Bayesian Updating: Continuous Priors

18.05 Spring 2018

Compute

$$\int_a^b f(x|\theta)\,f(\theta)\,d\theta$$

Vacation!

Next pset is due one week after vacation!

No office hours March 23–April 1!

Beta distribution

$$f(\theta) = \frac{(a+b-1)!}{(a-1)!(b-1)!}\theta^{a-1}(1-\theta)^{b-1}$$

We use a and b pos. integers but real a, $b > 0$ are allowed.

http://mathlets.org/mathlets/beta-distribution/

Observation:

Factorials are a normalizing factor, so if we have a pdf on $[0,1]$

$$f(\theta) = c\theta^{a-1}(1-\theta)^{b-1}$$

then

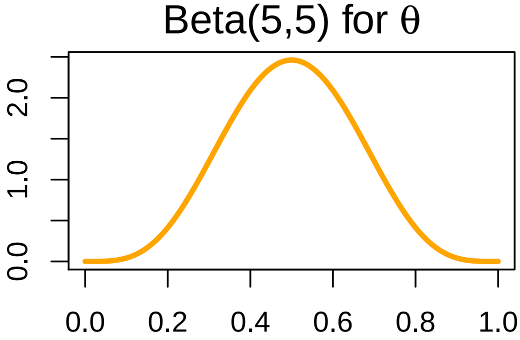
$$\theta \sim \text{beta}(a, b)$$

and

$$c = \frac{(a+b-1)!}{(a-1)!(b-1)!}$$

Board question preamble: beta priors

Suppose you are testing a new medical treatment with unknown probability of success θ. You don't know that θ, but your prior belief is that it's probably not too far from 0.5. You capture this intuition with a beta(5,5) prior on θ.



To sharpen this distribution you take data and update the prior.

Question on next slide.

Board question: beta priors

- $Beta(a, b)$: $f(\theta) = \dfrac{(a+b-1)!}{(a-1)!(b-1)!}\theta^{a-1}(1-\theta)^{b-1}$
- Treatment has prior $f(\theta) \sim \text{beta}(5, 5)$

1. Suppose you test it on 10 patients and have 6 successes. Find the posterior distribution on θ. Identify the type of the posterior distribution.

2. Suppose you recorded the order of the results and got S S S F F S S S F F. Find the posterior based on this data.

3. Using your answer to (2) give an integral for the posterior predictive probability of success with the next patient.

4. Use what you know about pdf's to evaluate the integral without computing it directly

Solution

1. Prior pdf is $f(\theta) = \frac{9!}{4!\,4!}\theta^4(1-\theta)^4 = c_1\theta^4(1-\theta)^4$.

hypoth.	prior	likelihood	Bayes numer.	posterior
θ	$c_1\theta^4(1-\theta)^4\,d\theta$	$\binom{10}{6}\theta^6(1-\theta)^4$	$c_3\theta^{10}(1-\theta)^8\,d\theta$	beta$(11, 9)$

We know the normalized posterior is a beta distribution because it has the form of a beta distribution ($c\theta^{a-}(1-\theta)^{b-1}$ on [0,1]) so by our earlier observation it must be a beta distribution.

2. The answer is the same. The only change is that the likelihood has a coefficient of 1 instead of a binomial coefficent.

3. The posterior on θ is beta$(11, 9)$ which has density

$$f(\theta\,|, \text{data}) = \frac{19!}{10!\,8!}\theta^{10}(1-\theta)^8.$$

Solution to (3) continued on next slide

Solution continued

The law of total probability says that the posterior predictive probability of success is

$$
\begin{aligned}
P(\text{success}\,|\,\text{data}) &= \int_0^1 f(\text{success}\,|\,\theta)\cdot f(\theta\,|\,\text{data})\,d\theta \\
&= \int_0^1 \theta\cdot\frac{19!}{10!\,8!}\theta^{10}(1-\theta)^8\,d\theta = \int_0^1 \frac{19!}{10!\,8!}\theta^{11}(1-\theta)^8\,d\theta
\end{aligned}
$$

