Independence, Covariance and Correlation
18.05 Spring 2018

$X\backslash Y$	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

Independence, Covariance, Correlation

Events A and B are independent if

$$P(A \cap B) = P(A)P(B).$$

Random variables X and Y are independent if

$$F(x, y) = F_X(x)F_Y(y).$$

Discrete random variables X and Y are independent if

$$p(x_i, y_j) = p_X(x_i)p_Y(y_j).$$

Continuous random variables X and Y are independent if

$$f(x, y) = f_X(x)f_Y(y).$$

Concept question: independence I

Roll two dice: X = value on first, Y = value on second

$X\backslash Y$	1	2	3	4	5	6	$p(x_i)$
1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
6	1/36	1/36	1/36	1/36	1/36	1/36	1/6
$p(y_j)$	1/6	1/6	1/6	1/6	1/6	1/6	1

Are X and Y independent? 1. Yes 2. No

answer: 1. Yes. Every cell probability is the product of the marginal probabilities.

Concept question: independence II

Roll two dice: X = value on first, T = sum

$X\backslash T$	2	3	4	5	6	7	8	9	10	11	12	$p(x_i)$
1	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	0	1/6
2	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	1/6
3	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	1/6
4	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	1/6
5	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	1/6
6	0	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	1/6
$p(y_j)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36	1

Are X and Y independent? 1. Yes 2. No

answer: 2. No. The cells with probability zero are clearly not the product of the marginal probabilities.

Concept Question

Among the following pdf's which are independent? (Each of the ranges is a rectangle chosen so that $\int\int f(x, y)\, dx\, dy = 1$.)

(i) $f(x, y) = 4x^2y^3$.

(ii) $f(x, y) = \frac{1}{2}(x^3y + xy^3)$.

(iii) $f(x, y) = 6e^{-3x-2y}$

Put a 1 for independent and a 0 for not-independent.

(a) 111 **(b)** 110 **(c)** 101 **(d)** 100

(e) 011 **(f)** 010 **(g)** 001 **(h)** 000

answer: *(c). Explanation on next slide.*

Solution

(i) $\boxed{\text{Independent.}}$ The variables can be separated: the marginal densities are $f_X(x) = ax^2$ and $f_Y(y) = by^3$ for some constants a and b with $ab = 4$.

(ii) $\boxed{\text{Not independent.}}$ X and Y are not independent because there is no way to factor $f(x, y)$ into a product $f_X(x)f_Y(y)$.

(iii) $\boxed{\text{Independent.}}$ The variables can be separated: the marginal densities are $f_X(x) = a\mathrm{e}^{-3x}$ and $f_Y(y) = b\mathrm{e}^{-2y}$ for some constants a and b with $ab = 6$.

Covariance

Measures the degree to which two random variables vary together, e.g. height and weight of people.

X, Y random variables with means μ_X and μ_Y.

$$\mathrm{Cov}(X, Y) =_{\mathrm{def}} E((X - \mu_X)(Y - \mu_Y)).$$

- Vary together might mean X is usually bigger than μ_X when Y is bigger than μ_Y, and vice versa. In this case $(X - \mu_X)(Y - \mu_Y)$ is usually positive, so $\mathrm{Cov}(X, Y)$ is positive.
- Vary together might mean X is usually bigger than μ_X when Y is smaller than μ_Y, and vice versa. In this case $(X - \mu_X)(Y - \mu_Y)$ is usually negative, so $\mathrm{Cov}(X, Y)$ is negative.
- If X and Y don't vary together, then sign of $(X - \mu_X)$ tells nothing about sign of $(Y - \mu_Y)$. In this case $(X - \mu_X)(Y - \mu_Y)$ can be both positive and negative, so $\mathrm{Cov}(X, Y)$ might be zero or small.

Properties of covariance

Properties

1. $\text{Cov}(aX + b, cY + d) = ac\text{Cov}(X, Y)$ for constants a, b, c, d.
2. $\text{Cov}(X_1 + X_2, Y) = \text{Cov}(X_1, Y) + \text{Cov}(X_2, Y)$.
3. $\text{Cov}(X, X) = \text{Var}(X)$
4. $\text{Cov}(X, Y) = E(XY) - \mu_X \mu_Y$.
5. If X and Y are independent then $\text{Cov}(X, Y) = 0$.
6. **Warning** The converse is not true: if covariance is 0, the variables might not be independent.

