dim

$$\underbrace{A_{i_0,i_1,\dots,i_n}}_{\text{traiettoria}} := \bigcap_{j=0,\dots,n} (X_j = i_j)$$

$$B_{i_n} := (X_n = i_n)$$

Vogliamo provare che (M) è equival. a

$$(*) \quad \boxed{\begin{gathered} \mathbb{P}(X_{n+1}=j \mid A_{i_0,\dots,i_n}) = \mathbb{P}(X_{n+1}=j \mid B_{i_n}) \\ \forall\, n \in \mathbb{N}_0, \quad \forall\, i_0,i_1,\dots,i_n,j \ : \ \mathbb{P}(A_{i_0,\dots,i_n}) > 0 . \end{gathered}}$$

- $\forall\, A_{i_0,\dots,i_n}$ t.c. $\mathbb{P}(A_{i_0,\dots,i_n}) > 0$

e $\forall\, \omega \in A_{i_0,\dots,i_n}$ valgono:

$$\mathbb{P}(X_{n+1}=j \mid X_n)(\omega) \overset{!}{\underset{\omega \in B_{i_n} \supset A_{i_0,\dots,i_n}}{=}} \mathbb{P}(X_{n+1}=j \mid B_{i_n})$$

$$(M) \Rightarrow \ \| \ \text{q.c.} \qquad\qquad \| \ \Leftarrow (*)$$

$$\mathbb{P}(X_{n+1}=j \mid \underbrace{\mathcal{F}^X_n}_{\sigma(X_0,X_1,\dots,X_n)})(\omega) \overset{\omega \in A_{i_0,\dots,i_n}}{=} \mathbb{P}(X_{n+1}=j \mid A_{i_0,\dots,i_n})$$

(M) $\Rightarrow$ (*) (<u>SEMPRE</u>, non solo nel caso discreto)

$(*) \Rightarrow (M)$: gli eventi $A_{i_0, \dots, i_n}$ formano una partizione numerabile di Ω.

- Data X catena di Markov discreta e omogenea

$$p_{ij} := \mathbb{P}(X_1 = j \mid X_0 = i)$$
$$= \mathbb{P}(X_{n+1} = j \mid X_n = i)$$

(quando $\exists$ almeno per un n)

la "matrice" $P = (p_{ij})_{i,j \in E}$ si chiama

MATRICE DI TRANSIZIONE

OSS. $(X_{n+1} = j)$ formano una partizione

$$\Downarrow$$

$$\sum_{j \in E} p_{ij} = 1$$

- OPERAZIONI :

- $A = (a_{ij})_{i,j \in E}$, $B = (b_{ij})_{i,j \in E}$
  - $A + B := (a_{ij} + b_{ij})_{i,j \in E}$
  - $A \cdot B := \left(\sum_{k \in E} a_{ik} \cdot b_{kj} \right)_{i,j \in E}$

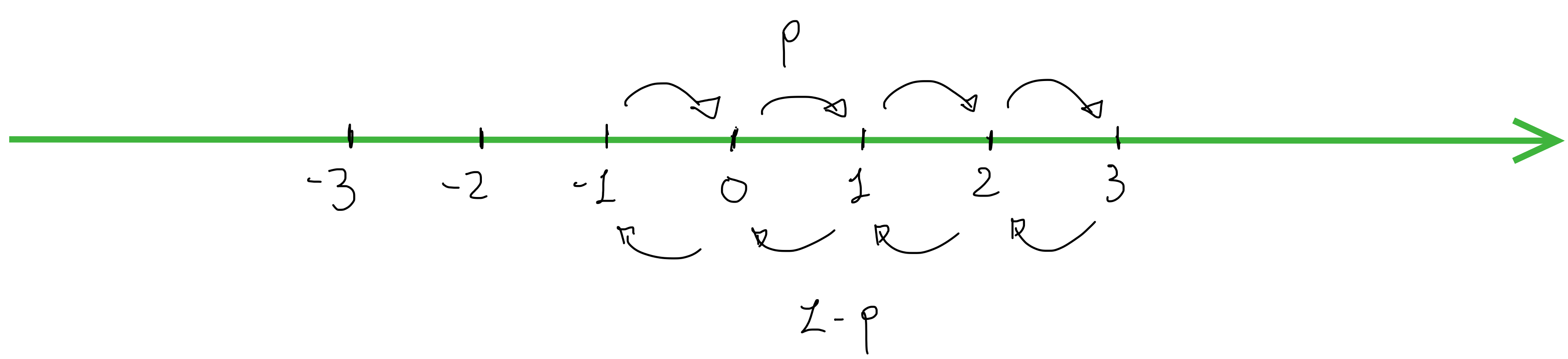
- $X = (x_i)_{i \in E}$ (vettore colonna)

