- Vogliamo provare:

$$X^{(N)} \xrightarrow[N \to +\infty]{d} X \sim \mathcal{N}_{(r-\sigma^2/2)T,\ \sigma^2 T}$$

$$X^{(N)} := \log \frac{S_N^N}{S_0}$$

$S_N^N \to$ prezzo sottostante in $t_N = T$ secondo il mod. binomiale a N periodi

$$= \sum_{n=1}^{N} Y_n^{(N)} \qquad \text{dove:}$$

$$Y_n^{(N)} = \log(1 + \mu_n^{(N)})$$

$\to$ indip. identic. distrib.

$$Q_N\left(Y_n^{(N)} = \sigma\sqrt{\delta_N} + \alpha\,\delta_n\right) = q_N$$

$$Q_N\left(Y_n^{(N)} = -\sigma\sqrt{\delta_N} + \beta\,\delta_n\right) = 1 - q_N$$

$$q_N = \frac{1 + r_N - d_N}{u_N - d_N}, \qquad \delta_N = \frac{T}{N}$$

dim (lemma sui valori attesi)

- $$\mathbb{E}^{Q_N}\left[Y_1^{(N)}\right] = q_N\left(\sigma\sqrt{\delta_N} + \alpha\,\delta_N\right) + (1-q_N)\left(-\sigma\sqrt{\delta_N} + \beta\,\delta_N\right)$$

$$= \sigma\sqrt{\delta_N}\,(2q_N - 1) + \delta_N\left(\alpha\, q_N + \beta(1-q_N)\right)$$

$$\underbrace{\qquad\qquad}_{\searrow\, 0 \ \text{lemma}} \qquad\qquad \underbrace{\qquad\qquad\qquad\qquad}_{\substack{N \mapsto +\infty \\ \downarrow \\ \frac{\alpha+\beta}{2}}} \ (\text{lemma})$$

$$= \left(r - \frac{\sigma^2}{2} - \cancel{\frac{\alpha+\beta}{2}} \right) \delta_N + o(\delta_N) + \cancel{\frac{\alpha+\beta}{2}} \delta_N + o(\delta_N)$$

$$= \left(r - \frac{\sigma^2}{2} \right) \delta_N + o(\delta_N) \quad \text{per } N \mapsto +\infty .$$

- $$\mathbb{E}^{Q_N}\left[\left(Y_1^{(N)} \right)^2 \right] = \left(\log u_N + \log d_N \right) \mathbb{E}^{Q_N}\left[Y_1^{(N)} \right] - \left(\log u_N \right)\left(\log d_N \right)$$

$$= (\alpha + \beta) \delta_N \cdot \mathbb{E}^{Q_N}\left[Y_1^{(N)} \right] - \left(\sigma \sqrt{\delta_N} + \alpha \delta_N \right)\left(- \sigma \sqrt{\delta_N} + \beta \delta_N \right)$$

$$= \sigma^2 \delta_N + o(\delta_N) \quad \text{per } N \mapsto +\infty$$

- $$\mathbb{E}^{Q_N}\left[\left| Y_3^{(N)} \right|^3 \right] = q_N \left| \sigma \sqrt{\delta_N} + \alpha \delta_N \right|^3 + (1 - q_N) \left| \sigma \sqrt{\delta_N} + \beta \delta_N \right|^3$$

$$= \sigma^3 \delta_N^{3/2} + \cdots$$

$= o(\delta_N)$ per $N \to +\infty$.

