Choosing Priors
Probability Intervals

18.05 Spring 2017

Two-parameter tables: Malaria

In the 1950s scientists injected 30 African "volunteers" with malaria.

S = carrier of sickle-cell gene

N = non-carrier of sickle-cell gene

$D+$ = developed malaria

$D-$ = did not develop malaria

	$D+$	$D-$	
S	2	13	15
N	14	1	15
	16	14	30

Model

$\theta_S =$ probability an injected S develops malaria.

$\theta_N =$ probability an injected N develops malaria.

Assume conditional independence between all the experimental subjects.

Likelihood is a function of both θ_S and θ_N:

$$P(\text{data}|\theta_S, \theta_N) = c\,\theta_S^2(1-\theta_S)^{13}\theta_N^{14}(1-\theta_N).$$

Hypotheses: pairs (θ_S, θ_N).

Finite number of hypotheses: θ_S and θ_N are each one of $0, .2, .4, .6, .8, 1$.

Color-coded two-dimensional tables

Hypotheses

$\theta_N \backslash \theta_S$	0	0.2	0.4	0.6	0.8	1
1	(0,1)	(.2,1)	(.4,1)	(.6,1)	(.8,1)	(1,1)
0.8	(0,.8)	(.2,.8)	(.4,.8)	(.6,.8)	(.8,.8)	(1,.8)
0.6	(0,.6)	(.2,.6)	(.4,.6)	(.6,.6)	(.8,.6)	(1,.6)
0.4	(0,.4)	(.2,.4)	(.4,.4)	(.6,.4)	(.8,.4)	(1,.4)
0.2	(0,.2)	(.2,.2)	(.4,.2)	(.6,.2)	(.8,.2)	(1,.2)
0	(0,0)	(.2,0)	(.4,0)	(.6,0)	(.8,0)	(1,0)

Table of hypotheses for (θ_S, θ_N)

Corresponding level of protection due to S:

red = strong, pink = some, orange = none,

white = negative.

Color-coded two-dimensional tables

Likelihoods (scaled to make the table readable)

$\theta_N \backslash \theta_S$	0	0.2	0.4	0.6	0.8	1
1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.8	0.00000	1.93428	0.18381	0.00213	0.00000	0.00000
0.6	0.00000	0.06893	0.00655	0.00008	0.00000	0.00000
0.4	0.00000	0.00035	0.00003	0.00000	0.00000	0.00000
0.2	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Likelihoods scaled by $100000/c$

$$p(\text{data}|\theta_S, \theta_N) = c\,\theta_S^2(1-\theta_S)^{13}\theta_N^{14}(1-\theta_N).$$

Color-coded two-dimensional tables

Flat prior

$\theta_N \backslash \theta_S$	0	0.2	0.4	0.6	0.8	1	$p(\theta_N)$
1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
0.8	1/36	1/36	1/36	1/36	1/36	1/36	1/6
0.6	1/36	1/36	1/36	1/36	1/36	1/36	1/6
0.4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
0.2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
0	1/36	1/36	1/36	1/36	1/36	1/36	1/6
$p(\theta_S)$	1/6	1/6	1/6	1/6	1/6	1/6	1

Flat prior $p(\theta_S, \theta_N)$: each hypothesis (square) has equal probability

Color-coded two-dimensional tables

Posterior to the flat prior

$\theta_N \backslash \theta_S$	0	0.2	0.4	0.6	0.8	1	$p(\theta_N \mid \text{data})$
1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.8	0.00000	0.88075	0.08370	0.00097	0.00000	0.00000	0.96542
0.6	0.00000	0.03139	0.00298	0.00003	0.00000	0.00000	0.03440
0.4	0.00000	0.00016	0.00002	0.00000	0.00000	0.00000	0.00018
0.2	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
$p(\theta_S \mid \text{data})$	0.00000	0.91230	0.08670	0.00100	0.00000	0.00000	1.00000

Normalized posterior to the flat prior: $p(\theta_S, \theta_N \mid \text{data})$

Strong protection: $P(\theta_N - \theta_S > .5 \mid \text{data})$ = sum of red = .88075

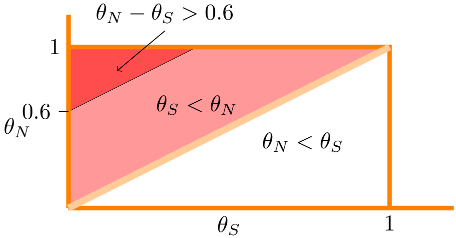
Some protection: $P(\theta_N > \theta_S \mid \text{data})$ = sum pink and red = .99995

Continuous two-parameter distributions

Sometimes continuous parameters are more natural.

