1} i(m) e^{-j2\pi\frac{um}{M}}, \quad u = 0 \ldots M-1$$

The inverse transform allows reconstructing signal samples from DFT samples:

$$i(m) = \frac{1}{M} \sum_{u=0}^{M-1} \tilde{I}(u) e^{j2\pi\frac{um}{M}}, \quad m = 0 \ldots M-1$$

DFT (2)

The M transform samples computed by the DFT consist in sampling the frequency interval $[0, 1/\Delta x]$ according to a step equal to $1/M\,\Delta x$

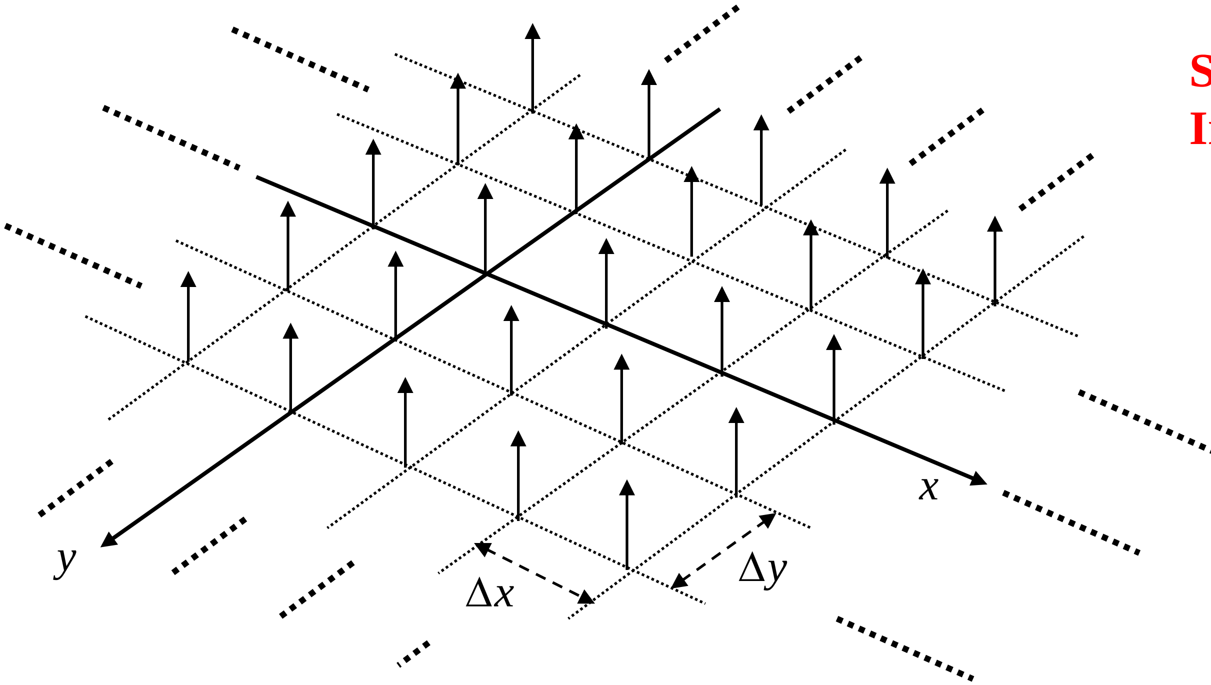


$$\tilde{I}(u) = \tilde{I}\left(f = \frac{u}{M\,\Delta x}\right)$$

$$\begin{cases} u = 0 \rightarrow \tilde{I}(0) \\ u = 1 \rightarrow \tilde{I}\left(\frac{1}{M\,\Delta x}\right) \\ \vdots \\ u = \frac{M}{2} \rightarrow \tilde{I}\left(\frac{M}{2M\,\Delta x}\right) = \tilde{I}\left(\frac{1}{2\Delta x}\right) \\ \vdots \\ u = M-1 \rightarrow \tilde{I}\left(\frac{M-1}{M\,\Delta x}\right) \end{cases}$$

$$u > \frac{M}{2} \rightarrow \tilde{I}(u) = \tilde{I}\left(u - \frac{1}{\Delta x}\right) = \tilde{I}\left(\frac{u}{M\Delta x} - \frac{1}{\Delta x}\right)$$

