

# A Focused Sequent Calculus System of Skew Monoidal Closed Categories

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# 1 Introduction

Recent discoveries on skew monoidal categories [4] [3] [2] (check references in previous papers, at least include papers cited by previous three papers) provide us a good reasons to study their corresponding proof systems. In previous work [7] [5] [6] [8], sequent calculus proof system for skew monoidal categories, prounital closed categories, partial normal monoidal categories, and skew symmetric monoidal categories are presented respectively. However, there is still lack of a proof analysis on skew monoidal closed categories. Therefore, in this paper we give sound and complete sequent calculus system with skew monoidal closed categories. Moreover, we use focusing strategy from [1] to solve the coherence problem of skew monoidal closed categories.

Interestingly, because of having two connectives in our sequent calculus system, the focusing strategy becomes subtle and involved. It cannot just divide derivations into invertible part and non-invertible part then fix their order. We discovered that it is no harm to arrange invertible rules in a fixed order, but in non-invertible rules, bad thing happens. The naive focused sequent calculus system cannot prove some provable sequents in original sequent calculus. We will see this involved focused sequent calculus system in (?) section.

This paper will go through in this order, in section two, we see the Hilbert style calculus of skew monoidal closed categories and its proof equivalences.

In third section, a cut-free sequent calculus of skew monoidal closed categories is presented. The key feature is that we have a special formula called stoup at the leftmost position in antecedent of a given sequent. In the same time, for any left rules, it can only be applied to the stoup formula. We will also see preliminarily relationship with Hilbert style calculus.

Next section, we will see a focused sequent calculus system and its connection with original sequent calculus system. We will see that all derivations in a same equivalence class under proof conversion relation, they will correspond to a unique derivation in focused system.

## 2 Skew monoidal closed categories

A *skew monoidal closed category*[3] is a category  $\mathcal{C}$  with a unit  $I$ , two bifunctors  $- \otimes - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  and  $- \multimap - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  satisfy  $\otimes \dashv \multimap$ , and six natural transformations

$$\begin{aligned} \lambda_A : I \otimes A &\Rightarrow A & \rho_A : A &\Rightarrow A \otimes I & \alpha_{A,B,C} : (A \otimes B) \otimes C &\Rightarrow A \otimes (B \otimes C) \\ i_A : I \multimap A &\Rightarrow A & j_A : I &\Rightarrow A \multimap A & L_{A,B,C} : B \multimap C &\Rightarrow (A \multimap B) \multimap (A \multimap C) \end{aligned}$$

Give a set  $\text{At}$  which contains countably infinite atomic formulae, we can generate a free skew monoidal closed category according to following rules:

$$\begin{aligned} & \frac{}{A \Rightarrow A} \text{id} \quad \frac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C} \text{comp} \\ & \frac{A \Rightarrow C \quad B \Rightarrow D}{A \otimes B \Rightarrow C \otimes D} \otimes \quad \frac{C \Rightarrow A \quad B \Rightarrow D}{A \multimap B \Rightarrow C \multimap D} \multimap \\ & \frac{}{I \otimes A \Rightarrow A} \lambda \quad \frac{}{A \Rightarrow A \otimes I} \rho \quad \frac{}{(A \otimes B) \otimes C \Rightarrow A \otimes (B \otimes C)} \alpha \\ & \frac{}{I \multimap A \Rightarrow A} i \quad \frac{}{I \Rightarrow A \multimap A} j \quad \frac{}{B \multimap C \Rightarrow (A \multimap B) \multimap (A \multimap C)} L \\ & \frac{}{(A \multimap B) \otimes A \Rightarrow B} \epsilon_{A,B} \quad \frac{}{A \Rightarrow B \multimap (A \otimes B)} \eta_{A,B} \end{aligned}$$

$$\frac{A \otimes B \Rightarrow C}{A \Rightarrow (B \multimap C)} \text{adj}_1 \quad \frac{A \Rightarrow (B \multimap C)}{A \otimes B \Rightarrow C} \text{adj}_2$$

**Remark 1** We do not need all of these rules to generate whole skew monoidal closed category. For example, suppose we have  $\lambda, \rho, \alpha, \epsilon$ , and  $\text{adj}_1$ , then  $i, j, \mathbf{L}, \eta$ , and  $\text{adj}_2$  are admissible.

$$\begin{aligned} i &=_{\text{df}} \text{comp}(\rho, (\text{adj}_2(\text{id}_{\mathbf{L} \multimap A}))) \\ j &=_{\text{df}} \text{adj}_1(\lambda_A) \\ \mathbf{L} &=_{\text{df}} \text{adj}_1(\text{adj}_1(\text{comp}(\alpha, (\text{comp}(\text{id}_{B \multimap C} \otimes \epsilon_{A,B}), \epsilon_{B,C})))) \\ \eta &=_{\text{df}} \text{adj}_1(\text{id}_{A \otimes B}) \\ \text{adj}_2 &=_{\text{df}} \text{comp}((f \otimes \text{id}_B), \epsilon_{B,C}) \end{aligned} \quad \text{given } f : A \Rightarrow B \multimap C$$