4. We compute the integral in (3) by relating it to the pdf of beta$(12, 9)$: $\frac{20!}{11!\,8!}\theta^{11}(1-\theta)^7$. Since the pdf of beta$(12, 9)$ integrates to 1 we have

$$
\int_0^1 \frac{20!}{11!\,8!}\theta^{11}(1-\theta)^7 = 1 \quad\Rightarrow\quad \int_0^1 \theta^{11}(1-\theta)^7 = \frac{11!\,8!}{20!}.
$$

Thus

$$
\int_0^1 \frac{19!}{10!\,8!}\theta^{11}(1-\theta)^8\,d\theta = \frac{19!}{10!\,8!}\cdot\frac{11!\,8!}{20!}. = \boxed{\frac{11}{20}}.
$$

Conjugate priors

We had

- Prior $f(\theta)\,d\theta$: beta distribution
- Likelihood $p(x|\theta)$: binomial distribution
- Posterior $f(\theta|x)\,d\theta$: beta distribution

The beta distribution is called a **conjugate prior** for the binomial likelihood.

That is, the beta prior becomes a beta posterior and updating is easy!
Only the parameters have been changed to reflect the data.

Concept Question

Suppose your prior $f(\theta)$ in the bent coin example is Beta$(6, 8)$. You flip the coin 7 times, getting 2 heads and 5 tails. What is the posterior pdf $f(\theta|x)$?

1. Beta(2,5)

2. Beta(11,10)

3. Beta(6,8)

4. Beta(8,13)

We saw in the previous board question that 2 heads and 5 tails will update a beta(a, b) prior to a beta$(a + 2, b + 5)$ posterior.

<u>**answer:**</u> (4) beta$(8, 13)$.

Reminder: predictive probabilities

Continuous hypotheses θ, discrete data $x_1, x_2, \ldots$
(Assume trials are independent given the hypothesis θ.)

Prior predictive probability

$$p(x_1) = \int p(x_1 \mid \theta) f(\theta)\, d\theta$$

Posterior predictive probability

$$p(x_2 \mid x_1) = \int p(x_2 \mid \theta) f(\theta \mid x_1)\, d\theta$$

Analogous to discrete hypotheses: $\mathcal{H}_1, \mathcal{H}_2, \ldots$.

$$p(x_1) = \sum_{i=1}^{n} p(x_1 \mid \mathcal{H}_i) P(\mathcal{H}_i) \qquad p(x_2 \mid x_1) = \sum_{i=1}^{n} p(x_2 \mid \mathcal{H}_i) p(\mathcal{H}_i \mid x_1).$$

Continuous priors, continuous data

Bayesian update tables:

hypoth.	prior	likelihood	Bayes numerator	posterior $f(\theta\|x)\,d\theta$
θ	$f(\theta)\,d\theta$	$\phi(x\,\|\,\theta)$	$\phi(x\,\|\,\theta)f(\theta)\,d\theta$	$\dfrac{\phi(x\,\|\,\theta)f(\theta)\,d\theta}{\phi(x)}$
total	1		$\phi(x)$	1

$$\phi(x) = \int \phi(x\,|\,\theta)f(\theta)\,d\theta$$

$\phi(x)dx$ is the prior predictive probability that the data belongs to a small interval of size dx around x.

Normal prior, normal data

$\mathsf{N}(\mu, \sigma^2)$ has density

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-(y-\mu)^2/2\sigma^2}.$$

Observation:
The coefficient is a normalizing factor, so if we have a pdf

$$f(y) = c\mathrm{e}^{-(y-\mu)^2/2\sigma^2}$$

then

$$y \sim \mathsf{N}(\mu, \sigma^2) \qquad \textbf{and} \qquad c = \frac{1}{\sigma\sqrt{2\pi}}$$

Better: All we need is $f(y) = c\mathrm{e}^{-ay^2+by}$

Board question: normal prior, normal data

- $\mathrm{N}(\mu, \sigma^2)$ has pdf: $\quad f(y) = \dfrac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-(y-\mu)^2/2\sigma^2}$.
- Suppose our data follows a $\mathrm{N}(\theta, 4)$ distribution with unknown mean θ and variance 4. That is

$$\phi(x \mid \theta) = \text{pdf of } \mathrm{N}(\theta, 4)$$

- Suppose our prior on θ is $\mathrm{N}(3, 1)$.

