Physics Labs

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Contents

Chapter 1

An Exercise in Making Measurements and Propagating Error

Date Performed:	September 18, 2017
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1.1 Introduction to Error

Fundamentally, nearly all knowledge is in some sense erroneous. That is to say, that with the exception of tautologies, provable mathematical statements, and other knowledge a priori, the truth or falsity of a claim cannot be known with 100% certainty. One may observe the occurrence of a certain event, say the release of an apple from a hand, one hundred times with the same result, that apple falling from that hand onto the floor, and it would be very reasonable to assume that when dropped for the one hundred and first time that apple would fall once again onto the floor. In fact, it would be rather unreasonable to assume the opposite. However, one cannot know for certain that the apple will not behave differently. Perhaps it will just float there in the air; Maybe it will fall up; Maybe all the previous falls were hallucinated and the apple never fell down in the first place. All of these assumptions would be unreasonable things to assume, but they remain possible.

Modern science is based on empiricism. From an epistemological perspective, empiricism relies on two primary assumptions. Firstly, that similar objects in similar circumstances tend to behave in similar ways. And secondly, that one can reason about the behavior of objects by observing them. These seem like very reasonable assumptions to make. However, accepting these assumptions does not mean that we have to put 100% of our faith into our observations. The scientists may realize that their observations may be flawed. However, their response is not to reject the observations, but instead to put their belief in the least flawed observations they can find. However, in order to be able to choose in which observation to believe, they must first be able to quantify just how flawed the observations could be.

One may estimate the most erroneous their observations could reasonably be and use this to figure estimate the quality of their findings. However, rarely is an interesting finding simply a matter of making an observation. Often calculations must be done regarding a series of differing observations to reach a meaningful conclusion. When estimating the error in a value one must make sure to bring the error of the initial observations along and consider them in judging the possible error in the conclusive observation.

As an exercise in judging the uncertainty in a given measurement, the area of tabletop and the density of a cylinder of metal were calculated and the estimated uncertainty in these respective values were found as well.

1.2 Procedure

1.2.1 Area of a Table Top

The length and width of a lab bench table top were found in order to find the area of that tabletop. 2 meter sticks were positioned on each end of the table such that the 10.00 cm mark was aligned with the edge of the table. Because the table was shorter than 180 cm in both length and width, the two meter sticks overlapped. A point was chosen on the 90.00 cm mark of one of the meter sticks and the distances from that point to each end were measured. This was done without removing the meter sticks. One of these distances was always 80.00 cm (90.00 cm mark - 10.00 cm initial starting position). This was done for both the length and width of the tables, and repeated 4 times each at different places on the table.

1.2.2 Density of a Metal Cylinder

The height and diameter of a small metal cylinder of tin were measured three times with calipers. The cylinder of tin was then massed with a digital

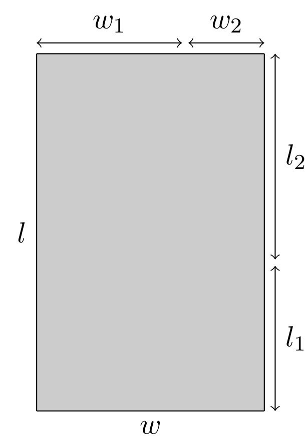

Figure 1.1: The length, l, of the table was measured as a sum of 2 intermediary lengths, l_1 and l_2, one of which was fixed to 80.00 cm. In the same sense, the width w was a sum of w_1 and w_2, one of which was also fixed to 80.00 cm

scale.

1.3 Data

1.3.1 Width of Table

The first and second intermediary widths of the tabletop were measured with meter sticks 4 times at a precision of ± 0.05 cm.

#	1st Width (cm)	2nd Width (cm)	Total Width (cm)
1	80.00	26.75	106.75
2	80.00	26.60	106.60
3	80.00	26.50	106.50
4	80.00	26.60	106.60
$\bar{x}$	80.00	26.61	106.61
δ_x	0.05	0.10	0.11

The measurements of 1st width displayed no variance, as its length was fixed to 80.00 cm, however, it had an instrument uncertainty of 0.05cm. The 2nd width was not quite as constant, and had a sample standard deviation of 0.10 cm. This, being greater than the instrument uncertainty, represents the uncertainty in w_2. Assuming the two measurement are independent, of each other the uncertainty in their sum is 0.11 cm.

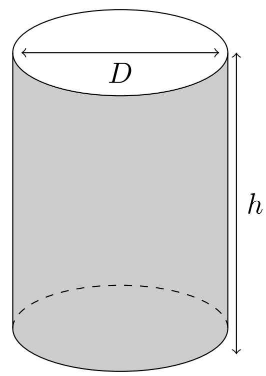

Figure 1.2: The height, h and the diameter, D of the tin cylinder were measured with calipers. The mass of the cylder was also taken with a digital scale

1.3.2 Length of Table

The first and second intermediary lengths of the tabletop were measured with meter sticks 4 times at a precision of ± 0.05 cm.

#	2st Length (cm)	2nd Length (cm)	Total Length (cm)
1	80.00	82.80	162.80
2	80.00	82.65	162.65
3	80.00	82.65	162.65
4	80.00	82.50	162.50
$\bar{x}$	80.00	82.65	162.65
δ_x	0.05	0.12	0.13

The 1st length also displayed no variance, as its length was also fixed to 80.00 cm, it had an instrument uncertainty of 0.05cm. The 2nd width was not held fixed and hence varied. It had a sample standard deviation of 0.12 cm. This, being greater than the instrument uncertainty, represents the uncertainty in l_2. Assuming the two measurement are independent of each other, the uncertainty in their sum is 0.13 cm.

1.3.3 Diameter of the Cylinder

The diameter of the cylinder was measured 3 times with callipers at a precision of ± 0.02cm.

#	Diameter of Cylinder (cm)
1	1.23
2	1.27
3	1.25
$\bar{x}$	1.25
δ_x	0.02

The sample standard deviation of the measurements is 0.02 cm, this is the uncertainty in the measurement of the diameter.

1.3.4 Height of the Cylinder

The height of the cylinder was measured 3 times with callipers at a precision of $\pm$0.02cm.

#	Height of Cylinder (cm)
1	3.20
2	3.21
3	3.21
$\bar{x}$	3.21
δ_x	0.02

The instrument uncertainty is greater than the standard deviation thus the overall uncertainty is equal to 0.02 cm.

1.3.5 Mass of the Cylinder

The mass of the cylinder was taken 3 times with a digital scale at a precision $\pm$0.01 g.

#	Mass of the Cylinder (g)
1	29.13
2	29.13
3	29.13
$\bar{x}$	29.13
δ_x	0.01

No variation existed in the measurements so the instrument uncertainty of 0.01 g was used to represent the overall uncertainty.

1.4 Analysis

1.4.1 Area of the Tabletop

The area of the rectangular table top is expressible simply as

$$A = l \cdot w. \tag{1.1}$$

However, both the length and width were measured in two parts and the expression for the area should reflect that. Altered the area of the tabletop is written as

$$A = (l_1 + l_2)(w_1 + w_2) \tag{1.2}$$

Evaluating with the average values for each measurement leaves the simple answer that the area of the table top is 17340. cm^2 or 1.7340 m^2. However, the uncertainty in our value must also be calculated.

1.4.2 Uncertainty in Area of Table

One may estimate the uncertainty in a computed value caused by an individual variable by taking the partial derivative of the computed with respect to the known value. Differentiating A in equation (1.2) with respect to l_1 leaves

$$\frac{\partial A}{\partial l_1} = w_1 + w_2. \tag{1.3}$$

Likewise,

$$\frac{\partial A}{\partial l_2} = w_1 + w_2, \tag{1.4}$$

$$\frac{\partial A}{\partial w_1} = l_1 + l_2, \tag{1.5}$$

$$\frac{\partial A}{\partial w_1} = l_1 + l_2, \tag{1.6}$$

all represent the change experienced in A provided a change in l_2, w_1, and w_2 respectively. The total change experience in A assuming the measurements are independent may be expressed as

$$\delta_A^2 \approx \left(\delta_{l_1}\frac{\partial A}{\partial l_1}\right)^2 + \left(\delta_{l_2}\frac{\partial A}{\partial l_2}\right)^2 + \left(\delta_{w_1}\frac{\partial A}{\partial w_1}\right)^2 + \left(\delta_{w_2}\frac{\partial A}{\partial w_2}\right)^2 .^{1} \tag{1.7}$$

[1]Note the use of $\approx$ as opposed to $=$. This is because the statement above effectively uses a linearization of the function above. This is a convention that will be held for the remainder of this report.

Plugging in the derivatives and simplifying,

$$\delta_A^2 \approx \left(\delta_{l_1}^2 + \delta_{l_2}^2\right)(w_1 + w_2)^2 + \left(\delta_{w_1}^2 + \delta_{w_2}^2\right)(l_1 + l_2)^2 \tag{1.8}$$

remains. Finally, the error in the area of the tabletop is expressible as

$$\delta_A \approx \sqrt{\left(\delta_{l_1}^2 + \delta_{l_2}^2\right)(w_1 + w_2)^2 + \left(\delta_{w_1}^2 + \delta_{w_2}^2\right)(l_1 + l_2)^2} \tag{1.9}$$

Plugging in the approximate errors and the average measured values, $d_A \approx 18.2\ \text{cm}^2$. Leaving us with the total area of the table being $17430 \pm 18\ \text{cm}^2$ or alternatively $1.743 \pm 0.0018\ \text{m}^2$. An error that is approximately 0.1% of the calculated value.

1.4.3 Density of Tin Cylinder

The density, ρ, of any solid with mass m and volume v is defined to be.

$$\rho = \frac{m}{v}. \tag{1.10}$$

And the volume of a cylinder is with a diameter D and a height h is simply

$$v = \frac{\pi}{4} D^2 h. \tag{1.11}$$

Therefore the density of any solid given D, h, and m is

$$\rho = \frac{4m}{\pi D^2 h} \tag{1.12}$$

Plugging in the average value for each of the measurements leaves us with the simple density, $\rho \approx 7.39\ \text{g/cm}^3$

1.4.4 Uncertainty in Density of Tin cylinder

Again the uncertainty in the density caused by the uncertainty of each measurement will be found using the partial derivatives of ρ.

$$\frac{\partial \rho}{\partial m} = \frac{4}{\pi D^2 h} \tag{1.13}$$

$$\frac{\partial \rho}{\partial D} = -\frac{8m}{\pi D^3 h} \tag{1.14}$$

$$\frac{\partial \rho}{\partial h} = -\frac{4m}{\pi D^2 h^2} \tag{1.15}$$

Again one may approximate the uncertainty in ρ using these partial derivatives.

$$\delta_\rho^2 \approx \left(\delta_m \frac{\partial \rho}{\partial m}\right)^2 + \left(\delta_D \frac{\partial \rho}{\partial D}\right)^2 + \left(\delta_h \frac{\partial \rho}{\partial h}\right)^2 \tag{1.16}$$

Plugging in the partial derivatives to the expression above and factoring out ρ,

$$\delta_\rho^2 \approx \left(\frac{4m}{\pi D^2 h}\right)^2 \left(\delta_m^2 \frac{1}{m^2} + \delta_D^2 \frac{4}{D^2} + \delta_h^2 \frac{1}{h^2}\right) \tag{1.17}$$

remains. Or alternatively,

$$\delta_\rho \approx \rho \sqrt{\delta_m^2 \frac{1}{m^2} + \delta_D^2 \frac{4}{D^2} + \delta_h^2 \frac{1}{h^2}} \tag{1.18}$$

Plugging in the uncertainties and mean values, δ_ρ comes to equal approximately 0.24 g/cm^3 an maximum possible error of 3.2%

1.5 Results and Conclusions

The area of the tabletop was found to be 17430 ± 18 cm^2 or alternatively 1.743 ± 0.0018 m^2. The uncertainty of the area of the table is 0.1 % of its value. One could presumably get even higher precision with more precise measuring instruments, or by taking more measurements and using the standard deviations of sample means to represent the uncertainty in each value [2]. but the precision is already rather high.

The density of the tin cylinder was found to be 7.39 ± 0.24 g/cm^3. The accepted density for tin is 7.31 g/cm^3. The accepted density falls well within the bounds of error for the measured density. In fact, the measurement is only 1 % different from the accepted value. However, the bounds of error remain rather large, being closer to 3 % of the total value; This is because the cylinder was rather small, so any error however small makes up a reasonably large proportion of the total measurement. With a larger cylinder and similarly precise instruments the error in density would have been significantly smaller. As always, taking more samples and using the standard deviation of sample means would also reduce the error in density

[2] Were the standard deviation of sample means used as opposed to the standard deviation of the random variables, the total found error would have been 11.5 cm^2. This number would decrease proportionally to $\frac{1}{\sqrt{n}}$ as the number of samples increased.

Chapter 2

A Verification of Newton's Second Law of Motion

Date Preformed :	October 6, 2017
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2.1 Introduction

Isaac Newton's three laws of motion, first codified in his seminal 1687 work *Philosophiæ Naturalis Principia Mathematica*, set the foundation for much of classical mechanics. The laws of motion describe the relationships between forces, velocity, mass and acceleration. The second law in particular describes the resultant acceleration of a body given a force applied to that body. More precisely, it states, firstly, that the acceleration of a body undergone as a result of a given force is directly proportional to the net force applied to that body. Secondly, that the acceleration underwent is inversely proportional to the mass of the body. And finally, just so that we remember that we are, in fact, dealing with vector quantities, the direction of acceleration is the same direction as the net force on the body. All of these statements may be neatly and concisely summarized in the equation

$$\vec{a} \propto \frac{\vec{F}}{m}. \tag{2.1}$$

In both the SI system of units and the US Customary system[1], the proportionality constant of this expression is just 1, allowing one to change

[1]In order for this to be true, one must be careful to use the appropriate units. If the slug is used as the unit of mass and the pound as the unit of force then the proportionality

the relationship to one of equality rather than proportionality. Additionally, with a simple bit of algebraic rearrangement, we may rewrite the above equation (10.14) into the familiar form

$$\vec{F} = m\vec{a}. \tag{2.2}$$

The above statement may be empirically verified by measuring the acceleration of a body of a certain mass undergoing a certain force and comparing the relationship of these values to the above statement.

2.2 Procedure

In order to measure the relationship between these values, a small cart, the body which will undergo the acceleration, was placed on a flat tabletop. Attached to this cart was a string thats other end dangled of the edge of the table by way of a small pulley. On the end of the string, off the end of the table, a mass m lay attached by a small hook. In order to be able to vary the mass of the system, another mass M rests on top of the cart. And in order to measure the distance the cart travels, and indirectly acceleration the car undergoes, an ultra sonic distance probe sits on the opposite end of the table.

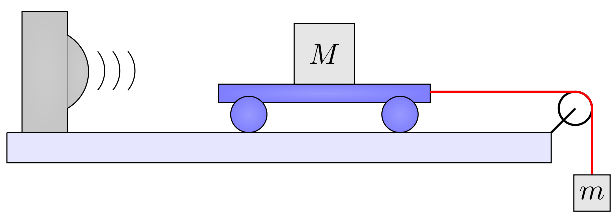


Figure 2.1: The mass of m causes the cart and M to accelerate.

In order to avoid the movement of the cart prior to collection of data, the cart is held back by hand. Once collection of data has begun, the cart is allowed to move forward, and due to the weight of the mass m, the cart begins to accelerate. Once the cart reaches a point close to the edge of the table, the cart is stopped and data collection ends.

Once the position data has been recorded, its first derivative with respect to time is taken numerically. This represents the velocity of the cart at a

constant remains 1. Alternatively, one may also choose to use the pound as the unit of mass, in which case the poundal must be used as as the unit of force. In both cases the foot remains the base unit of length and the second the base unit of time given time. The portion of time in which the velocity appears to be linear is the portion of time during which the cart is accelerating due to the weight of m. The slope of this is recorded and deemed to be the acceleration experienced by the system during the time of interest.

In order to verify the relation described in Newton's second law of motion, two separate experiments are carried out. In the first, the total mass of the system is kept constant, but the forced applied to the cart is increased by moving mass from the top of the cart to the free hanging portion. While in the second, the force applied is kept constant and the total mass of the system is varied by changing the mass M on top of the cart.

2.3 Data

2.3.1 Constant Mass, Changing Force

The first experiment aimed to find the relationship between force and acceleration. The total mass of the system was kept constant at 350 g on top of the mass of the cart and string. That is to say that the sum of M and m was fixed to that value. However, the force was varied by allowing mass to move from M to m in different trials, increasing the force due to gravity acting on the cart.

Table 2.1: Acceleration Measured given Constant Mass and Varying Force

M (g) $\pm$ 0.1	m (g) $\pm$ 0.1	Force on Cart (N)	Acceleration (m/s^2)
300.0	50.0	.490 $\pm$ 0.001	0.393 $\pm$ 0.0025
250.0	100.0	.980 $\pm$ 0.001	0.769 $\pm$ 0.01
200.0	150.0	1.47 $\pm$ 0.001	1.272 $\pm$ 0.008
150.0	200.0	1.96 $\pm$ 0.001	1.781 $\pm$ 0.005
100.0	250.0	2.45 $\pm$ 0.001	2.246 $\pm$ 0.004
50.0	300.0	2.94 $\pm$ 0.001	2.723 $\pm$ 0.007

2.3.2 Constant Force, Changing Mass

The second experiment aimed to relate mass and acceleration. The mass dangling of the edge of the table m remained constant and therefore the force felt due to gravity remained constant at mg as well. However, the mass M on top of the cart was allowed to vary.

Table 2.2: Acceleration Measured given Constant Force and Varying Mass

M (g) $\pm$ 0.1	m (g) $\pm$ 0.1	Force on Cart (N)	Acceleration (m/s^2)
0.0	200.0	1.96 $\pm$ 0.001	2.096 $\pm$ 0.018
50.0	200.0	1.96 $\pm$ 0.001	1.974 $\pm$ 0.003
100.0	200.0	1.96 $\pm$ 0.001	1.876 $\pm$ 0.003
150.0	200.0	1.96 $\pm$ 0.001	1.771 $\pm$ 0.006
200.0	200.0	1.96 $\pm$ 0.001	1.708 $\pm$ 0.005
250.0	200.0	1.96 $\pm$ 0.001	1.627 $\pm$ 0.005
300.0	200.0	1.96 $\pm$ 0.001	1.561 $\pm$ 0.004
350.0	200.0	1.96 $\pm$ 0.001	1.488 $\pm$ 0.003
500.0	200.0	1.96 $\pm$ 0.001	1.322 $\pm$ 0.004
1000.0	200.0	1.96 $\pm$ 0.001	0.976 $\pm$ 0.005

2.4 Analysis

2.4.1 Constant Mass, Changing Force

If equation (10.14), $\vec{a} \propto \vec{F}$, is true one would expect the acceleration of the cart to increase linearly as the force acting on the cart increases. Moreover, one would expect the slope of this line to be the reciprocal of the mass of the body being accelerated. This is more readily apparent if one chooses to write the expression in this form: $a = \frac{1}{M_{tot}} F$ where M_{tot} is the mass of the whole system: cart, string, mass M, and mass m. Plotting these values, one sees what one might expect. In Figure 2 one sees the direct proportionality rather clearly, as the force on the cart increases its acceleration increases as well. One may try to fit a line of the form $y = ax$ onto the data set, and if one does, ones finds an a value of 0.9069 ± 0.0015 kg. If we recall that the slope a should have been $1/M_{tot}$, we can compute the total mass of the system. However, one must be wary of the uncertainty in the values. One may state that

$$M_{tot} = \frac{1}{a}, \tag{2.3}$$

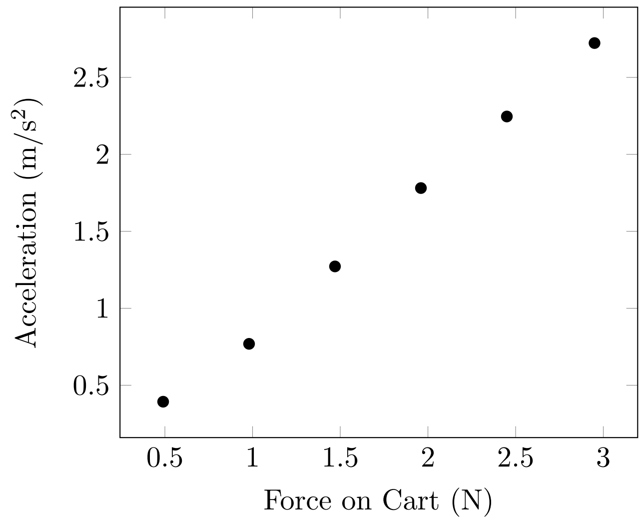
and therefore

$$\frac{\mathrm{d}M_{tot}}{\mathrm{d}a} = -\frac{1}{a^2}. \tag{2.4}$$

One may use this to estimate the error in M_{tot}

$$\delta_{M_{tot}} = \left|\delta_a \frac{\mathrm{d}M_{tot}}{\mathrm{d}a}\right| \tag{2.5}$$

Figure 2.2: Acceleration as a Function of Force with a Constant Mass



Substituting in the expression and simplifying one may quantify the error as

$$\delta_{M_{tot}} = \frac{\delta_a}{a^2} \tag{2.6}$$

If we recall that $M_{tot} = 1/a$ then we may simplify further to

$$\delta_{M_{tot}} = M_{tot}\frac{\delta_a}{a} \tag{2.7}$$

or alternatively

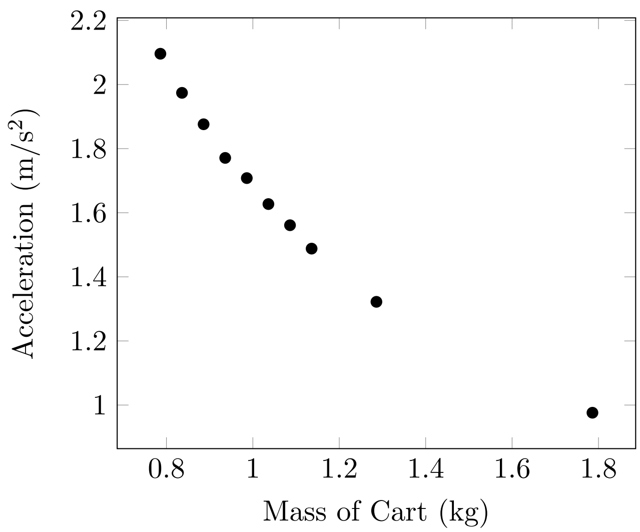
$$\frac{\delta_{M_{tot}}}{M_{tot}} = \frac{\delta_a}{a} \tag{2.8}$$

Finally, if one choose to calculate M_{tot}, the total mass of the system, one finds it to be 1.103 ± 0.0018 kg. This means that the cart, string, and masses M and m should weigh 1.103 ± 0.0018 kg.