### 3 Sequent calculus for skew monoidal closed categories

In this section, we introduce sequent calculus system for skew monoidal closed categories. Following the settings from [7], [5], and [6], the sequent  $S \mid \Gamma \vdash C$  in our calculus system splits in three parts. First  $S$  is called a stoup formula where we can apply to any left rule.  $\Gamma$  is a list of formulae where we can use  $\text{pass}$  to move stoup formula into context and no other rules can be applied.  $C$  is a formula just like in any single succedent sequent calculus sense. Sequent calculus system of skew monoidal closed categories:

$$\begin{aligned} &\frac{A \mid \Gamma \vdash C}{- \mid A, \Gamma \vdash C} \text{pass} \quad \frac{- \mid \Gamma \vdash C}{\mathbf{I} \mid \Gamma \vdash C} \text{IL} \quad \frac{A \mid B, \Gamma \vdash C}{A \otimes B \mid \Gamma \vdash C} \otimes \text{L} \\ &\frac{S \mid \Gamma \vdash A \quad - \mid \Delta \vdash B}{S \mid \Gamma, \Delta \vdash A \otimes B} \otimes \text{R} \quad \frac{S \mid \Gamma, A \vdash B}{S \mid \Gamma \vdash A \multimap B} \multimap \text{R} \\ &\frac{}{A \mid \vdash A} \text{ax} \quad \frac{}{- \mid \vdash \mathbf{I}} \text{IR} \quad \frac{- \mid \Gamma \vdash A \quad B \mid \Delta \vdash C}{A \multimap B \mid \Gamma, \Delta \vdash C} \multimap \text{L} \end{aligned}$$

If we interpret morphism  $A \Rightarrow C$  as a sequent  $A \mid \vdash C$ , then we can see the natural transformations are derivable in this sequent calculus. For example, natural transformations  $\lambda, \rho$ , and  $\alpha$  (subscripts are omitted when there is no ambiguity) are derivable:

$$\begin{aligned} &\frac{\frac{\frac{}{A \mid \vdash A} \text{ax}}{- \mid A \vdash A} \text{pass} \quad \frac{\frac{}{A \mid \vdash A} \text{ax} \quad \frac{}{- \mid \vdash \mathbf{I}} \text{IR}}{A \mid \vdash A \otimes \mathbf{I}} \text{IR}}{\mathbf{I} \otimes A \mid \vdash A} \otimes \text{L} \\ &\frac{\frac{\frac{}{B \mid \vdash B} \text{ax} \quad \frac{\frac{}{C \mid \vdash C} \text{ax}}{- \mid C \vdash C} \text{pass}}{B \mid C \vdash B \otimes C} \otimes \text{R}}{\frac{}{A \mid \vdash A} \text{ax} \quad \frac{}{- \mid B, C \vdash B \otimes C} \otimes \text{L}}{A \mid B, C \vdash A \otimes (B \otimes C)} \otimes \text{R} \\ &\frac{}{(A \otimes B) \otimes C \vdash \vdash A \otimes (B \otimes C)} \otimes \text{L} \end{aligned}$$

Thanks to the invertibility of  $\otimes L$  and  $\multimap R$  [7] [5], two adjoint rules are also admissible in sequent calculus system:

$$\frac{\frac{f}{A \otimes B \mid \vdash C} \otimes L^{-1} \quad \frac{A \mid \vdash B \multimap C}{A \mid \vdash B \multimap C} \multimap R}{A \mid \vdash B \multimap C} \multimap R^{-1} \quad \frac{\frac{f}{A \mid \vdash B \multimap C} \multimap R^{-1} \quad \frac{A \otimes B \mid \vdash C}{A \otimes B \mid \vdash C} \otimes L}{A \otimes B \mid \vdash C} \otimes L$$

Therefore, we can summarize two admissible rules in SMCC system:

$$\frac{f}{A \otimes B \mid \vdash C} \text{adj}_1 \quad \frac{f}{A \mid \vdash B \multimap C} \text{adj}_2$$

Another important thing is that we have to ensure this sequent calculus system could not prove the inverse of any natural transformations in section 1. Here is one example,  $\rho^{-1}$  is not derivable in our system:

$$\frac{??}{\frac{A \mid I \vdash A}{A \otimes I \mid \vdash A} \otimes L} \otimes L$$

We interpret  $\rho^{-1}$  into  $A \otimes I \mid \vdash A$ , then according to bottom-up proof search strategy, we first apply  $\otimes L$  but we get stuck immediately. Therefore,  $\rho^{-1}$  is not derivable in our sequent calculus. Other cases are similar.

Next we see cut-freeness of this sequent system:

**Theorem 2** Two cut rules **scut** and **ccut**

$$\frac{S \mid \Gamma \vdash A \quad A \mid \Delta \vdash C}{S \mid \Gamma, \Delta \vdash C} \text{scut} \quad \frac{- \mid \Gamma \vdash A \quad S \mid \Delta_0, A, \Delta_1 \vdash C}{S \mid \Delta_0, \Gamma, \Delta_1 \vdash C} \text{ccut}$$

are admissible in sequent calculus system for skew monoidal closed categories.