Discrete 2D Signals (1)



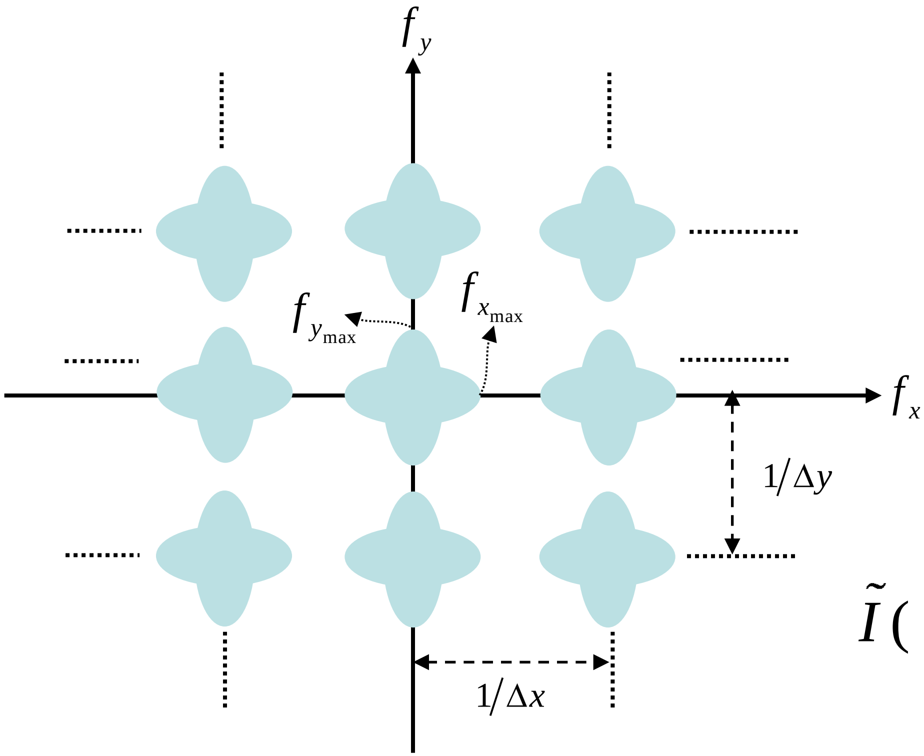
Sampling Impulses

$$\tilde{i}(x,y) = i(x,y) \cdot \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(x - j\Delta x, y - k\Delta y)$$

Sampled Signal

$$\tilde{i}(x,y) = i(x,y) \cdot s_{\Delta x,\Delta y}(x,y)$$

Discrete 2D Signals (2)



Fourier Transform of the sampled signal

$$\tilde{I}(f) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} I\left(f_x - \frac{j}{\Delta x}, f_y - \frac{k}{\Delta y} \right)$$

$$\begin{cases} \dfrac{1}{\Delta x} \geq 2 f_{x_{\max}} \\ \dfrac{1}{\Delta y} \geq 2 f_{y_{\max}} \end{cases}$$

the original signal can be reconstructed from its samples without any loss of information (*Sampling Theorem*).

2D DFT (1)

As already discussed for 1D signals, the previous formula does not allow the Fourier Transform to be computed from signal samples.

The 2D DFT allows computing $M \cdot N$ **Fourier Transform samples from** $M \cdot N$ **signal samples:**

