- (α, β) strategie ammissibili $((\alpha,\beta) \in \mathcal{A})$ se
  - α, β sono predicibili
  - (α, β) autofinanz.
- (α,β) autofin.

$$\Downarrow$$

$$(*) \quad \underset{\underset{V_n^{(\alpha,\beta)}}{\shortparallel}}{V_n} = V_{n-1}(1 + r_n) + \sum_{i=1}^{d} \alpha_n^i S_{n-1}^i (\mu_n^i - r_n)$$

$$\forall \; n = 1, \dots, N$$

dim $V_n - V_{n-1} = \alpha_n \cdot (S_n - S_{n-1})$

$$+ \beta_n (B_n - B_{n-1})$$

$$= \sum_{i=1}^{d} \alpha_n^i \mu_n^i S_{n-1}^i + \underbrace{\beta_n r_n B_{n-1}}_{\overset{\shortparallel \text{ (autofinanz.)}}{r_n (V_{n-1} - \alpha_n \cdot S_{n-1})}}$$

$$= \sum_{i=1}^{d} \alpha_n^i (\mu_n^i - r_n) S_{n-1}^i + r_n V_{n-1} \qquad \#$$

- Dati V_0 e $\alpha = (\alpha_n)_{n=1,\dots,N}$, $(V_n)_{n=1,\dots,N}$ è determinato.

Proposizione Dato $V_0 \in \mathbb{R}$ e $\alpha = (\alpha_n)_{n=1,\dots,N}$ a valori in $\mathbb{R}^d$, predicibile, $\exists$ $\beta = (\beta_n)_n$

t.c. $(\alpha,\beta)\in\mathcal{A}$ con $V_0^{(\alpha,\beta)} = V_0$.

dim)
$$\beta_n := \frac{V_{n-1} - \alpha_n \cdot S_{n-1}}{B_{n-1}} \; , \quad \text{dove}$$

V_{n-1} è dato da $(*)$. #

$\in m\,\mathcal{F}_{n-1}$

- Data $(\alpha_n)_{n=1,\dots,N}$, $V_0 \longrightarrow (\beta_n)_{n=1,\dots,N}$

  in particolare $V_0 = \alpha_1 S_0 + \beta_1$

- <u>NUMERAIRE</u>: un titolo di riferimento Y rispetto al quale scontiamo gli altri titoli.

- Dato $Y \in \{S^1, \dots, S^d, B\}$,

$$\tilde{S}_n^i := \frac{S_n^i}{Y_n} \;,\quad i=1,\dots,d \;; \qquad \tilde{B}_n := \frac{B_n}{Y_n}$$

- <u>Scelta comune</u>: $Y = B$

$$\Downarrow$$

$$\tilde{S}_n^i = \frac{S_n^i}{B_n} \;, \qquad \tilde{B}_n \equiv 1$$

- Data $(\alpha, \beta) \in \mathcal{A}$, definiamo

$$\tilde{V}_n^{(\alpha,\beta)} = \tilde{V}_n := \frac{V_n}{B_n} = \alpha_n \cdot \tilde{S}_n + \beta_n$$

- In particolare:

$$(\alpha, \beta) \text{ è autofin.}$$

$$\Big\Updownarrow \text{ def}$$

$$\frac{V_{n-1}}{B_{n-1}} = \alpha_n \cdot \frac{S_{n-1}}{B_{n-1}} + \beta_n \frac{B_{n-1}}{B_{n-1}}$$

$$\Big\Updownarrow$$

$$\tilde{V}_{n-1} = \alpha_n \cdot \tilde{S}_{n-1} + \beta_n$$

- Anche:

$$\tilde{V}_n - \tilde{V}_{n-1} = \alpha_n \cdot (\tilde{S}_n - \tilde{S}_{n-1})$$

- Data $(\alpha, \beta) \in \mathcal{A}$,

$$\begin{aligned} V_N - V_0 &= \sum_{k=1}^{N} V_k - V_{k-1} \\ &= \sum_{k=1}^{N} \left(\alpha_k \cdot (S_k - S_{k-1}) + \beta_k B_{k-1} \right) \\ &=: g^{(\alpha,\beta)} \longleftarrow \text{guadagno totale} \end{aligned}$$

