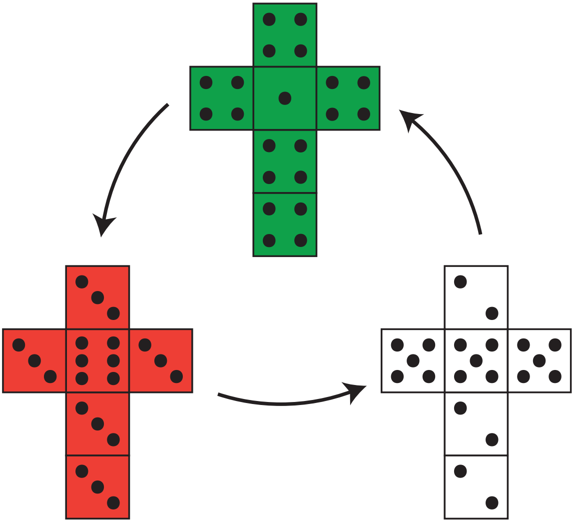
Probability: Terminology and Examples
18.05 Spring 2018



Probability Cast

Introduced so far

- Experiment: a repeatable procedure
- Sample space: set of all possible outcomes S (or Ω).
- Event: a subset of the sample space.
- Probability function, $P(\omega)$: gives the probability for each outcome $\omega \in S$
    1. Probability is between 0 and 1
    2. Total probability of all possible outcomes is 1.

Example (from the reading)

Experiment: toss a fair coin, report heads or tails.

Sample space: $\Omega = \{H,\ T\}$.

Probability function: $P(H) = .5, \quad P(T) = .5.$

Use tables:

Outcomes	H	T
Probability	$1/2$	$1/2$

(Tables can really help in complicated examples)

Discrete sample space

Discrete = listable

Examples:

$\{a, b, c, d\}$ (finite)

$\{0, 1, 2, \ldots\}$ (infinite)

{sun,cloud,rain,snow,fog}

{patient cured, unchanged, patient died}

Events

Events are sets:

- Can describe in words
- Can describe in notation
- Can describe with Venn diagrams

Experiment: toss a coin 3 times.

Event:

You get 2 or more heads = { HHH, HHT, HTH, THH}

Events, sets and words

Experiment: toss a coin 3 times.

Which of following equals the event "exactly two heads"?

$A = \{THH, HTH, HHT, HHH\}$
$B = \{THH, HTH, HHT\}$
$C = \{HTH, THH\}$

(1) A (2) B (3) C (4) A or B

answer: : 2) B.
The event "exactly two heads" determines a *unique subset*, containing *all* outcomes that have exactly two heads.

Events, sets and words

Experiment: toss a coin 3 times.

Which of the following describes the event $\{THH, HTH, HHT\}$?

(1) "exactly one head"
(2) "exactly one tail"
(3) "at most one tail"
(4) none of the above

answer: (2) "exactly one tail"
Notice that the same event $E \subset \Omega$ may be described in words in multiple ways ("exactly 2 heads" and "exactly 1 tail").

Events, sets and words

Experiment: toss a coin 3 times.

The events "exactly 2 heads" and "exactly 2 tails" are disjoint.

(1) True (2) False

answer: True: $\{THH, HTH, HHT\} \cap \{TTH, THT, HTT\} = \emptyset$.

Events, sets and words

Experiment: toss a coin 3 times.

The event "at least 2 heads" implies the event "exactly two heads".

(1) True (2) False

False. It's the other way around:
$\{THH, HTH, HHT\} \subset \{THH, HTH, HHT, HHH\}$.

Probability rules in mathematical notation

Sample space: $S = \{\omega_1, \omega_2, \dots, \omega_n\}$

Outcome: $\omega \in S$

Probability between 0 and 1: $0 \leq P(\omega) \leq 1$

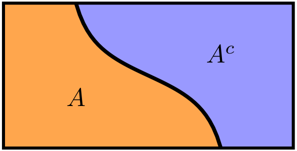
Total probability is 1: $\displaystyle\sum_{j=1}^{n} P(\omega_j) = 1, \quad \sum_{\omega \in S} P(\omega) = 1$

Event A: $\displaystyle P(A) = \sum_{\omega \in A} P(\omega)$

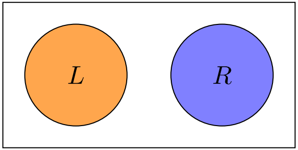
Probability and set operations on events

Events A, L, R

- Complements: $P(A^c) = 1 - P(A)$.
- Disjoint events: If L and R are disjoint then $P(L \cup R) = P(L) + P(R)$.
- Inclusion-exclusion principle: For any L and R: $P(L \cup R) = P(L) + P(R) - P(L \cap R)$.



$\Omega = A \cup A^c$, no overlap



$L \cup R$, no overlap

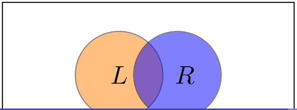


Table question

- Class has 50 students
- 20 male (M), 25 brown-eyed (B)

What is the range of possible values for $p = P(M \cup B)$?

(a) $p \leq .4$
(b) $.4 \leq p \leq .5$
(c) $.4 \leq p \leq .9$
(d) $.5 \leq p \leq .9$
(e) $.5 \leq p$

answer: (d) $.5 \leq p \leq .9$
Explanation on next slide.