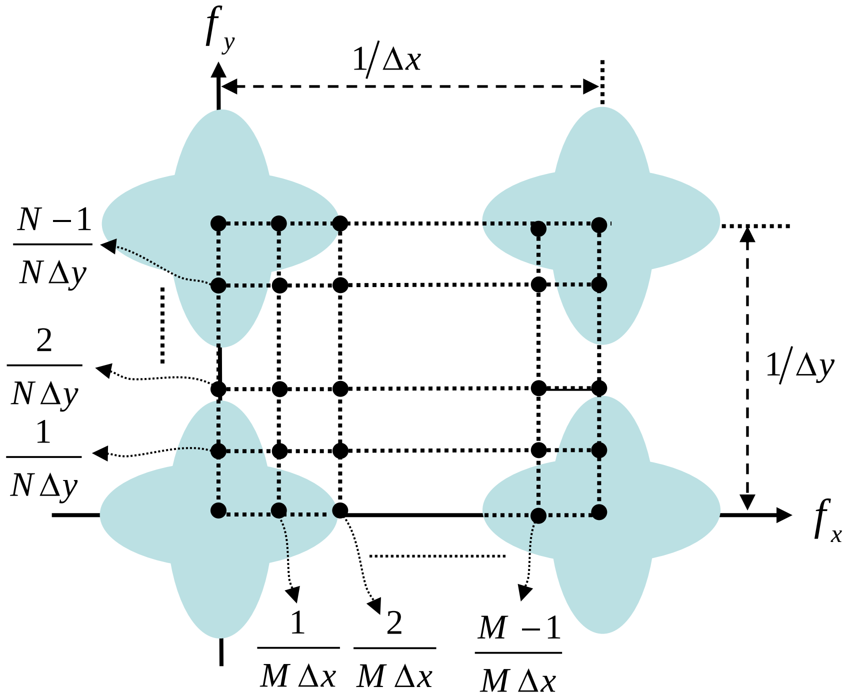
$$\tilde{I}(u,v) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} i(m,n) e^{-j2\pi\left(\frac{um}{M}+\frac{vn}{N}\right)}, \quad u = 0 \ldots M-1, \ v = 0 \ldots N-1$$

The inverse transform allows to reconstruct signal samples from Fourier Transform samples :

$$i(m,n) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \tilde{I}(u,v) e^{j2\pi\left(\frac{um}{M}+\frac{vn}{N}\right)}, \quad m = 0 \ldots M-1, \ n = 0 \ldots N-1$$

2D DFT (2)

The $M \cdot N$ **DFT samples consist in sampling the 2D frequency interval** $f_x \in [0, 1/\Delta x],\ f_y \in [0, 1/\Delta y]$ **according to horizontal and vertical steps given by** $1/M\,\Delta x$ **and** $1/N\,\Delta y$ **respectively.**



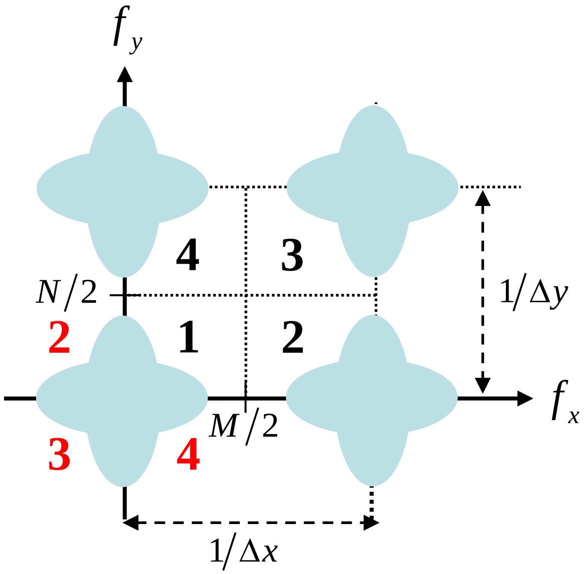
$$\tilde{I}(u,v) = \tilde{I}\left(f_x = \frac{u}{M\,\Delta x},\ f_y = \frac{v}{N\Delta y}\right)$$