Suppose we obtain data $x_1 = 5$.

1. Use the data to find the posterior pdf for θ.

Write out your tables clearly. Use (and understand) infinitesimals.

You will need to complete a square to do the updating!

Solution

We have:

Prior: $\theta \sim \mathrm{N}(3,1)$: $\quad f(\theta) = c_1 \mathrm{e}^{-(\theta-3)^2/2}$

Likelihood $x \sim \mathrm{N}(\theta, 4)$: $\quad f(x \,|\, \theta) = c_2 \mathrm{e}^{-(x-\theta)^2/8}$

For $x = 5$ the likelihood is $c_2 \mathrm{e}^{-(5-\theta)^2/8}$

hypoth.	prior	likelihood	Bayes numer.
θ	$c_1 \mathrm{e}^{-(\theta-3)^2/2}\, d\theta$	$c_2 \mathrm{e}^{-(5-\theta)^2/8}\, dx$	$c_3 \mathrm{e}^{-(\theta-3)^2/2} \mathrm{e}^{-(5-\theta)^2/8}\, d\theta\, dx$

A bit of algebraic manipulation of the Bayes numerator gives

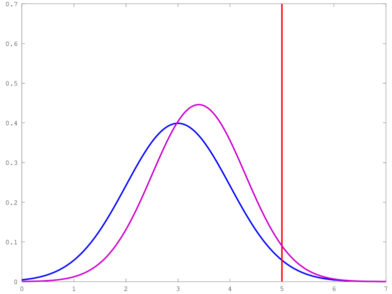
$$c_3 \mathrm{e}^{-(\theta-3)^2/2} \mathrm{e}^{-(5-\theta)^2/8}\, d\theta\, dx = c_3 \mathrm{e}^{-\frac{5}{8}[\theta^2 - \frac{34}{5}\theta + 61]} = c_3 \mathrm{e}^{-\frac{5}{8}[(\theta-17/5)^2 + 61 - (17/5)^2]}$$

$$= c_3 \mathrm{e}^{-\frac{5}{8}(61-(17/5)^2)} \mathrm{e}^{-\frac{5}{8}(\theta-17/5)^2}$$

$$= c_4 \mathrm{e}^{-\frac{5}{8}(\theta-17/5)^2} = c_4 \mathrm{e}^{-\frac{(\theta-17/5)^2}{2\cdot\frac{4}{5}}}$$

The last expression shows the posterior is $\mathrm{N}\left(\frac{17}{5}, \frac{4}{5}\right)$.

Solution graphs



prior = blue; posterior = purple; data = red

Data:	$x_1 = 5$	
Prior is normal:	$\mu_{\text{prior}} = 3$;	$\sigma_{\text{prior}} = 1$
Likelihood is normal:	$\mu = \theta$;	$\sigma = 2$
Posterior is normal	$\mu_{\text{posterior}} = 3.4$;	$\sigma_{\text{posterior}} = 0.894$

- Will see simple formulas for doing this update next time.

Board question: Romeo and Juliet

Romeo is always late. How late follows a uniform distribution uniform$(0, \theta)$ with unknown parameter θ in hours.

Juliet knows that $\theta \leq 1$ hour and she assumes a flat prior for θ on $[0, 1]$.

On their first date Romeo is 15 minutes late. Use this data to update the prior distribution for θ.

(a) Find and graph the prior and posterior pdfs for θ.

(b) Find the prior predictive pdf for how late Romeo will be on the first date and the posterior predictive pdf of how late he'll be on the second date (if he gets one!). Graph these pdfs.