Malaria example (from class notes):
  discrete prior table from the class notes.
  Similarly colored version for the continuous parameters (θ_S, θ_N) over range $[0, 1] \times [0, 1]$.

$\theta_N \backslash \theta_S$	0	0.2	0.4	0.6	0.8	1
1	(0,1)	(.2,1)	(.4,1)	(.6,1)	(.8,1)	(1,1)
0.8	(0,.8)	(.2,.8)	(.4,.8)	(.6,.8)	(.8,.8)	(1,.8)
0.6	(0,.6)	(.2,.6)	(.4,.6)	(.6,.6)	(.8,.6)	(1,.6)
0.4	(0,.4)	(.2,.4)	(.4,.4)	(.6,.4)	(.8,.4)	(1,.4)
0.2	(0,.2)	(.2,.2)	(.4,.2)	(.6,.2)	(.8,.2)	(1,.2)
0	(0,0)	(.2,0)	(.4,0)	(.6,0)	(.8,0)	(1,0)



The probabilities are given by double integrals over regions.

Treating severe respiratory failure*

*Adapted from *Statistics: a Bayesian Perspective* by Donald Berry

Two treatments for newborns with severe respiratory failure.

1. CVT: conventional therapy (hyperventilation and drugs)

2. ECMO: extracorporeal membrane oxygenation (invasive procedure)

In 1983 in Michigan:

$19/19$ ECMO babies survived and $0/3$ CVT babies survived.

Later Harvard ran a randomized study:

$28/29$ ECMO babies survived and $6/10$ CVT babies survived.

Board question: updating two parameter priors

Michigan: 19/19 ECMO babies and 0/3 CVT babies survived.

Harvard: 28/29 ECMO babies and 6/10 CVT babies survived.

θ_E = probability that an ECMO baby survives
θ_C = probability that a CVT baby survives

Consider the values 0.125, 0.375, 0.625, 0.875 for θ_E and θ_S

1. Make the 4×4 prior table for a flat prior.
2. Based on the Michigan results, create a reasonable informed prior table for analyzing the Harvard results (unnormalized is fine).
3. Make the likelihood table for the Harvard results.
4. Find the posterior table for the informed prior.
5. Using the informed posterior, compute the probability that ECMO is better than CVT.
6. Also compute the posterior probability that $\theta_E - \theta_C \geq 0.6$.
(The posted solutions will also show 4-6 for the flat prior.)

Solution

Flat prior

		θ_E			
		0.125	0.375	0.625	0.875
	0.125	0.0625	0.0625	0.0625	0.0625
θ_C	0.375	0.0625	0.0625	0.0625	0.0625
	0.625	0.0625	0.0625	0.0625	0.0625
	0.875	0.0625	0.0625	0.0625	0.0625

Informed prior (this is unnormalized)

		θ_E			
		0.125	0.375	0.625	0.875
	0.125	18	18	32	32
θ_C	0.375	18	18	32	32
	0.625	18	18	32	32
	0.875	18	18	32	32

(Rationale for the informed prior is on the next slide.)

Solution continued

Since 19/19 ECMO babies survived we believe θ_E is probably near 1.0
That 0/3 CVT babies survived is not enough data to move from a uniform distribution. (Or we might distribute a little more probability to larger θ_C.)
So for θ_E we split 64% of probability in the two higher values and 36% for the lower two. Our prior is the same for each value of θ_C.

Likelihood

Entries in the likelihood table are $\theta_E^{28}(1-\theta_E)\theta_C^6(1-\theta_C)^4$. We don't bother including the binomial coefficients since they are the same for every entry.

		θ_E			
		0.125	0.375	0.625	0.875
	0.125	1.012e-31	1.653e-18	1.615e-12	6.647-09
θ_C	0.375	1.920e-29	3.137e-16	3.065e-10	1.261-06
	0.625	5.332e-29	8.713e-16	8.513e-10	3.504e-06
	0.875	4.95e-30	8.099e-17	7.913e-11	3.257e-07

(Posteriors are on the next slides).

Solution continued

Flat posterior

The posterior table is found by multiplying the prior and likelihood tables and normalizing so that the sum of the entries is 1. We call the posterior derived from the flat prior the flat posterior. (Of course the flat posterior is not itself flat.)