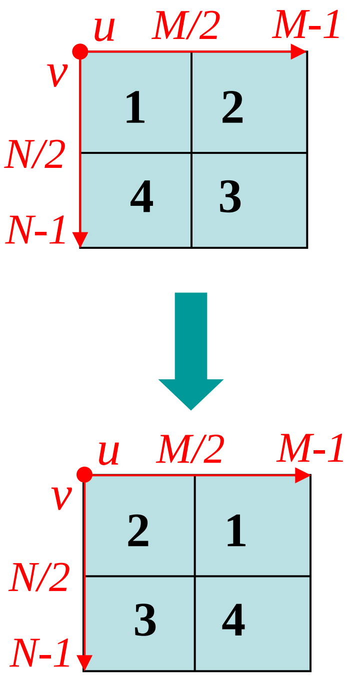
$$\begin{cases} u = 0, v = 0 \rightarrow \tilde{I}(0,0) \\ u = 1, v = 1 \rightarrow \tilde{I}\left(\frac{1}{M\,\Delta x}, \frac{1}{N\,\Delta y}\right) \\ \vdots \\ u = \frac{M}{2}, v = \frac{N}{2} \rightarrow \tilde{I}\left(\frac{M}{2M\,\Delta x}, \frac{N}{2N\,\Delta y}\right) = \tilde{I}\left(\frac{1}{2\Delta x}, \frac{1}{2\Delta y}\right) \\ \vdots \\ u = M-1, v = N-1 \rightarrow \tilde{I}\left(\frac{M-1}{M\,\Delta x}, \frac{N-1}{N\,\Delta y}\right) \end{cases}$$

$$u > \frac{M}{2}, v > \frac{N}{2} \rightarrow \tilde{I}(u,v) = \tilde{I}\left(u - \frac{1}{\Delta x}, v - \frac{1}{\Delta y}\right) = \tilde{I}\left(\frac{u}{M\Delta x} - \frac{1}{\Delta x}, \frac{v}{N\Delta y} - \frac{1}{\Delta y}\right)$$

2D DFT (3)

The 2D DFT is often displayed as an image. To ease interpretation, it is convenient to rearrange DFT samples so as to display the origin of the frequency plane at the centre of the image.





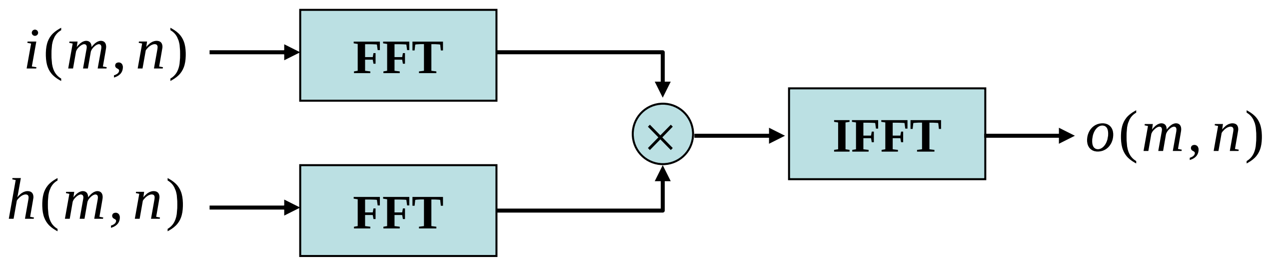
Applications (1)

- ***Fast Convolution/Correlation***

There exist very efficient algorithms (FFT: *Fast Fourier Transform*) to compute the DFT

$$(MN)^2 \rightarrow MN \cdot \log_2 MN$$

so that, sometimes, it may turn out faster to carry out convolution/correlation in the frequency rather than signal domain:



This is typically the case when the spatial filter has a large-size kernel. It is worth pointing out, though, that convolution/correlation in the frequency domain requires floating-point calculations.

Applications (2)

• ***Filtering in the frequency domain***

An LSI operator can dually be defined through its frequency response, *H(u,v)*. Accordingly, it is possible to define diverse kinds of image filters (e.g. *low-pass*, *high-pass*…). Frequency domain design is particularly convenient in order to define selective filters operating on certain frequencies or frequency ranges, such as e.g. *notch filters* for suppression of periodic interference.

• ***Restoration***

Let us assume the available, corrupted image, $i_P(m,n)$, to be generated from the ideal image, *i(m,n)*, by an LSI nuisance process, *P*, whose impulse response is denoted as *P(u,v)* :

$$I_p(u,v) = \Im\left[i_p(m,n)\right],\ I(u,v) = \Im\left[i(m,n)\right] \rightarrow i_P(m,n) = \Im^{-1}\left[I_P(u,v)\right] = \Im^{-1}\left[I(u.v)\cdot P(u,v)\right]$$

If *P(u,v)* can be determined, either analytically or empirically:

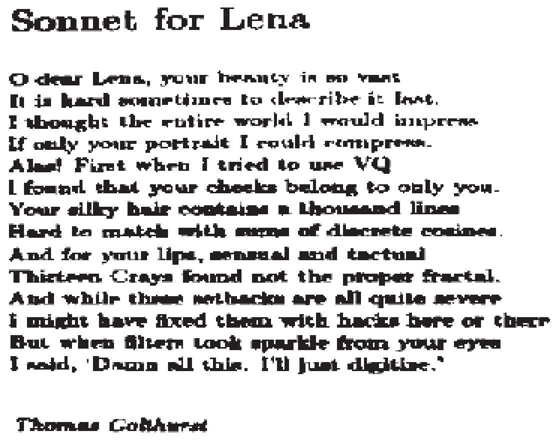
$$\Im^{-1}\left[I_p(u,v)\cdot 1/P(u,v)\right] = \Im^{-1}\left[\left(I(u,v)\cdot P(u,v)\right)\cdot 1/P(u,v)\right] = \Im^{-1}\left[I(u,v)\right] = i(m,n)$$

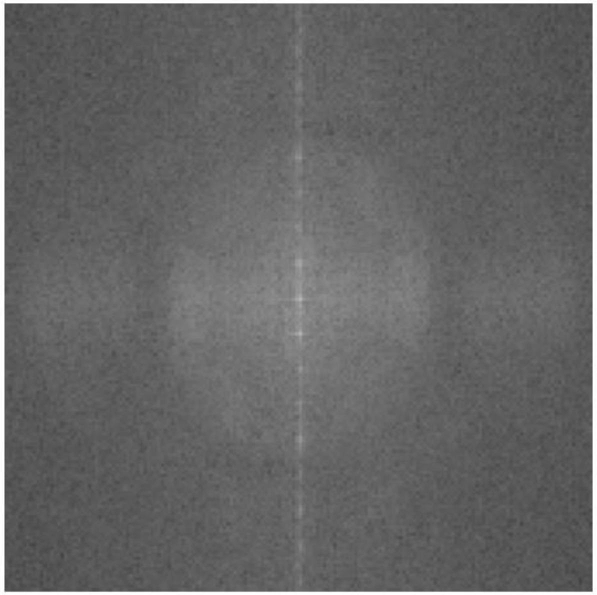
e.g. : restoration of an image corrupted by *blur* caused by translational motion of the camera during exposure time.

Applications (3)

- *Compensation of unknown rotation in images featuring periodic patterns*

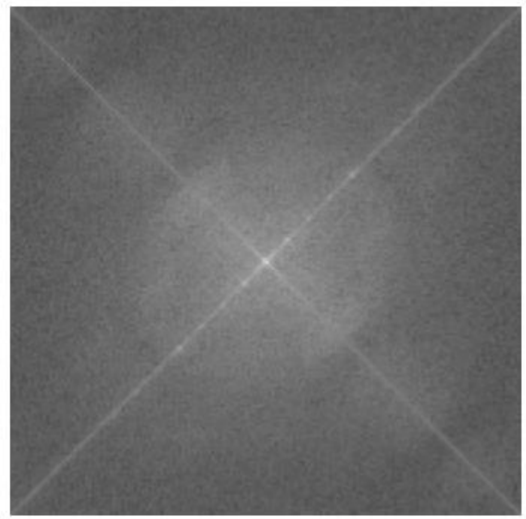
From left to right: a binary text image, the corresponding DFT (amplitude spectrum), the main harmonics highlighted by thresholding the amplitude spectrum.





Applications (4)

Upon rotation of the original image:



the main harmonics appear rotated alike. Therefore, analysis of the direction of the main harmonics allows for estimation (and thus compensation) of the unknown image rotation.

Main References

1) R. Gonzales, R. Woods, “Digital Image Processing – Third Edition”, Pearson Prentice-Hall, 2008.

2) R. Fisher, S. Perkins, A. Walker, E. Wolfart, “Hypermedia Image Processing Reference”, Wiley, 1996 (http://homepages.inf.ed.ac.uk/rbf/HIPR2/)

3) E. O. Brigham, “The Fast Fourier Transform and its applications”, Prentice-Hall, 1988.