See next slides for solution

Solution

Parameter of interest: $\theta =$ upper bound on R's lateness.
Data: $x_1 = 0.25$.
Goals: (a) Posterior pdf for θ
(b) Predictive pdf's—requires pdf's for θ
In the update table we split the hypotheses into the two different cases $\theta < 0.25$ and $\theta \geq 0.25$:

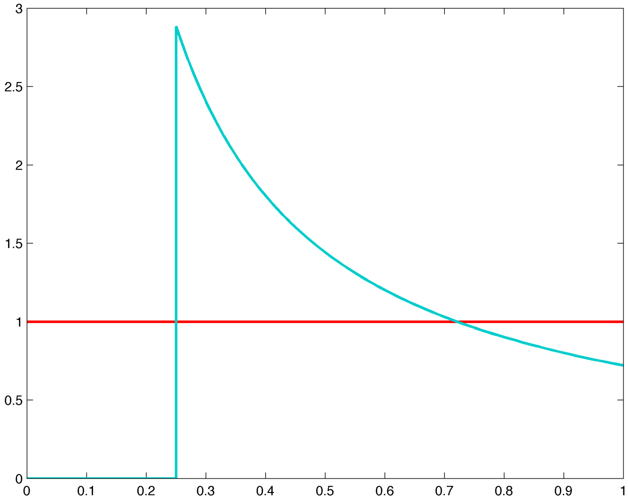
hyp.	prior $f(\theta)$	likelihood $\phi(x_1\|\theta)$	Bayes numerator	posterior $f(\theta\|x_1)$
$\theta < 0.25$	$d\theta$	0	0	0
$\theta \geq 0.25$	$d\theta$	$\frac{1}{\theta}$	$\frac{d\theta}{\theta}$	$\frac{c}{\theta}\, d\theta$
Tot.	1		T	1

The normalizing constant c must make the total posterior probability 1, so

$$c \int_{0.25}^{1} \frac{d\theta}{\theta} = 1 \;\Rightarrow\; c = \frac{1}{\ln(4)}.$$

Continued on next slide.

Solution graphs



Prior and posterior pdf's for θ.

Solution graphs continued

(b) Prior prediction: The likelihood function falls into cases:

$$f(x_1|\theta) = \begin{cases} \frac{1}{\theta} & \text{if } \theta \geq x_1 \\ 0 & \text{if } \theta < x_1 \end{cases}$$

Therefore the prior predictive pdf of x_1 is

$$\phi(x_1) = \int \phi(x_1|\theta) f(\theta)\, d\theta = \int_{x_1}^{1} \frac{1}{\theta}\, d\theta = -\ln(x_1).$$

continued on next slide

Solution continued

Posterior prediction:
The likelihood function is the same as before:

$$\phi(x_2|\theta) = \begin{cases} \frac{1}{\theta} & \text{if } \theta \geq x_2 \\ 0 & \text{if } \theta < x_2. \end{cases}$$

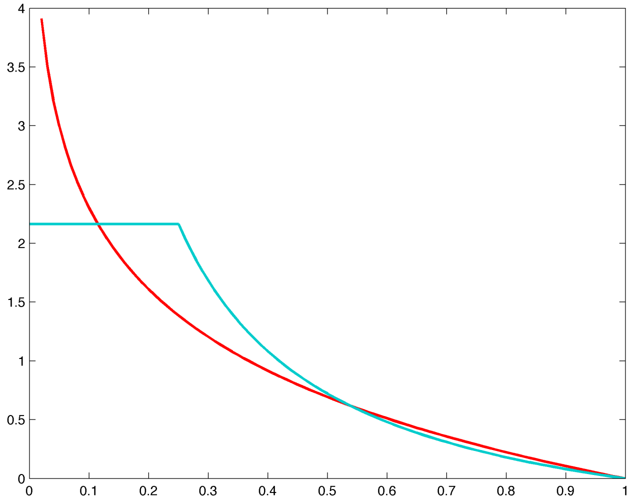
The posterior predictive pdf $\phi(x_2|x_1) = \int \phi(x_2|\theta) f(\theta|x_1)\, d\theta$. The integrand is 0 unless $\theta > x_2$ and $\theta > 0.25$. There are two cases:

$$\text{If } x_2 < 0.25: \qquad \phi(x_2|x_1) = \int_{0.25}^{1} \frac{c}{\theta^2}\, d\theta = 3c = 3/\ln(4).$$

$$\text{If } x_2 \geq 0.25: \qquad \phi(x_2|x_1) = \int_{x_2}^{1} \frac{c}{\theta^2}\, d\theta = (\frac{1}{x_2} - 1)/\ln(4)$$

Plots of the predictive pdf's are on the next slide.