$X^T \longrightarrow$ vettore riga

- $A \cdot X := \left(\sum_{k \in E} a_{ik} \cdot x_k \right)_{i \in E}$

- $X^T \cdot A := \left(\sum_{k \in E} x_k \cdot a_{ki} \right)_{i \in E}$

esempio (RANDOM-WALK 1-DIM)



- Sia X_0 v.a. su $\mathbb{Z} = E$
- Sia $(Z_n)_{n \in \mathbb{N}}$ succ. di v.a. i.i.d t.c.

$$Z_n \sim p\, \delta_1 + (1-p)\, \delta_{-1} \;, \qquad p \in \,]0,1[$$

- Poniamo:

$$X_{n+1} = X_n + Z_{n+1} \;, \qquad n \in \mathbb{N}_0$$

$$\longrightarrow X_n = X_0 + \sum_{j=1}^{n} Z_j$$

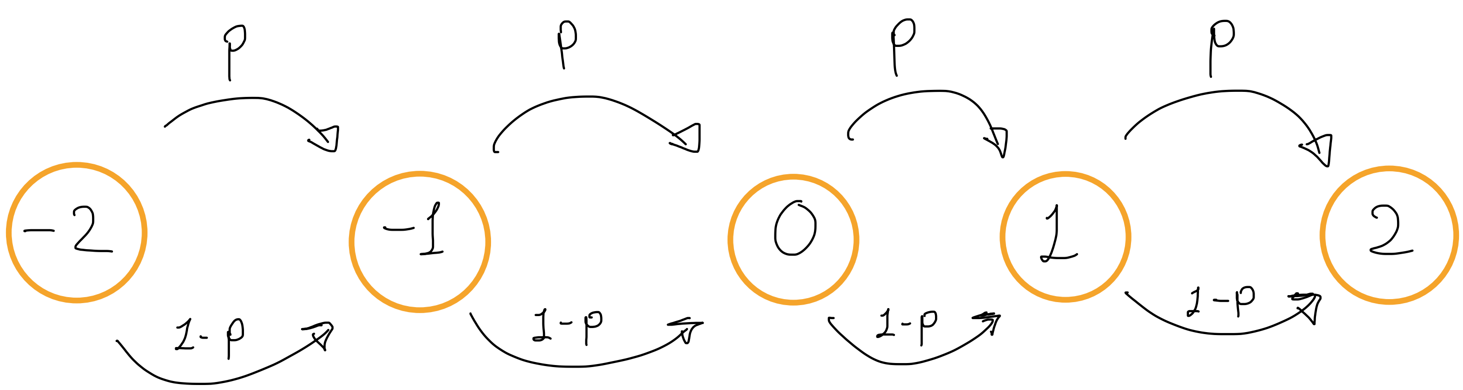
(M): già vista.

$$p_{ij} = \begin{cases} p & \text{se } j = i+1 \\ 1-p & \text{se } j = i-1 \\ 0 & \text{altrimenti} \end{cases}$$

$$\mathbb{P}(\underbrace{X_{n+1}}_{= X_n + Z_{n+1}} = i+1 \mid X_n = i) = \mathbb{P}(\cancel{i} + Z_{n+1} = \cancel{i} + 1 \mid X_n = i)$$

$$\underset{\substack{\uparrow \\ Z_{n+1} \text{ indip. da } X_n}}{=} \mathbb{P}(Z_{n+1} = 1) = p$$

- GRAFO:



- $\nu := \mu_{X_0}$

OSS. La legge di X catena di Markov discreta è data da ν e P.

$$\boxed{\mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = \nu(\{i_0\}) \cdot p_{i_0 i_1} \cdot p_{i_1 i_2} \cdot \ldots \cdot p_{i_{n-1} i_n}}$$

es:

$$\mathbb{P}(X_0 = i_0, X_1 = i_1) = \mathbb{P}(X_1 = i_1 \mid X_0 = i_0) \cdot \mathbb{P}(X_0 = i_0)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel \qquad\qquad\qquad\quad \shortparallel$$

