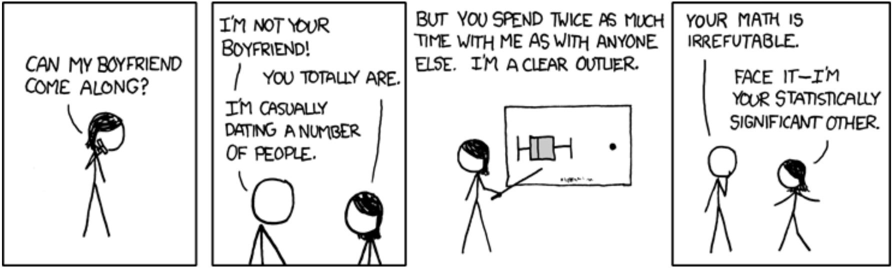
Frequentist Statistics and Hypothesis Testing

18.05 Spring 2018



http://xkcd.com/539/

Agenda

- Introduction to the frequentist way of life.
- What is a statistic?

- NHST ingredients; rejection regions
- Simple and composite hypotheses
- z-tests, p-values

Frequentist school of statistics

- Dominant school of statistics in the 20$^{\text{th}}$ century.
- *p*-values, *t*-tests, χ^2-tests, confidence intervals.
- Defines probability as long-term frequency in a repeatable random experiment.
  - Yes: probability a coin lands heads.
  - Yes: probability a given treatment cures a certain disease.
  - Yes: probability distribution for the error of a measurement.
- Rejects the use of probability to quantify incomplete knowledge, measure degree of belief in hypotheses.
  - No: prior probability for the probability an unknown coin lands heads.
  - No: prior probability on the efficacy of a treatment for a disease.
  - No: prior probability distribution for the unknown mean of a normal distribution.

The fork in the road

Probability (mathematics)

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)}$$

Everyone uses Bayes' formula when the prior $P(H)$ is known.

Bayesian path

Frequentist path

Statistics (art)

$$P_{\text{Posterior}}(H|D) = \frac{P(D|H)P_{\text{prior}}(H)}{P(D)}$$

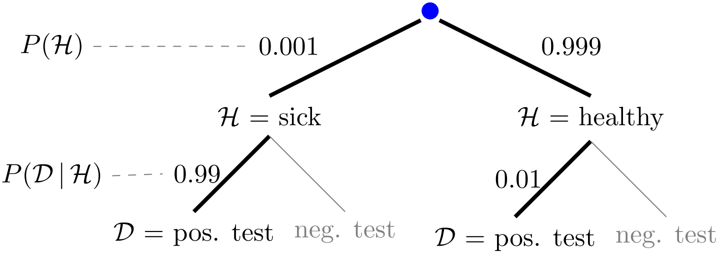
Bayesians require a prior, so they develop one from the best information they have.

Likelihood $L(H;D) = P(D|H)$

Without a known prior frequentists draw inferences from just the likelihood function.

Disease screening redux: probability

The test is positive. Are you sick?

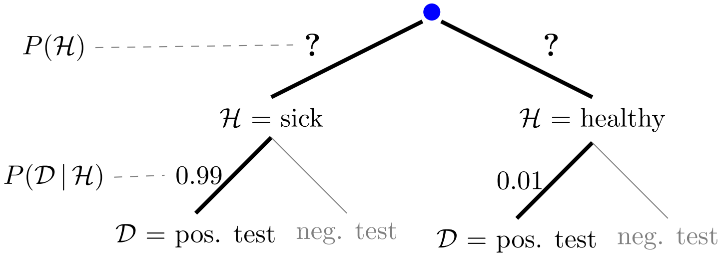


The prior is known so we can use Bayes' Theorem.

$$P(\text{sick}\,|\,\text{pos. test}) = \frac{0.001 \cdot 0.99}{0.001 \cdot 0.99 + 0.999 \cdot 0.01} \approx 0.1$$

Disease screening redux: statistics

The test is positive. Are you sick?



The prior is not known.
Bayesian: use a subjective prior $P(\mathcal{H})$ and Bayes' Theorem.