Concept question

Suppose we have the following joint probability table.

$Y\backslash X$	-1	0	1	$p(y_j)$
0	0	$1/2$	0	$1/2$
1	$1/4$	0	$1/4$	$1/2$
$p(x_i)$	$1/4$	$1/2$	$1/4$	1

At your table work out the covariance Cov(X, Y).

Because the covariance is 0 we know that X and Y are independent

1. True
2. False

Key point: covariance measures the linear relationship between X and Y. It can completely miss a quadratic or higher order relationship.

Board question: computing covariance

Flip a fair coin 12 times.

Let $X =$ number of heads in the first 7 flips

Let $Y =$ number of heads on the last 7 flips.

Compute $\mathrm{Cov}(X, Y)$,

Solution

Use the properties of covariance.

X_i = the number of heads on the i^{th} flip. (So $X_i \sim$ Bernoulli(.5).)

$$X = X_1 + X_2 + \ldots + X_7 \quad \text{and} \quad Y = X_6 + X_7 + \ldots + X_{12}.$$

We know $\text{Var}(X_i) = 1/4$. Therefore using Property 2 (linearity) of covariance

$$\begin{aligned}\text{Cov}(X, Y) &= \text{Cov}(X_1 + X_2 + \ldots + X_7,\ X_6 + X_7 + \ldots + X_{12}) \\ &= \text{Cov}(X_1, X_6) + \text{Cov}(X_1, X_7) + \text{Cov}(X_1, X_8) + \ldots + \text{Cov}(X_7, X_{12})\end{aligned}$$

Since the different tosses are independent we know

$$\text{Cov}(X_1, X_6) = 0,\ \text{Cov}(X_1, X_7) = 0,\ \text{Cov}(X_1, X_8) = 0, \text{ etc.}$$

Looking at the expression for $\text{Cov}(X, Y)$ there are only two non-zero terms

$$\text{Cov}(X, Y) = \text{Cov}(X_6, X_6) + \text{Cov}(X_7, X_7) = \text{Var}(X_6) + \text{Var}(X_7) = \boxed{\frac{1}{2}}.$$

Correlation

Like covariance, but removes scale.

The *correlation coefficient* between X and Y is defined by

$$\text{Cor}(X, Y) = \rho = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}.$$

1. $\rho =$ covariance of standardized versions of X and Y.

2. ρ is dimensionless (it's a ratio).

3. $-1 \leq \rho \leq 1$.

4. $\rho = 1$ if and only if $Y = aX + b$ with $a > 0$.

5. $\rho = -1$ if and only if $Y = aX + b$ with $a < 0$.

Real-life correlations

- Over time, amount of ice cream consumption is correlated with number of pool drownings.
- In 1685 (and today) being a student is the most dangerous profession.
- In 90% of bar fights ending in a death the person who started the fight died.
- Hormone replacement therapy (HRT) is correlated with a lower rate of coronary heart disease (CHD).

Discussion is on the next slides.

Real-life correlations discussion

- Ice cream does not cause drownings. Both are correlated with summer weather.

- In a study in 1685 of the ages and professions of deceased men, it was found that the profession with the lowest average age of death was "student." But, being a student does not cause you to die at an early age. Being a student means you *are* young. This is what makes the average of those that die so low.

- A study of fights in bars in which someone was killed found that, in 90% of the cases, the person who started the fight was the one who died.

  Of course, it's the person who survived telling the story.

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(continued)

- In a widely studied example, numerous epidemiological studies showed that women who were taking combined hormone replacement therapy (HRT) also had a lower-than-average incidence of coronary heart disease (CHD), leading doctors to propose that HRT was protective against CHD. But randomized controlled trials showed that HRT caused a small but statistically significant increase in risk of CHD. Re-analysis of the data from the epidemiological studies showed that women undertaking HRT were more likely to be from higher socio-economic groups (ABC1), with better-than-average diet and exercise regimens. The use of HRT and decreased incidence of coronary heart disease were coincident effects of a common cause (i.e. the benefits associated with a higher socioeconomic status), rather than cause and effect, as had been supposed.

Correlation is not causation

Edward Tufte: "Empirically observed covariation is a necessary but not sufficient condition for causality."