**Proof :** According to previous works [7] and [5] we know two cut rules, **scut** and **ccut** are admissible in each system separately. As our new sequent calculus system is a union of two previous systems, we can just check the  $\otimes$  and  $\multimap$  interaction cases.

Dealing with **scut** first, same as proof strategy in previous papers, we proof by induction on left premise of **scut** rule.

1. First case is **scut**(( $\otimes L$ ),  $g$ ) where  $f : A' \mid B', \Gamma \vdash A, g : A \mid \Delta \vdash C$ , then we do subinduction on  $g$  to obtain two subcases:

- a.  $g = \multimap Rg'$ , then we let **scut**(( $\otimes L$ ), ( $\multimap Rg'$ )) =<sub>df</sub>  $\multimap R(\text{scut}((\otimes L), g'))$ .
- b.  $g = \multimap Lg'$ , then we let **scut**(( $\otimes L$ ), ( $\multimap Rg'$ )) =<sub>df</sub>  $\otimes L(\text{scut}(f, (\multimap Rg')))$

2. Other cases are similar.

Next we see the cut-freeness of **ccut** rule. Similarly, we prove it by induction on the second premise.

1. **ccut**( $f, (\multimap L(g, h))$ ) where  $f : - \mid \Gamma \vdash A, g : - \mid \Delta_1 \vdash A', h : B' \mid \Delta_2 \vdash C$ , then we do subinduction on  $f$ :

- a. The only possibility is  $A = A' \otimes B'$  and  $f = \otimes R(f_1, f_2)$  where  $f_1 : - \mid \Gamma_1 \vdash A'$ ,  $f_2 : - \mid \Gamma_2 \vdash B'$ , then depending on  $A' \otimes B'$  in  $\Delta_1$  or  $\Delta_2$ , we let  $\text{ccut}(f, (\multimap L(g, h))) =_{\text{def}} \multimap L(\text{ccut}(f, g), h)$  or  $\multimap L(g, (\text{ccut}(f, h)))$ , respectively.
2.  $\text{ccut}(f, (\otimes R(g, h)))$  where  $f : - \mid \Gamma \vdash A$ ,  $g : S \mid \Delta_1 \vdash A'$ ,  $h : - \mid \Delta_2 \vdash B'$  is similar as above.
3.  $\text{ccut}(f, (\otimes Lg))$  and  $\text{ccut}(f, (\multimap Rh))$  cases are similar, we permute  $\text{ccut}$  up, where  $g : A' \mid B', \Delta_0, A, \Delta_1 \vdash C$ ,  $h : S \mid \Delta_0, A, \Delta_1, A' \vdash B'$ .

## 4 Focusing

Before getting into focused sequent calculus system, we need to define new proof equivalences in original sequent calculus system. In [7] and [5], Uustalu et al. provided proof equivalences between  $\otimes$  only and  $\multimap$  only systems respectively. Again, similar as cut elimination proof above, we have to give  $\otimes$  and  $\multimap$  interaction cases here:

For  $f : - \mid \Gamma, A \vdash C$ ,  $\text{IL}$ :

$$\frac{\frac{f}{- \mid \Gamma, A \vdash C} \multimap R}{\frac{- \mid \Gamma \vdash A \multimap C}{\text{IL}}} \multimap R \stackrel{\circ}{=} \frac{\frac{f}{- \mid \Gamma, A \vdash C} \text{IL}}{\frac{\text{I} \mid \Gamma, A \vdash C}{\text{IL}}} \multimap R$$

For  $f : A \mid B, \Gamma, C \vdash D$ :

$$\frac{\frac{f}{A \mid B, \Gamma, C \vdash D} \multimap R}{\frac{A \mid B, \Gamma \vdash C \multimap D}{A \otimes B \mid \Gamma \vdash C \multimap D} \otimes L} \multimap R \stackrel{\circ}{=} \frac{\frac{f}{A \mid B, \Gamma, C \vdash D} \otimes L}{\frac{A \otimes B \mid \Gamma, C \vdash D}{A \otimes B \mid \Gamma \vdash C \multimap D} \multimap R} \multimap R$$

For  $f : - \mid \Gamma \vdash A$ ,  $g : B \mid \Delta \vdash C$ , and  $h : - \mid \Delta' \vdash D$ :

$$\frac{\frac{f}{- \mid \Gamma \vdash A} \quad \frac{\frac{g}{B \mid \Delta \vdash C} \quad \frac{h}{- \mid \Delta' \vdash D}}{B \mid \Delta, \Delta' \vdash C \otimes D} \otimes R}{\frac{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash C \otimes D}{\multimap L}} \multimap L \stackrel{\circ}{=} \frac{\frac{f}{- \mid \Gamma \vdash A} \quad \frac{g}{B \mid \Delta \vdash C}}{A \multimap B \mid \Gamma, \Delta \vdash C} \multimap L \quad \frac{h}{- \mid \Delta' \vdash D} \otimes R$$