Frequentist: the likelihood is all we can use: $P(\mathcal{D} \mid \mathcal{H})$

Concept question

Each day Jane arrives X hours late to class, with $X \sim \text{uniform}(0, \theta)$, where θ is unknown. Jon models his initial belief about θ by a prior pdf $f(\theta)$. After Jane arrives x hours late to the next class, Jon computes the likelihood function $\phi(x|\theta)$ and the posterior pdf $f(\theta|x)$.

Which of these probability computations would the frequentist consider valid?

1. none
2. prior
3. likelihood
4. posterior
5. prior and posterior
6. prior and likelihood
7. likelihood and posterior
8. prior, likelihood and posterior.

Concept answer

answer: 3. likelihood

Both the prior and posterior are probability distributions on the possible values of the unknown parameter θ, i.e. a distribution on hypothetical values. The frequentist does not consider them valid.

The likelihood $\phi(x|\theta)$ is perfectly acceptable to the frequentist. It represents the probability of data from a repeatable experiment, i.e. measuring how late Jane is each day. Conditioning on θ is fine. This just fixes a model parameter θ. It doesn't require computing probabilities of values of θ.

Statistics are computed from data

Working definition. A statistic is anything that can be computed from random data.

A statistic cannot depend on the true value of an unknown parameter.

A statistic can depend on a hypothesized value of a parameter.

Examples of point statistics

- Data mean
- Data maximum (or minimum)
- Maximum likelihood estimate (MLE)

A statistic is random since it is computed from random data.

We can also get more complicated statistics like **interval statistics.**

Concept questions

Suppose $x_1, \ldots, x_n$ is a sample from $\mathrm{N}(\mu, \sigma^2)$, where μ and σ are unknown.

Is each of the following a statistic?

1. Yes 2. No

1. The median of $x_1, \ldots, x_n$.
2. The interval from the 0.25 quantile to the 0.75 quantile of $\mathrm{N}(\mu, \sigma^2)$.
3. The standardized mean $\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$.
4. The set of sample values less than 1 unit from $\bar{x}$.

Concept answers

1. Yes. The median only depends on the data $x_1, \ldots, x_n$.

2. No. This interval depends only on the distribution parameters μ and σ. It does not consider the data at all.

3. No. this depends on the values of the unknown parameters μ and σ.

4. Yes. $\bar{x}$ depends only on the data, so the set of values within 1 of $\bar{x}$ can all be found by working with the data.

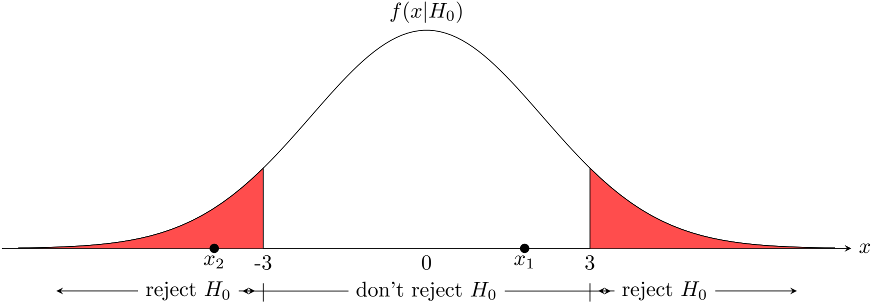
NHST ingredients

Null hypothesis: H_0

Alternative hypothesis: H_A

Test statistic: x

Rejection region: reject H_0 in favor of H_A if x is in this region



$p(x|H_0)$ or $f(x|H_0)$: null distribution

Choosing rejection regions

Coin with probability of heads θ.

Test statistic $x =$ the number of heads in 10 tosses.

H_0: 'the coin is fair', i.e. $\theta = 0.5$

H_A: 'the coin is biased, i.e. $\theta \neq 0.5$

Two strategies:

1. Choose rejection region then compute significance level.

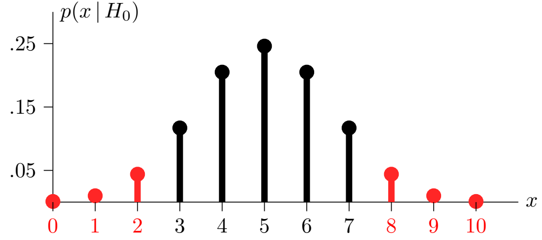
2. Choose significance level then determine rejection region.