2.4.2 Constant Mass, Changing Force

Alternatively, if one chooses to keep the force acting on the system constant but change the mass of the system one would expect the relationship to be $a \propto 1/M_{tot}$. Note that each M_{tot} was computed by summing the masses of M, m, and the cart and string. The cart and string together were measured to be 0.5860 ± 0.0005kg. This time, one would expect the proportionality constant to be the force acting on the cart. This is made clear if one represents the acceleration in the form $a = F\frac{1}{M_{tot}}$. The data collected support this notion, the graph in Figure 3 clearly shows an inverse relation between the total mass of the system and the acceleration undergone by the cart.

Figure 2.3: Acceleration as a Function of Mass with a Constant Force



Again, one may fit a curve to this dataset. This time the appropriate form is $y = a/x$ where a is some constant. However, recall that in the context of the problem this may be written as $a = F/M_{tot}$ where if is the constant that allows us to fit the curve. We can find this F to be 1.672 ± 0.009N.

2.5 Results and Conclusions

All of the data collected supported Newton's Second law of motion. The relationship between acceleration and force is very clearly one of direct proportionality. In fact, the R^2 value of the proportional fit was the incredibly high value of 0.994. The slope of this line was used to find the total mass of the system. This was found to be 1.103 ± 0.0018 kg. This is an incredibly precise value, with the error bounds falling within 0.16 % of the computed value. However, the measured value of the mass of the whole system is equal to 0.936 ± 0.005 kg. This rather large inaccuracy manifests itself as a percent error of 17.8 %.

The second experiment also supports Newton's second law. One can clearly see that the relationship between the acceleration and the mass of the accelerated body is one of inverse proportionality. Again, the fit of the inverse curve is incredibly tight, with an R^2 value of 0.995. This time the force is calculated and found to be 1.672 ± 0.009N. This also has very tight error bounds, accounting for only 0.5 % of variation in the calculated force. But once again there is a high amount of inaccuracy. The force measured was 1.96 ± 0.001N. This means there was a percent error of 14.6 %.

Somewhere in the experiment an error of about 17% is lurking. One might conjecture that friction was the cause of this consistent error. After all, not accounting for friction would cause one to conclude that either the car was heavier than it truly was or that the force acting upon it was smaller than it truly was, both of which were conclusions that the data would seem to support. However, in order for the error we see of 17 % to occur the coefficient of rolling friction between the wheels and the table would have to be around 0.3. This value is much too large for wheels on a flat table, and therefore friction cannot truly be the main culprit. Also if one corrects for this amount of friction, the fit of the curves dramatically worsens which confirms that friction cannot have been the primary cause.

Alternatively, one might think that perhaps the distance sensor was angled in some sense and therefore the distance measured was smaller than the true distance traveled by the car. Having done this, the cart would have appeared to moved slower, and therefore the mass would have been overestimated and force underestimated. However, in order for the difference to be so extreme as it is, the angle of the sensor would have had to have been around 25°. An angle as large as 25° would have been immediately noticeably and hence the angle cannot be the source of such a large error.

Another possible explanation of this error involves the sensing of the distance traveled by the cart. The distance sensor collected distance data at 40 Hz, however the sensor was only rated to collect at 30 Hz. This may also have contributed to some sort of systematic error.

Overall the 17% error is likely a cause of a combination of theses factors and perhaps some others. However, the data are still in concordance the relationship described by Newton's second law of motion.

Chapter 3

Demonstrating Conservation of Momentum in Two Dimensions with Colliding Bodies

Date Performed:	October 18, 2017
Partners :	Vincent Du
	Matthew Harvey
Instructor:	Dr. Brad Miller

3.1 Introduction

Momentum is the quantity, a property of a body, defined as being the product of its velocity and its mass.

$$\vec{p} = \vec{v}m \tag{3.1}$$

Momentum is a vector quantity, in that it has both magnitude and direction. And furthermore, momentum is always conserved. No matter what happens, bar the action of any external force, the net momentum of a system remains both of the same magnitude and in the same direction. This one of the most fundamental notions in all of classical mechanics.

The first discussion of momentum came from Byzantine philosopher John Philoponus, who, in 530 AD, wrote a commentary refuting Aristotelian ideas of impetus. The Aristotelian viewpoint, which remained prevalent until the time of Galileo, thought that everything must be kept in motion by some agent. For example, a ball flying through the air was kept moving by currents in the air. Philoponus wrote that the Aristotelean view was absurd, and that the ball's impetus instead came from the hand throwing it which in turn lost its impetus. This was precursor to the modern notion of momentum and its conservation. In 1350, French Philosopher Jean Buridan first developed an expression for this momentum, writing that this impetus was proportional to a body's speed multiplied by its weight. And yet later Descartes wrote that the total "Quantity of Motion" in the universe was constant.

Eventually, Isaac Newton formalized the modern notion of momentum. In fact, conservation of momentum is a corollary of Newtons 3rd law of Motion,

$$F_a - F_b = 0. \tag{3.2}$$

As a vector quantity, both the direction and the magnitude of this quantity should be maintained in n dimensions. We will attempt to demonstrate this by allowing balls to collide and computing the momentum before and after the collisions. We will restrict ourselves to discussing momentum on the plane parallel to the floor. Because momentum is a vector quantity the components on the this plane should be conserved as well. This allows us to simplify many of the computations, and assume that no forces are acting on the balls. Of course gravity is acting on the balls, but because gravity is orthogonal to this plane we may imagine that it does not exist.

3.2 Procedure

Initially a single steel marble was placed on top of a ramp which rested on top of a table at a height of 92.5cm above the ground. This marble was allowed to hit the ground on top of which rested a large sheet of paper with a single sheet of carbon paper on top of that. This configuration resulted in a small dot on the sheet of paper at the location of each landing. This single marble will be used to measure the momentum sans collision. In a second round a second marble was placed at the bottom of the ramp and the two were allowed to collide. The locations in which these two colliding marbles were measured as well. In a third and final version the second marble was replaced with a steal marble. Each test was run a total of 25 times. And the standard deviation of both the angle and radius were estimated.

Figure 3.1: The average radii and angles of each of the cases

	Distance to Origin (cm)	Angle (°)
Single Steel Marble	46.1 ± 1.0	88.1 ± 0.5
Steel that hit Steel	28.3 ± 1.3	131.9 ± 3.9
Steel hit by Steel	30.7 ± 0.9	59.5 ± 0.9
Steel that hit Glass	37.4 ± 0.7	103.5 ± 0.9
Glass hit by Steel	36.6 ± 1.5	49.8 ± 2.8

Figure 3.2: Other Relevant Information

Mass of First Steel Marble	8.32 g $\pm$ 0.01 g
Mass of Second Steel Marble	8.33 g $\pm$ 0.01 g
Mass of Glass Marble	2.69 g $\pm$ 0.01 g
Height of Table	92.5 cm $\pm$ 0.1 cm

3.3 Data

3.4 Analysis

3.4.1 Function for Momentum

Recall that momentum is defined as being the product of velocity and mass.

$$\vec{p} = \vec{v}m \tag{3.3}$$

If momentum is truly conserved, then this value must remain constant in any system provided no external agent acts upon the system. In our case we may define our system as consisting the two marbles that collide with each other. In stating that momentum is conserved, we state that

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2' \tag{3.4}$$

Where the momenta of each of the balls are represented by the expressions $\vec{p}_1$ and $\vec{p}_2$ and $\vec{p}'$ represents momentum after the event of interest. Alternatively one may substitute the instances of $\vec{p}$ with its expression in terms of masses and velocities. For the sake of simplicity, we will assume that the collision does not cause a change in mass and will neglect from using m' and use m wherever mass is written.

$$\vec{v}_1 m_1 + \vec{v}_2 m_2 = \vec{v}_1' m_2 + \vec{v}_2' m_2 \tag{3.5}$$

We may further simplify the expression by assuming that the second ball beg an with a velocity of zero, allowing us to totally remove the term from the expression.

$$\vec{v}_1 m_1 = \vec{v}_1{}' m_2 + \vec{v}_2{}' m_2 \tag{3.6}$$

If this equality holds we may presume that momentum has been conserved. However one finds it difficult to directly mention the velocity of the objects. But because, at least after the time of collision, the balls move in predicable parabolic arcs one may compute the velocity of the balls as a function of the distance they travel and the height from which they fall. Because we only concern ourselves with momentum on the plane parallel to the floor, we may compute the horizontal velocity of the ball, which should remain constant, as a function of the change in position $\vec{r}$ and the time in the air t.

$$\vec{v} = \frac{\vec{r}}{t} \tag{3.7}$$

Again, we find it difficult to directly measure the time spent in the air. However as a falling projectile we may compute the time in the air with a simple expression, as a function of the height and the gravitational acceleration.

$$t = \sqrt{\frac{2h}{g}} \tag{3.8}$$

Substituting these expressions into the definition of momentum.

$$\vec{p} = m\vec{r}\sqrt{\frac{g}{2h}} \tag{3.9}$$

Because the change in position is measured as a distance and angle. We rewrite the expression as

$$\vec{p} = mr\left(\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}\right)\sqrt{\frac{g}{2h}} \tag{3.10}$$

3.4.2 Error in Momentum

One may find the error in the magnitude of the momentum by writing a sum in quadrature of the error in each each of th respective facts multiplied by the partial derivative of the momentum with respect to that factor. In the case of momentum these factors are the mass, the height, and the radius. This is made easier if one considers uncertainty as a fractional quantity.

$$\frac{\delta_{|p|}}{|p|} = \sqrt{\frac{\delta_m^2}{m^2} + \frac{\delta_r^2}{r^2} + \frac{\delta_h^2}{4h^2}} \tag{3.11}$$

3.4.3 Sum of Momenta

Replicating the expression in equation (5.4) we write an expression discussing the sum of the momenta. However we substitute in the expression for $\vec{p}$ as computed in equation (3.10).

$$\vec{p} = m_1 r_1 \left(\cos(\theta_1)\hat{\imath} + \sin(\theta_1)\hat{\jmath}\right) \sqrt{\frac{g}{2h}} + m_2 r_2 \left(\cos(\theta_2)\hat{\imath} + \sin(\theta_2)\hat{\jmath}\right) \sqrt{\frac{g}{2h}} \tag{3.12}$$

We may express $\vec{p}$ as a magnitude and an angle.

$$|p| = \sqrt{\frac{g}{2h}} \sqrt{2m_1 m_2 r_1 r_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + m_1^2 r_1^2 + m_2^2 r_2^2} \tag{3.13}$$

$$\theta_p = \arctan\left(\frac{m_1 r_1 \sin(\theta_1) + m_2 r_2 \sin(\theta_2)}{m_1 r_1 \cos(\theta_1) + m_2 r_2 \cos(\theta_2)}\right) \tag{3.14}$$

3.4.4 Error in Sum of Momenta

We may recall that one may estimate the error in a quantity by taking a sum of product of the estimated errors in each of the quantities upon which the desired quantity is determined and the amount by which an error in one quantity affects the final one. The amount by which an error in a single factor affects the final result is proportional to the partial derivative of the final result with respect to the individual factor. However, as we assume that each of these individuals factors are independent of each other rather than taking a simple sum, we take the square root of the sum of the squares. For the case of magnitude of the sum of momenta, the value depends on 7 different values each with their own respective errors. The magnitude of the sum of momenta depends on the measured height of the ramp, the masses of each of the balls, the distances traveled by each of the balls, and the angles of the paths of each of the balls. So one may find the partial derivatives of momentum with respect to each of these values.

$$\delta_{|p|} \approx \sqrt{\sum \left(\delta_{x_i} \frac{\partial p}{\partial x_i}\right)^2} \tag{3.15}$$

3.4.5 Too many Partial Derivatives

$$\frac{\partial p}{\partial h} = \frac{-p}{2h} \tag{3.16}$$

$$\frac{\partial p}{\partial \theta_1} = p\frac{r_1 r_2 m_1 m_2 \left(-2\sin\left(\theta_1\right)\cos\left(\theta_2\right) + 2\cos\left(\theta_1\right)\sin\left(\theta_2\right)\right)}{r_1 r_2 m_2 m_1 \left(2\cos\left(t_1\right)\cos\left(\theta_2\right) + 2\sin\left(\theta_1\right)\sin\left(\theta_2\right)\right) + {m_1}^2{r_1}^2 + {m_2}^2{r_2}^2} \tag{3.17}$$

$$\frac{\partial p}{\partial \theta_2} = p\frac{r_1 r_2 m_1 m_2 \left(2\sin\left(\theta_1\right)\cos\left(\theta_2\right) - 2\cos\left(\theta_1\right)\sin\left(\theta_2\right)\right)}{r_1 r_2 m_2 m_1 \left(2\cos\left(t_1\right)\cos\left(\theta_2\right) + 2\sin\left(\theta_1\right)\sin\left(\theta_2\right)\right) + {m_1}^2{r_1}^2 + {m_2}^2{r_2}^2} \tag{3.18}$$

$$\frac{\partial p}{\partial m_1} = p\frac{r_1 m_2 r_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + m_1 r_1^2}{2m_1 m_2 r_1 r_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + m_1^2 r_1^2 + m_2^2 r_2^2} \tag{3.19}$$

$$\frac{\partial p}{\partial m_2} = p\frac{r_1 m_1 r_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + m_2 r_2^2}{2m_1 m_2 r_1 r_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + m_1^2 r_1^2 + m_2^2 r_2^2} \tag{3.20}$$

$$\frac{\partial p}{\partial r_1} = p\frac{m_1 r_2 m_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + r_1 m_1^2}{2m_1 m_2 r_1 r_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + m_1^2 r_1^2 + m_2^2 r_2^2} \tag{3.21}$$

$$\frac{\partial p}{\partial r_2} = p\frac{m_1 r_1 m_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + r_2 m_2^2}{2m_1 m_2 r_1 r_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + m_1^2 r_1^2 + m_2^2 r_2^2} \tag{3.22}$$

3.4.6 Error in Angle of Sum of Momenta

One may do the same thing for the expression for the angle this time with only six parameters.

$$\frac{\partial p}{\partial \theta_1} = \frac{m_1 r_1 m_2 r_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + m_1^2 r_1^2}{2m_1 m_2 r_1 r_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + m_1^2 r_1^2 + m_2^2 r_2^2} \tag{3.23}$$

$$\frac{\partial \theta_p}{\partial \theta_2} = \frac{m_1 r_1 m_2 r_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + m_2^2 r_2^2}{2m_1 m_2 r_1 r_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + m_1^2 r_1^2 + m_2^2 r_2^2} \tag{3.24}$$

$$\frac{\partial \theta_p}{\partial m_1} = \frac{r_1 m_2 r_2 \left(\sin(\theta_1)\cos(\theta_2) + \sin(\theta_2)\cos(\theta_1)\right)}{2m_1 m_2 r_1 r_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + m_1^2 r_1^2 + m_2^2 r_2^2} \tag{3.25}$$

$$\frac{\partial \theta_p}{\partial r_1} = \frac{m_1 m_2 r_2 \left(\sin(\theta_1)\cos(\theta_2) + \sin(\theta_2)\cos(\theta_1)\right)}{2m_1 m_2 r_1 r_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + m_1^2 r_1^2 + m_2^2 r_2^2} \tag{3.26}$$

$$\frac{\partial \theta_p}{\partial r_2} = \frac{r_1 m_1 m_2 \left(\sin(\theta_1)\cos(\theta_2) + \sin(\theta_2)\cos(\theta_1)\right)}{2m_1 m_2 r_1 r_2 \left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) + m_1^2 r_1^2 + m_2^2 r_2^2} \tag{3.27}$$

3.4.7 On Monstrous Expressions

Out of fear of a mathematically induced heart attack on the part of the reader, the authors choose to neglect to write either the error in the magnitude or angle as a single expression, and have chosen to separate it into the parts listed above. One is left to substitute the above expressions (3.16) all the way through (3.22) and (3.23) through (3.27) into the summation described in equation (3.15). This allows one to compute the error in both of the quantities.

3.5 Conclusion

With the expressions written above we may find the total momentum of each of the three cases in the experiment.

Figure 3.3: The Total Measured Momentum of Each System.

	Magnitude of Momentum (N s)	Angle of Momentum (°)
Single Steel Marble	0.0088 ± 0.0002	88.1 ± 0.5
Two Steel Marbles	0.0086 ± 0.0003	91.4 ± 2.6
One Steel One Glass	0.0087 ± 0.0004	86.0 ± 0.9

For one, the figures seem to indicate rather strongly that momentum is conserved. The momentum of the single ball system is found to be 0.0088 ± 0.0002 N·s at the angle $88.1^\circ \pm 0.5$. The other two cases, with the collision of the balls have very similar calculated momenta, with the first having a momentum of 0.0086 ± 0.0003 N·s and an angle of $91.4^\circ \pm 2.2$. These two values are remarkably close; comparing the magnitude of the momenta reveals a percent error of only 2.2 % falling just inside of the bounds of the uncertainty of the measurements. The angles are somewhat father apart with a percent error 3.7 %, this time falling a bit outside the uncertainty bounds of both measurements. When one considers the other case, the collision of the balls of different masses, the evidence remains strongly in support of conservation of momentum. The percent error this time is the smaller 1.1%, and once again momentum is underestimated. The magnitude falls well within the bounds of uncertainty. The error in the angle is also larger this time, falling at 2.4% and falling outside of the bounds of uncertainty.

The computed values for momentum are all remarkably close to each other and thus strongly support the notion that momentum is conserved regardless of action. However the deviations we do see in the momentum likely stem from several sources. Firstly one must recall that the ball was simply placed on top of the ramp. Perhaps when placing the ball there, there was deviation in exactly where it was placed. Or perhaps it was occasionally given some initial velocity as it was released. Or maybe the ramp itself was somewhat depressed as the ball was held on to the ramp. All of these would have an effect on the velocity with which the marble left the ramp, and thus their momentum. All of these factors could result in imprecision.

Another factor to consider is the second ball's initial position. The ball was placed on top of a screw protruding from the base of the ramp. The screw's angle with respect to the plane of the ramp was meant to be constant, however it is likely that some of the momentum of the balls went into the screw. This would have two effects. It would both decrease the momentum of the two ball system, as observed in the experiments. And it would slowly move the screw farther to the right and change the angle of intersection ever so slightly. This would introduce some variation to the measured momenta.

One may also consider that the ball as having some rotation. It is likely that as the collision of the balls was not head on, and as the ball was already rolling down the ramp, that some of the balls translational momentum became rotational as it hit the second ball. This may acount for some of the small loss of momentum measured.

Another property of the balls that may be computed is the elasticity of the collision. We may define a collision to be more elastic if it preserves more of its kinetic energy. Where kinetic energy is

$$K = \frac{1}{2}mv^2 \tag{3.28}$$

We may compute v to be a function of height and range.

$$v = r\sqrt{\frac{g}{2h}} \tag{3.29}$$

With these two expressions one may compute the elasticity as

$$\frac{K_1' + K_2'}{K} \tag{3.30}$$

If one does this, one finds the elasticity of the steel on steel collision to be around 82% while the steel of glass comes closer to 86%. These percentages represent the amount of energy that was conserved in the collision.

