Proof of the Theorems of Relationships between Propositions

The theoretical foundation in `Denfet` offer several readily available theorems, which can greatly simplify the analysis. Firstly, it is easy to see that the strong refinement relationships are transitive:

Theorem 1. *If there exist a $p_i \rightrightarrows p_j$ (or $p_i \leftleftarrows p_j$) and a $p_j \rightrightarrows p_k$ (or $p_j \leftleftarrows p_k$), then we have $p_i \rightrightarrows p_k$ (or $p_i \leftleftarrows p_k$).*

Proof. The proof is straightforward: $p_i \rightrightarrows p_j$ (or $p_i \leftleftarrows p_j$) and $p_j \rightrightarrows p_k$ (or $p_j \leftleftarrows p_k$) mean that $\forall y_a \leq y_b : y_a \in p_i \& y_b \in p_j$ while $\int_\alpha^\beta g_i(y)dy < \int_\alpha^\beta g_j(y)dy$; at the same time, $\forall y_b \leq y_c : y_b \in p_j \& y_c \in p_k$ while $\int_\alpha^\beta g_j(y)dy < \int_\alpha^\beta g_k(y)dy$. Therefore, we have $\forall y_a \leq y_c : y_a \in p_i \& y_c \in p_k$ while $\int_\alpha^\beta g_i(y)dy < \int_\alpha^\beta g_k(y)dy$, i.e., $p_i \rightrightarrows p_k$ (or $p_i \leftleftarrows p_k$). □

A similar theorem exists for multi-dimensional refinement:

Theorem 2. *If there exist a $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \rightrightarrows \overline{\boldsymbol{p}}_{\boldsymbol{j}}$ (or $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \leftleftarrows \overline{\boldsymbol{p}}_{\boldsymbol{j}}$) and a $\overline{\boldsymbol{p}}_{\boldsymbol{j}} \rightrightarrows \overline{\boldsymbol{p}}_{\boldsymbol{k}}$ (or $\overline{\boldsymbol{p}}_{\boldsymbol{j}} \leftleftarrows \overline{\boldsymbol{p}}_{\boldsymbol{k}}$), then we have $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \rightrightarrows \overline{\boldsymbol{p}}_{\boldsymbol{k}}$ (or $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \leftleftarrows \overline{\boldsymbol{p}}_{\boldsymbol{k}}$).*

Proof. This can be proved as that of Theorem 1. □

It is also clear that there can be some straightforward correlations between single- and multi-dimensional refinement. Specifically, for a given set of measured configurations $\boldsymbol{\mathcal{Y}}$, we have:

Theorem 3. *Given $\overline{\boldsymbol{p}}_{\boldsymbol{i}}$, if we solely perform an arbitrary number of $p_\alpha \rightrightarrows p_\beta$ (or $p_\alpha \leftleftarrows p_\beta$) $: p_\alpha \in \overline{\boldsymbol{p}}_{\boldsymbol{i}}$ & $p_\beta \in \overline{\boldsymbol{p}}_{\boldsymbol{j}}$, then there will be $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \rightrightarrows \overline{\boldsymbol{p}}_{\boldsymbol{j}}$ (or $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \leftleftarrows \overline{\boldsymbol{p}}_{\boldsymbol{j}}$).*

Proof. This can be proved by contradiction. Suppose that under such a case it is possible to have $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \rightarrow \overline{\boldsymbol{p}}_{\boldsymbol{j}}$ (or $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \leftarrow \overline{\boldsymbol{p}}_{\boldsymbol{j}}$), it means that $\exists \overline{\boldsymbol{y}}_{\boldsymbol{a}} \in \boldsymbol{\mathcal{Y}} : \nexists \overline{\boldsymbol{y}}_{\boldsymbol{b}} \in \boldsymbol{\mathcal{Y}} : \overline{\boldsymbol{g}}_{\boldsymbol{j}}(\overline{\boldsymbol{y}}_{\boldsymbol{b}}) \prec \overline{\boldsymbol{g}}_{\boldsymbol{i}}(\overline{\boldsymbol{y}}_{\boldsymbol{a}})$, that is, at least one point $\overline{\boldsymbol{y}}_{\boldsymbol{k}}$ over $\overline{\boldsymbol{p}}_{\boldsymbol{i}}$ is nondominated to other points over $\overline{\boldsymbol{p}}_{\boldsymbol{j}}$. Clearly, for this to be true, it requires at least one dimension of $\overline{\boldsymbol{y}}_{\boldsymbol{k}}$ to have higher satisfaction on $p_\alpha \in \overline{\boldsymbol{p}}_{\boldsymbol{i}}$ than that on $p_\beta \in \overline{\boldsymbol{p}}_{\boldsymbol{j}}$, i.e., $p_\alpha \rightarrow p_\beta$, $p_\alpha \leftarrow p_\beta$, or $p_\alpha \leftleftarrows p_\beta$ (or lower in the case of $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \leftarrow \overline{\boldsymbol{p}}_{\boldsymbol{j}}$ which implies $p_\alpha \leftarrow p_\beta$, $p_\alpha \rightarrow p_\beta$, or $p_\alpha \rightrightarrows p_\beta$). This, however, contradicts the fact that we solely perform an arbitrary number of $p_\alpha \rightrightarrows p_\beta$ (or $p_\alpha \leftleftarrows p_\beta$). The same proof procedure is applicable for proving the impossibility of $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \equiv \overline{\boldsymbol{p}}_{\boldsymbol{j}}$. □