Solution continued



Prior and posterior predictive pdf's for x_2

From discrete to continuous Bayesian updating

Bent coin with unknown probability of heads θ.

Data x_1: heads on one toss.

Start with a flat prior and update:

hyp.	prior	likelihood	Bayes numerator	posterior
θ	$d\theta$	θ	$\theta\, d\theta$	$2\theta\, d\theta$
Total	1		$\int_0^1 \theta\, d\theta = 1/2$	1

Posterior pdf: $f(\theta \mid x_1) = 2\theta$.

Approximate continuous by discrete

- approximate the continuous range of hypotheses by a finite number of hypotheses.
- create the discrete updating table for the finite number of hypotheses.
- consider how the table changes as the number of hypotheses goes to infinity.

Chop $[0, 1]$ into 4 intervals

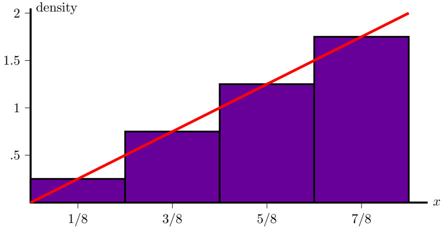
hypothesis	prior	likelihood	Bayes num.	posterior
$\theta = 1/8$	$1/4$	$1/8$	$(1/4) \times (1/8)$	$1/16$
$\theta = 3/8$	$1/4$	$3/8$	$(1/4) \times (3/8)$	$3/16$
$\theta = 5/8$	$1/4$	$5/8$	$(1/4) \times (5/8)$	$5/16$
$\theta = 7/8$	$1/4$	$7/8$	$(1/4) \times (7/8)$	$7/16$
Total	1	–	$\sum_{i=1}^{n} \theta_i \, \Delta\theta$	1

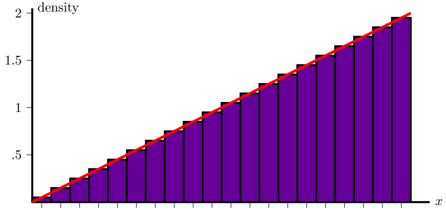
Chop $[0, 1]$ into 12 intervals

hypothesis	prior	likelihood	Bayes num.	posterior
$\theta = 1/24$	$1/12$	$1/24$	$(1/12) \times (1/24)$	$1/144$
$\theta = 3/24$	$1/12$	$3/24$	$(1/12) \times (3/24)$	$3/144$
$\theta = 5/24$	$1/12$	$5/24$	$(1/12) \times (5/24)$	$5/144$
$\theta = 7/24$	$1/12$	$7/24$	$(1/12) \times (7/24)$	$7/144$
$\theta = 9/24$	$1/12$	$9/24$	$(1/12) \times (9/24)$	$9/144$
$\theta = 11/24$	$1/12$	$11/24$	$(1/12) \times (11/24)$	$11/144$
$\theta = 13/24$	$1/12$	$13/24$	$(1/12) \times (13/24)$	$13/144$
$\theta = 15/24$	$1/12$	$15/24$	$(1/12) \times (15/24)$	$15/144$
$\theta = 17/24$	$1/12$	$17/24$	$(1/12) \times (17/24)$	$17/144$
$\theta = 19/24$	$1/12$	$19/24$	$(1/12) \times (19/24)$	$19/144$
$\theta = 21/24$	$1/12$	$21/24$	$(1/12) \times (21/24)$	$21/144$
$\theta = 23/24$	$1/12$	$23/24$	$(1/12) \times (23/24)$	$23/144$
Total	1	–	$\sum_{i=1}^{n} \theta_i \, \Delta\theta$	1

Density histogram

Density histogram for posterior pmf with 4 and 20 slices.





The original posterior pdf is shown in red.