Chapter 4

Computing the Moments of Inertia of Spinning Bodies Accelerated by Falling Masses

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4.1 Introduction

The moment of inertia of some body quantifies that body's resistance to an angular acceleration. Any object will tend to keep its velocity constant provided no external force acts upon it and causes an acceleration. This is simply inertia as described by Newton's first law of motion. However, forces do tend to act on objects, and it is useful to be able to describe how an object's velocity will change function of some force acting upon the object. This is not particularly difficult to do. One may, without much difficulty, conclude that a body will accelerate more as a larger force acts upon it, and that a body will provide a greater resistance to acceleration if it has a larger mass. This relationship is quantified in Newton's second law of motion,

$$a \propto \frac{F}{m} \tag{4.1}$$

The very same relationship occurs with regards to rotational motion. That is, firstly that rotating objects have inertia. An object in rotation will tend to continue rotating just as a linearly moving body will continue to move with the same velocity. Likewise, a torque acting upon a body will impart an acceleration. However, a rotating body's mass is not necessarily directly proportional to its resistance to rotational acceleration. This quantity is called a moment of inertia and is denoted by the symbol I. If one uses this information to create an analogue of the expression in equation (10.14), one is left with the expression

$$\alpha \propto \frac{\tau}{I} \tag{4.2}$$

This resistance to angular acceleration, this moment of inertia is distinct from mass because it depends on the distribution and placement of the mass within the spinning object. If the mass of an object is concentrated close to the axis of rotation, inducing an angular acceleration is much easier than if the mass of the object is concentrated farther away from the center. The moment of inertia of a single point is proportional to the mass of the point multiplied by the square of the distance between it and the axis of acceleration.

$$I = mr^2 \tag{4.3}$$

For more complex masses one may simply sum the moments of inertia of each of the points within the complex figure.

$$I = \int r^2 \mathrm{d}m \tag{4.4}$$

However for even more complex solids, solids that perhaps even exist in the physical world, one may, in place of using the geometry of figure, compute the moment of inertia by actually applying some quantifiable toque on the body and measuring the angular acceleration that the torque induces in the spinning object. The following experiment will serve as a method for computing the moment of inertia of a spinning body accelerated by the constant toque of a falling mass. And will serve to confirm that a body's resistance to acceleration is indeed proportional to the sum its mass and to the square of the distances to the axis of rotation.

4.2 Procedure

The experiment was divided into two portions. In the first, two masses were placed on a metal rod such that the center of their masses remained 10cm

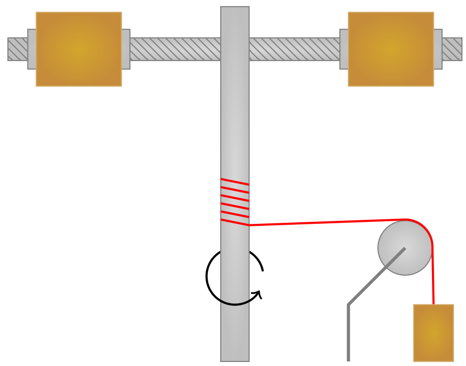


Figure 4.1: As hanging mass is permitted to fall it causes the apparatus to begin spinning.

from the center of the rod. The masses were held in place by two wing nuts. The center of this rod was suspended on the top of a second vertical rod which was permitted to rotate. This entire apparatus, the two rods, the masses suspended at a constant distance from the axis of rotation, and the wing nuts holding the masses in place, was the rotating mass whose moment of inertia intended to be found. The constant torque applied to the apparatus was supplied by a falling mass. A long string was positioned with one end wrapped around the circumference of the vertical rod the center passing through a small pulley and the opposite end with a small mass allowed to dangle above the ground. As the mass would begin to fall it would pull on the rod and the whole apparatus would begin to spin. The mass of the individual masses on the rod was permitted to vary throughout the experiment while the distance to the center was fixed at 10cm. In the second experiment, the only 200 grams were ever placed on either ends of the rods, but the distance at which they were placed was permitted to vary between 15cm and 3cm.

The time for the masses to traverse the distance between the tabletop on which the apparatus was placed and the floor was measured and will be used to compute the acceleration at which mass falls and eventually the moment of inertia of the spinning object.

4.3 Data

4.3.1 Constant Radius, Varying Masses

As the mass was allowed to vary, the time required to fall the fixed distance was recorded a total of four times. For each trial, the time was recorded by hand with a simple stopwatch. Inevitably, there was variation in the recorded times for falling, and often one of the four trials would be significantly distant from the other three. Because of this, the value farthest from the mean was cast out and not used in the calculation of either of the mean or range of the time. The cast out values have been *emphasized* in the data set in figures 4.2 and 4.3.

Figure 4.2: Time Required to Fall function of Mass

Masses (g)	Measured Times (t)				Average Time (t)	Time Range (t)
10	7.79	*8.00*	7.82	7.82	7.81	0.03
20	*8.35*	8.75	8.71	8.74	8.72	0.04
50	10.33	10.40	*10.05*	10.48	10.40	0.15
100	12.61	12.63	12.83	*12.29*	12.71	0.21
150	14.58	14.56	*14.50*	14.55	14.56	0.03
200	*16.70*	16.21	16.55	16.33	16.36	0.03
300	20.24	20.02	19.99	*20.37*	20.08	0.25
500	24.97	25.05	*25.25*	25.09	25.04	0.12

4.3.2 Constant Mass, Varying Radius

In the second series of measurements, rather than allowing the amount of mass on the spinning apparatus to vary, the distance at which the masses stood from the axis of rotation was varied. Again, the time required for the hanging mass to fall the fixed distance was measured a total of four times. And once again, the value farthest from the mean was *cast out* and not used in the computation of the mean or range.

Figure 4.3: Time Required to Fall function of Radius

Distance	Measured Times				Average Time	Time Range
3	8.20	*8.13*	8.21	8.22	8.21	0.02
4	9.10	*8.95*	9.02	9.06	9.06	0.08
5	10.08	10.08	*10.07*	10.08	10.08	0.01
6	11.24	11.25	*11.06*	11.31	11.27	0.07
7	12.32	12.38	*12.66*	12.52	12.41	0.20
8	13.94	*13.78*	13.85	13.93	13.90	0.08
9	15.11	15.03	15.16	*15.29*	15.10	0.13
10	16.44	16.53	*16.43*	16.53	16.50	0.09
11	*18.02*	17.32	17.31	17.38	17.34	0.07
12	18.99	19.04	18.99	*18.62*	19.01	0.05
13	20.63	20.64	*20.47*	20.60	20.62	0.04
14	22.60	22.43	22.20	*21.98*	22.41	0.4
15	23.35	23.36	23.44	*23.18*	23.83	0.09

4.3.3 Other Relevant Information

4.4 Analysis

4.4.1 Finding Moment of Inertia

One may compute the moment of inertia of the spinning apparatus as a function of the time required for the small hanging mass to fall. One may begin by first finding the tension in the string holding up the hanging mass. This tension is the force acting upon the rod and causing it to spin. Were the hanging mass, m, to be at rest then the tension in the string would simply be the mass' weight mg, however, the mass is indeed accelerating at some rate a. This means the tension in the string is in fact smaller than the weight and in fact equal to the mass multiplied by the difference between the gravitational acceleration and the true acceleration of the mass.

$$F = m\,(g - a) \tag{4.5}$$

This force is being applied on the vertical rod and causing it to spin. This force is inducing some angular acceleration into the rod and is hence causing some torque.

$$\tau = \vec{F} \times \vec{r} \tag{4.6}$$

Figure 4.4: Other Relevant Measurements

Height of Fall	88.35 $\pm$ 0.1 cm
Diameter of Rod (Bare)	1.27 $\pm$ 0.01 cm
Diameter of Rod (With String)	1.31 $\pm$ 0.01 cm
Mass of Wing Nut	6.04 $\pm$ 0.03 g
Mass of Horizontal Rod	62.19 $\pm$ 0.02 g
Length of Horizontal Rod	34.2 $\pm$ 0.1 cm
Local Gravitational Acceleration	9.798 $\pm$ 0.001 m/s^2

Because the tension in the string is acting tangent to the rod, the magnitude of the torque is simply equal to the product of the radius of the rod and the tension in the rope.

$$\tau = mr\,(g - a) \tag{4.7}$$

This torque causes an angular acceleration in the rods and masses proportional to the torque and inversely proportional to the moment of inertia of the spinning body

$$\alpha = \frac{\tau}{I} \tag{4.8}$$

This angular acceleration is related to a linear acceleration of the string and hanging mass.

$$\alpha = \frac{a}{r} \tag{4.9}$$

Combining the above two expressions allows one to write an expression for the moment of inertia function of the linear acceleration of the falling mass, torque acting on the rod, and radius of the rod.

$$I = \frac{\tau r}{a} \tag{4.10}$$

One may then substitute the expression for torque as found in (4.7).

$$I = \frac{mr^2\,(g - a)}{a} \tag{4.11}$$

However, it is often difficult to accurately measure acceleration and because the acceleration in this system is constant it is sufficient to simply measure the time required to fall a fixed distance from rest. Recall that for bodies falling from rest the distance fallen, h, may be written as a simple function of acceleration and time.

$$h = \frac{1}{2}at^2 \tag{4.12}$$

One may rearrange this to find acceleration.

$$a = \frac{2h}{t^2} \tag{4.13}$$

If one substitutes this into equation (4.11) one may reach an expression for moment of inertia solely in terms of easily measurable quantities.

$$I = \frac{mr^2t^2}{2h}\left(g - \frac{2h}{t^2}\right) \tag{4.14}$$

This simplifies to the somewhat more manageable expression.

$$I = \frac{mr^2\left(gt^2 - 2\,h\right)}{2h} \tag{4.15}$$

4.4.2 Errors In Moments of Inertia

In order to find the quantity of error in the moment of inertia as a result of the error in each of the individual measurements in the experiments it is useful to find the derivatives of the moment of inertia with respect to each of the measurements. However, not all of the errors in measurements are significant enough to the create a large change in the moment of inertia. For example, both the mass of the hanging mass and the height of the table are known to within less than a tenth of a percent and hence will contribute rather little to the error in the calculated moment of inertia. In fact, only the time required to fall and the radius of the rod generate significant error. Fortunately both of these are relatively easy to differentiate.

$$\frac{\partial I}{\partial r} = \frac{2I}{r} \tag{4.16}$$

$$\frac{\partial I}{\partial t} = \frac{2Itg}{gt^2 - 2h} \tag{4.17}$$

One may simply multiply these quantities by the estimated error in the respective quantities to find that particular measurements contribution to the error in the moment of inertia. If one assumes that the errors are uncorrelated the square estimated total error may be expressed as the sum of the square contributions from each of the measurements.

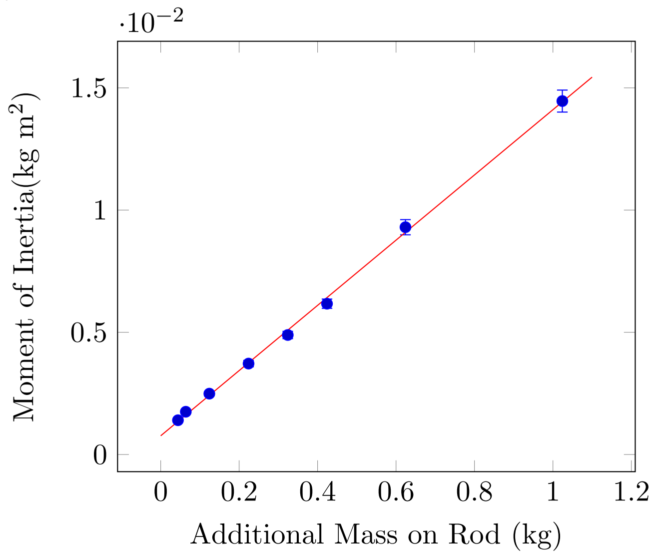
4.5 Results and Conclusions

4.5.1 Interpretation of Results

One may use the above expressions and the measured times to compute the moments of inertia of the apparatus in various positions.

One may begin to analyze the relations between the measured moment of inertia of the spinning apparatus and the additional masses placed on the horizontal threaded rod. If moment of inertia is truly a sum of products of mass and radius squared then, because the radius was kept constant in the first experiment, the relationship between mass and moment of inertia should appear to be linear.

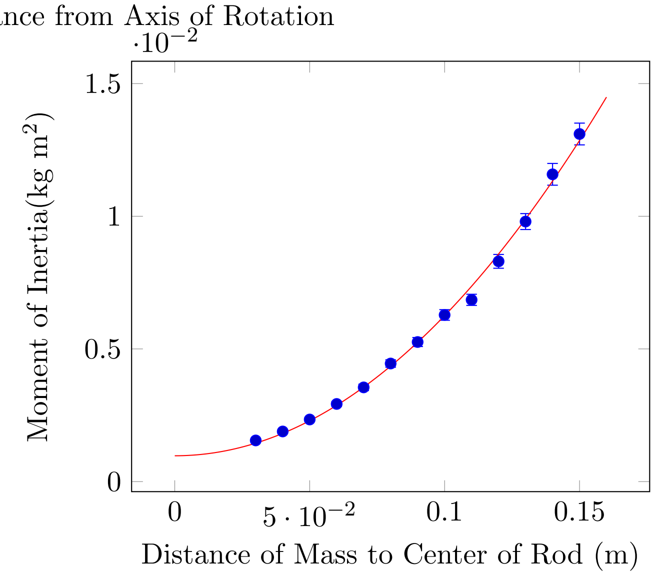
Figure 4.5: Measured Moment of Inertia Function of Additional Mass



This is, in fact, the relationship observed. Fitting a line to the dataset yields a remarkably close fit. So good of a fit that the R^2 value is the remarkably high value of 0.9988. The particular curve yielded by the regression is $I = 0.01334m + 0.000766$ However, there remain a few notable things about the linear regression. The first is its reluctance to pass through the origin. One might imagine that if there is no mass on the apparatus then the calculated moment of inertia ought to be zero as well. However, this is not the relationship suggested by the graph. The graphs suggestion is actually correct, even when there is no additional mass placed on the apparatus the two rods have mass and are positioned in such a way that they have a moment of inertia of their own. The y intercept of this curve, $7.66 \cdot 10^{-4} \pm 9 \cdot 10^{-6}$ kg m^2, is an estimate for the moment of inertia of the bare system sans any additional mass. One might also wonder if the a parameter in the linear regression $a\,x + b$ holds any significance. If moment of inertia of a point mass truly is of the form $m\,r^2$ and the x parameter in our linear regression represents mass than one might predict that the a parameter is simply the square distance of the masses from the center of rotation. Or alternatively that the distance to the axis of rotation is simply the square root of this a term. One may compute that the square root of 0.01334 ± 0.0002 m^2 and yield the distance 0.1155 ± 0.0009m^2. This distance is relatively close to the true distance of the masses from the axis of rotation of 10 ± 0.1 cm

One may conduct a similar analysis of the data in the second experiment. This time the data has a remarkably quadratic flair. A relationship which is reasonable given the expression for moment of inertia $m\,r^2$.

Figure 4.6: Measured Moment of Inertia Function of Distance of Additional Mass' Distance from Axis of Rotation



The equation for the best quadratic fit for this data is the curve $I = 0.5281r^2 + 0.00973$. And just as in the previous example, each of these parameters have a physical meaning. Again the y intercept of the function represents what would happen if the two masses were positioned directly on top of the axis of rotation. We have been treating the masses like point masses thus far, and if the masses were truly point masses then the y intercept would represent the moment of inertia of the bare apparatus. This particular experiments estimates the moment of inertia of the bare apparatus as $9 \cdot 10^{-4} \pm 2 \cdot 10^{-4}$ kg m^2. The a portion of the expression $a\ x^2 + c$ also has important physical meaning. Again, if one assumes that moment of inertia is truly $m\ r^2$ than the a value should simply represent the mass of the added mass. This would estimate the added mass to be 530 ± 30 g value that is reasonably close but decidedly larger than the true added mass of 424 g (two 200 g masses + four 6 gram wing nuts).

4.5.2 An Analysis of Error

One might attempt to compare the measured moments of inertia to an estimate of the moment of inertia using the geometry of the apparatus. The apparatus was composed of two rods one vertical and one horizontal. The horizontal rod spun perpendicular to its axis about its center. There is a simple expression for the moment of inertia of such a rod.

$$I = \frac{1}{12}\, m\, L^2 \tag{4.18}$$

Substituting the measured length of the rod, 34.2 ± 0.1cm, and the measured mass of the rod, 62.19 ± 0.02 g , yields the expression for moment of inertia, $6.06 \cdot 10^{-4} \pm 3.6 \cdot 10^{-6}$ kg m^2. One might also consider the moment of inertia of the vertical rod which was much smaller. No direct measurements for the geometry of this rod were taken but estimates will be given. Because the small moment of inertia of the vertical rod even very rough estimate do not contribute greatly to error. The mass of the vertical rod was estimated to be 200 ± 100 g and the radius was measured to be 0.655 ± 0.01 cm. Computation yields the vertical rod's contribution to moment of inertia to be the rather small value of $4 \cdot 10^{-6} \pm 2 \cdot 10^{-6}$ kg m^2. Summing these two values shows that the moment of inertia, as nucleated by the physical geometry of the apparatus is estimated to be $6.10 \cdot 10^{-4} \pm 4 \cdot 10^{-6}$ kg m^2.

Comparing this value to the measured moments of inertia shows that they serve as a reasonable estimate for the moment of inertia of the bare apparatus. The first intercept estimates the moment of inertia to be $7.66 \cdot 10^{-4} \pm 9 \cdot 10^{-6}$ kg m^2. There is a 25% error between the value that had been geometrically obtained and the value that has been obtained by measurements. However the error bars on each are terribly small. This suggests that some systematic error has caused either a systematic over estimation of the error on the part of the falling mass method or a systematic underestimation on the part of the geometric method. It is more to be likely the former of the two. Recall that in all the calculations the masses were treated as point masses. However in reality the masses have width and hence their moments of inertia are affected by that width. This effect causes the an overestimate as, because the radius is squared, the father away points make a greater contribution and hence pull the masses 'effective' center of mass father out. It is also work noting that the pulley used to suspend the string was not in the best of shape string would slip on the pulley as it turned. This friction would cause the hanging mass to take longer to fall and hence lead to an overestimate in the moment of inertia.

The second value is yet father away from the geometric model at $9 \cdot 10^{-4} \pm 2 \cdot 10^{-4}$ kg m^2 or nearly 50% away. In this case both of the previously mentioned sources of error exist and contribute to the overestimate in moment of inertia. But there is yet another source of error. The y intercept of the function represents what would happen if the masses were placed at $r = 0$. If they were a point mass this would be a representation of the bare apparatus, however the masses have width and thus their placement at the center would contribute to an increase in moment of inertia.

The same overestimates contribute to the 24% overestimate in mass from the second experiment and the 11% overestimate in the radius from the first experiment.

Chapter 5

Properties of Damped Oscillations

Date Performed:	November 27, 2017
Instructor:	Dr. Bradley Miller

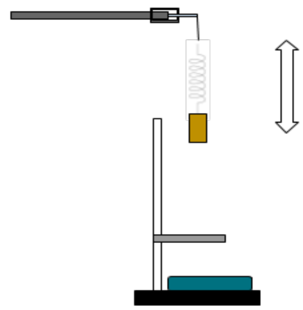
5.1 Introduction

Suppose one has a mass resting on the end of a spring, the other end of which is fixed in place. Several things might happen. The spring might be in its equilibrium position, in which case the mass would not move. Alternatively, the spring might be over-compressed, in which case the spring, seeking to relieve its tension, would extend and accelerate the mass towards equilibrium. The spring may also be overextended, in which case the spring would retract, once again accelerating towards equilibrium. However, even with this perpetual acceleration toward equilibrium, the mass and spring still find it difficult to come to rest. Instead, the mass, seeking to reach equilibrium, overshoots and must be once again drawn back. If allowed to move freely, the spring will bob back and forth. The spring will oscillate. Oscillations are a type of motion which is periodic, meaning it repeats in a predictable pattern. Objects like pendulums and springs can experience oscillations. In the simplest of cases, the oscillator will return exactly to its initial condition after one period. However, this is not always the case. The motion of these oscillators may be affected when the springs motion is damped being resisted in some way proportional to the speed at which the mass moves. The source of the damping usually comes from some kind of spring friction. This damping causes the oscillator to slow down, and thus means that the amplitude reach in progressive periods will be smaller. The elasticity constant of a spring, mass on the end of a spring, and the damping constant all affect the motion of the springs oscillation. The damping constant will affect the rate at which the oscillators amplitude decreases. If the damping constant is very large, the oscillator will begin to swing at rather small amplitudes rather quickly, while if the opposite is the case the amount after a long time will be rather similar to the initial amplitude. Therefore, working backwards, the damping constant may be found by analyzing the motion of oscillators over a period of time. In this lab, that relationship will be found and the angular velocity will also be found using the movements of the spring.

5.2 Procedure

In this experiment a spring/clamp apparatus was constructed from which the oscillator would hang. A clamp was attached to the side of a lab workstation. A paper clip was then attached the side of the clamp. A spring was hung from the paper clip, with a hanging mass hung from the other end of the spring. A ring stand apparatus was constructed by attaching an o-ring to the stand and placing a wire mesh on top of the ring clamp in order to protect the motion sensor from falling objects. The motion sensor was placed on the base of the stand so that it was directly under the hanging mass and protective mesh. First, the spring constant of two springs was calculated by placing slotted masses ranging from 100.0 g to 900.0 g on the spring and measuring the displacement. Approximately 4-5 trials were taken for each spring. After calculating the spring constant, the spring was attached to the clamp apparatus, and a mass was hung on the end of the spring. The spring and hanging mass were held such that the spring was in its natural equilibrium position, then the hanging mass was dropped in such a way so that there was little sway in the bouncing spring and data collection was started. The distance of the mass to the motion sensor was measured continuously for 30 seconds of damped oscillation for each trial. Four separate spring/mass combinations were used. This was repeated with the same 4 spring/mass combinations and an index card (m = 2.02g) taped to the bottom of the hanging mass, altering the damping constant of the oscillation.