The focused sequent calculus here has four phases with special annotations on  $\vdash$ . We explain the intuition of this focused system from bottom-up proof search perspective. Every proof in focused calculus is a tag-free sequent ended at phase  $\text{RI}$ .

$$\frac{S \mid \Gamma, A^x \vdash_{\text{RI}}^x B}{S \mid \Gamma \vdash_{\text{RI}}^x A \multimap B} \multimap R \quad \frac{S \mid \Gamma \vdash_{\text{LI}}^x P}{S \mid \Gamma \vdash_{\text{RI}}^x P} \text{LI2RI}$$

$$\begin{array}{c}
\frac{- \mid \Gamma \vdash_{\text{LI}}^x P}{\text{I} \mid \Gamma \vdash_{\text{LI}}^x P} \text{IL} \quad \frac{A \mid B, \Gamma \vdash_{\text{LI}}^x P}{A \otimes B \mid \Gamma \vdash_{\text{LI}}^x P} \otimes \text{L} \quad \frac{T \mid \Gamma \vdash_{\text{P}}^x P}{T \mid \Gamma \vdash_{\text{LI}}^x P} \text{P2LI} \\
\\
\frac{A^\circ \mid \Gamma \vdash_{\text{LI}}^x P}{- \mid A^x, \Gamma \vdash_{\text{P}}^x P} \text{pass} \quad \frac{T \mid \Gamma \vdash_{\text{F}}^x P}{T \mid \Gamma \vdash_{\text{P}}^x P} \text{F2P} \\
\\
\frac{T \mid \Gamma \vdash_{\text{RI}}^\bullet A \quad - \mid \Delta \vdash_{\text{RI}} B}{T \mid \Gamma, \Delta \vdash_{\text{F}}^x A \otimes B} \otimes \text{R} \\
\\
\frac{- \mid \Gamma^\circ \vdash_{\text{RI}} A \quad B \mid \Delta \vdash_{\text{LI}} P \quad \text{if } x = \bullet, \text{ then } \exists F^\bullet \in \Gamma}{A \multimap B \mid \Gamma, \Delta \vdash_{\text{F}}^x P} \multimap \text{L} \\
\\
\frac{}{X \mid \vdash_{\text{F}} X} \text{ax} \quad \frac{}{- \mid \vdash_{\text{F}} \text{I}} \text{IR}
\end{array}$$

- In RI, only right invertible rules can be applied, in particular,  $\multimap \text{R}$ . If endsequent of a RI application is black, then we add a tag to keep tracking the formulae decomposed from RI. In the end, we decompose  $C$  until it becomes a positive  $P$  where  $P \neq A \multimap B$ , then move to phase LI.
- Next, we destruct  $S$  by left invertible rules  $\otimes \text{L}$  and  $\text{IL}$  until we obtain an irreducible stoup  $S'$  where  $S'$  is not in  $A' \otimes B'$  or  $\text{I}$ .
- Then we move to passivation phase P, where only **pass** can be applied to sequents. Notice that for the active formula in each decomposition of **pass**, we remove its tag then whole sequent goes back to phase LI to do left invertible rules again because the passivated formula  $A$  could be reducible.
- In phase F, we have four rules,  $\otimes \text{R}$ ,  $\multimap \text{L}$ , **ax**, and **IR**. Here we have special restrictions on  $\otimes \text{L}$  and  $\multimap \text{L}$ . In general, we prefer to decompose  $\multimap$  in stoup prior then  $\otimes$  in conclusion. However, if there are some formula are packed into conclusion, then whole bottom-up proof strategy fails. We will see comparasion between involed and naive focused system later. For rule  $\otimes \text{R}$ , we let its left and right premises back to phase RI because  $A$  and  $B$  could be negative formulas. Especially, the left premise becomes a tagged sequent. In a  $\multimap \text{L}$  application, if conclusion sequent is tagged, then at least exists a formula  $F$  is black in the context of left premise. Remembering that we prefer  $\multimap \text{L}$  prior to  $\otimes \text{R}$ , so if there is any  $\otimes \text{R}$  before  $\multimap \text{L}$ , then we have to ensure that we cannot do  $\multimap \text{L}$  first. Black tags produced by  $\multimap \text{R}$  play essential roles here because if there is no black formula in  $\Gamma$ , then it implies that we cannot do  $\multimap \text{L}$  after  $\otimes \text{R}$ . In this situation, we must decompose  $\multimap$  in stoup prior than  $\otimes$  in conclusion.