***** Everything is computed assuming H_0 *****

Table question

Suppose we have the coin from the previous slide.

1. The rejection region is bordered in red, what's the significance level?



x	0	1	2	3	4	5	6	7	8	9	10
$p(x\|H_0)$	.001	.010	.044	.117	.205	.246	.205	.117	.044	.010	.001

2. Given significance level $\alpha = .05$ find a two-sided rejection region.

Solution

1. $\alpha = 0.11$

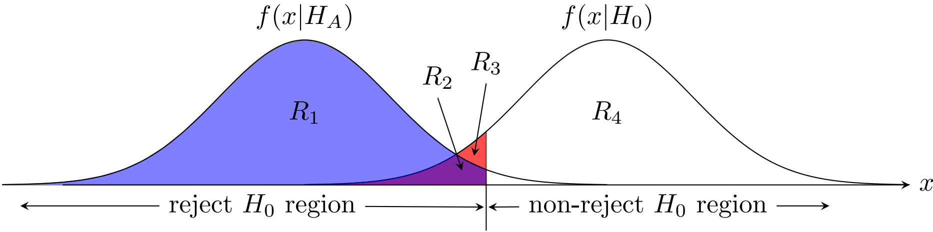
x	0	1	2	3	4	5	6	7	8	9	10
$p(x\|H_0)$	.001	.010	.044	.117	.205	.246	.205	.117	.044	.010	.001

2. $\alpha = 0.05$

x	0	1	2	3	4	5	6	7	8	9	10
$p(x\|H_0)$	.001	.010	.044	.117	.205	.246	.205	.117	.044	.010	.001

Concept question

The null and alternate pdfs are shown on the following plot



The significance level of the test is given by the area of which region?

1. R_1
2. R_2
3. R_3
4. R_4
5. $R_1 + R_2$
6. $R_2 + R_3$
7. $R_2 + R_3 + R_4$.

<u>**answer:**</u> 6. $R_2 + R_3$. This is the area under the pdf for H_0 above the rejection region.

z-tests, p-values

Suppose we have independent **normal Data:** $x_1, \ldots, x_n$; with unknown mean μ, known σ

Hypotheses: H_0: $x_i \sim N(\mu_0, \sigma^2)$

H_A: Two-sided: $\mu \neq \mu_0$, or one-sided: $\mu > \mu_0$

z-**value:** standardized $\overline{x}$: $z = \dfrac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$

Test statistic: z

Null distribution: Assuming H_0: $z \sim \mathrm{N}(0, 1)$.

p-**values:** Right-sided *p*-**value:** $p = P(Z > z \mid H_0)$

(Two-sided *p*-**value:** $p = P(|Z| > z \mid H_0)$)

Significance level: For $p \leq \alpha$ we reject H_0 in favor of H_A.

Note: Could have used $\overline{x}$ as test statistic and $\mathrm{N}(\mu_0, \sigma^2)$ as the null distribution.

Visualization

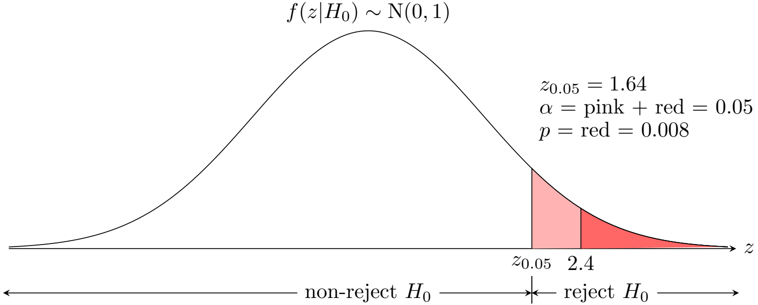
Data follows a normal distribution $\mathrm{N}(\mu, 15^2)$ where μ is unknown.