- $g_n^{(\alpha,\beta)} := V_n - V_0 \quad \forall\, n$

- $$\tilde{V}_n - \tilde{V}_0 = \sum_{k=1}^{n} \left(\tilde{V}_k - \tilde{V}_{k-1} \right)$$
  $\tilde{V}_0 = V_0$

  dipende solo da α

  $$= \sum_{k=1}^{n} \alpha_k \cdot \left(\tilde{S}_k - \tilde{S}_{k-1} \right) =: G_n^{(\alpha)}$$

oss. $G_n^{(\alpha)} \neq \dfrac{g_n^{(\alpha,\beta)}}{B_n}$

- **Arbitraggi:**

- P.N.A.: $X_N = Y_N$ certam. $\Rightarrow X_n = Y_n$ cert. $\forall\, n < N$

def $(\alpha,\beta) \in \mathcal{A}$ si dice un arbitraggio se

(i) $V_0 = 0$ (certamente)

(ii) $V_{\bar{n}} \geq 0$ $\mathbb{P}$-q.c. per un certo $\bar{n} \in \{1, \dots, N\}$

(iii) $\mathbb{P}(V_{\bar{n}} > 0) > 0$

def. Un mercato (S, B) si dice arbitrage-free se $\nexists$ arbitraggi.

Proposizione $\exists\, (\alpha,\beta)$ ARBITRAGGIO

$\Downarrow$

$\exists \ (\bar{\alpha}, \bar{\beta})$ ARBITRAGGIO t.c.

$$(ii)' \quad V_n^{(\bar{\alpha},\bar{\beta})} \geq 0 \quad \mathbb{P}\text{-q.c.} \quad \forall \ n = 1, \ldots, N$$

dim $\underline{\times}$ $\underline{\text{ipotesi}}$: $V_0^{(\alpha,\beta)} = 0$, $\exists \ \bar{n} \in \mathbb{N}$ t.c.

$$\begin{cases} V_{\bar{n}}^{(\alpha,\beta)} \geq 0 & \mathbb{P}\text{-q.c.} \\ \mathbb{P}\left(V_{\bar{n}}^{(\alpha,\beta)} > 0 \right) > 0 \end{cases}$$

assumiamo $\bar{n} = N$, SENZA PERDERE GENERALITÀ

$\ulcorner$

$\bar{n} < N$, $\tilde{\alpha}_{\bar{n}+1} := 0 \Rightarrow \tilde{\beta}_{\bar{n}+1} \underset{\text{autofinanz.}}{\overset{\Uparrow}{=}} \dfrac{V_{\bar{n}}}{B_{\bar{n}}}$

$$\Rightarrow \tilde{V}_{\bar{n}+1} = \underset{\substack{\vee\!\!| \\ 0}}{V_{\bar{n}}} \cdot \underbrace{\frac{B_{\bar{n}+1}}{B_{\bar{n}}}}_{>0} \geq 0$$

$$\tilde{\alpha}_{\bar{n}+2} := 0 \Rightarrow \tilde{\beta}_{\bar{n}+2} = \frac{\tilde{V}_{\bar{n}+1}}{B_{\bar{n}+1}}$$

$$\Rightarrow \tilde{V}_{\bar{n}+2} = \underset{\substack{\vee\!\!| \\ 0}}{\tilde{V}_{\bar{n}+1}} \cdot \underbrace{\frac{B_{\bar{n}+2}}{B_{\bar{n}+1}}}_{>0} \geq 0$$

$\vdots$

$$\tilde{V}_N = \underset{\substack{\vee\!\!| \\ 0}}{\tilde{V}_{N-1}} \cdot \underbrace{\frac{B_N}{B_{N-1}}}_{>0} \geq 0$$

$\lrcorner$

- Se vale (iii) niente da provare, altrimenti

$\exists\ k < N$ t.c. $\exists\ F \in \mathcal{F}_k$: $\mathbb{P}(F) > 0$ e

$m\mathcal{F}_k$

$$\begin{cases} V_k = \alpha_k \cdot S_k + \beta_k B_k < 0 & \text{su } F \\ V_n = \alpha_n S_n + \beta_n B_n \underset{\text{q.c.}}{\geq} 0 & k < n \leq N \end{cases}$$

Definiamo :