Figure 5.1: The mass would bob above and below its equilibrium position.



5.3 Data

The first data collection involved finding the physical properties of the system. The spring constant of both of the two springs was found by placing different masses on the bottom of the spring and measuring their extension.

Spring 1			Spring 2		
Mass (g)	Force (N)	Displacement (cm)	Mass (g)	Force (N)	Displacement (cm)
20.0	0.196	1.06	400.0	3.92	0.80
50.0	0.490	2.63	500.0	4.90	1.76
100.0	0.980	5.44	600.0	5.88	2.76
150.0	1.47	8.11	700.0	6.86	3.82
200.0	1.96	10.79	1000.0	9.80	6.92

If these points are graphed, the slopes of the lines represent the spring constants of the two springs. The first was found to be 18.07 ± 0.10 N/m while the second was found to be 95.7 ± 0.8 N/m. One may see these graphs on the attached graph page.

The motion of each of the oscillators was recorded with the motion sensor. The graphs of this as well as the curve fit and its parameters are on the attached graph pages. One should note that the fits of the curves are very tight, with r^2 values well into 99%s. This means that the model for damped harmonic motion reached in the analysis (See next section) accurately models the physical phenomenon.

5.4 Analysis

Lets suppose we have a mass on the end of a spring resting off the end of the table. The mass will Suppose there is a mass at the end of a spring which resting off the edge of the table. The mass has relatively few forces acting upon it. One of these forces is the result of gravity, which draws the mass down towards the earth. This force will simply be $-m\,g$ where m is the mass of the mass allowed to rest at the base of the spring and g is the acceleration due to gravity. The spring will naturally exert a force on the spring. This force is proportional to the negative distance of the mass to the springs equilibrium position. We may express this force as $-k\,x$, where k is the spring constant and x is the distance that the spring has been extended. There is one remaining force that will model the damping that the mass experiences. We will imagine some force slowing the mass down that is proportional to the speed at which the mass moves. We will express this as $-b\,x'(t)$ where b is some constant related to the damping and $x'(t)$ the speed of the mass. Thus the forces on the mass at any given time may be written as

$$F = -m\,g - k\,x - b\,x'(t)$$

Of course we recall that by Newton's second law of motion, $F = m\,a$, the force may be substituted by the product of the mass and the acceleration.

$$m\,a = -m\,g - k\,x - b\,x'(t)$$

We may write this such that the terms all fall on one side, and for consistency with the remainder of the equation replace the term a with $x''(t)$. This leaves us with the simple differential equation.

$$m\,x''(t) + b\,x'(t) + k\,x + m\,g = 0$$

One might solve this by converting the above differential equation into its characteristic equation.

$$m\,r^2 + b\,r + k = 0$$

This is rather simple to solve, and may be done as a simple quadratic equation.

$$r = \frac{-b}{2\,m} \pm \frac{\sqrt{4\,mk - b^2}}{2\,m} i$$

Of course the solutions are almost certainly not real but, this is what gives simple harmonic oscillations their characteristic sinusoidal motion.[1] Now,

[1] Note that the solutions are only imaginary if $b^2 - 4mk < 0$. This only occurs in underdamped oscillation which is what was measured.

with this information it is relatively easy to model the position $x(t)$ without much effort. The any sum of these two

$$x(t) = c_1\,\mathrm{e}^{r_1} + c_2\,\mathrm{e}^{r_2} - \frac{m\,g}{k}$$

Where r_1 and r_2 are the two solutions to the characteristic equation. is often useful to us to separate r into its real and imaginary parts. The real part of r is the rather simple to find, $b/2m$. The imaginary part remain a little bit more complicated, we will refer to the imaginary portion of the solutions as $\pm\omega$ for reasons that will hopefully become clear. This allows us to rewrite the expression.

$$x(t) = c_1\,\mathrm{e}^{-bt/2m}\,\mathrm{e}^{i\omega t} c_2\,\mathrm{e}^{-bt/2m}\,\mathrm{e}^{i\omega t} - \frac{m\,g}{k}$$

Both sides of the sum, that with c_1 and with c_2 have a $e^{-bt/2m}$ term which may be factored out. The remaining portions, those with imaginary exponents are sine waves and thus so is there sum. They will have a sum with some amplitude A and some phase ϕ. This rids us of the need to use the constants c_1 and c_2

$$x(t) = A\,\mathrm{e}^{-bt/2m}\,\sin(\omega\,t + \phi) - \frac{m\,g}{k}$$

If one measures the position of one of these damped as a function of time, one may try to fit such a curve onto the data. The most function to fit is of the form.

$$x(t) = A\mathrm{e}^{-Bt}\,\sin(C\,t + D) + E$$

Each of these parameters have a meaningful physical interpretation. A represents the initial amplitude of the oscillation. If the mass was dropped from rest (it was) this value represents the distance from its initial potion to its equilibrium position. Bs value is a little bit more involved. As in equation (5.4), the exponent is $b/2m$. B is proportional to the damping force and inversely proportional to the mass of the object. C is the same as ω, the angular speed. This may be calculated as a function of b, m, and k . D represents the phase shift. This should represent the position at which the mass begins. In reality the masses were all released from the top of their period, but the beginning of the data collection period did not necessarily coincide with the beginning of the oscillation and thus D has rather little meaning in this context. The final parameter E represents the average position of the mass. This ought to be the height of the bottom of the mass on the unextended spring to the distance sensor minus the elongation of the spring when it is in equilibrium, $m\,g/k$.

We may calculate the damping coefficient for the two situations (with and without index card) by comparing the calculated parameter B with mass, m, recall that

$$B = \frac{b}{2m}$$

. Solving for b,

$$b = \frac{B}{2m}$$

Likewise we may try to find angular speed function of both the parameters and the measured physical properties of the system.

$$C = \omega = \frac{\sqrt{4\,m\,k - b^2}}{2\,m}$$

5.4.1 Error Propagation

In order to represent the error in the measurement of the damping constant. One might find the error contribution with respect to each parameter simply by taking partial derivatives with respect to each parameter.

$$\frac{\partial b}{\partial B} = \frac{b}{B}$$

$$\frac{\partial b}{\partial m} = -\frac{b}{m}$$

This means that the total error in b may be expressed as

$$\delta_b = b\sqrt{\left(\frac{\delta_B}{B}\right)^2 + \left(\frac{\delta_m}{m}\right)^2}$$

A similar process may be used to find the angular speed. Again, one takes the partial derivatives with respect to each parameter

$$\frac{\partial \omega}{\partial b} = \omega \frac{b}{b^2 - 4\,m\,k}$$

$$\frac{\partial \omega}{\partial k} = -2\,\omega \frac{m}{b^2 - 4\,mk}$$

$$\frac{\partial \omega}{\partial m} = \omega \frac{-b^2 + 2\,mk}{m\,(b^2 - 4\,mk)}$$

Again the sum of square the above statements may be used to find the total error in the above measurement. If one wishes to expand the above one finds that the error in angular speed is

$$\delta_\omega = \omega \frac{\sqrt{b^4\delta_m^2 - 4\,b^2km\delta_m^2 + \delta_b^2b^2m^2 + 4\,k^2m^2\delta_m^2 + 4\,\delta_k^2m^4}}{m\,(b^2 - 4\,mk)}$$

5.5 Conclusion

One may use the above expressions to compute the damping coefficients for each of the arrangements as well as the angular speeds of the harmonic oscillators. These end up being

Figure 5.2: Physical Properties as measured for Damping Constant, and as calculated for Angular Speed

Arrangement	Damping Constant ($\mathrm{N \cdot s/m}$)	Angular Speed (rad / s)
100 g Mass, Spring 1	0.0456 ± 0.002	13.44 ± 0.04
200 g Mass, Spring 1	0.0375 ± 0.003	9.51 ± 0.03
500 g Mass, Spring 2	0.0273 ± 0.006	13.83 ± 0.06
1000 g Mass, Spring 2	0.0195 ± 0.002	9.78 ± 0.04
100 g, Index card (+2.1 g)	0.211 ± 0.001	13.26 ± 0.04
200 g, Index card (+2.1 g)	0.0803 ± 0.0004	9.45 ± 0.03
500 g, Index card (+2.1 g)	0.0298 ± 0.0004	13.80 ± 0.06
1000 g, Index card (+2.1 g)	0.0185 ± 0.0001	9.77 ± 0.04

One thing to note is that the damping constant doesn't seem to very constant at all. This may seem troubling, however it is relatively easy to explain. The masses hung on the end of the spring were all different sizes, as of course is natural. However this means that their cross section and hence damping do to air friction would be different. The smaller masses seem to have categorically larger damping constants. This again makes sense, the cross section of the masses grows as a function of the square of its dimensions, while the mass grows as a cube of the mass' volume. Because the mass grows more quickly than the cross section. It is natural that the damping constant should be smaller for larger masses. Likewise, for the case in which an index card was attached to the base of the mass, the cross section was constant while the mass grew. This caused the damping constant to fall yet faster.

Likewise, it is also worth noting that the damping provided by air resistance is not truly proportional to the speed at which the mass moves. In reality it is likely more proportional to the square of the mass' speed. However, doing this would make the differential non-linear and hence needlessly difficult for the scope of this experiment.

Even for all the criticism of the index card, it is worth noting that it did effective increase the damping constant, especially for small masses. For the 100g mass the addition of the index card increased the damping constant by a factor of 5. The difference is not nearly as extreme for larger masses, but it still does have an effect. For the 200g mass, its addition doubles the damping constant. For the large masses its difference is negligible. For 500g it leads to a small increase and for 1000g it seems to have the effect of decreasing the damping constant. This is likely due to some error in measurement as will be discussed later.

The angular speeds of motion as estimated from the physical parameters (k, m, b) are rather close the the fit parameters C which are meant to model the angular speed as well. For example, angular speed of the first arrangement was predicted to be 13.44 ± 0.04. While the fit parameter C for the same arrangement was measured to be 13.35 ± 0.004. This is a difference of merely 0.6%. Similar differences occur in each of the other arrangements.[2] One should also note that the calculated values tend to be somewhat larger than the fit parameters especially for the lighter spring. Perhaps this is because not all of the motion of the oscillator was vertical and some of the energy was used to move left and right, this would mean that the mass would be moving vertically at a slower rate that it possibly could and thus the parameter would be larger than the calculated value, just as observed. It is easier for the lighter spring to move horizontally as the masses are also lighter and hence less stable in their motions. It is worth noting that the b parameter has truly very little influence on the angular speed. This means that $\sqrt{k/m}$ is really a rather good approximation of the angular speed, even if the b is rather large. This is especially true for larger masses.

Major identifiable sources of error include the horizontal movement of the mass and spring. The non-linearity of the damping due to friction which influences both the calculated damping factor and the angular speed, as well as the fact that the spring has mass itself which would affect the movement of the oscillator marginally

[2] All th percent errors were [0.67%,0.42%,1.0%,0.71%,0.45%,0.32%,0.86%,0.91%] in the order of the table above.

Chapter 6

Electric Fields and Electric Potential

Partners:	Eashwar Mahadevan
	Alex Hoerler
Date Performed:	January 11, 2018
Instructor:	Dr. Bradley Miller

6.1 Introduction

Coulomb's Law describes the force acting between statically charged particles.

$$\mathbf{F} = \frac{kq_1q_2}{r^2}\hat{\mathbf{r}} \tag{6.1}$$

That is, if two charged particles were to be placed together, at a distance r from each other. They would feel a force proportional to the product of their charges and inversely proportional to the square of the distance between them. This relation conveniently allows one to describe the force acting on a pair of particles, but often is is convenient to describe the forces in an alternative way.

Consider gravity.

$$\mathbf{F} = \frac{GmM}{r^2}\hat{\mathbf{r}} \tag{6.2}$$

Newton describes the force due to gravity in a remarkably similar way Coulomb: In this case proportional to the product of the masses and the inverse of the square of the distance. But when describing events on the surface of the earth, the mass of the object acted on need not be considered. One may simply discuss a constant acceleration. Effectively, by dividing both sides of the expression by the mass of the on may see a body as creating some sort of gravitational field of force per mass (acceleration) around it that exists with or without another body to act upon.

It is natural to do the same with electrostatic relationships as well. However, rather than dividing both sides by the mass, consider dividing by a charge. This allows us to consider each charged particle as creating electric field irrespective of any neighboring particles. This field is one of force per charge. That is the force acting on any charged particle within a field is equal to the product of the field and the particle's charge.

$$\mathbf{F} = q\mathbf{E} \tag{6.3}$$

This electric field, $\mathbf{E}$, is simply a vector field generated by each particle.

$$\mathbf{E} = \frac{k_e q}{r^2}\hat{\mathbf{r}} \tag{6.4}$$

When confronted with any vector field, particularly one that deals with forces it is natural to try and fit a potential function to it. That is, to find some scalar function f such that $\nabla f = \mathbf{F}$. Of course, this may only be done if the vector field is conservative, but this happens to be the case. This potential function would allow one to find the work to get from one point to another without the need of any line integrals and allow one to associate a potential energy to each point.

However, we would like to keep this in terms of an electric field and not a force. So we shift out perspective to find a function V such that $\nabla V = \mathbf{E}$. This scalar field V is called the electric potential. This field should represent energy per charge.

Visually one should be able to represent the electric field as a standard vector field. The electric potential could be visualized as a 3d graph, but is often more convenient to view it as a series of contours. Note that the contours will always be perpendicular to the field lines.

6.2 Procedure

An experiment was conducted to see that electric potential and dipole behave as posited by Coulomb's Law, and by their relationship as potential functions and gradients for one another. To do this, the electric potential difference was measured between various points.

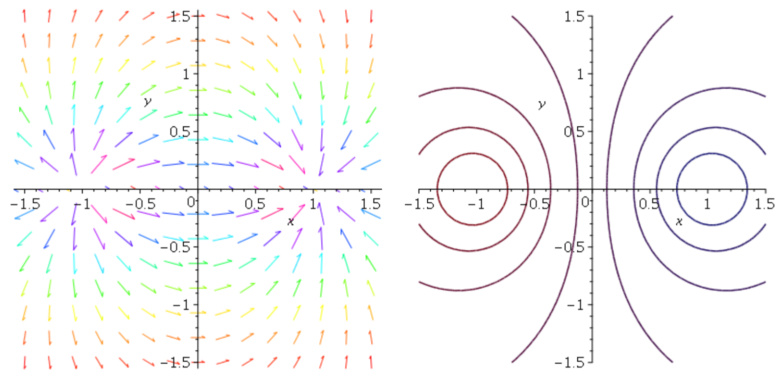


Figure 6.1: Field Lines and Contours of Electric potential for a particular charge configuration (Dipole)

6.2.1 Dipole Configuration

In order to generate a dipole configuration, two small conductive dots were placed on the gridded paper. These two dots were placed at the 8 and 17 cm marks on the paper (9 cm apart). A small pin was placed into each of these dots. A power supply which outputted a constant 10 V was attached to the configuration. One of leads was attached to each of the pins on the paper. A digital multi-meter was then used to measure voltage differences across the paper. One end of the multimeter was attached to one end of the power supply while the other was attached to a small probe which was placed at various points along the paper. This measured the difference in voltage between the power supply and that particular point on the paper.

The voltage difference was measured at 1cm intervals along the horizontal lines on which the two points of charge fell. And Lines of equipotential were traced by moving the probe while attempting to keep the voltage difference constant.

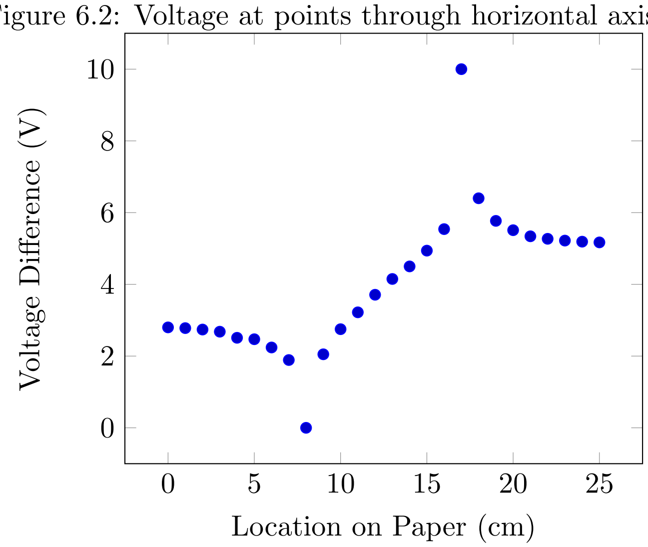
6.2.2 Alternative Configuration

A similar procedure was followed for an alternative charge configuration. This charge configuration consisted of a small line of charge and lying opposite that line a teardrop shape of charge. Both were made of conductive ink. Once again equipotential were traced.

6.3 Data

The voltage difference between one of the poles and various points were measured and recorded. Among these points were the those on the line passing through the two points of charge. The paper was marked with a 1 cm grid and the two points of charge were placed at points 8 and 17.

Figure 6.2: Voltage at points through horizontal axis



The equipotential curves were also traced out for each of the configurations. See the attached page.

6.4 Analysis

One might attempt to fit a model for the electric potential of a dipole over the data collected. Doing this proves relatively simple. Consider a dipole with point charges q and $-q$ respectively at points a and b. We simply wish to find the electric potential at a point x on the line through the dipole. We begin by finding the electric field at x. Recall that the electric field due to a point charge is

$$\mathbf{E} = \frac{kq}{r^2}\hat{\mathbf{r}} \tag{6.5}$$

In this case, $\mathbf{r}$ simply refers to the distance to the point charge. The distances between the point charges and x are simply $|x-a|$ and $|x-b|$. However, this is not the only $\mathbf{r}$ based term in the expression. The force acts in the direction of $\mathbf{r}$, $\hat{\mathbf{r}}$. This we represent, somewhat more frustratingly with $\text{sgn}(x-a)$ and $\text{sgn}(x-b)$ where $\text{sgn}(x)$ is the sign function, returning 1 for positive values and -1 for negative inputs. Substituting these terms and summing the two relevant expressions (for each a and b). We are left with the rather unsatisfying result.

$$\mathbf{E} = \frac{kq}{(x-a)^2}\text{sgn}(x-a) - \frac{kq}{(x-b)^2}\text{sgn}x - b \tag{6.6}$$

We are now left to find the potential function. Because this is in one dimension we must simply take a simple integral. Conveniently, this allows us to clean up our expression quite a bit. Recall that sign function is expressed (and occasionally defined) as the derivative of the absolute value function. This makes the integral relatively easy and leaves one with the much more satisfying potential function.

$$V = \frac{kq}{|x-a|} - \frac{kq}{|x-b|} \tag{6.7}$$

One might even try to fit this curve to the data collected above. However, it requires some modification. There is no convenient way to separate the k and q parameters, which always appear in a pair, so it is convenient to replace it with a single parameter C which represents their product. Also, like with energy, with voltage only differences may be measured. Thus the fit will be offset by some constant V_0 which represents the limit voltage difference measure as one gets arbitrarily far away from the dipole configuration. This value should also be the voltage at the exact middle of the dipole. Intuitively, its value should be the mean of voltage at the two poles.

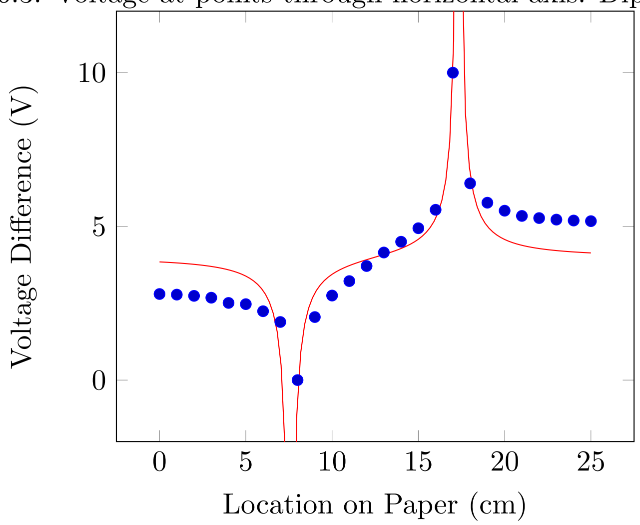
$$V = \frac{C}{|x-a|} - \frac{C}{|x-b|} + V_0 \tag{6.8}$$

With these adjustments in mind one may fit a curve to the data collected.

