MAT 211: Homework #3

Your Name

Spring 2026

Combinatorics

1. Prove $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. *Hint:* Use either the Binomial Theorem or an argument based on counting subsets.

2. Prove $1 + 2 + 3 + 4 + \cdots = \binom{n+1}{2}$.

3. The identity $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$ is a special case of Vandermonde's Convolution. In the following questions, we explore this equality:

   (a) For $n = 5$, compute $\sum_{k=0}^{n} \binom{n}{k}^2$ and $\binom{2n}{n}$.

   (b) How does the sum $\sum_{k=0}^{n} \binom{n}{k}^2$ relate to Pascal's triangle?

   (c) Explain why $\binom{n}{k} = \binom{n}{n-k}$ for any values $n \geq k \in \mathbb{N}$.

   (d) Compute the sum $\sum_{k=0}^{5} \binom{5}{k}\binom{5}{5-k}$.

   (e) Let A and B be disjoint sets with $|A| = |B| = 5$. How many ways are there to choose 5 elements from $A \cup B$?

4. Muriel Bristol famously claimed she could tell the difference between cups of tea in which tea had been poured into milk versus cups of tea where milk was poured into tea. To test her ability to distinguish between them, 10 cups of tea are set out in which 5 cups were prepared with the tea poured first, while 5 cups were prepared with the milk poured first. Assume Muriel then randomly sorts the cups into two groups of 5 each.

   (a) What is the probability that she correctly sorts the cups into the 5 cups that were poured tea-first and the 5 cups that were poured milk-first?

   (b) What is the probability that Muriel correctly identifies exactly 4 of the 5 that were poured milk-first?

Direct and Contrapositive Proofs

Use a direct or contrapositive method of proof for each of the following.

5. Prove for all $n \in \mathbb{N}$, if n is even, then n^2 is even.

6. Prove $\forall x, y \in \mathbb{R}$ with $x, y \geq 0$, $x \leq y$ implies $x^2 \leq y^2$.

7. Let $u, v, x, y, A, B \in \mathbb{R}$. Suppose $0 < A \leq 1$ and $B \geq 1$. Further, suppose $x, y \leq B, 0 \leq u < A$ and $0 \leq v < A$. Show $uy + vx + uv < 3AB$.

8. Let $a, b \in \mathbb{Z}$. Prove: If a is even or b is odd, then $a(b + 3)$ is even.

9. For $n \in \mathbb{Z}$, if $3n^3$ is even, then n is even.

10. Show $\forall x \in \mathbb{R}$, $x^3 - x > 0$ implies $x > -1$.

11. Suppose $x, y, z \in \mathbb{Z}$ with $x \neq 0$. Show if $x \nmid yz$ then $x \nmid y$ and $x \nmid z$.