- $\bar{\alpha}_n = \bar{\beta}_n \equiv 0 \quad \forall\ n = 1, \ldots, k$

$\left(V_n^{(\bar{\alpha}, \bar{\beta})} \equiv 0 \quad \forall\ n = 1, \ldots, k \right)$

- $\bar{\alpha}_n := \begin{cases} 0 & \text{su } F^c \\ \alpha_n & \text{su } F \end{cases} \quad \forall\ n = k+1, \ldots, N$

- $\bar{\beta}_n := \begin{cases} 0 & \text{su } F^c \\ ? & \text{su } F \end{cases} \quad \forall\ n = k+1, \ldots, N$

- su F:

$$\bar{\beta}_{k+1} = \frac{1}{B_k} \Big(\underbrace{V_k^{(\bar{\alpha}, \bar{\beta})}}_{= 0} - \underbrace{\bar{\alpha}_{k+1}}_{= \alpha_{k+1}} S_k \Big) = - \frac{\alpha_{k+1} \cdot S_k}{B_k}$$

$$= \frac{- \alpha_{k+1} \cdot S_k - \beta_{k+1} B_k + \beta_{k+1} B_k}{B_k}$$

$$= \frac{-V_k^{(\alpha,\beta)} + \beta_{k+1} B_k}{B_k} = -\frac{V_k^{(\alpha,\beta)}}{B_k} + \beta_{k+1}$$

$$\Rightarrow V_{k+1}^{(\bar{\alpha},\bar{\beta})} \overset{\text{def.}}{=} \bar{\alpha}_{k+1} \cdot S_{k+1} + \bar{\beta}_{k+1} B_{k+1}$$

$$= \alpha_{k+1} \cdot S_{k+1} + \beta_{k+1} B_{k+1}$$

$$- \underbrace{V_k^{(\alpha,\beta)}}_{<0} \cdot \underbrace{\frac{B_{k+1}}{B_k}}_{>0}$$

$$= \underbrace{V_{k+1}^{(\alpha,\beta)}}_{\geq 0} \underbrace{- V_k^{(\alpha,\beta)} \cdot \frac{B_{k+1}}{B_k}}_{\geq 0} \geq 0$$

- Reiterando, su F:

$$\forall\ n = k+1, \dots, N, \quad \bar{\beta}_n = \beta_n - \frac{V_{n-1}^{(\bar{\alpha},\bar{\beta})} - V_{n-1}^{(\alpha,\beta)}}{B_{n-1}}$$

$$\Rightarrow V_n^{(\bar{\alpha},\bar{\beta})} = V_n^{(\alpha,\beta)} + \left(V_{n-1}^{(\bar{\alpha},\bar{\beta})} - V_{n-1}^{(\alpha,\beta)} \right) \frac{B_n}{B_{n-1}}$$

$$= \underbrace{V_n^{(\alpha,\beta)}}_{\geq 0} - \underbrace{V_k^{(\alpha,\beta)}}_{<0} \underbrace{\frac{B_n}{B_k}}_{>0} \geq 0$$

$$\Rightarrow V_n^{(\bar{\alpha},\bar{\beta})} \geq 0 \quad \mathbb{P}\text{-q.c.} \quad \forall\ n = 1, \dots, N \qquad \#$$

- **Misura Martingala equivalente :**

  - <u>Modelli uniperiodali</u>:

    - $\exists !$ $\mathbb{Q}$ neutrale al rischio $\Leftrightarrow$ $\exists$ prezzo di replicaz.
    - $\exists$ più $\mathbb{Q}$ neutrali al rischio $\Leftrightarrow$ $\nexists$ " " "

oss $\mathbb{Q}$ era la soluzione di

$$S_0 \overset{(\Delta)}{=} B_1^{-1}\, \mathbb{E}^{\mathbb{Q}}[S_1] = \mathbb{E}^{\mathbb{Q}}[\tilde{S}_1]$$

$$= \mathbb{E}^{\mathbb{Q}}[\tilde{S}_1 \mid \mathcal{F}_0]$$

$$\mathcal{F}_0 = (\Omega, \emptyset)$$

($q, 1-q$, 2 eventi)