$p_{i_0 i_1}$ $\nu(\{i_0\})$

$$\mathbb{P}(X_0 = i_0,\ X_1 = i_1,\ X_2 = i_2)$$

$$\parallel$$

$$\underbrace{\mathbb{P}(X_2 = i_2 \mid X_1 = i_1,\ X_0 = i_0)}_{\parallel} \cdot \underbrace{\mathbb{P}(X_1 = i_1,\ X_0 = i_0)}_{\parallel}$$

$$\mathbb{P}(X_2 = i_2 \mid X_1 = i_1) \qquad p_{i_0 i_1} \cdot \nu(\{i_0\})$$

$$\parallel$$

$$p_{i_1 i_2}$$

- $\nu_n := \mu_{X_n}$

$$\nu_{n+1}(\{j\}) = \mathbb{P}(X_{n+1} = j)$$

$$= \sum_{i \in E} \mathbb{P}(X_{n+1} = j,\ X_n = i)$$

$$= \sum_{\substack{i \in E: \\ \mathbb{P}(X_n = i) > 0}} \underbrace{\mathbb{P}(X_{n+1} = j \mid X_n = i)} \cdot \underbrace{\mathbb{P}(X_n = i)}$$

$$= \sum_{i \in E} p_{ij} \cdot \nu(\{i\})$$

$$\Rightarrow \quad \gamma_{n+1}^T = \gamma_n^T \cdot P$$

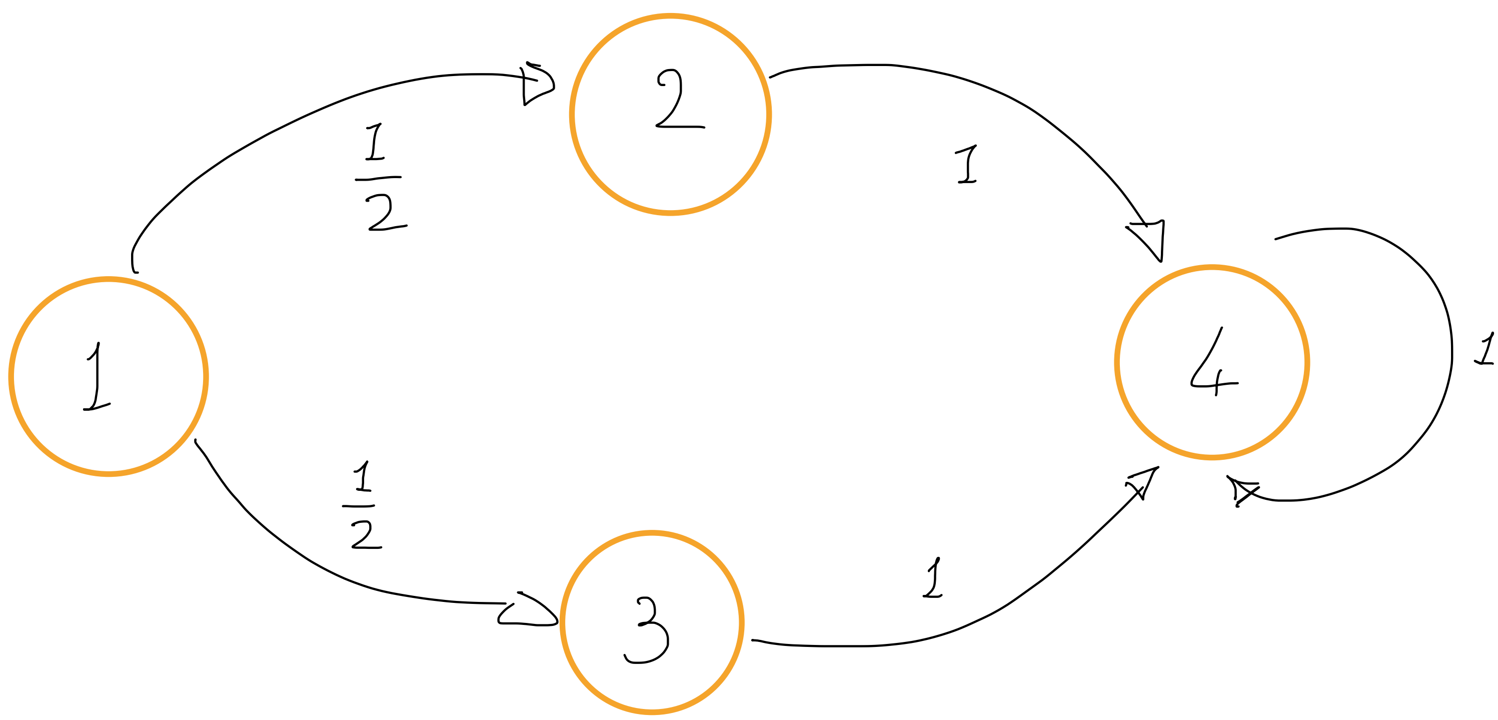
$\times$ induz.