Assume that we would prefer to decompose  $\multimap$  in stoup prior than  $\otimes$  in succedent without tag setting above. In this naive focused sequent calculus, we cannot prove  $X \multimap Y \mid Z \vdash_{\text{RI}} (X \multimap Y) \otimes Z$ , where  $X, Y$ , and  $Z$  are atomic formulae:

	ax
	$\frac{Z \mid \vdash_F Z}{F2P}$
	$\frac{Z \mid \vdash_P Z}{P2LI}$
	$\frac{Z \mid \vdash_{LI} Z}{pass}$
	$\frac{- \mid Z \vdash_P Z}{P2LI}$
	$\frac{- \mid Z \vdash_{LI} Z}{LI2RI}$
??	$\frac{- \mid Z \vdash_{RI} Z}{\otimes R}$
$\frac{Y \mid X \vdash_{RI} Y}{Y \mid \vdash_{RI} X \multimap Y} \multimap R$	
	$\frac{Y \mid Z \vdash_F (X \multimap Y) \otimes Z}{F2P}$
	$\frac{Y \mid Z \vdash_P (X \multimap Y) \otimes Z}{P2LI}$
??	$\frac{Y \mid Z \vdash_{LI} (X \multimap Y) \otimes Z}{\multimap L}$
$- \mid \vdash_{RI} X$	
$\frac{X \multimap Y \mid Z \vdash_F (X \multimap Y) \otimes Z}{X \multimap Y \mid Z \vdash_P (X \multimap Y) \otimes Z} F2P$	
$\frac{X \multimap Y \mid Z \vdash_P (X \multimap Y) \otimes Z}{X \multimap Y \mid Z \vdash_{LI} (X \multimap Y) \otimes Z} P2LI$	
$\frac{X \multimap Y \mid Z \vdash_{LI} (X \multimap Y) \otimes Z}{X \multimap Y \mid Z \vdash_{RI} (X \multimap Y) \otimes Z} LI2RI$	

The key point is that  $X$  in  $(X \multimap Y) \otimes Z$  is locked, so that we could not move it to the context of left premise to close proof tree. However, in tagged setting, we can avoid this awkward situation:

$\frac{X \mid \vdash_F X}{X \mid \vdash_P X}$	$\frac{X \mid \vdash_P X}{X \mid \vdash_{LI} X}$	$\frac{X \mid \vdash_{LI} X}{- \mid X \vdash_P X}$	$\frac{- \mid X \vdash_P X}{- \mid X \vdash_{LI} X}$
ax	F2P	P2LI	pass
$\frac{X \mid \vdash_F X}{X \mid \vdash_P X}$	$\frac{X \mid \vdash_P X}{X \mid \vdash_{LI} X}$	$\frac{- \mid X \vdash_P X}{- \mid X \vdash_{LI} X}$	$\frac{- \mid X \vdash_{LI} X}{- \mid X \vdash_{RI} X}$
F2P	P2LI	LI2RI	ax
$\frac{Y \mid \vdash_F Y}{Y \mid \vdash_P Y}$	$\frac{Y \mid \vdash_P Y}{Y \mid \vdash_{LI} Y}$	$\frac{Y \mid \vdash_{LI} Y}{- \mid X \vdash_P X}$	$\frac{- \mid X \vdash_P X}{- \mid X \vdash_{LI} X}$
F2P	P2LI	LI2RI	ax
$\frac{Z \mid \vdash_F Z}{Z \mid \vdash_P Z}$	$\frac{Z \mid \vdash_P Z}{Z \mid \vdash_{LI} Z}$	$\frac{Z \mid \vdash_{LI} Z}{- \mid Z \vdash_P Z}$	$\frac{- \mid Z \vdash_P Z}{- \mid Z \vdash_{LI} Z}$
F2P	P2LI	LI2RI	ax
$\frac{X \multimap Y \mid X^\bullet \vdash_F Y}{X \multimap Y \mid X^\bullet \vdash_P Y}$	$\frac{X \multimap Y \mid X^\bullet \vdash_P Y}{X \multimap Y \mid X^\bullet \vdash_{LI} Y}$	$\frac{X \multimap Y \mid X^\bullet \vdash_{LI} Y}{X \multimap Y \mid X^\bullet \vdash_{RI} Y}$	$\frac{X \multimap Y \mid X^\bullet \vdash_{RI} Y}{X \multimap Y \mid \vdash_{RI} X \multimap Y}$
F2P	P2LI	LI2RI	$\multimap L$
$\frac{X \multimap Y \mid X^\bullet \vdash_F Y}{X \multimap Y \mid X^\bullet \vdash_P Y}$	$\frac{X \multimap Y \mid X^\bullet \vdash_P Y}{X \multimap Y \mid X^\bullet \vdash_{LI} Y}$	$\frac{X \multimap Y \mid X^\bullet \vdash_{LI} Y}{X \multimap Y \mid X^\bullet \vdash_{RI} Y}$	$\frac{X \multimap Y \mid X^\bullet \vdash_{RI} Y}{X \multimap Y \mid \vdash_{RI} X \multimap Y}$
F2P	P2LI	LI2RI	$\multimap R$
$\frac{X \multimap Y \mid Z \vdash_F (X \multimap Y) \otimes Z}{X \multimap Y \mid Z \vdash_P (X \multimap Y) \otimes Z}$	$\frac{X \multimap Y \mid Z \vdash_P (X \multimap Y) \otimes Z}{X \multimap Y \mid Z \vdash_{LI} (X \multimap Y) \otimes Z}$	$\frac{X \multimap Y \mid Z \vdash_{LI} (X \multimap Y) \otimes Z}{X \multimap Y \mid Z \vdash_{RI} (X \multimap Y) \otimes Z}$	$\frac{X \multimap Y \mid Z \vdash_{RI} (X \multimap Y) \otimes Z}{X \multimap Y \mid Z \vdash_{RI} (X \multimap Y) \otimes Z}$
F2P	P2LI	LI2RI	$\otimes R$