$(\Delta) \Leftrightarrow \tilde{S}$ è una martingale

def Una "misura martingala equivalente (EMM) rispetto ad un numeraire Y" è una probabilità $\mathbb{Q}$ su $(\Omega, \mathcal{F})$ t.c. :

(i) $\mathbb{Q} \sim \mathbb{P}$ (equivalente)

(ii) $\tilde{S}^1, \ldots, \tilde{S}^d, \tilde{B}$ (rispetto a Y)

sono martingale secondo $\mathbb{Q}$

- (i) $\mathbb{Q} \sim \mathbb{P} \overset{\text{def}}{\Leftrightarrow} \left(\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0 \right)$, $A \in \mathcal{F}$

• (ii) significa:

$$\tilde{S}^i_n = \frac{S^i_n}{Y_n} = \mathbb{E}^Q\left[\frac{S^i_{n+1}}{Y_{n+1}} \,\Big|\, \mathcal{F}_n \right] = \mathbb{E}^Q\left[\tilde{S}^i_{n+1} \,|\, \mathcal{F}_n \right]$$

$i = 1, \dots, d$

$$\tilde{B}_n = \frac{B_n}{Y_n} = \mathbb{E}^Q\left[\frac{B_{n+1}}{Y_{n+1}} \,\Big|\, \mathcal{F}_n \right] = \mathbb{E}^Q\left[\tilde{B}_{n+1} \,|\, \mathcal{F}_n \right]$$

$\Downarrow$ (se $Y = B$)

$$\boxed{B^{-1}_k \cdot S^i_k = B^{-1}_n \, \mathbb{E}^Q\left[S^i_n \,|\, \mathcal{F}_k \right] \qquad k < n}$$

deterministico

$\Downarrow$ ($k = 0$)

$$S^i_0 = B^{-1}_n \, \mathbb{E}^Q\left[S^i_n \right]$$

N.B (i) è l'unico legame tra $\mathbb{P}$ e $\mathbb{Q}$

teorema (FIRST FUND. TH. OF ASSET PRICING - FFTAP)

Un mercato discreto (S, B) è

ARBITRAGE - FREE $\Longleftrightarrow$ $\exists$ $\mathbb{Q}$ E.M.M.

rispetto ad un certo numeraire

• <u>cambi di misura</u>: $(\Omega, \mathcal{F}, \mathbb{P})$ sp. di probabilità

e X v.a t.c.

$$X \geq 0 \quad \mathbb{P}\text{-q.c.} \quad , \quad \mathbb{E}[X] = 1$$

$$\Downarrow$$

la $\mathbb{P}^X$ definita da

$$\mathbb{P}^X(A) := \int_A X \, d\mathbb{P} \quad , \quad A \in \mathcal{F}$$

è una probabilità su $(\Omega, \mathcal{F})$

La funz. X si chiama DERIVATA DI RADON-NIKODYM di $\mathbb{P}^X$ risp. a $\mathbb{P}$.

$$\left(X =: \frac{d\mathbb{P}^X}{d\mathbb{P}} \right)$$

• <u>analogia</u>: dx misura di Lebesgue

$$f \in L^1(\mathbb{R}) \quad , \quad \int_{\mathbb{R}} f = 1 \quad , \quad f \geq 0$$

$$\mu(A) := \int_A f(x) \, dx \quad , \quad A \in \mathcal{B}$$

$\mu \in AC$ con densità f

teorema Sia (S, B) mercato discreto, e $\mathbb{Q}$

E.M.M. risp. a Y. Per ogni $X \in \{S^1, \dots, S^d, B\}$ positivo, la misura $\mathbb{Q}^X$ data da

$$\frac{d\mathbb{Q}^X}{d\mathbb{Q}} = \frac{X_N}{X_0} \left(\frac{Y_N}{Y_0} \right)^{-1} \quad \text{è t.c.}$$

$$Y_n \, \mathbb{E}^{\mathbb{Q}} \left[\frac{Z}{Y_N} \,\Big|\, \mathcal{F}_n \right] = X_n \, \mathbb{E}^{\mathbb{Q}^X} \left[\frac{Z}{X_N} \,\Big|\, \mathcal{F}_n \right]$$

$$\forall \; n \leq N \qquad \forall \; Z \text{ v.a}$$

Corollario $\mathbb{Q}^X$ è E.M.M. risp. a X.