$$\Rightarrow \quad \gamma_n^T = \gamma_0^T \cdot \underbrace{P^n}_{} \quad , \quad n \in \mathbb{N}$$

↳ "matrice" potenza

es.

$$E = \{1, 2, 3, 4\} , \quad \gamma_0(\{j\}) = \frac{1}{4} \quad \forall j = 1, \dots, 4$$



$$\mathbb{P}(X_1 = 2) = \mathbb{P}(X_1 = 2, X_0 = 1) = p_{12} \cdot \gamma_0(\{1\})$$

$$= \frac{1}{2} \cdot \frac{1}{4} > 0$$

$$\overset{\shortparallel}{\sum_{i=1}^{4} \mathbb{P}(X_1 = 2 , X_0 = i)}$$

es:

$$\mathbb{P}(X_4 = j \mid X_1 = 2) \quad \nexists$$

$$\shortparallel$$

$$p_{24}$$

$$\mathbb{P}(X_2 = 2) = 0 = \mathbb{P}(X_n = 2) \quad \forall\ n \geq 2$$

es: $\mathbb{P}(X_5 = j \mid X_2 = 2)$ $\not\exists$

oss. $A := (X_{n+1} = j_1) \cap (X_{n+2} = j_2) \cap \dots \cap (X_{n+k} = j_{n+k})$ → futuro

$B := (X_{n-1} = i_{n-1}) \cap \dots \cap (X_1 = i_1) \cap (X_0 = i_0)$ → passato

$$(M) \Rightarrow \mathbb{P}(A \mid \underbrace{X_n = i_n}_{\text{valore presente}}, B) = \mathbb{P}(A \mid \underbrace{X_n = i_n}_{\text{valore presente}})$$

$$\Downarrow$$

$$\mathbb{P}(A \cap B \mid X_n = i_n) = \mathbb{P}(A \mid X_n = i_n) \cdot \underbrace{\mathbb{P}(B \mid X_n = i_n)}_{= \frac{\mathbb{P}(B \cap (X_n = i_n))}{\mathbb{P}(X_n = i_n)}}$$

$\Downarrow$

PASSATO E FUTURO SONO

INDIPENDENTI CONDIZIONATAM.

AL VALORE PRESENTE

teorema Sia $(Z_n)_{n \in \mathbb{N}}$ succ. di v.a i.i.d

a valori in $(G, \mathcal{G})$ spazio misurabile.

Sia $E \subset \mathbb{R}^d$ numerabile e

$f : E \times G \longrightarrow E \quad \in m(\mathcal{P}(E) \otimes \mathcal{G})$.

Allora, $\forall X_0$ v.a. a valori in E indip. da $(Z_n)_{n \in \mathbb{N}}$ il processo

$$X_{n+1} := f(X_n, Z_{n+1}), \qquad n \in \mathbb{N}_0$$

è una catena di Markov omogenea, e

$$p_{ij} = \mathbb{P}(f(i, \underset{\underset{Z_n}{\sim}}{Z_1}) = j), \qquad i,j \in E$$

es.