H_0: $\mu = 100$

H_A: $\mu > 100$ (one-sided)

Collect 9 data points: $\bar{x} = 112$. So, $z = \dfrac{112 - 100}{15/3} = 2.4$.

Can we reject H_0 at significance level 0.05?



Board question

- H_0: data follows a $N(5, 10^2)$
- H_A: data follows a $N(\mu, 10^2)$ where $\mu \neq 5$.
- Test statistic: $z =$ standardized $\overline{x}$.
- Data: 64 data points with $\overline{x} = 6.25$.
- Significance level set to $\alpha = 0.05$.

(i) Find the rejection region; draw a picture.

(ii) Find the z-value; add it to your picture.

(iii) Decide whether or not to reject H_0 in favor of H_A.

(iv) Find the p-value for this data; add to your picture.

(v) What's the connection between the answers to (ii), (iii) and (iv)?

Solution

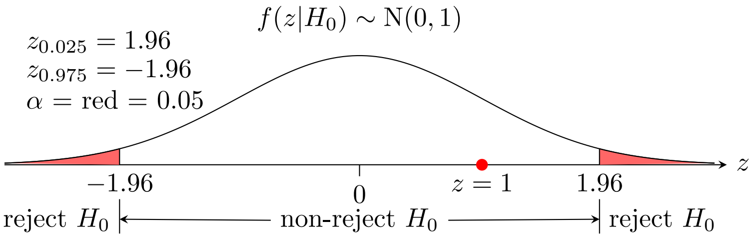
The null distribution $f(z \mid H_0) \sim N(0,1)$

(i) The rejection region is $|z| > 1.96$, i.e. 1.96 or more standard deviations from the mean.

(ii) Standardizing $z = \dfrac{\overline{x} - 5}{5/4} = \dfrac{1.25}{1.25} = 1.$

(iii) Do not reject since z is not in the rejection region.

(iv) Use a two-sided p-value $p = P(|Z| > 1) = .32.$



Solution continued

(v) The z-value not being in the rejection region tells us exactly the same thing as the p-value being greater than the significance, i.e., don't reject the null hypothesis H_0.

Board question

Two coins: probability of heads is 0.5 for C_1; and 0.6 for C_2.

We pick one at random, flip it 8 times and get 6 heads.

1. $H_0 =$ 'The coin is C_1' $\quad H_A =$ 'The coin is C_2'

Do you reject H_0 at the significance level $\alpha = 0.05$?

2. $H_0 =$ 'The coin is C_2' $\quad H_A =$ 'The coin is C_1'

Do you reject H_0 at the significance level $\alpha = 0.05$?

3. Do your answers to (1) and (2) seem paradoxical?

Here are binomial(8,θ) tables for $\theta = 0.5$ and 0.6.

k	0	1	2	3	4	5	6	7	8
$p(k\|\theta = 0.5)$	.004	.031	.109	.219	.273	.219	.109	.031	.004
$p(k\|\theta = 0.6)$	.001	.008	.041	.124	.232	.279	.209	.090	.017

Solution

1. Since $0.6 > 0.5$ we use a right-sided rejection region.
Under H_0 the probability of heads is 0.5. Using the table we find a one sided rejection region $\{7, 8\}$. That is we will reject H_0 in favor of H_A only if we get 7 or 8 heads in 8 tosses.
Since the value of our data $x = 6$ is not in our rejection region we do not reject H_0.

2. Since $0.6 > 0.5$ we use a left-sided rejection region.
Now under H_0 the probability of heads is 0.6. Using the table we find a one sided rejection region $\{0, 1, 2\}$. That is we will reject H_0 in favor of H_A only if we get 0, 1 or 2 heads in 8 tosses.
Since the value of our data $x = 6$ is not in our rejection region we do not reject H_0.

3. The fact that we don't reject C_1 in favor of C_2 or C_2 in favor of C_1 reflects the asymmetry in NHST. The null hypothesis is the cautious choice. That is, we only reject H_0 if the data is extremely unlikely when we assume H_0. This is not the case for either C_1 or C_2.