		θ_E			
		0.125	0.375	0.625	0.875
	0.125	.984e-26	3.242e-13	3.167e-07	0.001
θ_c	0.375	.765e-24	6.152e-11	6.011e-05	$\boxed{0.247}$
	0.625	1.046e-23	1.709e-10	1.670e-04	$\boxed{0.687}$
	0.875	9.721e-25	1.588e-11	1.552e-05	0.0639

The boxed entries represent most of the probability where $\theta_E > \theta_C$.

All our computations were done in R. For the flat posterior:
Probability ECMO is better than CVT is

$$P(\theta_E > \theta_C \,|\, \text{Harvard data}) = 0.936$$
$$P(\theta_E - \theta_C \geq 0.6 \,|\, \text{Harvard data}) = 0.001$$

Solution continued

Informed posterior

		θ_E			
		0.125	0.375	0.625	0.875
	0.125	1.116e-26	1.823e-13	3.167e-07	0.001
θ_C	0.375	2.117e-24	3.460e-11	6.010e-05	0.2473
	0.625	5.882e-24	9.612e-11	1.669e-04	0.6871
	0.875	5.468e-25	8.935e-12	1.552e-05	0.0638

For the informed posterior:

$$P(\theta_E > \theta_C \,|\, \text{Harvard data}) = 0.936$$
$$P(\theta_E - \theta_C \geq 0.6 \,|\, \text{Harvard data}) = 0.001$$

Note: Since both flat and informed prior gave the same answers we gain confidence that these calculations are robust. That is, they are not too sensitive to our exact choice of prior.

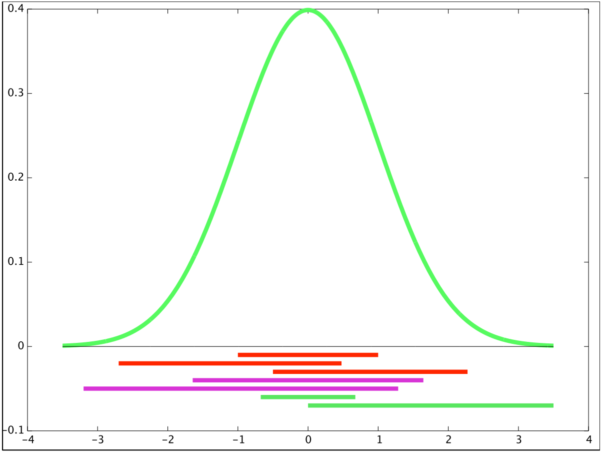
Probability intervals

- **Example.** If $P(a \leq \theta \leq b) = 0.7$ then $[a, b]$ is a 0.7 probability interval for θ. We also call it a 70% probability interval.
- **Example.** Between the 0.05 and 0.55 quantiles is a 0.5 probability interval. Another 50% probability interval goes from the 0.25 to the 0.75 quantiles.
- **Symmetric probability intevals.** A symmetric 90% probability interval goes from the 0.05 to the 0.95 quantile.
- **Q-notation.** Writing q_p for the p quantile we have 0.5 probability intervals $[q_{0.25}, q_{0.75}]$ and $[q_{0.05}, q_{0.55}]$.
- **Uses.** To summarize a distribution; to help build a subjective prior.

Probability intervals in Bayesian updating

- We have p-probability intervals for the prior $f(\theta)$.
- We have p-probability intervals for the posterior $f(\theta|x)$.
- The latter tend to be smaller than the former. Thanks data!
- Probability intervals are good, concise statements about our current belief/understanding of the parameter of interest.
- We can use them to help choose a good prior.

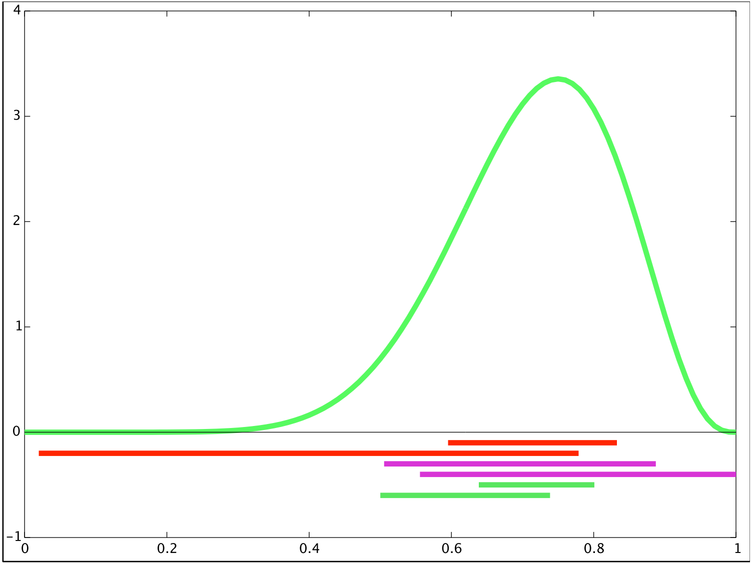
Probability intervals for normal distributions



Red = 0.68, magenta = 0.9, green = 0.5

68% of the probability for a standard normal is between -1 and 1.