$$G = \{-1, 1\}, \qquad f(i, z) = i + z$$

$$\Downarrow$$

$$X_{n+1} = X_n + Z_n, \qquad Z_n \sim p\,\delta_1 + (1-p)\,\delta_{-1}$$

dim

$$\mathbb{P}(X_{n+1} \in A \mid \mathcal{F}_n^X)$$

$$\overset{\cdot\cdot}{=}$$

$$\mathbb{E}[\mathbb{1}_{(X_{n+1} \in A)} \mid \mathcal{F}_n^X]$$

$X_{n+1} = f(X_n, Z_{n+1})$

$$= \mathbb{E}[\mathbb{1}_A(f(\underbrace{X_n}_{m\mathcal{F}_n^X}, \underbrace{Z_{n+1}}_{\text{indip. da } \mathcal{F}_n^X})) \mid \mathcal{F}_n^X]$$

$$\begin{pmatrix}\text{Lemma}\\ \text{freezing}\end{pmatrix} = \mathbb{E}\left[\mathbb{1}_A \left(f(i, Z_{n+1}) \right) \right] \Big|_{i = X_n}$$

$$\|$$

$$\mathbb{P}\left(f(i, Z_{n+1}) \in A \right) \Big|_{i = X_n}$$

$\Downarrow$ (<u>funzione di X_n</u>)

(M) e

$$p_{ij} := \mathbb{P}(X_{n+1} = j \mid X_n = i) \underset{\uparrow}{=} \mathbb{P}\left(f(i, Z_{n+1}) = j \right)$$

prendendo $A = \{j\}$

$$\begin{pmatrix}Z_n \text{ identic.}\\ \text{distribuite}\end{pmatrix} \longrightarrow = \mathbb{P}\left(f(i, Z_1) = j \right)$$

#

oss. Se $(Z_n)_n$ sono solo indip., X come sopra è una catena di Markov discreta (in generale non homogenea)

es. (MODELLO DI MERCATO A TEMPO DISCRETO)

$$S_{n+1} = S_n (1 + \mu_{n+1}), \quad n \in \mathbb{N}_0$$

$= f(S_n, \mu_{n+1})$, con

$f(i,\mu) = i(1+\mu)$

- se $(\mu_n)_n$ sono indip. S catena di Markov
- se $(\mu_n)_n$ anche i.d. la catena è omogenea

$\to$ MODELLO BINOMIALE

- COSTRUZIONE CANONICA:

dati ν_0 e $P = (p_{ij})_{i,j \in E}$

assumiamo $E \leftrightarrow \mathbb{N}$

- $Z_n \sim \text{Unif}_{[0,1]}$, indip.
- $X_0 \sim \nu$ indip da $(Z_n)_n$
- $f : \mathbb{N} \times [0,1] \longrightarrow \mathbb{N}$

$$f(i,z) := \sum_{j \in \mathbb{N}} j \cdot \mathbb{1}_{A_{ji}}(z) \quad ,$$

$$A_{ji} := \left[\sum_{k=0}^{j-1} p_{ik} \, , \, \sum_{k=0}^{j} p_{i,k} \right] \subset [0,1]$$

Definendo $X_{n+1} = f(X_n, Z_{n+1})$, $n \in \mathbb{N}_0$

$X = (X_{n+1})$ catena di Markov omogenea

$$\mathbb{P}(X_1 = j \mid X_0 = i)$$

$$\parallel$$

$$\mathbb{P}(f(i, Z_1) = j) = \mathbb{P}(Z_1 \in A_{ji})$$

$$\boxed{Z_n \sim \text{Unif}_{[0,1]}} \longrightarrow \; = \text{Unif}_{[0,1]}(A_{ji})$$

$$= p_{ij}$$

ej. (Urna di Ehrenfest)

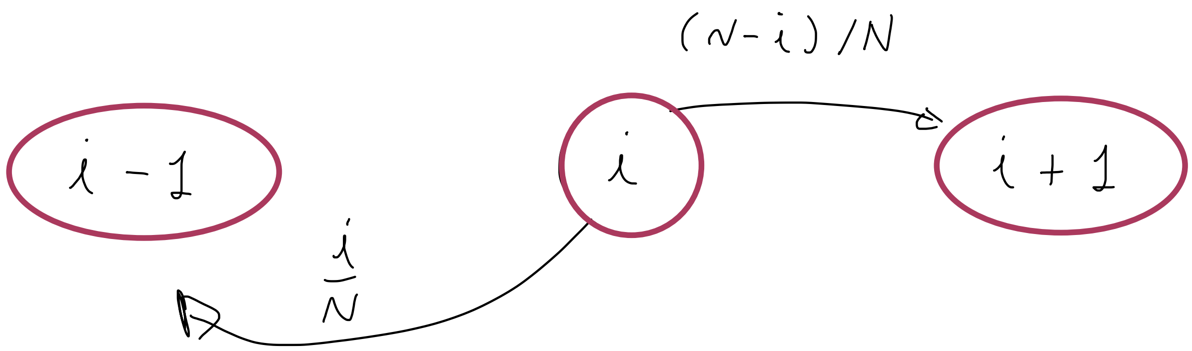
- N palline totali
- $X_n :=$ n. palline nell'urna A al tempo n