6.5 Conclusions

The fit is decidedly disappointing. One can clearly see that shape of the graph is generally correct, however we may clearly see quite a few short comings. The correlation of the graph is reasonably high at 0.913 but the RMSE is a rather elephantine 0.87 V. That means that on average we deviate from out predictions by nearly 9%. Upon further examination of our graph we notice that we predict the potential to die away as we leave the dipole

Figure 6.3: Voltage at points through horizontal axis: Dipole fit



much more quickly than it truly does. Examining the parameters provides some clarity. The a and b parameters are found to be 7.58 ± 0.07 cm and 17.32 ± 0.05 cm respectively. Both relatively close to their true values of 8 and 17 cm. Percent errors are not particularly meaningful values to compare here, but the absolute error indicates that the predictions are off by a 0.42 and 0.32 cm respectively. The parameter for V_0 indicates that the base voltage should have been the same as the middle value between the two nodes. The fit indicated that V_0 should be 3.99 ± 0.2 V. This is close to the voltage measured at the central points which averages 3.93 V. However, this may seem distressingly far from 5 V which we might expect from being halfway in between 0 V and 10 V. This may be explained to some degree. Notice that the 10 V point is relatively much father away from its neighbors than the 0 V point. In fact, the average difference between the 0 V point and its neighbors is 1.97 V, while the 10 V point is a substantial 4.03 V away from either of its neighbors. It seems as if the 10 V point would much rather be an 8V point. One might also note that when measured right outside of the point of charge the voltage dropped to around 8 V instantly, while the same behavior was not noted around the 0 V point. Perhaps there was some bad connection that prevented all 10V from entering the paper. The initial coefficient C was fit to be $.02 \pm 0.3\,\mathrm{N\,m^2/C}$ this should be equivalent to kq. If one is to assume that k is simply Coulomb's constant then must one conclude that the charge on each point is C/k or $2.2 \times 10^{-12} \pm 3 \times 10^{-13}$ C. However, things are not this simple. Coulomb's constant is useful for describing these events in a vacuum. But the events of the experiment were conducted in a decidedly non-vacuum environment. The permittivity, the resistance to the formation of an electric field, changes function of the medium. For example, the same charge configuration would produce a field nearly 80 times weaker if placed in underwater as opposed to in the air. Presumably one would need to take into account the the permittivity of the paper transmitting the charge to accurately compute the charge on each point. Paper typically has a dielectric constant of around 2. But, this particular sheet of paper was covered in some carbon based coating. Carbon typically has a dielectric constant between 2.5 and 3. Given these estimates, we will assume that the paper has a dielectric constant of 2.5 ± 0.5. This would imply that charge on each point is truly closer to 5.5 ± 1.3pC. [1]

6.5.1 An Alternative Fit

Consider that the point charges in the dipole are not truly points in th real world. Rather, pins were placed into the points and attached to the pins were wires which also carried charge. This may have shifted the results by a substantial amount. The potential function for this situation, two parallel lines of opposite charge may be expressed as

$$V = \frac{\lambda}{2\pi\epsilon} \ln \left| \frac{x-a}{x-b} \right| + V_0 \tag{6.9}$$

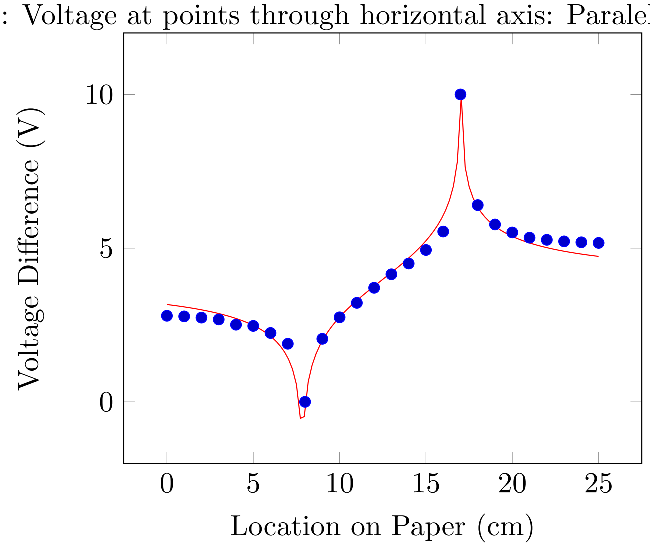
Where λ is the linear charge density. We might rewrite this using a constant k as opposed to in terms of a permissibility *epsilon*.

$$V = 2\lambda k \ln \left| \frac{x-a}{x-b} \right| + V_0 \tag{6.10}$$

Again we choose to swap the initial coefficient This fit is rather refreshingly better than the previous one. The R is a substantial 99.2%, and the RMSE is nearly of a third of the previous fit at 0.27 V. We can be rather confident that it makes more sense to model the dipole configuration as a set of oppositely charged parallel lines as opposed to two point charges. The parameters relay a similar story. The peaks of the function,a and b were predicted to be 7.83 ± 0.05 cm and 17.02 ± 0.01 cm. These are relatively closer to the true positions than the previous fit. Once again the leftmost point a is predicted to be father left than it truly is, perhaps the pin was not particularly well

[1] https://www.kabusa.com/Dilectric-Constants.pdf

Figure 6.4: Voltage at points through horizontal axis: Paralell Line Fit



centered on its grid mark. The middle voltage was found to be 3.95 ± 0.05 V, which is close to the voltage measured at the middle point 3.93 V but once again distant from the assumed 5 V. This follows from the same explanation as before. The constant C represents $2k\lambda$ rather than qk as before. If we assume the dielectric constant of the carbon coated paper to be 2.5 ± 0.5. We conclude that λ, the linear charge density,to be $1.4 \times 10^{-10} \pm 0.3 \times 10^{-11}$ C$/m$

If one uses this estimate for the linear charge density and the charge estimate found earlier, one can attempt to find the length of the parallel rod.

$$l = \frac{q}{\lambda} \tag{6.11}$$

Dividing these two yields an estimate for line length of 3.9 ± 1.2cm. The value for this is incredibility rough but it is about the length of the pin and alligator clip attached to the paper.

6.5.2 Equipotentials and Field Lines

The lines of equipotential were traced on copies of the gridded paper. These lines of equipotential were traced in green. One would expect the equipotentials of the dipole to take the characteristics of **(FIG 1)**. However, the measured potentials seem to extend father outward when outside of the dipole than they do in the model. This is because the dipole is better modeled by the parallel line fit. A logarithm is involved in the parallel line fit and thus it should die away much more slowly than standard dipole fit. Notice

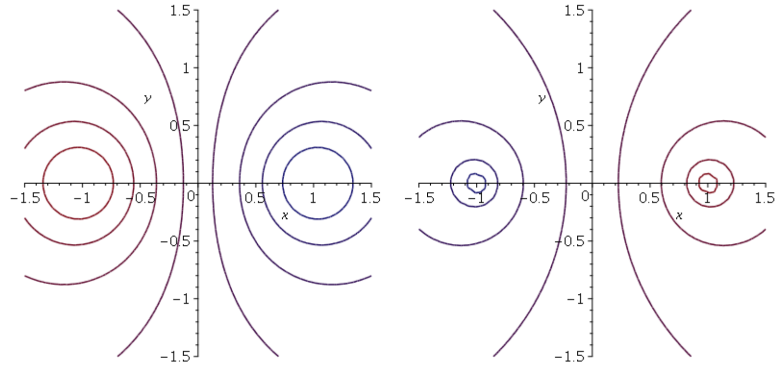


Figure 6.5: Contours of Electric potential for charge configurations. Dipole (Left) and Parallel Line (Right)

how the contours traced much more closely match the parallel line fit, with much smaller curvature as contours get far away from the center.

An approximation of the field lines on the dipole has been drawn in red. Note that the red field lines always intersect perpendicular to green equipotentials and vice versa. Notice that the density of the field lines on the plane corresponds to the density of the equipotentials. That is, if there are many equi-potentials close to each other we expect the gradient of the electric potential to be large and thus we expect many field lines to pass through the area. This is well demonstrated at the center of the dipole. We also expect an equal number of field lines to pass into and out of each point (or any contour), as they have the same charge. This follows from Gauss' law which states that the electric flux through a surface is proportional to the charge enclosed. Because the charge for the two points is merely a sign change away, both will have an equal number of lines passing through, but one will that many field lines entering while the other will have that many leaving.

6.5.3 Alternate Equipotentials

The second configuration has similar characteristics. The field lines also cross perpendicular to the measured equipotentials as they would in any other diagram of field lines and equipotential. Ultimately this configuration will look like a dipole from a distance as it is comprised of a relatively compact pair of positively and negatively charged particles. Locally it is a bit more complicated than the dipole. At some points there were issues with some mal-connectivity. This explains the intersection of the green equipotential and the blue charge configuration at (5,12). In fact the very end of the charge configuration was not attached at all. One might also assume that there exists some larger accumulation of charge at the ends of the line and at the corner of the tear drop. This is not immediately evident but it may be occurring.

Chapter 7

Ohm's and Kirchoff's Laws

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Date Performed:	January 31, 2018
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7.1 Omh's Law

7.1.1 Introduction

Electrical potential measures the potential energy of a unit positive charge within a particular electric field. Electrical current is a flow of electric charge, that is current measures the amount of charge passing through a point per unit time. Electrical resistance measures the difficulty of an electrical current to pass though a conductor. Ohm's law predicts a relationship between these 3 quantities.

$$I = \frac{V}{R} \tag{7.1}$$

where I is the electrical current, V is the electrical potential, and R is resistance. In essence, Ohm's law says that the rate at which charge flows is proportional to the difference in electrical potential with some proportionality constant, or conductivity, $1/R$.

If this proportionality constant R is truly constant, one would expect to the relationship between the current I and the voltage V to be linear. If this is relationship holds, we call the conductor *ohmic*. If this resistance R is not truly constant we say that the resistor is *non-ohmic*.

7.1.2 Procedure

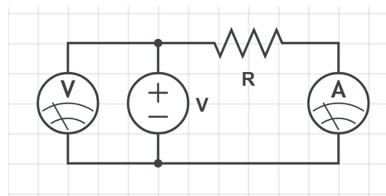


Figure 7.1: Depiction of Method of Measuring Current and Voltage

In order to verify that the relationship between voltage and current are indeed linear one may measure the current at through a resistor at various different voltages. A variable power supply was attached to a fixed resistor. The current through the resistor was measured with a digital ammeter connected in series with the resistor. The voltage drop across the power supply was also measured with a digital voltmeter. The variable power supply initially provided a potential difference of 0 V, this voltage was gradually increased from 0 V to 5 V over the course of about 50 seconds. The voltage was then decreased gradually from 5 V to 0 V over the course of another 50 s. Throughout the process, 3 data points (current and voltage) were collected every other second. A light bulb was used instead of a fixed resistance resistor for model a *non-ohmic* resistor. The bulb was positioned in the place where the resistor had been previously, and the voltage was slowly raised from 0 V. In order to avoid damaging the bulb, it was only raised to approximately 2.5 V.

7.1.3 Data

The procedure above was used to collect about 180 data points for each of the resistors and bulbs. Examining these shows that at least for 2 of the resistors the data seems extraordinarily linear. In fact, for the resistors with resistance 555 Ω and 2.63 kΩ the correlations for a linear fit are 0.998 and 0.997 respectively. Both also seem to be close to a proportional fit as well, that is, when the the voltage is zero the current seems to be zero as well. Examining the linear regression confirms this, the y-intercepts of the graphs are $-8 * 10^{-5} \pm 2 * 10^{-5}$ A and $-9 * 10^{-5} \pm 5 * 10^{-5}$ A respectively. This small deviation may be explained by a slight miscalibration in the current measuring probe.

The data for the other two schemes is not nearly as nice. For the largest resistor (4.68 kΩ) the current measured for voltages under 1.5 V fluctuated wildly. Perhaps because the current was so small (under 0.3 mA) the current probe was not able to accurately measure the current through the resistor. However, this range did not prove to be a major issue for the second resistor which also briefly included very small currents. After this point the data proved to be linear.

The bulb, as predicted was *non-ohmic*, the currents measured did not related proportionally with the voltage rather, the rate at which current increased decreased as voltage increase.

For the resistors, the slopes of the graphs and their inverses may be used to compute the overall resistance

Resistor	Slope (Ω^{-1})	Resistance (Ω)
555 $\pm 2\Omega$	$1.834 * 10^{-3} \pm 8 * 10^{-6}\ \Omega^{-1}$	$545 \pm 2\ \Omega$
2.63 $\pm 0.2k\Omega$	$3.76 * 10^{-4} \pm 2 * 10^{-6}\ \Omega^{-1}$	$2.66 \pm 0.01\ k\Omega$
4.68 $\pm 0.2k\Omega$	$1.94 * 10^{-4} \pm 7 * 10^{-6}\ \Omega^{-1}$	$5.1 \pm .2\ k\Omega$

7.1.4 Analysis

Ohm's Law, $V = IR$ effectively models the current-voltage relationship for the purpose built resistors. However, being *non-ohmic*, the bulb is more difficult to model. Ultimately the non-linearity of the bulb's curve stems from its constant change in temperature. A change in temperature brings about a change in resistance. For most materials this means an increase in temperature will result in an increase in resistance. Typically this relationship is described as

$$R = R_0(1 + \alpha(T - T_0)) \tag{7.2}$$

where T is temperature and α is some proportionality constant called the temperature coefficient of resistance.

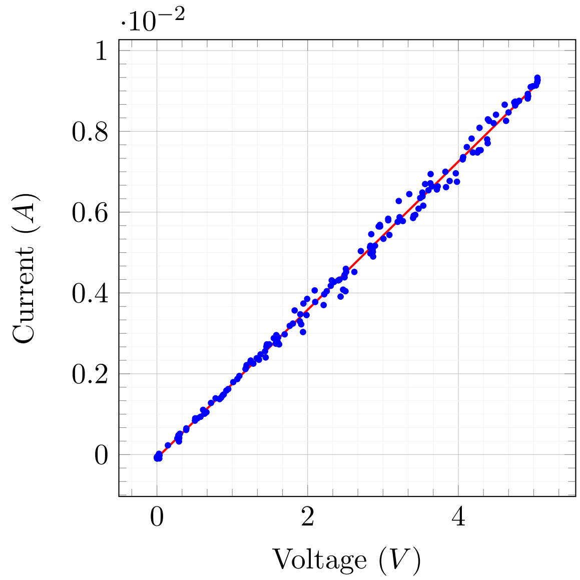
One might assume that the increase in temperature of the bulb is proportional to its power. The power dissipated by the bulb is

$$P \propto I^2 R \tag{7.3}$$

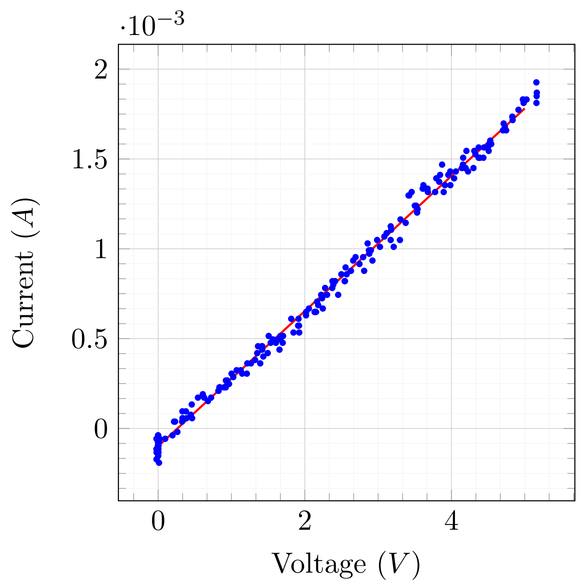
This means that the resistance should be expressible as

$$R = R_0(1 + \alpha\, I^2\, R) \tag{7.4}$$

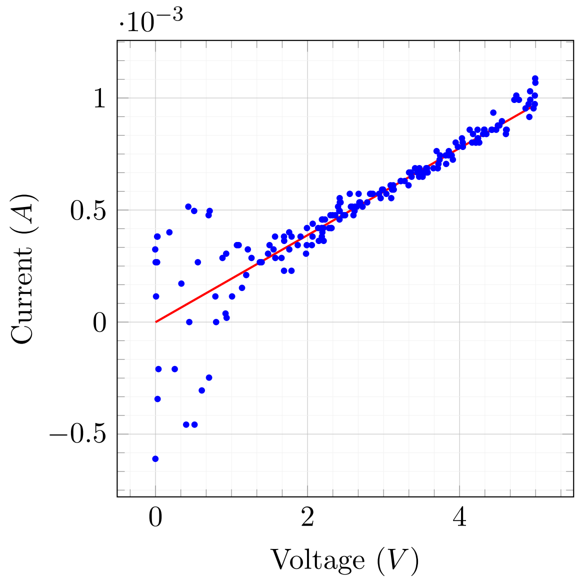
Current through Resistor (555 Ω)
at variable voltage



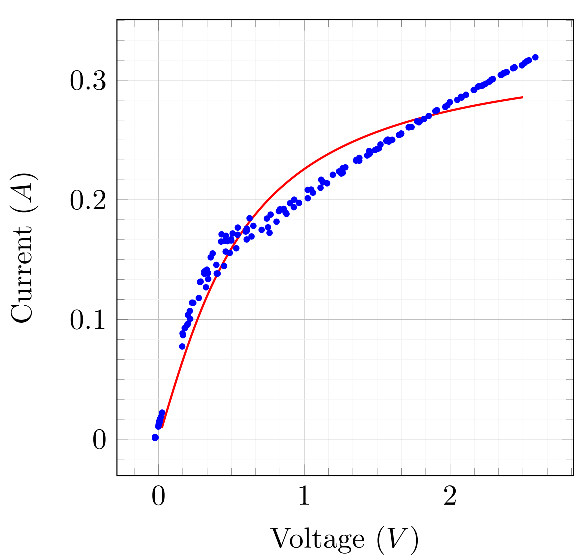
Current through Resistor (2.63 kΩ)
at variable voltage



Current through Resistor (4.68 kΩ)
at variable voltage



Current through Bulb
at variable voltage



It is convenient to solve for current in terms of voltage rather than resistance. Thus we may subsistent R for V/I.

$$\frac{V}{I} = R_0(1 + \alpha\, V\, I) \tag{7.5}$$

Solving for current yields the somewhat clunky expression

$$I = \frac{-R_0 + \sqrt{4\, R_0\, \alpha\, V^2 + R_0^2}}{2\, R_0\, \alpha\, V} \tag{7.6}$$

This should match the data collected for the bulb, however the fit seems quite disappointing. It generally captures the trend of graph but seems to curve more than the true data. The coefficients are found to be

$$R_0 = 2.42 \pm 0.6\Omega$$

$$\alpha = 3.6 \pm 0.2\ C^{-1}$$

7.1.5 Conclusions

The three resistors measured followed the trend of Ohm's law incredibly well. The current related to voltage in a linear relation. The resistances calculated using the linear fit to the data come relatively close to the measured resistances for each of the resistors. Resistor 1 was measured to be $555 \pm 2\Omega$ while the linear fit predicted it to be $545 \pm 2\Omega$. This constitutes an error of about 1.8 %. The second resistor had similarly good results. It was measured to be $2.63 \pm 0.02\ k\Omega$ while the data suggested a more appropriated value of $2.66 \pm 0.02\, k\Omega$, constituting an error of closer to 1.1%. While these values are quite close to their expected values, the third resistor is much farther off, measured at $4.68 \pm 0.02\ k\Omega$ while the data would suggest that the resistance is closer to $5.1 \pm 0.2\ k\Omega$ is error is a much larger 8.9%. Part of the increased error in resistor 3 may be explained by the issues with current measurement. While the other issues in measurement that account for the 1 - 2 % error in the others may be explained by some small internal resistance in the ammeter, some small amounts of current passing through the voltmeter and other issues in measurement.

The fit of the curve on the bulb is somewhat more troubling, however some of its short comings may be explained by the model. The model used for the fit assumes that the temperature of the bulb is directly proportional to the power dissipated. This is a naive interpretation as the bulb will of course have some sort of thermal inertia. That is, the bulb will take time to reach its stable temperature at any given voltage, however the voltage was changed rather rapidly and the bulb was not given sufficient time to adjust after each change in voltage.

The parameters for the fit have some meaning as well.R_0 is meant to represent the resistance of the bulb at room temperature. This would imply that the resistance of the bulb at this temperature is $2.42 \pm 0.6\Omega$. This value seems to agree visually with the inverse slope of the current voltage curve around zero which appears to be in the 2.5-3 Ω range. The calculated 2.42 value does not fall within this range bt has a very broad uncertainty which certainty pushes the value into this range. The α parameter was meant to represent the temperature coefficient of the material, however one might notice that $3.6 \pm 0.2\ C^{-1}$ is nowhere near the temperature coefficient of tungsten, $0.0044C^{-1}$ [1] However, this was based on the naive assumption that one Watt of power would result in a degree increase in temperature, this is simply not the case. Instead, alpha is the product of the temperature coefficient of tungsten and the ratio of watt power to degree Celsius. Dividing α by the temperature coefficient of tungsten should yield the amount of power required to increase the temperature of the resistor by one degree Celsius. This value is $810 \pm 50W/C$

7.2 Kirchoff's Laws

7.2.1 Introduction

Kirchoff's Laws are a series of equabilities that describe the distribution of current and electric potential within a circuit. Essentially they state that the voltage across any closed loop is 0, and that the current through any junction is 0. One may use these principles to determine the resistance, current, and voltage of any given circuit. For example, consider 3 resistors in series, R_1, R_2, R_3 and an ideal voltage source V. Because the current through a closed loop must be equal to 0, we know that the equality $V - IR_1 - IR_2 - IR_3 = 0$ must hold. This may be factored to show that $V - I(R_1 + R_2 + R_3)$. This shows that resistors effectively add in series. Similarly consider two resistors R_1, R_2 attached in parallel to an ideal voltage source V. Because currents must sum to zero in a junction

$$\frac{V}{R} = \frac{V}{R_1} + \frac{V}{R_2}. \tag{7.7}$$

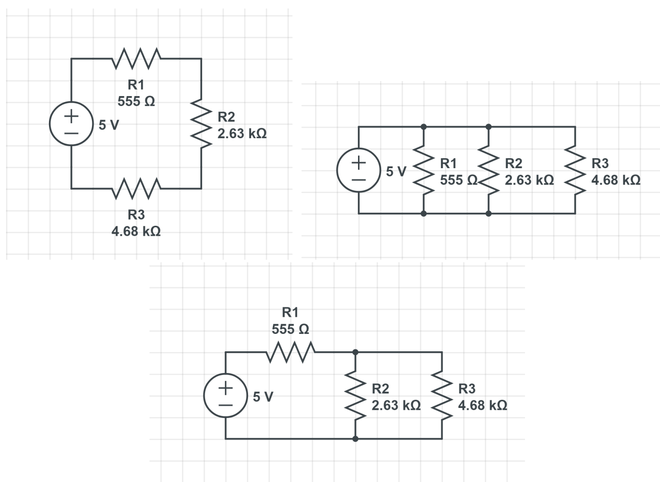
[1] https://www.allaboutcircuits.com/textbook/direct-current/chpt-12/temperature-coefficient-resistance/

This shows that in parallel resistors are the inverse of the sum of the inverses.