Probability intervals for beta distributions



Red = 0.68, magenta = 0.9, green = 0.5

Concept question

To convert an 80% probability interval to a 90% interval should you shrink it or stretch it?

1. Shrink **2.** Stretch.

answer: 2. Stretch. A bigger probability requires a bigger interval.

Reading questions

The following slides contain bar graphs of 2015 responses to the reading questions. Each bar represents one student's estimate of their own 50% probability interval (from the 0.25 quantile to the 0.75 quantile).

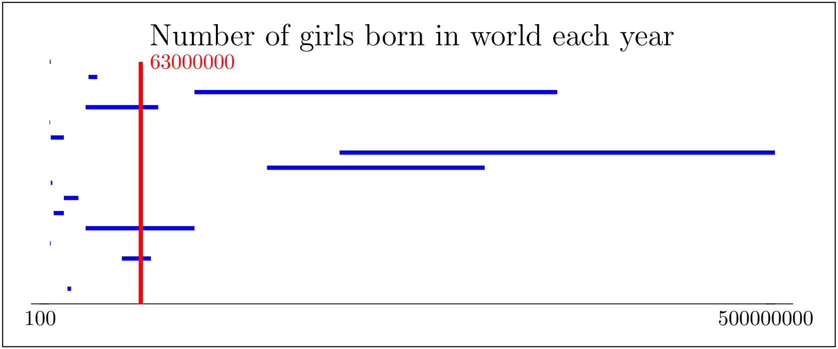
Here is what we found for answers to the questions:

1. Number of girls born in the world each year: I had trouble finding a reliable source. Wiki.answers.com gave the number of 130 million births in 2005. If we take what seems to be the accepted ratio of 1.07 boys born for every girl then $130/2.07 = 62.8$ million baby girls.

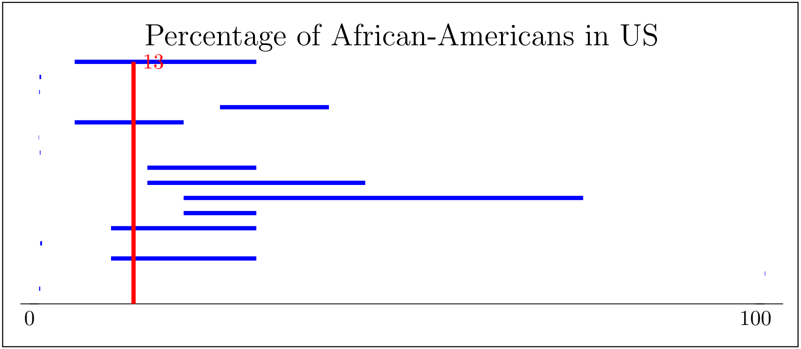
2. Percentage of African-Americans in the U.S.: 13.1% (`http://quickfacts.census.gov/qfd/states/00000.html`)

3. Percentage of African-Americans in the U.S.: 13.1% (`http://quickfacts.census.gov/qfd/states/00000.html`)

Subjective probability 1 (50% probability interval)

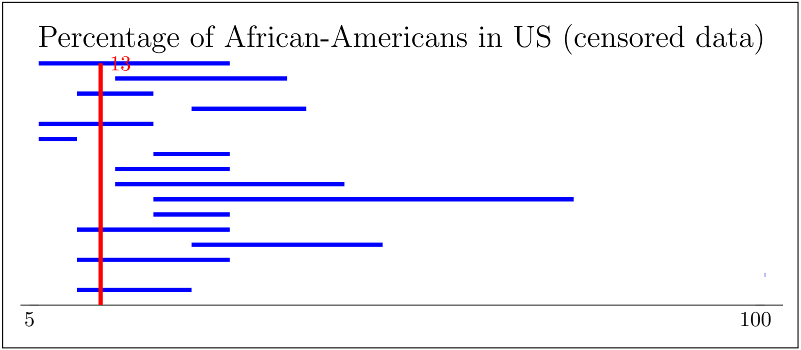


Subjective probability 2 (50% probability interval)