Solution to CQ

The easy way to answer this is that $A \cup B$ has a minumum of 25 members (when all males are brown-eyed) and a maximum of 45 members (when no males have brown-eyes). So, the probability ranges from .5 to .9
Thinking about it in terms of the inclusion-exclusion principle we have

$$P(M \cup B) = P(M) + P(B) - P(M \cap B) = .9 - P(M \cap B).$$

So the maximum possible value of $P(M \cup B)$ happens if M and B are disjoint, so $P(M \cap B) = 0$. The minimum happens when $M \subset B$, so $P(M \cap B) = P(M) = .4$.

Table Question

Experiment:

1. Your table should make 9 rolls of a 20-sided die (one each if the table is full).
2. Check if two rolls at your table match.

Repeat the experiment five times and record the results.

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Repeat the experiment five times and record the results.

For this experiment, how would you define the sample space, probability function, and event?

Compute the true probability that two rolls (in one trial) match and compare with your experimental result.

<u>**answer:**</u> $1 - (20 \cdot 19 \cdots 13 \cdot 12)/(20^9) = 0.881$. (The explanation is on the next frame.)

Board Question Solution

Sample space S is all sequences of 9 numbers between 1 and 20. (We are assuming there are 9 people at table.) We find the size of S using the rule of product. There are 20 ways to choose the first number in the sequence, followed by 20 ways to choose the second, etc. Thus, $|S| = 20^9$.
It is sometimes easier to calculate the probability of an event indirectly by calculating the probability of the complement and using the formula

$$P(A) = 1 - P(A^c).$$

In our case, A is the event 'there is a match', so A^c is the event 'there is no match'. We can use the rule of product to compute $|A^c|$ as follows. There are 20 ways to choose the first number, then 19 ways to choose the second, etc. down to 12 ways to choose the ninth number. Thus, we have

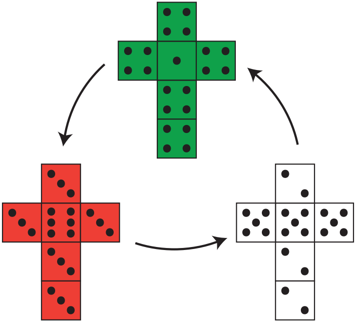
$$|A^c| = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12$$

That is $|A^c| = {}_{20}P_9$. Putting this all together

$$P(A) = 1 - P(A^c) = 1 - \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{20^9} = .881$$

Jon's dice

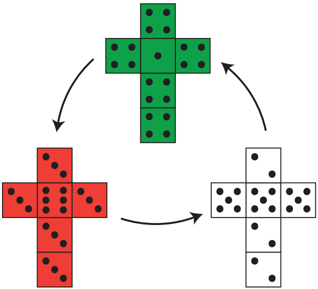
Jon has three six-sided dice with unusual numbering.



A game consists of two players each choosing a die. They roll once and the highest number wins.

Which die would you choose?

Board Question



1. Make probability tables for the red and white dice.
2. Make a probability table for the product sample space of red and white.
3. Compute the probability that red beats white.

4. Pair up with another group. Have one group compare red vs. green and the other compare green vs. white. Based on the three comparisons rank the dice from best to worst.

Computations for solution

	Red die		White die		Green die	
Outcomes	3	6	2	5	1	4
Probability	5/6	1/6	3/6	3/6	1/6	5/6

- The 2×2 tables show pairs of dice.
- Each entry is the probability of seeing the pair of numbers corresponding to that entry.
- The color gives the winning die for that pair of numbers. (We use black instead of white when the white die wins.)

		White		Green	
		2	5	1	4
Red	3	15/36	15/36	5/36	25/36
	6	3/36	3/36	1/36	5/36
Green	1	3/36	3/36		
	4	15/36	15/36		

Answer to board question continued

		White		Green	
		2	5	1	4
Red	3	15/36	15/36	5/36	25/36
	6	3/36	3/36	1/36	5/36
Green	1	3/36	3/36		
	4	15/36	15/36		

The three comparisons are:

$$\begin{aligned} \text{P(red beats white)} &= 21/36 = 7/12 \\ \text{P(white beats green)} &= 21/36 = 7/12 \\ \text{P(green beats red)} &= 25/36 \end{aligned}$$

Thus: red is better than white is better than green is better than red.

There is no best die: the relation 'better than' is not transitive.

Concept Question

Lucky Larry has a coin that you're quite sure is not fair.

- He will flip the coin twice
- It's your job to bet whether the outcomes will be the same (HH, TT) or different (HT, TH).

Which should you choose?

1. Same
2. Different
3. It doesn't matter, same and different are equally likely

Board Question

Lucky Larry has a coin that you're quite sure is not fair.

- He will flip the coin twice
- It's your job to bet whether the outcomes will be the same (HH, TT) or different (HT, TH).

Which should you choose?

1. Same 2. Different 3. Doesn't matter

Question: Let p be the probability of heads and use probability to answer the question.

(If you don't see the symbolic algebra try p = .2, p=.5)

Solution

answer: 1. Same (same is more likely than different)
The key bit of arithmetic is that if $a \neq b$ then

$$(a-b)^2 > 0 \iff a^2 + b^2 > 2ab.$$

To keep the notation cleaner, let's use $P(T) = (1-p) = q$.
Since the flips are independent (we'll discuss this next week) the probabilities multiply. This gives the following 2×2 table.

		second flip	
		H	T
first flip	H	p^2	pq
	T	qp	q^2

So, $P(\text{same}) = p^2 + q^2$ and $P(\text{diff}) = 2pq$. Since the coin is unfair we know $p \neq q$. Now we use our key bit of arithmetic to say

$$p^2 + q^2 > 2pq \quad \Rightarrow \quad P(\text{same}) > P(\text{different}).$$

QED