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{7.8}$$

7.2.2 Procedure

In order to demonstrate that Kirchoff's laws do in fact hold a series of circuits were arranged and the currents, voltages and resistances were measured at a variety of points. Three resistors, 555, 2630 and 4680 Ω were arranged in three separate schemes. First they were arranged in series, then all in parallel, and eventually in a combined scheme were resistors were both in series and in parallel. These were all connected to a 5 V ideal voltage supply. The three arrangements are pictured below.



7.2.3 Data

The voltage current and resistance were measured for each circuit.
The resistances under 2000 Ω are measured to the nearest 2 Ω while those above 2000 are measured to the nearest 20 Ω. Currents are accurate within 0.5 % of the stated value. Voltages under 2 V measure to the nearest millivolt while those above 2 V are accurate the the nearest 10 mV.

Figure 7.2: Measurements of Series Circuit

Location	Resistance kΩ	Current (mA)	Voltage (V)
R_1	0.555	6.38	0.357
R_2	2.63	6.38	1.69
R_3	4.68	6.38	3.03
Total	7.92	6.38	5.07

Figure 7.3: Measurements of Parallel Circuit

Location	Resistance kΩ	Current (mA)	Voltage (V)
R_1	0.555	8.97	5.07
R_2	2.63	1.862	5.07
R_3	4.68	1.057	5.07
Total	0.419	11.85	5.07

7.2.4 Analysis

Series Circuits

Kirchoff's Laws make several statements about properties that should hold in a series circuit. The most obvious of switch is simply the loop rule, that is, the voltages through each of the resistors should sum to the voltage drop across the power supply. Simply put

$$V = V_1 + V_2 + V_3. \tag{7.9}$$

One may extend this notion to the resistances as was done in the introduction. Alternatively, the effective resistance across the circuit is equal to the sum of the sub resistors.

$$R = R_1 + R_2 + R_3. \tag{7.10}$$

Because there exist no junctions in the circuit the current through each of the components should remain constant.

Parallel Circuits

In a parallel circuit voltages should be constant as each of the components form their own loop with the voltages source. This may be expressed as

$$V = V_1 = V_2 = V_3. \tag{7.11}$$

Figure 7.4: Measurements of Combined Circuit

Location	Resistance kΩ	Current (mA)	Voltage (V)
R_1	0.555	2.24	1.254
R_2	2.63	1.140	3.81
R_3	4.68	0.795	3.81
Total	2.27	2.24	5.07

Each of these voltages should be equal to the product of the current through the circuit and the resistance.

In parallel, resistances sum as the inverse of the sum of the inverses, this is simply a corollary of the loop rule and Ohm's law.

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \tag{7.12}$$

From these one may deduce the currents through each of the 3 resistors. By Ohm's law the current through a resistor with a resistance R and voltage V is a V/R.

$$I_1 = \frac{V}{R_1}, I_2 = \frac{V}{R_2}, I_3 = \frac{V}{R_3} \tag{7.13}$$

Combined Circuit

This circuit is a little bit more complicated than the previous two.

Being in parallel, the voltages through R_2 and R_3 should be equal and these should sum with V_1 to get the sum V.

$$V = V_1 + V_2 = V_1 + V_3 \tag{7.14}$$

The overall resistances are summed both as in series and as in parallel.

$$R = R_1 + \frac{1}{\frac{1}{R_2} + \frac{1}{R_3}} \tag{7.15}$$

The junction present in the circuit should obey Kirchoff's junction rule.

$$I = I_1 = I_2 + I_3 \tag{7.16}$$

7.2.5 Conclusions

All of the rules in the above section are obeyed by the three circuits.

Series Circuit

Resistance In a series circuit the resistances sum to give a single effective resistance. Summing the measured resistances yield a effective resistance of $7.87 \pm 0.03 \mathrm{k}\Omega$. This summed value is just under the measured total resistance $7.92 \pm 0.02 \mathrm{k}\Omega$. These values are just within each others uncertainty and have a total difference of 0.7 %.

Current Having computed the overall total resistance and measured the voltage though the power supply one may find the current through the whole system. Dividing the measured voltage by the calculated resistance yields a calculated current of 6.44 ± 3 mA. This is under measured current of 6.38 ± 0.03 mA and off by about 0.9%. This current should and did remain constant across the whole circuit.

Voltage The voltage drop through each resistor should be equal to the current though the resistor multiplied by its resistance. The calculated voltages are 0.355 ± 0.02 V, 1.69 ± 0.02 V, and 3.01 ± 0.02 V. These are remarkably close to the measured voltages. In fact, 2 of these values are the exact voltages measured. The only one that is off is the third voltage which is of by a mere 0.6 %. These voltages sum to 5.06 ± 0.03 V. This is an error of merely 0.2%. This confirms Kirchoff's law.

Parallel Circuit

Resistance In a parallel circuit, the resistances sum as the inverse of the sum of the inverses. The total effective resistance is computed to be $417 \pm 2\Omega$. This is within 0.5 % of the measured value.

Current One may use Ohm's law to compute the ideal currents across each of the resistors. This calculation shows that the currents through each of the resistors should be 9.14 ± 0.05 mA, 1.93 ± 0.02 mA, 1.083 ± 0.007 mA. These are generally good estimates but represent a overestimation of the measured currents by 1.9%, 3.6%, and 2.7%.

Voltage In parallel, voltage is equal, thus it makes sense that all measured voltages were the same as the total voltage measured.

Combined Circuit

Resistance In this combined circuit, the effective total resistance may be found by combining the methods for parallel and series circuits. Doing this will yield an calculated effective resistance of 2.24 ± 0.01 kΩ. This is 1.3 % below the measured value of 2.27 kΩ.

Current The current through the first resistor should be the same as the current through the whole system. Thus one can find this simply using Ohm's law. This calculation would conclude the current though the system to be 2.26 ± 0.01 mA, 0.9% higher than the measured value of 2.24mA. The other two currents should sum to get this previous current (Loop Rule) and should also be inversely proportional to the resistance through the resistors (Ohm's Law). This rules may be rearranged to say that

$$I_2 = \frac{IR_3}{R_2 + R_3},\ I_3 = \frac{IR_2}{R_2 + R_3} \tag{7.17}$$

These equations predict the currents through the resistors to be 1.447 ± 0.008 mA and 0.813 ± 0.006mA. The later of these values is an overestimation of 2.2%. But the first value is more troubling. Taken as written these predict an error of over 25%. This is troubling and doest fit well with the other data taken. It is possible and very likely that some kind of error in transcription occurred and the measured value was perhaps 1.410 mA or 1.440 mA which would better fit with the remaining data. However, the original manuscript is no longer in the possession of the author who can do no better than guess at the true value. These guessed values provide errors of 2.5% (for 1.410 mA) or 0.3% for (1.440 mA). The previous errors indicate that 1.410 is the most likely answer.

Voltage With calculated currents and measured resistances one need only take a product to find the voltages across each resistor. One should expect the second two voltages to be the same. The first voltage was found to be 1.254 ± 0.07 V, exactly the measured voltage. The second was found to be 3.81 ± 0.04 V, also exceptionally close to the measured voltage. The third and final was found to be 3.81 ± 0.05 V, like the other 3 accurate to within the significant figures of the measurement.

General Conclusions

The measurements gathered here agreed with both Kirchoff's laws and with Ohm's law. All values predicted were exceptionally close to their measured values, and the variability apparent is rather easily explained. Resistance was consistently underestimated, this means that there is some agent acting to increase the resistance of the system without our knowledge. This stems from the fact that the voltages source, resistance probe, and wires have some small unaccounted for resistance. The underestimation in resistance accounts for the constant over estimation in current. Voltage was consistently accurate.

Chapter 8

Measuring the Fundamental Charge of An Electron

8.1 Introduction

In 1897, J.J. Thomson discovered the electron and measured its charge-to-mass ratio. But no one knew how to measure the charge of the electron. Thomson used clouds of charged water droplets and observed how fast they fell when gravity and an electric field were present. However, Thomsons experiments produced only a very crude estimate of the electrons charge.

In 1906 Millikan, who was already a successful educator and textbook writer, was eager to make his mark also as a scientific researcher. Millikan saw the opportunity to make a significant contribution by improving upon Thomsons results. Millikan believed that instead of measuring the charge on whole clouds of water like Thomson did, he could try to determine the charge on individual droplets. When he began his experiment, however, he noticed that water droplets evaporated too quickly for accurate measurement.

Fortunately, working with his graduate student Harvey Fletcher, Millikan figured out how to conduct his experiment with a substance that would evaporate more slowly: oil droplets produced by an atomizer. The oil droplets, injected into an air-filled chamber, fall or rise under the combined influence of gravity, viscosity of the air, and an electric field. The droplets pick up charge from the ionized air between the plates. By adjusting the electric field, Millikan could watch the droplets through a microscope and measure the time it takes for a droplet to fall or rise. After obtaining fall and rise time measurements, Millikan was able to calculate the charge on the droplet.

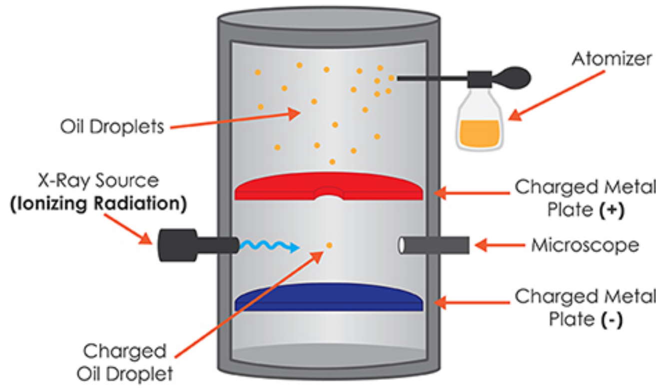


Figure 8.1: Milikan's Experiment found the charge of very small drops of Oil

Countless rounds of rigorous data collection later, Millikan reported the elementary charge to be 1.592 x 10-19 C, which is only slightly lower than the currently accepted value of 1.602 x 10-19 C. Millikan went on to win the 1923 Nobel Prize for his contribution to the field of electricity. The purpose of our experiment is to duplicate Millikans oil drop experiment, and hopefully verify his conclusion about the charge of the electron.

8.2 Procedure

ll materials were gathered before beginning experimentation. The materials included a high-voltage DC power source which can generate a minimum potential difference of 500V, a PASCO Millikan Oil Drop Apparatus (Model AP-8210) which includes a low-viscosity mineral oil and an atomizer, a timer, and a stand to position the Apparatus at eye level.

The Apparatus was placed on the stand and secured to prevent it from sliding as shown in the figure below. Afterwards, the DC power source was connected to an AC adapter connected to a wall socket and turned on to 0V. The power source was then connected to the Apparatus through the included plate charging switch, then the voltage on the Apparatus was slowly increased to 500V. Ambient light was then reduced by turning off the room lights and placing the device away from windows before turning on the halogen lamp which illuminates the droplet area. Oil was then poured

Figure 8.2: Setup for the PASCO Millikan Oil Drop Apparatus

into the atomizer and filled to approximately half of its maximum volume. To fill the droplet area with oil, the tip of the atomizer was placed over the hole in the covering of the droplet chamber and the bulb of the atomizer was given one full squeeze to release the oil. The oil was then seen either slowly rising or falling at a relatively constant velocity. The droplets were either put to rest by flipping the switch to its middle setting or allowed to change the direction of their velocities by flipping the switch to its opposite setting.

Data was taken by recording the amount of time it took for one particular droplet to rise a chosen amount of distance on the order of one millimeter, then the time it took for the same droplet to fall the same distance. The data was assigned to the date it was taken, the ambient air pressure, the resistance of the thermistor, the separation of the electric plates, and the density of the oil as cited in the manual as well as its viscosity. Materials were put away after a total of eight rise and fall recordings.

8.3 Data

The time for small drops of ionized oil to fall and rise (under the influence of an electric field) were calculated these

Figure 8.3: Properties of Drops

Drop #	Date	Fall dist (mm)	Fall Time (s)	Rise dist (mm)	Rise Time (s)
0	3-Feb-2013	0.50	33.0 ± 0.8	0.50	2.64 ± 0.04
1	7-Feb-2018	1.00	18.9 ± 0.5	1.00	9.13 ± 0.3
2	8-Feb-2018	0.50	15 ± 0.8	0.50	3.35 ± 0.15
3	8-Feb-2018	0.50	37.9 ± 3.5	1.00	4.85 ± 0.03
4	8-Feb-2018	0.50	37.9 ± 2.2	0.50	6.19 ± 0.17
5	8-Feb-2018	1.00	25.6 ± 0.4	1.00	6.44 ± 0.11
6	3-Feb-2018	0.50	106 ± 10	0.50	2.31 ± 0.6
7	8-Feb-2018	0.50	41.2 ± 5.4	0.50	4.72 ± 0.19

Figure 8.4: Properties of Days When Drops Were Collected

Date	Atmospheric Pressure (bar)	Thermistor Resistance (MΩ)
3-Feb-2013	1017	2.04
7-Feb-2018	1018	1.98
8-Feb-2018	1030	1.95

These may be used to find the velocities of rise and fall which characterize the charge on each oil drop. It is also necessary to characterize the environmental conditions of the chamber in which the drops fall and rise. To do this, the pressure is measured, and the temperature inside the thermistor is estimated through a resistor inside. This may be also used to give an estimate of the viscosity of the air in the droplet chamber. Note that pressure may be adjusted slightly as the temperature within the thermistor is higher than the ambient temperature at which the pressure was measured, yielding a somewhat higher pressure within the thermistor. Generally this effect is relatively minimal and accounts for less that 0.5% of the variance in the calculated charges

8.4 Analysis

[1] Density of Oil Viscosity of Air at measured temperature taken from model in AP-821A0 manual.
Gravitational acceleration based on EMG2008 12th order model

Figure 8.5: Other Relevant Information

Voltage	500 ± 2 V
Viscosity of Air	$1.844 \cdot 10^{-5} \frac{\mathrm{N \cdot s}}{\mathrm{m^2}}$
Density of Oil	$866 \ \frac{\mathrm{kg}}{\mathrm{m^3}}$
Gravitational Acceleration [1]	$9.797 \ \frac{\mathrm{m}}{\mathrm{s^2}}$

In order to determine the charge of the elemental charge particle, the charge on many small particles are computed and the distribution of the measured charges are compared. Of course, to do this one must first be able to find the charge on a single oil drop. This can be done simply using the velocities of the fall and the rise of the oil drop.

Both in free fall and when rising due to an electric charge the particles, being very small, reach a terminal velocity incredibly quickly. A terminal velocity is, of course, constant and thus the particle is experiencing no acceleration and has no net force acting upon it.

In free fall, the only forces acting upon the small drop of oil are the one due to gravity, $m\,g$, and the one due to the coefficient of friction with the air. The relationship between the force of friction and the speed of free fall are assumed to be linearly related, with this frictional force, $k\,v_f$ being proportional to the speed of freefall, v_f, and some coefficient of friction with the air k. These expressions give the equality,

$$m\,g = k\,v_f \tag{8.1}$$

The rising oil drops are governed by a similar, if a bit more complicated, expression. These drops feel a third force, that of an electric field $\mathbf{E}$. This force is simply $\mathbf{E}\,q$, the product of the field and the charge of the particle q And, as rising instead of falling the direction of the force due to friction is downward rather than upward as before. This force due to friction is proportional to the speed of rising, v_r. These give the equality

$$m\,g + k\,v_r = \mathbf{E}\,q. \tag{8.2}$$

The coefficient of friction with the air is unknown and difficult to measure, it is thus necessary to remove it from the expression for charge. This may be accomplished by means of a simple substitution.

$$k = \frac{m\,g}{v_f} \tag{8.3}$$

Substituting this give a simple expression for the charge on an oil drop

$$q = \frac{m\,g\,(v_f + v_r)}{v_f\,E}. \tag{8.4}$$

This expression is simple, clean, and concise. But, it is also not particularly useful. It relies on the mass of the drops of oil, this is unknown, unmeasured, and by most conventional methods unmeasurable. This may seem problematic. But, If one assumes the oil drop to be a sphere of uniform density, then one may find the mass of the drop in terms of its radius, a, and density, ρ.

$$m = \frac{4\pi}{3}\,\rho\,a^3. \tag{8.5}$$

This may seem no more useful than the previous expression, if more convoluted. For the radius of the drops of oil is also not well known. However, there exists a relationship, Stokes Law, which attempts to model the falling speed of a droplet function of its radius that may be used. Stokes law claims that

$$a = \sqrt{\frac{9\,\eta\,v_f}{2\,\rho\,g}}, \tag{8.6}$$

where η is the viscosity of the medium through which the drop falls. However, Stokes law is best applicable for droplets falling at a rate of greater than 0.1 cm/s, while the drops of oil that have been observed fall at a rate closer to 10^{-5}m/s. This error may be fixed with a correction factor applied to the viscosity of air.

$$\eta_{eff} = \eta\left(\frac{1}{1 + \frac{b}{p\,a}}\right) \tag{8.7}$$

Where p is the atmospheric pressure and b is a constant equal to $8.2 \times 10^{-3}\mathrm{Pa}\cdot\mathrm{m}$. Substituting this into the expression for a yields the clunky and self referential formula :

$$a = \sqrt{\frac{9\,\eta\,v_f}{2\,\rho\,g}\left(1 + \frac{b}{p\,a}\right)^{-1}}. \tag{8.8}$$

With some rearrangements, this may be solved as a quadratic equation for a. This is not too clean, but it yields an expression for a.

$$a = \sqrt{\left(\frac{b}{2\,p}\right)^2 + \frac{9\,\eta\,v_f}{2\,\rho\,g}} - \frac{b}{2\,p} \tag{8.9}$$

This may be substituted into equation (5) to find the mass of the oil drop

$$m = \frac{4\,\pi}{3}\,\rho\,\left(\sqrt{\left(\frac{b}{2\,p}\right)^2 + \frac{9\,\eta\,v_f}{2\,\rho\,g}} - \frac{b}{2\,p}\right)^3 \tag{8.10}$$

And this into equation (4) along with the electric field for parallel plates $\mathbf{E} = V/d$.

$$q = \frac{4\,\pi}{3}\,\rho\,\left(\sqrt{\left(\frac{b}{2\,p}\right)^2 + \frac{9\,\eta\,v_f}{2\,\rho\,g}} - \frac{b}{2\,p}\right)^3 \frac{d\,g\,(v_f + v_r)}{v_f\,V} \tag{8.11}$$

8.4.1 ~~Erorrs~~ ~~Erors~~ Errors

It is necessarily and useful to quantify the error in the calculated charge. This may be estimated by estimating the error in each parameter and calculating the error contribution. This may be done my multiplying the error in each parameter by the partial derivative of the charge with respect to each parameter.

Partial Derivatives

$$\frac{\partial q}{\partial d} = \frac{q}{d} \tag{8.12}$$

$$\frac{\partial q}{\partial V} = -\frac{q}{V} \tag{8.13}$$

$$\frac{\partial q}{\partial p} = \frac{3\,q\,b\sqrt{\rho\,g}}{p\sqrt{b^2\,\rho\,g + 18\,\eta\,v_f\,p}} \tag{8.14}$$

$$\frac{\partial q}{\partial v_r} = \frac{q}{v_r + v_f} \tag{8.15}$$

$$\frac{\partial q}{\partial v_f} = q\,\frac{\left(b^2 g\rho\,v_r - 27\,\eta\,{v_f}^2 p^2 - 9\,\eta\,v_f p^2 v_r - \sqrt{b^2\rho\,g + 18\,\eta\,v_f p^2}\sqrt{\rho\,g}bv_r\right)\sqrt{\rho\,g}}{v_f\,(v_f + v_r)\left(b\rho\,g - \sqrt{b^2\rho\,g + 18\,\eta\,v_f p^2}\sqrt{\rho\,g}\right)\sqrt{b^2\rho\,g + 18\,\eta\,v_f p^2}} \tag{8.16}$$

Each of these may be multiplied with the errors in their respective values and used to find their error contribution to the total calculated charge. If one is to assume independence in the parameters, one may choose to use the square root of the sum of the squares of these errors.

8.4.2 Calculated Charges

One may use the above sections to estimate the charge on each of the observed oil drops. Doing this yields the calculated charges

Figure 8.6: Charges on drops of oil

Drop #	Charge (C) $\cdot 10^{-19}$
0	3.01 ± 0.07
1	5.16 ± 0.36
2	4.42 ± 0.26
3	2.96 ± 0.22
4	1.25 ± 0.07
5	5.15 ± 0.25
6	1.43 ± 0.53
7	1.50 ± 0.05

This information is useful, but it is not immediately evident how to deduce the charge of a single electron simply from glancing at these numbers. It becomes beneficial to display these numbers in a means that best illustrates the integer multiples that one would expect to see in a world where charges come in discrete chunks. One may do this, simply by plotting the charges measured on each drop of oil from least to greatest such that near by charges may be compared to those with similar charges.