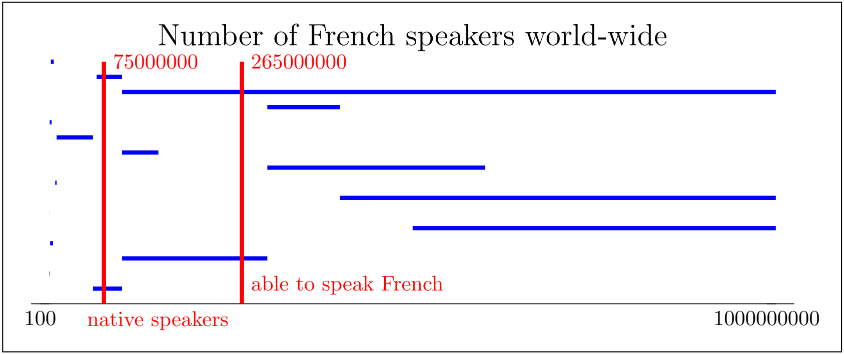


Subjective probability 2 censored (50% probability interval)

Censored by changing numbers less than 1 to percentages and ignoring numbers bigger than 100.



Subjective probability 4 (50% probability interval)



Meteor!

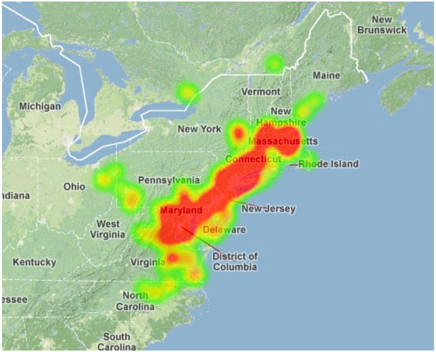
On March 22, 2013, a meteor lit up the skies.
It passed almost directly over NYC.

Board question: Meteor! No data.

Draw a pdf $f(\theta)$ for the meteor's direction.

Draw a 0.5-probability interval. How long is it?

Board question: Meteor! Heat map.

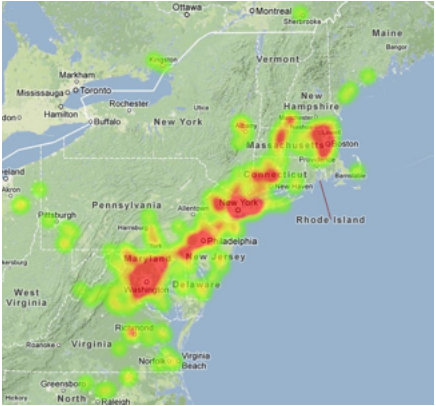


Heat map of the number of reported sightings

Draw a pdf $f(\theta|x_1)$ for the meteor's direction.

Draw a 0.5-probability interval. How long is it?

Board question: Meteor! Finer heat map.

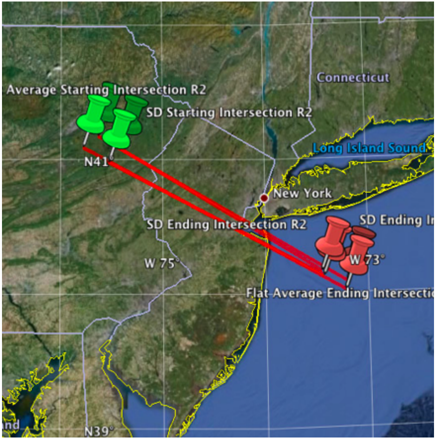


Heat map of the number of reported sightings

Draw a pdf $f(\theta|x_2)$ for the meteor's direction.

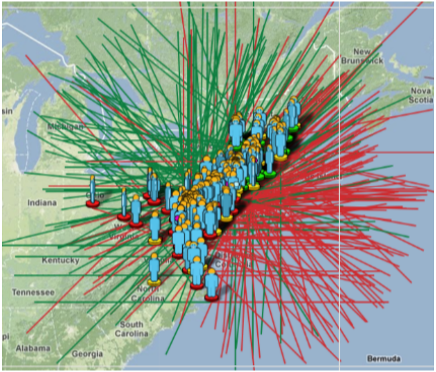
Draw a 0.5-probability interval. How long is it?

Discussion: Meteor! Actual direction.



Discussion: how good is the data of the heat map for determining the direction of the meteor?

Discussion: Meteor! Better data.



Here's the actual data they used to calculate the direction:
1236 reports of location and orientation

`http://amsmeteors.org/2013/03/update-for-march-22-2013-northeast-fireball/`