$$X_{n+1} = X_n + Z_{n+1}$$

$$Z_{n+1} \in \{-1, 1\}$$

$$\left.\begin{aligned} \mathbb{P}(Z_{n+1} = -1 \mid X_n = i) &= i/N \\ \mathbb{P}(Z_{n+1} = +1 \mid X_n = i) &= (N-i)/N \end{aligned}\right] \longrightarrow P$$

! $(Z_n)_n$ non sono indipendenti

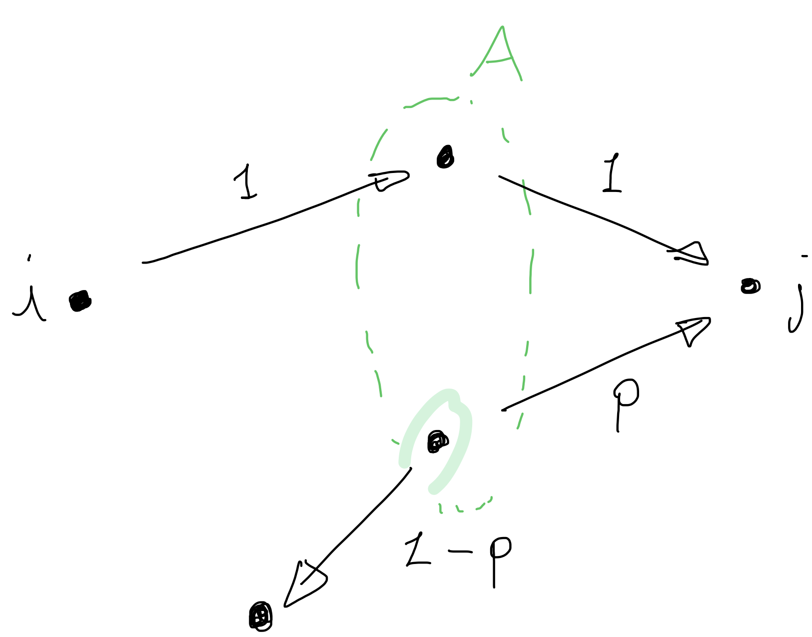


, $i = 1, \dots, N-1$

oss.

$$\mathbb{P}(X_{n+1} = j \mid X_n \in A_n, \dots, X_0 \in A_0) \neq \mathbb{P}(X_{n+1} = j \mid X_n \in A_n)$$

es.



$$\mathbb{P}(X_2 = j \mid X_1 \in A, X_0 = i) = 1$$

$$\mathbb{P}(X_2 = j \mid X_1 \in A) < 1$$

corollario | Data $\varphi \in m\mathcal{B}$ e X catena di Markov come nel teorema ($X_{n+1} = f(X_n, Z_{n+1})$, Z_n i.i.d), allora

$$\mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n^X] = \mathbb{E}[\varphi(f(x, Z_1))]\big|_{x = X_n}$$

dim |

$$\mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n^X] = \mathbb{E}[\varphi(X_{n+1}) \mid X_n]$$

$$= \sum_{i \in E} \mathbb{1}_{(X_n = i)}(X_n) \cdot \underbrace{\mathbb{E}[\varphi(X_{n+1}) \mid X_n = i]}$$

$$= \int_\Omega \varphi(X_{n+1}) \, \mathbb{P}|_{X_n = i}(d\omega)$$

$$= \int \varphi(x) \, \mu_{X_{n+1} \mid X_n = i}(dx)$$

$$\text{(Teorema)} \longrightarrow =$$

$$\mathbb{E}[\varphi(f(i, Z_{n+1}))]$$

$$= \mathbb{E}\left[\varphi\left(f(i, \underbrace{z_{n+1}}_{\sim Z_1})\right)\right]_{i=X_n} \qquad \#$$