Figure 8.7: Charges on Droplets of Oil

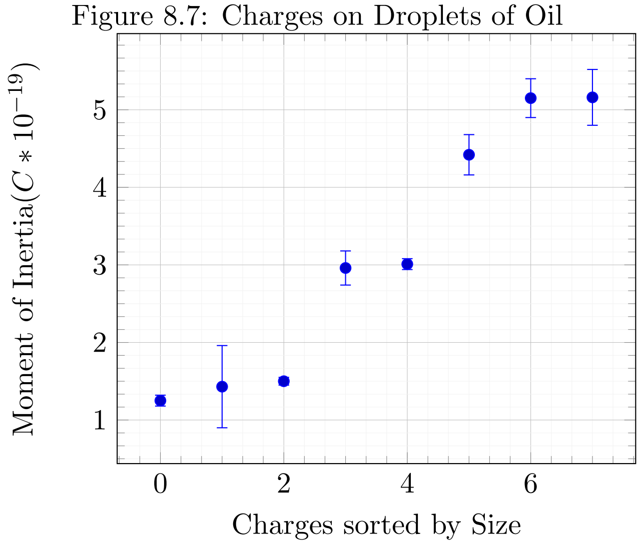


Figure 8.8: Estimation of Fundamental Charge Based on Each Electron

Drop #	Charge (C $\cdot 10^{-19}$)
0	1.50 ± 0.04
1	1.72 ± 0.12
2	1.47 ± 0.09
3	1.48 ± 0.07
4	1.25 ± 0.02
5	1.47 ± 0.08
6	1.42 ± 0.53
7	1.50 ± 0.05

From this interpretation, one may rather quickly see that the charges are distributed on three strata. Droplets 4, 6 and 7 all appear to have roughly the same amount of charge on them, as do droplets 0 and 3. Droplets 1, 2. and 5 also appear to be in one grouping. One might also notice that the strata seem to be at charges roughly relating by integer multiples. Stratum 2 has roughly 2.1 times the charge as stratum 1, who has roughly a third (0.31) the charge of Stratum 3. This behavior suggests that the droplets of strata 1, 2 and 3 have 1, 2 and 3 fundamental charges on them respectively.

If one wishes to find the fundamental charge most accurately, one may choose to divide each measured droplet's charge by the estimated number of fundamental charges on that little droplet. Each drop should thus provide its own estimate for the electron's charge. And, A mean of this should provide a more accurate estimate for the fundamental charge on the electron and proton.

The mean of these values yields an average estimation of the fundamental charge of an electron to be $1.48 \pm 0.12 \cdot 10^{-19}$C.

8.5 Conclusion

This experiment estimated the fundamental charge of the electron to be $1.48 \pm 0.12 \cdot 10^{-19}$C. This is relatively close to its true value of $1.60 \cdot 10^{-19}$C and the calculated value constitutes an error of 7.5%. The true value of the charge of the electron is just within 1 standard deviation of the calculated charge for the drops. It is also worth noting the rather large amount of variance present in the charge estimate for each drop. This is partly because of the large amount of variance in the measured speeds. A At least qualitatively, the drops did not always appear to fall at a constant speed.

Especially when falling, the drops appeared to rather randomly slow down, stop, or even drift upward for short periods of times. This might have to do with small disturbances and un-uniformities present in the distribution of the air.

The effects of these are exacerbated because the oil drop is very small and thus more susceptible to these disturbances. This, somewhat random motion also, to a certain extent explains the underestimation in the charge on the drops. This stems from a overestimation of the fall speed and a underestimation of the time to fall.

Consider a droplet which has a probability of falling down a little bit in every small period of time, and a smaller probability of staying still or going up in each small period of time. Naturally, this drop will trend downward, but will bob up and down anyway. It is very likely that this drop will pass through the same height quite a few times as it bobs up and down, but the impatient stop watch timer will chose to stop the drop as soon as the drop crosses the line, rather than the more difficult to estimate average line crossing time.

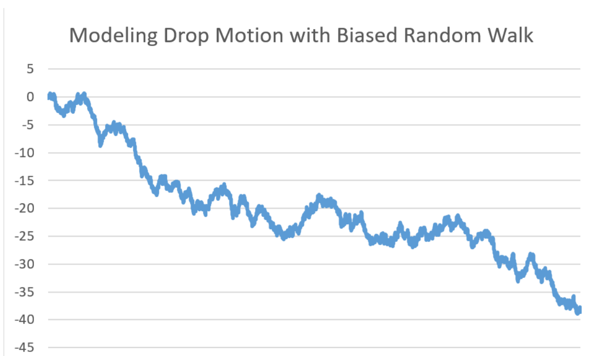


Figure 8.9: Randomly generated drop motion. Notice how the drop passes through the point -25 several times, the stopwatch would be stopped at the first and thus give a shorter time, higher speed, and smaller charge than if it had been stopped at the second.

Also, an impatient stop watch timer may chose to call time right before the drop has crossed the line rather than right after. This would also contribute to an underestimation of time.

This, among other things, contributes to then underestimation of the charge on an electron.

8.6 References

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Chapter 9

Measuring the Charge-to-Mass Ratio of an Electron

Date Performed:	February 9, 2018
Instructor:	Dr. Bradley Miller
Partners:	Emilio Folley, Mathew Harvey

9.1 Introduction

An electron is a subatomic particle. It has a charge, called the elemental charge. It also has mass. Both of these are very small and thus not particularly easy to measure with standard measuring apparatus. That is, one might find it rather difficult to measure the mass of an electron on a standard kitchen scale. Rather, it is more natural to find some more easily measurable occurrence which depends on the properties of electrons and from them infer these properties. Chief among the properties of an electron that one might want to measure are the aforementioned charge and mass.

Historically, the first property of an electron to be measured was actually neither charge nor mass. In 1897, J. J. Thomson, discoverer of the electron, first characterized the charge-to-mass of this newly discovered particle. About 10 years later, American physicist Robert Millikan calculated the charge, and thus, using Thomson's charge-to-mass ratio, the mass, of an electron. Later experiments yielded similar values for other electrons.

Thomson's experiment used a cathode ray tube, a vacuum tube containing an electron gun. This cathode ray tube produces a stream quickly moving electrons. Being charged particles under motion these electrons are liable to be acted upon by a magnetic field. Actually, this is a method by which one may show that electrons have negative charge. If one is careful, and one chooses to apply a magnetic field which is perpendicular to the direction of the electrons' motion, then one may assume a constant force due to magnetism acting on the electrons perpendicular to the direction of both the electron beam and the magnetic field. This will bend the electron beam into a circular arc whose radius is dependent on several properties, one of which is the charge-to-mass ratio of the electron.

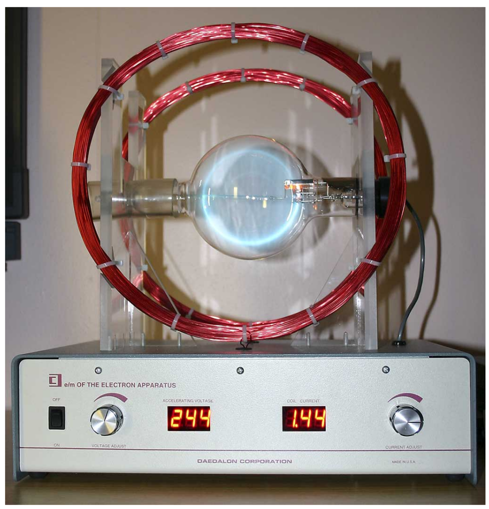


Figure 9.1: The magnetic field from a pair of Helmholtz coils causes the electron beam to move in a circle

9.2 Procedure

An electron beam was placed within a glass tube of low pressure, in side of which was helium gas which ionized and thus became visible when passed through by the electron beam. This glass tube had a spherical bulb in which the electron beam was to rotate. The magnetic field was generated by a pair of Helmholtz coils with 130 loops and a current passed through them. This bent the electron beam into a circle of diameter between 5 and 11 cm. A small glass ruler was inside of the glass bulb to measure the diameter of the electron beam radius. The diameter of the loop was measured at various

Figure 9.2: Fitting Parameter for Inverse Relationship

Trial	Orientation 1 (A m)	Orientation 2 (A m)
(200 V E-W)	0.1255 ± 0.005	0.1233 ± 0.008
(200 V N-S)	0.1224 ± 0.008	0.1282 ± 0.004
(300 V N-S)	0.1558 ± 0.003	0.1611 ± 0.004
(150 V N-S)	0.1058 ± 0.005	0.1094 ± 0.004

currents voltages and orientations. Currents ranged from approximately 1 to 2.5 A, while the experiment was done with voltages of 200, 250, and 300 V. These were done in easterly, westerly, northerly, and southerly directions to account for variation in the magnetic field of the earth.

9.3 Data

Initially the current through the Helmholtz coils was varied such that the diameter of the electron circle fell on each half inch interval. After this, the apparatus was rotated 180 degrees and the diameter was measured at the currents used in the previous orientation.

These points seem to signal an inverse relationship between the diameter of the loop of current and the current through the coil. One might try to fit an inverse curve of the sort A/x to the data.

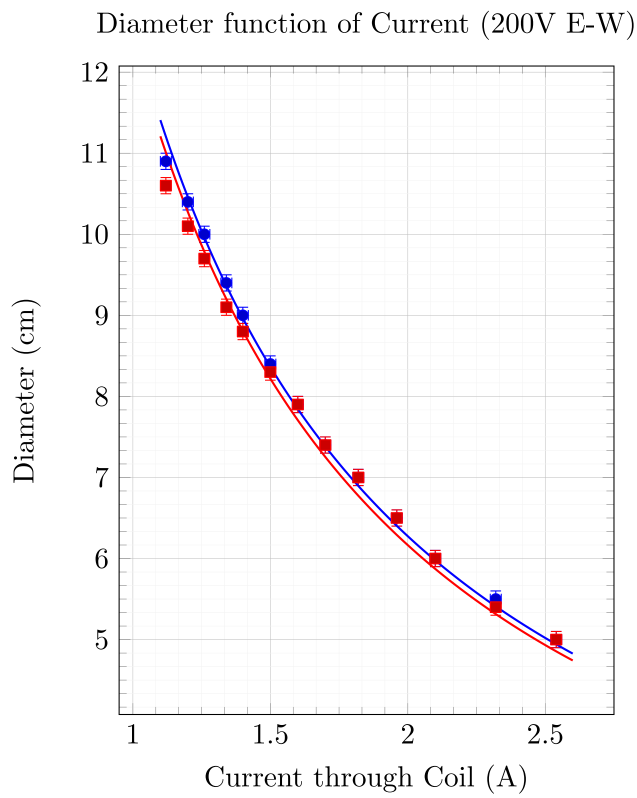
These might be used to find the charge-to-mass ratio as demonstrated later on.

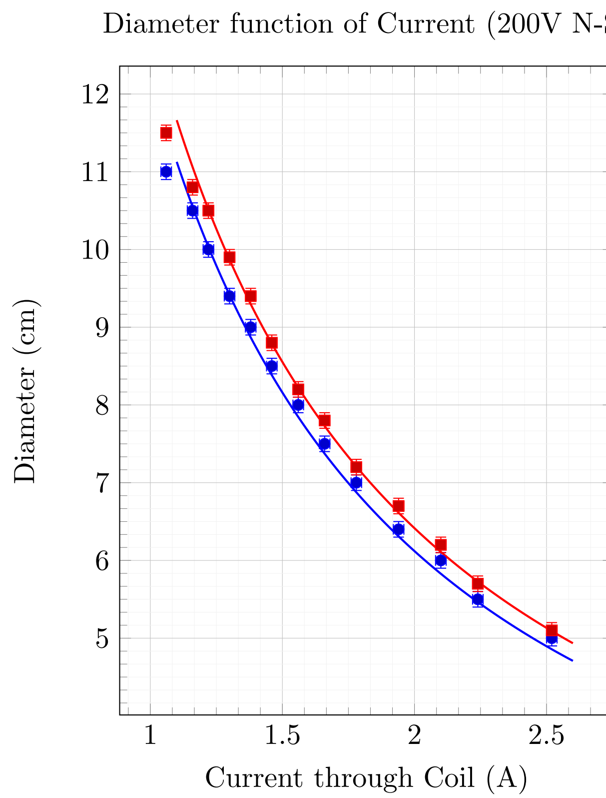
The radius and separation between the Helmholtz coils was measured to be 15.3 ± 0.3 cm.

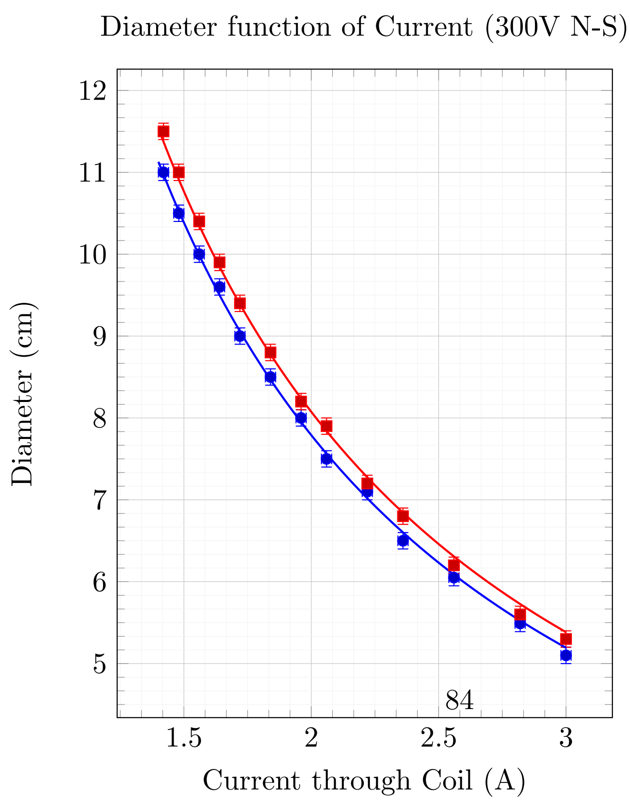
9.4 Analysis

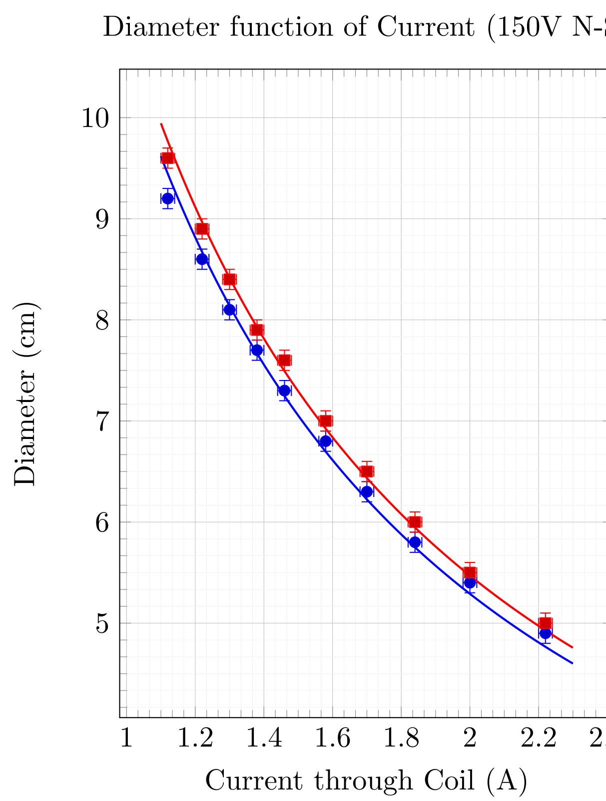
One should be able to find the charge-to-mass ratio function of the diameters and currents measured above. Consider a beam of electrons at velocity $\mathbf{v}$ moving perpendicular to a magnetic field $\mathbf{B}$. Each of these electrons should have some unknown charge q. The force from the magnetic field acting on these electrons should hence be

$$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B} \tag{9.1}$$


Diameter function of Current (200V E-W)
Diameter (cm)
Current through Coil (A)


Diameter function of Current (200V N-
Diameter (cm)
Current through Coil (A)


Diameter function of Current (300V N-S)
Diameter (cm)
Current through Coil (A)


Diameter function of Current (150V N-
Diameter (cm)
Current through Coil (A)

Notice that these electrons are moving in a circular path. Recall that the force required perpendicular to the direction of motion is described as

$$\mathbf{F} = \frac{m\,\mathbf{v}^2}{r}\hat{\mathbf{r}} \tag{9.2}$$

One might set these forces equal to each other to conclude that

$$q\,\mathbf{v} \times \mathbf{B} = \frac{m\,\mathbf{v}^2}{r}. \tag{9.3}$$

And from there one is left to solve for the charge-to-mass ratio.

$$\frac{q}{m} = \frac{v}{r\,\mathbf{B}} \tag{9.4}$$

This is a wonderful little expression. The trouble is that it isn't particularly useful, for one the speed at which the electrons move is unknown. However it may be rearranged to be more useful. Consider the energy of the electrons. The gain of energy from an electron is simply the product of the potential difference V and the charge q. While the kinetic energy is $m\,v^2/2$ setting these equal allows one to solve for the velocity.

$$V\,q = \frac{m\,v^2}{2} \tag{9.5}$$

$$v = \sqrt{\frac{2Vq}{m}} \tag{9.6}$$

substituting this into the previous expression yields the somewhat nasty expression.

$$\frac{q}{m} = \frac{1}{r\,\mathbf{B}}\sqrt{\frac{2Vq}{m}} \tag{9.7}$$

This expression is unfortunately in terms of q/m the very ratio we wish to find. Fortunately we may square this equality and simplify.

$$\frac{q^2}{m^2} = \frac{2\,V\,q}{r^2\,\mathbf{B^2}\,m} \tag{9.8}$$

$$\frac{q}{m} = \frac{2\,V}{r^2\,\mathbf{B^2}} \tag{9.9}$$

This is a nice, convenient expression, but it relies on knowledge of $\mathbf{B}$. This magnetic field comes from a pair of Helmholtz coils, luckily there is a expression for the magnetic field from this configuration.

$$\mathbf{B} = \left(\frac{4}{5}\right)^{3/2}\frac{\mu_0\,n\,I}{R} \tag{9.10}$$

Where μ_0 is the permeability of free space, n is the number of loops in the Helmholtz coil, I is the current passing through the coils, and R is both the radius of the coils and the separation between the two. Substituting this into the expression for the charge-to-mass ratio yields .

$$\frac{q}{m} = \frac{125\, V\, R^2}{32\, r^2\, \mu_0^2\, n^2\, I^2} \tag{9.11}$$

Now, one might try look at the function of current to radius, as these were the quantities directly measured in the experiment. Solving for radius yields

$$r = \frac{R}{\mu_0\, n\, I}\sqrt{\frac{125\, V\, m}{32\, q}} \tag{9.12}$$

And hence the diameter would be

$$d = \frac{R}{\mu_0\, n\, I}\sqrt{\frac{125\, V\, m}{8\, q}} \tag{9.13}$$

This would mean that the current through the Helmholtz coils and the diameter of circle measured will be related by an inverse relationship. One could attempt to fit the data on a current diameter graph with a simple relationship

$$d = \frac{A}{I} \tag{9.14}$$

where A is some constant whose value should be equal to

$$A = \frac{R}{\mu_0\, n}\sqrt{\frac{125\, V\, m}{8\, q}} \tag{9.15}$$

solving for the the ratio q/m

$$\frac{q}{m} = \frac{125\, R^2\, V}{8A^2\, \mu_0^2\, n^2} \tag{9.16}$$

9.4.1 Uncertainty

One must also attempt to quantify the uncertainty in the above expression. The natural way to do this is to look at the contribution each of the variable parameters makes to the above expressions. This is done by taking the partial derivative of q/v with respect to the variable parameters A,R, and V.

$$\frac{\partial q/m}{\partial A} = \frac{-2\,q}{A\,m} \tag{9.17}$$

$$\frac{\partial q/m}{\partial R} = \frac{2\,q}{R\,m} \tag{9.18}$$

$$\frac{\partial q/m}{\partial V} = \frac{q}{V\,m} \tag{9.19}$$

These may be combined with and the square root of the sum of their squares may be used to quantify the uncertainty in the overall expression

$$\delta_{q/m} = \frac{q}{m}\sqrt{\frac{4\delta_A^2}{A^2} + \frac{4\delta_R^2}{R^2} + \frac{\delta_V^2}{V^2}} \tag{9.20}$$

9.5 Conclusion

The above expression as well as the fitting parameters calculated for the graphs in section 3 may be used to quickly find the charge-to-mass ratio for each of the orientations and voltages. These figures seem relatively close

Figure 9.3: Charge-to-mass ratio

Trial	Orientation 1 (C/kg $\cdot 10^{11}$)	Orientation 2 (C/kg $\cdot 10^{11}$)
(200 V E-W)	1.74 ± 0.15	1.80 ± 0.24
(200 V N-S)	1.83 ± 0.25	1.67 ± 0.12
(300 V N-S)	1.69 ± 0.09	1.59 ± 0.1
(150 V N-S)	1.84 ± 0.19	1.72 ± 0.14

to the $1.76 \cdot 10^{11}$C/kg[1] accepted value for the charge to mass ratio of the electron. However, it is useful to be able to point to a single number for the calculated charge-to-mass ratio of the electron. To do this, one may take a simple average of the above estimates. However, it is useful to take a weighted average and to give larger weights to estimates with smaller variance. In this case, a weight proportional to the inverse variance was assigned to each of the estimates. This process yields an estimate of $1.69 \pm 0.05 \cdot 10^{11}$C/kg, an error of 3.4%. One might notice that the estimates for the charge-to-mass ratio done while facing in a northerly direction are consistently larger than those measured when facing a southerly direction.