#

dim (teo principale)

poniamo:

$$\varphi_N(\eta) := \varphi_{X^{(N)}}(\eta) = \mathbb{E}^{\mathbb{Q}_N}\left[e^{i\eta X^{(N)}}\right]$$

proviamo:

$$\varphi_N(\eta) \xrightarrow[N \to +\infty]{} \underbrace{\exp\left(i\eta T\left(r-\frac{\sigma^2}{2}\right) - \frac{\eta^2}{2}\sigma^2 T\right)}_{\substack{\| \\ \varphi_X(\eta) \\ \backsim \\ N\left(\left(r-\frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right)}}$$

$\forall\ \eta \in \mathbb{R}$

$$\varphi_N(\eta) = \mathbb{E}^{\mathbb{Q}_N}\left[\prod_{n=1}^{N} e^{i\eta Y_n^{(N)}}\right]$$

$$\overset{(Y_n^{(N)}\ i.i.d)}{=} \left(\mathbb{E}^{\mathbb{Q}_N}\left[\underbrace{e^{i\eta Y_1^{(N)}}}_{\|\ \text{Taylor - Lagrange in } \eta}\right]\right)^N$$

$$1 + i\eta Y_1^{(N)} - \frac{\eta^2\left(Y_1^{(N)}\right)^2}{2} - i\frac{\eta^3\left(Y_1^{(N)}\right)^3}{3!}e^{i\theta\eta Y_1^{(N)}}$$

θ v.a.

$$\underbrace{}_{|\cdot| \leq 1}$$

$$= \Big(1 + i\eta \Big[(r - \tfrac{\sigma^2}{2}) \delta_N + o(\delta_N) \Big] - \frac{\eta^2}{2} \Big[\sigma^2 \delta_N + o(\delta_N) \Big] + \frac{i\,\eta^3}{3!}\, o(\delta_N) \Big)^N \qquad \Big(\delta_N = \frac{T}{N}\Big)$$

$$= \Big(1 + \frac{1}{N} \Big[\underbrace{i\eta (r - \tfrac{\sigma^2}{2}) T - \frac{\eta^2}{2} \sigma^2 T + o(1)} \Big] \Big)^N$$

$$\Big\downarrow N \to +\infty$$

$$\varphi_X(\eta) \qquad \#$$

teorema Sia F il payoff di un derivato Europeo t.c. $F \in C_b$, vale:

$$\lim_{N \to +\infty} H_0^{(N)} = H_0 := e^{-rT}\, \mathbb{E}\left[F(S_0 e^X) \right]$$

con $X \sim \mathcal{N}_{(r - \frac{\sigma^2}{2})T,\, \sigma^2 T}$

dim segue da $X^{(N)} \xrightarrow{d} X$

e

$$y \mapsto F(S_0 e^y) \in C_b \,. \quad \#$$

corollario (Formula di Black-Scholes-Merton)

Se $F(s) = (K - s)^+$ (opz. Put)

$$\boxed{P_0 := H_0 = K e^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1)}$$

dove:

- Φ è la funz. di rip. di $\mathcal{N}_{0,1}$

- $d_1 = \dfrac{\log(S_0/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

- $d_2 = d_1 - \sigma\sqrt{T}$

$$= \frac{\log(S_0/K) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

dim Dobbiamo calcolare $\mathbb{E}\left[(K - S_0 e^X)^+\right]$

con $X \sim \mathcal{N}_{(r-\frac{\sigma^2}{2})T,\, \sigma^2 T}$.

$$X = \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\, Z, \qquad Z \sim \mathcal{N}_{0,1}$$

$$\bullet \quad (K - S_0 e^x)^+ = (K - S_0 e^x)\,\mathbb{1}_{\{\underbrace{K \ge S_0 e^x}_{\Updownarrow \; z \le -d_2}\}}$$

$$P_0 = e^{-rT} K \underbrace{\mathbb{E}\left[\mathbb{1}_{\{z \le -d_2\}}\right]}_{= \Phi(-d_2)}$$

$$- \cancel{e^{-rT}} S_0 \underbrace{\mathbb{E}\left[e^x \,\mathbb{1}_{\{z \le -d_2\}}\right]}$$

$$= \int_{-\infty}^{-d_2} e^{(r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}\, z} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} \, dz$$

$$= e^{rT} \int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\underbrace{\sigma^2 T - 2\sigma\sqrt{T}\, z + z^2}_{(\underbrace{z - \sigma\sqrt{T}}_{= y})^2}\right)} dz$$

$$= e^{rT} \int_{-\infty}^{-d_2 - \sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2} \, dy$$

$$= \cancel{e^{rT}} \, \Phi\left(-d_2 - \sigma\sqrt{T}\right)$$

$= -d_1$

oss. x la PUT-CALL PARITY:

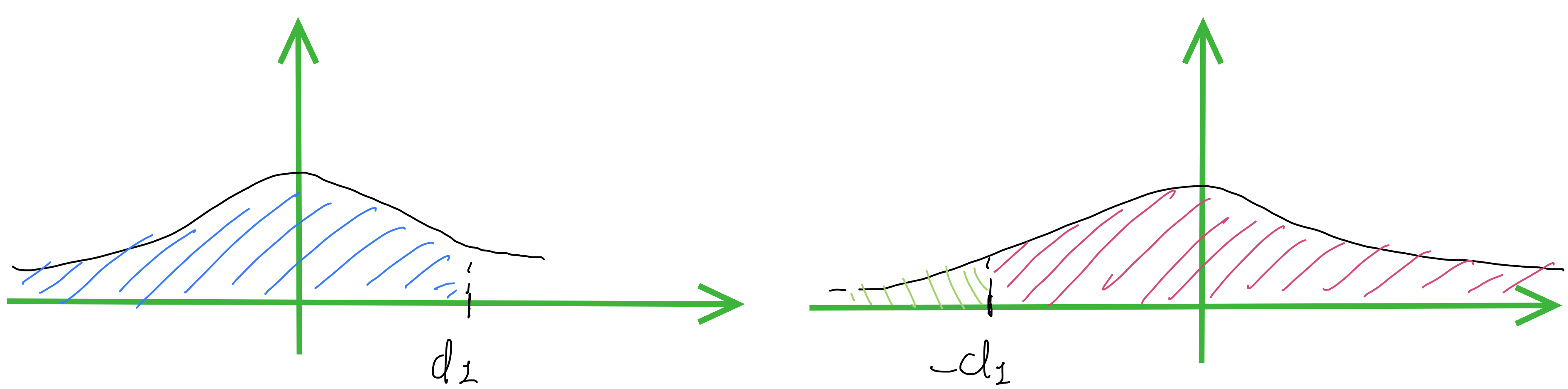
$$C_0 = P_0 + S_0 - K e^{-rT}$$

prezzo Call

$$= -K e^{-rT} \left(1 - \Phi(-d_2) \right)$$

$$+ S_0 \left(1 - \Phi(-d_1) \right)$$

$$= \boxed{S_0 \, \Phi(d_1) - K e^{-rT} \, \Phi(d_2)}$$



oss. $\mathbb{E}^{P}\left[\varphi\left(S_N^{(N)}\right) \right] \xrightarrow[N \to +\infty]{} \; ?$

se $p = \frac{1}{2}$, dato che $q_N \longrightarrow \frac{1}{2}$,

allora $\mathbb{E}^{P}\left[\varphi\left(S_N^{(N)}\right) \right] \longrightarrow P_0$? <u>NO</u>!

In realtà:

$$X^{(N)} \xrightarrow[N \mapsto +\infty]{(\mathbb{P})} \mathcal{N}\left(\underbrace{\frac{\alpha+\beta}{2} T}_{\downarrow\ \left(r-\frac{\sigma^2}{2}\right)T \text{ sotto } \mathbb{Q}}, \sigma^2 T\right)$$

oss. Definendo $\left(X_n^{(N)}\right)_{n=1,\dots,N}$ come

$$X_n^{(N)} := \log \prod_{k=1}^{n} \left(1+\mu_k^{(N)}\right)$$

$$= \sum_{k=1}^{n} Y_k^{(N)} \quad , \quad \text{si vede che:}$$

$$X_\cdot^{(N)} \xrightarrow[N \mapsto +\infty]{} X_\cdot \quad \text{dove}$$

$$X_t = \underbrace{\left(r-\frac{\sigma^2}{2}\right)t}_{\text{“drift”}} + \underbrace{W_t}_{\text{moto Browniano}} \quad , \quad t \geq 0 \ !$$