Theorem 4. *Given $\overline{\boldsymbol{p}}_{\boldsymbol{i}}$, if we perform an arbitrary number of mixed $p_\alpha \rightrightarrows p_\beta$ and $p_\alpha \leftleftarrows p_\beta : p_\alpha \in \overline{\boldsymbol{p}}_{\boldsymbol{i}}$ & $p_\beta \in \overline{\boldsymbol{p}}_{\boldsymbol{j}}$, then there will never be $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \rightrightarrows \overline{\boldsymbol{p}}_{\boldsymbol{j}}$ or $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \leftleftarrows \overline{\boldsymbol{p}}_{\boldsymbol{j}}$.*

Proof. Again, we can prove this theorem by contradiction. Suppose that it is possible to have $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \rightrightarrows \overline{\boldsymbol{p}}_{\boldsymbol{j}}$ (or $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \leftleftarrows \overline{\boldsymbol{p}}_{\boldsymbol{j}}$), it implies $\forall \overline{\boldsymbol{y}}_{\boldsymbol{a}} \in \boldsymbol{\mathcal{Y}} : \exists \overline{\boldsymbol{y}}_{\boldsymbol{b}} \in \boldsymbol{\mathcal{Y}} : \overline{\boldsymbol{g}}_{\boldsymbol{j}}(\overline{\boldsymbol{y}}_{\boldsymbol{b}}) \prec \overline{\boldsymbol{g}}_{\boldsymbol{i}}(\overline{\boldsymbol{y}}_{\boldsymbol{a}})$ (or $\overline{\boldsymbol{g}}_{\boldsymbol{j}}(\overline{\boldsymbol{y}}_{\boldsymbol{b}}) \succ \overline{\boldsymbol{g}}_{\boldsymbol{i}}(\overline{\boldsymbol{y}}_{\boldsymbol{a}})$ for $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \leftleftarrows \overline{\boldsymbol{p}}_{\boldsymbol{j}}$), that is, all points over $\overline{\boldsymbol{p}}_{\boldsymbol{i}}$ needs to be dominated by (or dominates for $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \leftleftarrows \overline{\boldsymbol{p}}_{\boldsymbol{j}}$) at least one point $\overline{\boldsymbol{y}}_{\boldsymbol{k}}$ on $\overline{\boldsymbol{p}}_{\boldsymbol{j}}$. Therefore, it requires all dimensions of $\overline{\boldsymbol{y}}_{\boldsymbol{k}}$ to have higher (or at least equal) satisfaction on $p_\beta \in \overline{\boldsymbol{p}}_{\boldsymbol{j}}$ than that on $p_\alpha \in \overline{\boldsymbol{p}}_{\boldsymbol{i}}$, i.e., there is only sole $p_\alpha \rightrightarrows p_\beta$ (or lower in the case of $\overline{\boldsymbol{p}}_{\boldsymbol{i}} \leftleftarrows \overline{\boldsymbol{p}}_{\boldsymbol{j}}$ which implies sole $p_\alpha \leftleftarrows p_\beta$). This clearly contradicts the fact that we perform an arbitrary number of mixed $p_\alpha \rightrightarrows p_\beta$ and $p_\alpha \leftleftarrows p_\beta$. □

The other actions of refinements, e.g., solely performing weak refinement on single propositions, could lead to any possible results, i.e., weak, strong, or equally-satisfiable refinement.