[1] 2014 CODATA

This is consistent with the wildly held belief that compasses tend to point north. That is, the earth has a magnetic field which points northward. This magnetic field contributes to that of the Helmholtz coil, and when facing north results in a larger magnetic field. This will result in an overestimation of the charge to mass ratio when facing north, and an underestimate when facing south. Interestingly this isn't held when comparing the east-west orientations. There actually is a small easterly component to the magnetic field in Chapel Hill, however this isn't represented in the data. However, it is worth noting that these values are very close to each other and each have a very large uncertainty.

The relatively small error present in the estimate, however it may be explained by some of the qualities if the experiment and apparatus. For example, the coils used to create the magnetic field were not quite circular and were actually rather squished. This will have an effect of the estimate for the charge-to-mass ratio. Also, the electron beam was not actually a thin beam, and thus measuring the point at which the beam hits the glass rod is not easy to accurately measure. It is also possible that the viewer didn't look at the rod and thus had some parallax distortion. Also, it is possible that the moving electron beam, a current, created its own small magnetic field which had a very small effect on the path of them.

Chapter 10

Focusing Light with a Lens

[Optics]

Date Performed:	March 15, 2018
Instructor:	Dr. Bradley Miller
Partners:	Emilio Folley, Mathew Harvey

10.1 Introduction

People generally feel that they have some intuitive understanding of light; they have lived around it their entire ng ones eyes, even though stars are very far away. Euclid concluded that light must travel at an infinite speed.

10.1.1 Electro-Magnetic Radiation

In reality light is not made up of beams spouting from one's eyes, rather what we call light is simply a flulives, vision guiding most all daily activities, but when pressed to describe light most would struggle. Perhaps light is most commonly thought as the agent that permits vision. However, this seems a rather anthropocentric perspective. Presumably, there should exist a description of light that exists independent of human observance.

10.1.2 A Historical Perspective

Ancient Greek philosophers explained light not as an agent to aid human vision, but rather as a method by which humans saw. There is a key distinction here. Rather than being a tool for humans, light was a product of humans. Empedocles postulated that there was a small fire in the human eye witch illuminated the world and permitted sight, but he failed to adequately explain why vision was impossible in the dark. Around 300 BC Euclid wrote his treatise on the geometry of vision *Optica*. He was troubled by the explanation of light beams streaming from the eyes, noting that one sees the stars immediately after openictuation in the electric and magnetic fields. Luckily, these fields are governed by a set of relatively simple rules, and we may manipulate these rules to more accurately describe light. Consider the differential forms of Maxwell's equations.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0},\ \nabla \cdot \mathbf{B} = 0,\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},\ \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{10.1}$$

Most of the time, light travels in a simple environment: no charges and no currents need be involved. We will simplify things by letting ρ and $\mathbf{J}$ go to zero.

$$\nabla \cdot \mathbf{E} = 0,\ \nabla \cdot \mathbf{B} = 0,\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},\ \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{10.2}$$

If possible, we would like to disentangle $\mathbf{E}$ and $\mathbf{B}$. We will do this by taking the curl of both of the curl expressions above. Recall the curl of the curl identity: $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$.

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{E}) &= -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t} \right), & \nabla \times (\nabla \times \mathbf{B}) &= \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) & (10.3)\\
\nabla \times (\nabla \times \mathbf{E}) &= -\frac{\partial}{\partial t} \nabla \times \mathbf{B}, & \nabla \times (\nabla \times \mathbf{B}) &= \mu_0 \epsilon_0 \frac{\partial}{\partial t} \nabla \times \mathbf{E} & (10.4)\\
\nabla \times (\nabla \times \mathbf{E}) &= -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, & \nabla \times (\nabla \times \mathbf{E}) &= -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}, & (10.5)\\
\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, & \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} &= -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}, & (10.6)\\
\nabla^2 \mathbf{E} &= \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, & \nabla^2 \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}, & (10.7)
\end{aligned}$$

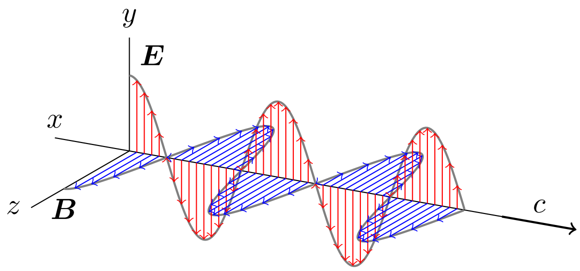
These final equations effective mean that the local acceleration at a given point is proportional to the local curvature at that same point. This is the same kind of relationship one might see with say, a sheet of rubber.

The astute might recognize these as wave equations. This means that, like a rope held taught at two points, one can introduce a perturbation into the electric or magnetic fields and it will propagate forward. Recall that the general wave equation is the form

$$\nabla^2 \mathbf{F} = \frac{1}{c^2} \frac{\partial^2 \mathbf{F}}{\partial t^2} \tag{10.8}$$

where c is the wave speed. These imply that both the electric and magnetic fields are waves which propagate forth at a fixed speed : $1/\sqrt{\epsilon_0 \mu_0}$ – light speed. This demonstrates that light is a wave that will generally move forth at a fixed speed in straight lines.

Figure 10.1: Linearly Polarized Light Propagates Forward



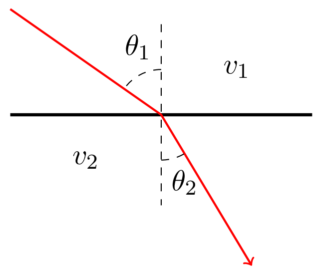
10.1.3 Passing Through Media

Taking into account the description of light as a wave propagating forward, it makes sense to conclude that once traveling in a certain direction, light will continue to travel in that straight line. It thus makes sense to describe light as a ray, a straight line emerging from a single point. However, we erred in claiming that the speed of light was constant. We stated that the speed of light was $1/\sqrt{(}\epsilon_0 \mu_0)$, but those constants, the permittivity and permeability of free space respectively, only make sense in free space. Light, like electricity and magnetism, need not travel through a vacuum, and the respective constants and wave speeds are different for different media. This shift in speed can mean that a change in media will result in a change in direction. This means that light can appear to bend as it passes through media. This relationship is described by Snell's law.

When entering a boundary between media at a given angle θ_1 with respect to the surface normal, a ray of light will exit at an angle θ_2 such that

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_2}{v_1} \tag{10.9}$$

Figure 10.2: Light bends at interface between surfaces with differing speeds of light

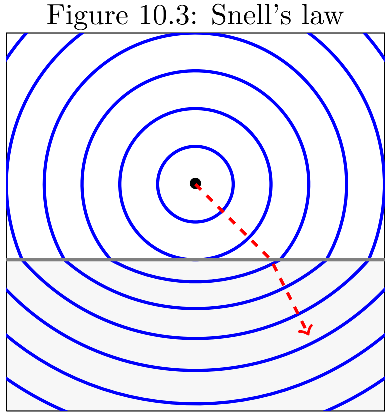


Where v_1 and v_2 are the speeds of light in the respective media. Often we choose to express this relation in terms of refractive indices. A refractive index is a number n that represents the relative speed of light within a medium. It is defined as the ration c/v where c is the speed of light in a vacuum and v the speed of light the the medium of interest. This means we may write snell's law in the form.

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \tag{10.10}$$

Snell's law follows from Fermat's principle of least time, which states that light travels in the path that will most quickly bring it from one point to another. The algebraic proof of this is beyond the scope of this text, but a visual explanation is provided in figure 3. Note that the waves of light leaving the source in figure 3 slow down once they enter the medium with a higher optical density. If light bends when traveling through media, it

Figure 10.3: Snell's law

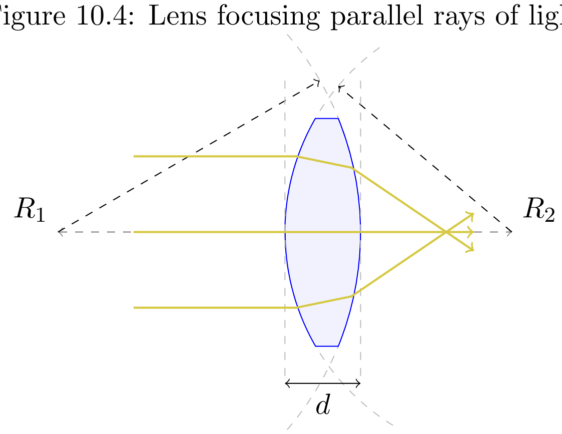


follows that with strategic positioning of materials we may bend light to our choosing. Most notably this is done with a lens.

10.1.4 Lens

We would like to describe the change in path like undergoes when passing through a lens. We consider a spherical lens of thickness d and radii R_1 and R_2.

Figure 10.4: Lens focusing parallel rays of light



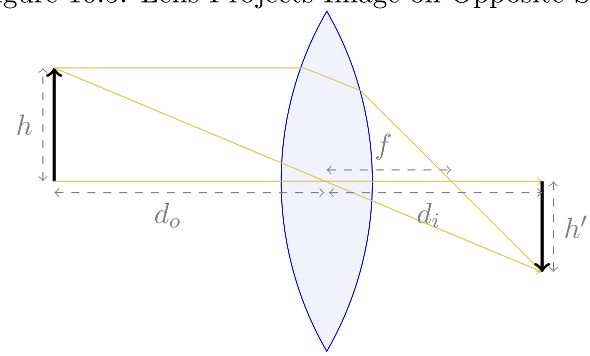
Lenses have the property of focusing light. In that, parallel rays of light entering a lens will exit intersecting at a fixed point called the focal point. The distance of this focal point from the lens is called the focal distance, and is a useful way to describe the actions of a lens. We would like to find an expression for this focal distance in terms of the geometry of a lens.

This expression is called the lens makers equation and its derivation is beyond the scope of this text, but gives the power of the lens to be

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{10.11}$$

Knowing the focal length of a lens allows one to predict the location at which an image will appear. Consider an object placed in front of a thin lens. We consider 2 rays. One called the chief ray runs through the center of the lens. For thin lenses, this means that it will travel through unmolested in a straight line. The other ray, begins parallel to the optic axis and after exiting passes through the focal point. The figure above clearly illustrates two pairs of similar triangles that allow us to set up side ratios. These two

Figure 10.5: Lens Projects Image on Opposite Side



are

$$\frac{d_o}{d_i} = \frac{h}{h'}, \text{ and } \frac{h}{h'} = \frac{f}{d_i - f} \tag{10.12}$$

equating these yields

$$\frac{d_o}{d_i} = \frac{f}{d_i - f} \tag{10.13}$$

which may be simplified to the rather elegant

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i} \tag{10.14}$$

This equation, paired with measurements of object and image distance allow one to compute the charectaristic focal distance of a given lens.

10.2 Procedure

Atop a table lay a large metal rail upon which the optical apparatus rest. A light source, a large lens, and an index card onto which an image shone lay suspended atop this metal rail. The light source lay at the far end of the rail, shining its light across the whole of its span. The light source was a bright bulb kept inside a metal enclosure. One end of this enclosure was open as to permit the light to shine outward. This opening was covered in a fine metal mesh and a small metal arrow which served as the object to be projected. Further down the rail lay a large glass lens to focus the light,and an index card was placed at the end of the rail. The index card was placed at a point such that projected image, of the metal mesh and small arrow was sharp and in focus. The distance from the source to the lens, the object distance, and the distance from the lens to the projection, the image distance were recorded.

In a final experiment the light source was removed and a nearby window was opened. This light was meant to be the case of light that is arbitrarily far away.

10.3 Data

The image distance d_i at which a clear projection of the small arrow and metal mesh was visible was measured at a variety of object distances d_o. The height of the projected object was also measured and recorded. All

Figure 10.6: Measured Object and Image Distances

d_0 (cm)	d_i (cm)	h (cm)	d_0 (cm)	d_i (cm)	h (cm)
27.76	68.30	3.62	49.01	32.26	0.99
29.05	60.95	3.15	51.00	31.70	1.00
31.29	53.74	2.52	53.00	31.30	0.95
32.21	48.51	2.15	54.26	30.67	0.90
34.95	44.88	1.90	59.00	29.48	0.78
37.00	41.71	1.61	64.00	28.35	0.69
41.00	37.20	1.36	LARGE	19.81	
44.00	35.03	1.25	LARGE	19.91	
46.00	33.58	1.07	LARGE	19.95	

distances were measured to an accuracy of 0.01 cm.

Characteristics of the environment are to be noted as well.

Object height	1.55 ± 0.01 cm
Estimated LARGE Distance	600 ± 100 cm
Lens Radii	20.4 ± 0.4 cm
Lens Height	7.41 ± 0.01
Lens Thickness	0.78 ± 0.01
Lens Top Thickness	0.10 ± 0.01

10.4 Analysis

Taking the measured object and image distances and computing the focal length of the lens proves incredibly easy. A simple manipulation of equation

(10.14) will suffice.

$$f = \left(\frac{1}{d_o} + \frac{1}{d_i}\right)^{-1} \tag{10.15}$$

Doing this for all of the measured object and image distances yields 15 separate estimates for the focal distance, all of which are about 19.6 cm.

The individual estimates are given below in the order listed above. (top to bottom then left to right)

Figure 10.7: Estimates of focal distance of lens

19.74	19.61	19.55
19.67	19.50	19.68
19.78	19.50	19.59
19.36	19.41	19.66
19.65	19.45	19.65

These figure estimate an average focal distance of 19.59 cm with a sample standard deviation of 0.12 cm.

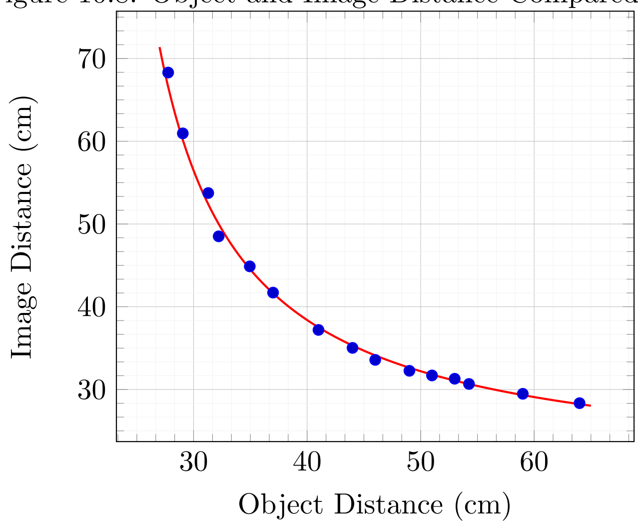
Another approach to finding the focal distance is to plot the image distance d_i function of the object distance d_o and use the focal distance f as a fitting parameter. This time the alternate equation:

$$d_i = -\frac{f d_o}{f - d_0} \tag{10.16}$$

is used. Doing this yields the somewhat higher but generally similar estimate $f = 19.64$ cm. It is also worth noting that the R^2 correlation of the graph above is the incredibly high value of 99.6%, suggesting that the model for image distance function of object distance is in fact accurate.

This focal distance should extend to the images which used light coming from outside as well as from the window. These were intended to have represented an object that was infinitely far away, and in that case the focal distance should merely be the object distance. However, the object used as a reference for being in focus was a nearby tree, which was, by no fault of its own, not quite infinitely far away. This tree was estimated to have been 6 meters away. Using this estimates for distance yields calculated focal lengths of 19.30, 19.27, 19.21 cm. These are decidedly smaller than other calculated focal distances. Perhaps the tree outside was truly farther than 6 meters away?

Figure 10.8: Object and Image Distance Compared.



10.4.1 Error in Focal Lengths

One might find the error associated with the estimate of the focal distance given the error in each of the measurements. Begin by differentiating the expression for f with respect to it's two parameters, d_i and d_o. With a little bit of cleverness on can reduce these derivatives to the convenient expressions :

$$\frac{\partial f}{\partial d_i} = \frac{f^2}{d_i^2}, \quad \frac{\partial f}{\partial d_o} = \frac{f^2}{d_o^2} \tag{10.17}$$

These lend themselves to the convenient expression for uncorrelated error

$$\delta_f = f^2 \sqrt{\frac{\delta_{d_o}^2}{d_o^4} + \frac{\delta_{d_i}^2}{d_i^4}} \tag{10.18}$$

However, because the error in measurements for image and object distance are rather small, the overall error in the focal distance is insignificant when compared to the standard deviation of the calculated values.[1]

[1] Standard deviation uncertainty is about 0.1 cm, uncertainty propagated is typically 0.005 cm

10.4.2 Index of Refraction

Pairing the lens makers equation with the estimate for focal length above gives a value for the index of refraction of the lens used. Solve the lens maker's equation for index of refraction. Doing this yields the simple, but somewhat clunky expression for index of refraction.

$$n = \frac{f\,R_2 + f\,R_2 + R_1\,R_2}{f\,R + 1 + f\,R_2} \tag{10.19}$$

However, assuming a symmetrical lens simplifies the expression greatly. Let $R_1 = R_2 = r$.

$$n = \frac{2\,f + r}{2\,f} \tag{10.20}$$

or simply,

$$n = 1 + \frac{r}{2\,f} \tag{10.21}$$

Evaluating this expression with the calculated radius and focal length gives an estimated refractive index of 1.52

10.4.3 Uncertainty in Index of Refraction

Once again, it is useful to give an estimate of the uncertainty in the calculation for refractive index. Once again take partial derivatives.

$$\frac{\partial n}{\partial f} = -\frac{r}{2\,f^2}, \quad \frac{\partial n}{\partial r} = -\frac{1}{2\,f}, \tag{10.22}$$

These yield the general expression for error.

$$\delta_n = \frac{1}{2\,f}\sqrt{\frac{\delta_f^2\,r^2}{f^2} + \delta_r^2} \tag{10.23}$$

This expression gives the uncertainty in the calculation for index of refraction to be 0.01.

10.4.4 Magnification

Recall equation (10.12). It stated that the ratio of heights of image to object, the magnification if you will, should have been the same ratio of image distance to object distance.

$$\frac{d_i}{d_o} = \frac{h'}{h} \tag{10.24}$$

However, this is a bit misleading. Recall that in figure 10.5 the image is inverted, this means that there is a very real sense in which its height is negative. Taking this into account gives an expression for magnification m.

$$m = -\frac{d_i}{d_o} = \frac{h'}{h} \tag{10.25}$$

To prove that this relationship holds, one may compare the estimates for magnification using both measured heights, and measured distances.

Doing this for the first few images distances yields a set of relatively similar magnifications. Using the image and object distances gives values -2.46, -2.10, and -1.72 while doing the same with the ratio of heights gives values -2.34, -2.03, -1.63. The values are generally in the same area, as they should be. But there is a non-trivial difference between the two estimates, in fact, the magnification calculated based on height was up to 8% smaller than that based on distances. On average the magnification calculated using distances was 2% higher than the one using heights.

10.5 Results and Conclusions

The focal length of the lens used in the experiment was found to be 19.59 ± 0.12 cm, however, this is not particularly easy to compare to any measured physical dimension. Although it does generally agree with the estimate for focal distance with an object that is arbitrarily far away, which was 19.90 when no correction for tree distance was made, and 19.26 using the correction. Interestingly, the 19.59 estimate of the focal length falls nearly exactly in-between the corrected and uncorrected values. It is worth noting that with a large estimate for the 'arbitrarily far away case' the uncertainty in the focal length grows with the object distance, so the uncertainty in the corrected value has a higher uncertainty of about 1 mm. This still places the corrected value out of the range of 19.59 ± 0.12 but it explains some variation. The other discrepancy are explained by the difficultly in a accurately focusing on one part of a rather small image. Although, consider that the values for close object distances rely on para-axial approximations, but, being close by, perhaps this is misleading. Aberrant rays will push the focal distance inward, giving a lower effective focal distance than the focal distance for truly para-axial rays. This might mean that the true focal distance is in fact a little bit larger than the calculated 19.59.

The index of refraction calculated is somewhat easier to compare physical values. Recall that it was found to have been 1.52 ± 0.01. This value is a rather sensible index of refraction. For example, it is larger than 1, not implying light travels faster in the medium than in a vacuum. Glass typically has an index of refraction of around 1.5. And the value of 1.52 is closely associated with crown glass [2], the glass most typically used in optical apparatus. This provides strong evidence to conclude that the lens was made of some sort of crown glass.

The magnification calculated by the two methods, by measuring heights and measuring distances, were generally around the same values, however, there was some notable deviation in the two values. This might be explained by the difficulty of accurately measuring the image and object height. The heights were often in awkward positions and the starting place for measuring was often not well defined. This makes accurately measuring the heights difficult and thus makes their respective

[2]http://hyperphysics.phy-astr.gsu.edu/hbase/Tables/indrf.html