In the mean time, our focused system does not contain redundant non-determinacy. For example, as the proof equivalences above,  $\otimes R$  and  $\multimap L$  rule applications could be swapped when  $\Gamma, \Delta$ , and  $\Delta'$  separated properly. Without loose of generality, we assume  $\Gamma, \Delta$ , and  $\Delta'$  are sufficient and necessary in proving each sequent respectively. In tagged focused calculus, such exchangeable situation would not happen. Given a sequent  $A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{RI} C \otimes D$  and supposed that  $B$  is irreducible,  $C = A' \multimap B'$ , we can construct a proof:

$$\begin{array}{c}
\vdots \\
\frac{B \mid \Delta \vdash_{\text{RI}}^{\bullet} A' \multimap B' \quad - \mid \Delta' \vdash_{\text{RI}} D}{B \mid \Delta, \Delta' \vdash_{\text{F}} (A' \multimap B') \otimes D} \otimes \text{R} \\
\frac{B \mid \Delta, \Delta' \vdash_{\text{F}} (A' \multimap B') \otimes D}{B \mid \Delta, \Delta' \vdash_{\text{P}} (A' \multimap B') \otimes D} \text{F2P} \\
\frac{B \mid \Delta, \Delta' \vdash_{\text{P}} (A' \multimap B') \otimes D}{B \mid \Delta, \Delta' \vdash_{\text{LI}} (A' \multimap B') \otimes D} \text{P2LI} \\
\frac{B \mid \Delta, \Delta' \vdash_{\text{LI}} (A' \multimap B') \otimes D}{B \mid \Delta, \Delta' \vdash_{\text{RI}} (A' \multimap B') \otimes D} \text{LI2RI} \\
\vdots \\
\frac{- \mid \Gamma \vdash_{\text{RI}} A \quad B \mid \Delta, \Delta' \vdash_{\text{RI}} (A' \multimap B') \otimes D}{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{\text{F}} (A' \multimap B') \otimes D} \multimap \text{L} \\
\frac{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{\text{F}} (A' \multimap B') \otimes D}{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{\text{P}} (A' \multimap B') \otimes D} \text{F2P} \\
\frac{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{\text{P}} (A' \multimap B') \otimes D}{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{\text{LI}} (A' \multimap B') \otimes D} \text{P2LI} \\
\frac{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{\text{LI}} (A' \multimap B') \otimes D}{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{\text{RI}} (A' \multimap B') \otimes D} \text{LI2RI}
\end{array}$$

However, the other way around (decompose  $\otimes$  prior to  $\multimap$ ) is impossible. If we want to apply  $\multimap \text{L}$ , because the sequent is black, supposing that  $C = A' \multimap B'$  and put at least  $\Gamma, \Delta, A'$  to left premise. In the end, the proof tree cannot close:

$$\begin{array}{c}
\begin{array}{c} ?? \\ - \mid \Gamma, \Delta, A' \vdash_{\text{RI}} A \end{array} \quad \begin{array}{c} ?? \\ B \mid \vdash_{\text{LI}} B' \end{array} \\
\frac{}{A \multimap B \mid \Gamma, \Delta, A' \vdash_{\text{F}}^{\bullet} B'} \multimap \text{L} \\
\frac{A \multimap B \mid \Gamma, \Delta, A' \vdash_{\text{F}}^{\bullet} B'}{A \multimap B \mid \Gamma, \Delta, A' \vdash_{\text{P}}^{\bullet} B'} \text{F2P} \\
\frac{A \multimap B \mid \Gamma, \Delta, A' \vdash_{\text{P}}^{\bullet} B'}{A \multimap B \mid \Gamma, \Delta, A' \vdash_{\text{LI}}^{\bullet} B'} \text{P2LI} \\
\frac{A \multimap B \mid \Gamma, \Delta, A' \vdash_{\text{LI}}^{\bullet} B'}{A \multimap B \mid \Gamma, \Delta, A' \vdash_{\text{RI}}^{\bullet} B'} \text{LI2RI} \\
\frac{A \multimap B \mid \Gamma, \Delta, A' \vdash_{\text{RI}}^{\bullet} B'}{A \multimap B \mid \Gamma, \Delta \vdash_{\text{RI}}^{\bullet} A' \multimap B'} \multimap \text{R} \\
\vdots \\
\frac{A \multimap B \mid \Gamma, \Delta \vdash_{\text{RI}}^{\bullet} A' \multimap B' \quad - \mid \Delta' \vdash_{\text{RI}} D}{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{\text{F}} (A' \multimap B') \otimes D} \otimes \text{R} \\
\frac{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{\text{F}} (A' \multimap B') \otimes D}{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{\text{P}} (A' \multimap B') \otimes D} \text{F2P} \\
\frac{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{\text{P}} (A' \multimap B') \otimes D}{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{\text{LI}} (A' \multimap B') \otimes D} \text{P2LI} \\
\frac{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{\text{LI}} (A' \multimap B') \otimes D}{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash_{\text{RI}} (A' \multimap B') \otimes D} \text{LI2RI}
\end{array}$$