- Un moto Browniano è t.c.:

1 - traiettorie continue

2 - adattato

3 - incrementi indipend (come $X_n^{(N)}$)

4 - stazionarietà degli increm. :

$$W_t - W_s \sim N_{0,t-s} \quad \forall\ 0 \leq s < t$$

. Metodo di programmazione dinamica :

- Sia $(\Omega, \mathcal{F}, \mathbb{P})$ spazio di probabilità finito.
- Consideriamo :

$$\mathbb{R}^d \ni X_n = G_n (X_{n-1} , \underset{\mathbb{R}^q}{\underset{\cup}{\mu_n}} ; \underset{\mathbb{R}^p}{\underset{\cup}{\eta_{n-1}}} (X_{n-1}))$$

$$\forall\ n = 1, \ldots, N$$

- $\mu_1, \ldots, \mu_N$ v.a. indipendenti
- $\eta_0, \ldots, \eta_{N-1}$ sono funzioni determin. (controlli)
- $G_n : \mathbb{R}^d \times \mathbb{R}^q \times \mathbb{R}^p \longrightarrow \mathbb{R}^d$

Funzioni deterministiche

oss. Tale $X = (X_n)_{n=0,\dots,N}$ è un processo di Markov

esempio (S, B) mercato discreto $(d=1)$

$(\alpha, \beta) \in \mathcal{A}$ t.c. $\alpha_n = \alpha_n(S_{n-1})$

$\beta_n = \beta_n(S_{n-1})$

$\downarrow$ (visto in precedenza)

$$V_n = V_{n-1}(1 + r_n) + \alpha_n(S_{n-1}) \cdot S_{n-1}(\mu_n - r_n)$$

- $X_n = (V_n, S_n)$
- $\eta_{n-1}(v, s) = \alpha_n(s)$
- $G_n(v, s, \mu; \eta) = \left(s(1+\mu),\ v(1+r_n) + \eta \cdot s(\mu - r_n) \right)$

$\left(S_n = S_{n-1}(1 + \mu_n) \right)$

- Alternativa: $\pi_n := \dfrac{\alpha_n \cdot S_{n-1}}{V_{n-1}}$

proporz. di ricchezza investita in S nel periodo $[t_{n-1}, t_n]$

$$\Rightarrow V_n = V_{n-1}(1+r_n) + \pi_n(S_{n-1}) \cdot V_{n-1}(\mu_n - r_n)$$

$$= V_{n-1}\left[1 + r_n + \pi_n(S_{n-1})(\mu_n - r_n)\right]$$

$$X_n = V_n$$

$$G_n(v, \mu; \eta) = v\left(1 + r_n + \eta(\mu - r_n)\right)$$

notazione

Dato x, $n = 0, \dots, N$,

dato anche $\eta = (\eta_0, \dots, \eta_{N-1})$

$\hookrightarrow$ vettore di controlli

$$\begin{cases} X_n^{n,x,\eta} = x \\ X_k^{n,x,\eta} = G_k\left(X_{k-1}^{n,x}, \mu_k ; \eta_{k-1}(X_{k-1}^{n,x})\right) \end{cases}$$

$k = n+1, \dots, N$

$\Rightarrow$ FAMIGLIA DI PROCESSI

$$U^{n,x}(\eta) = U^{n,x}(\eta_n, \dots, \eta_{N-1}) := \mathbb{E}\left[\mathcal{U}(X_N^{n,x,\eta})\right]$$

- **Problema** : trovare

$$\sup_{\eta_0, \dots, \eta_{N-1}} U^{0,x}(\eta_0, \dots, \eta_{N-1})$$

e anche $\operatorname{arg\,max}_{\eta_0, \dots, \eta_{N-1}}$ se $\exists$