Overlapping sums of uniform random variables

We made two random variables X and Y from overlapping sums of uniform random variables

For example:

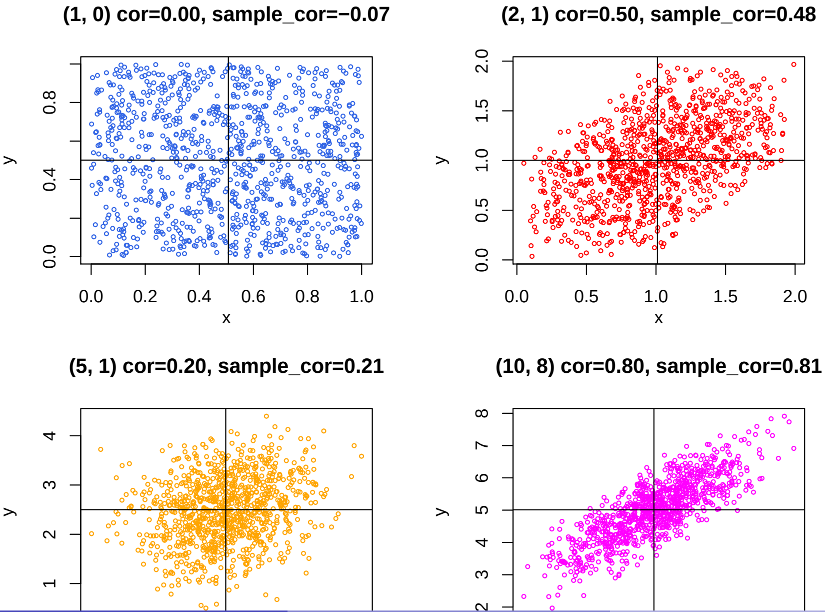
$$X = X_1 + X_2 + X_3 + X_4 + X_5$$
$$Y = X_3 + X_4 + X_5 + X_6 + X_7$$

These are sums of 5 of the X_i with 3 in common.

If we sum r of the X_i with s in common we name it (r, s).

Below are a series of scatterplots produced using R.

Scatter plots



Concept question

Toss a fair coin $2n+1$ times. Let X be the number of heads on the first $n+1$ tosses and Y the number on the last $n+1$ tosses.

If $n = 1000$ then $\text{Cov}(X, Y)$ is:

(a) 0 (b) $1/4$ (c) $1/2$ (d) 1

(e) More than 1 (f) tiny but not 0

answer: 2. $1/4$. This is computed in the answer to the next table question.

Board question

Toss a fair coin $2n+1$ times. Let X be the number of heads on the first $n+1$ tosses and Y the number on the last $n+1$ tosses.

Compute $\text{Cov}(X, Y)$ and $\text{Cor}(X, Y)$.

As usual let $X_i =$ the number of heads on the i^{th} flip, i.e. 0 or 1. Then

$$X = \sum_{1}^{n+1} X_i, \qquad Y = \sum_{n+1}^{2n+1} X_i$$

X is the sum of $n+1$ independent Bernoulli$(1/2)$ random variables, so

$$\mu_X = E(X) = \frac{n+1}{2}, \quad \text{and} \quad \text{Var}(X) = \frac{n+1}{4}.$$

Likewise, $\mu_Y = E(Y) = \dfrac{n+1}{2}$, and $\text{Var}(Y) = \dfrac{n+1}{4}$.

Continued on next slide.

Solution continued

Now,

$$\text{Cov}(X, Y) = \text{Cov}\left(\sum_{1}^{n+1} X_i \sum_{n+1}^{2n+1} X_j\right) = \sum_{i=1}^{n+1} \sum_{j=n+1}^{2n+1} \text{Cov}(X_i X_j).$$

Because the X_i are independent the only non-zero term in the above sum is $\text{Cov}(X_{n+1}X_{n+1}) = \text{Var}(X_{n+1}) = \dfrac{1}{4}$ Therefore,

$$\text{Cov}(X, Y) = \frac{1}{4}.$$

We get the correlation by dividing by the standard deviations.

$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{1/4}{(n+1)/4} = \frac{1}{n+1}.$$

This makes sense: as n increases the correlation should decrease since the contribution of the one flip they have in common becomes less important.