The situation is worse if  $C$  is positive, we cannot produce tags from RI application to ensure applicability of  $\multimap \text{L}$ . In conclusion, tagged focused system here is not only sound but also without redundant non-determinacy.

**Lemma 3** Following rules are admissible in tagged focused system:

$$\begin{array}{c}
\frac{- \mid \Gamma \vdash_{\text{RI}} C}{\mid \Gamma \vdash_{\text{RI}} C} \text{IL}^{\text{RI}} \quad \frac{A \mid B, \Gamma \vdash_{\text{RI}} C}{A \otimes B \mid \Gamma \vdash_{\text{RI}} C} \otimes \text{L}^{\text{RI}} \quad \frac{A \mid \Gamma \vdash_{\text{RI}} C}{- \mid \Gamma \vdash_{\text{RI}} C} \text{pass}^{\text{RI}} \\
\\
\frac{- \mid \Gamma \vdash_{\text{RI}} A \quad B \mid \Delta \vdash_{\text{RI}} C}{A \multimap B \mid \Gamma, \Delta \vdash_{\text{RI}} C} \multimap \text{L}^{\text{RI}} \quad \frac{}{A \mid \vdash_{\text{RI}} A} \text{ax}^{\text{RI}} \quad \frac{}{- \mid \vdash_{\text{RI}} \mid} \text{IR}^{\text{RI}} \\
\\
\frac{S \mid \Gamma, \Gamma' \vdash_{\text{RI}} A \quad - \mid \Delta \vdash_{\text{RI}} B}{S \mid \Gamma, \Delta \vdash_{\text{RI}} [\Gamma' \mid A] \multimap B} \otimes \text{R}^{\text{RI}}
\end{array}$$

Inductive definition of formula  $[\Gamma \mid C]_{\multimap}$  is from [5]:

$$[\mid C]_{\multimap} = C \quad [A, \Gamma \mid C]_{\multimap} = A \multimap [\Gamma \mid C]_{\multimap}$$

**Theorem 4** 1. For any derivation  $f : S \mid \Gamma \vdash_{\text{RI}} C$ , there is a derivation  $\text{emb } f : S \mid \Gamma \vdash C$ .



2. For any derivation  $f : S \mid \Gamma \vdash C$ , there is a derivation **focus**  $f : S \mid \Gamma \vdash_{\text{Rl}} C$ .

**Proof :** 1. First theorem is straightforward, we just erase labels in a focused derivation then it would become a derivation in original sequent calculus system with some duplicated step.

2. By induction on  $f$  and lemma 3.

## 5 Soundness and completeness

**Theorem 5** For any derivation  $f : S \mid \Gamma \vdash C$ , there is a derivation **sound**  $f : \llbracket S \mid \Gamma \rrbracket_{\otimes} \Longrightarrow C$ .

**Proof :** Proof of soundness is by induction on  $f : S \mid \Gamma \vdash C$ . We can obtain two repeated application version of adjoint rules **Gadj<sub>1</sub>** and **Gadj<sub>2</sub>**(we use  $\llbracket - \rrbracket_{\otimes}$  and  $\llbracket - \rrbracket_{\multimap}$  to indicate the interpretation):

$$\frac{\llbracket S \mid \Gamma \rrbracket_{\otimes} \Longrightarrow A}{S \Longrightarrow \llbracket \Gamma \mid C \rrbracket_{\multimap}} \text{Gadj}_1 \quad \text{and} \quad \frac{A \Longrightarrow \llbracket \Gamma \mid C \rrbracket_{\multimap}}{\llbracket A \mid \Gamma \rrbracket_{\otimes} \Longrightarrow C} \text{Gadj}_2$$

We can obtain the soundness proof of  $\multimap \text{L}$  immediately. For any  $f : - \mid \Gamma \vdash A$  and  $g : B \mid \Delta \vdash C$ :

$$\frac{\frac{\text{Lemma 4.3 in [7]}}{\llbracket A \multimap B \mid \Gamma \rrbracket_{\otimes} \Longrightarrow A \multimap B \otimes \llbracket - \mid \Gamma \rrbracket_{\otimes}} \quad \frac{\frac{\frac{\text{sound } f}{\llbracket - \mid \Gamma \rrbracket_{\otimes} \Longrightarrow A} \quad \frac{\frac{\text{sound } g}{\llbracket B \mid \Delta \rrbracket_{\otimes} \Longrightarrow C}}{B \Longrightarrow \llbracket \Delta \mid C \rrbracket_{\multimap}} \text{Gadj}_1}{A \multimap B \Longrightarrow \llbracket - \mid \Gamma \rrbracket_{\otimes} \multimap \llbracket \Delta \mid C \rrbracket_{\multimap}} \multimap}{\llbracket A \multimap B \otimes \llbracket - \mid \Gamma \rrbracket_{\otimes} \rrbracket_{\otimes} \Longrightarrow \llbracket \Delta \mid C \rrbracket_{\multimap}} \text{adj}_2}{\llbracket A \multimap B \mid \Gamma \rrbracket_{\otimes} \Longrightarrow \llbracket \Delta \mid C \rrbracket_{\multimap}} \text{comp} \quad \frac{\llbracket A \multimap B \mid \Gamma \rrbracket_{\otimes} \Longrightarrow \llbracket \Delta \mid C \rrbracket_{\multimap}}{\llbracket A \multimap B \mid \Gamma, \Delta \rrbracket_{\otimes} \Longrightarrow C} \text{Gadj}_2$$

**Theorem 6** (Weak completeness) For any derivation  $f : A \Longrightarrow C$ , there is a derivation **complt**  $f : A \mid \vdash C$ .

**Proof :** In previous studies from Uustalu et al. ([7] and [5]), main cases of completeness are proved. Therefore, we only have to show new cases in skew monoidal closed categories. In particular, new cases are unit, counit, and two adjoint rules.

- (Counit)Case  $f = \epsilon : (A \multimap B) \otimes A \Longrightarrow B$ . We define:

$$\text{complt} \left( \frac{}{(A \multimap B) \otimes A \Longrightarrow B} \epsilon \right) =_{\text{df}} \frac{\frac{\frac{}{A \mid \vdash A} \text{ax}}{- \mid A \vdash A} \text{pass} \quad \frac{}{B \mid \vdash B} \text{ax}}{A \multimap B \mid A \vdash B} \multimap \text{L}}{(A \multimap B) \otimes A \mid \vdash B} \otimes \text{L}$$

- (Unit)Case  $f = \eta : A \Longrightarrow B \multimap (A \otimes B)$ . We define:

$$\text{complt} \left( \frac{}{A \Longrightarrow B \multimap (A \otimes B)} \eta \right) =_{\text{df}} \frac{\frac{\frac{}{A \mid \vdash A} \text{ax}}{A \mid B \vdash A \otimes B} \otimes \text{R} \quad \frac{\frac{}{B \mid \vdash B} \text{ax}}{- \mid B \vdash B} \text{pass}}{A \mid \vdash B \multimap (A \otimes B)} \multimap \text{R}$$

Two adjoint rules need invertibility of  $\otimes \text{L}$  and  $\multimap \text{R}$  which are proved respectively in [7] and [5].

- Case  $f = \text{adj}_1 g$  where  $g : A \otimes B \Longrightarrow C$ . We define:

$$\text{complt} \left( \frac{\vdots g}{\frac{A \otimes B \Longrightarrow C}{A \Longrightarrow B \multimap C} \text{adj}_1} \right) =_{\text{df}} \frac{\vdots \text{complt } g}{\frac{\frac{A \otimes B \mid \vdash C}{A \mid B \vdash C} \otimes L^{-1}}{A \mid \vdash B \multimap C} \multimap R}$$

- Case  $f = \text{adj}_2 g$  where  $g : A \Longrightarrow B \multimap C$ . We define:

$$\text{complt} \left( \frac{\vdots g}{\frac{A \Longrightarrow B \multimap C}{A \otimes B \Longrightarrow C} \text{adj}_2} \right) =_{\text{df}} \frac{\vdots \text{complt } g}{\frac{\frac{A \mid \vdash B \multimap C}{A \mid B \vdash C} \multimap R^{-1}}{A \otimes B \mid \vdash C} \otimes L}$$

## References

- [1] Jean-Mark Andreoli. Logic Programming with Focusing Proofs in Linear Logic. *Journal of Logic and Computation*, 2(3):297–347, June 1992.
- [2] Stephen Lack and Ross Street. Triangulations, orientals, and skew monoidal categories. *Advances in Mathematics*, 258:351–396, 2014. Publisher: Elsevier BV.
- [3] Ross Street. Skew-closed categories. *Journal of Pure and Applied Algebra*, 217(6):973–988, June 2013.
- [4] Kornél Szlachányi. Skew-monoidal categories and bialgebroids. 231(3-4):1694–1730, 2012. Publisher: Elsevier BV.
- [5] Tarmo Uustalu, Niccolò Veltri, and Noam Zeilberger. Deductive Systems and Coherence for Skew Prounital Closed Categories. page 19.
- [6] Tarmo Uustalu, Niccolò Veltri, and Noam Zeilberger. Proof Theory of Partially Normal Skew Monoidal Categories. *EPTCS*, 333:230–246.
- [7] Tarmo Uustalu, Niccolò Veltri, and Noam Zeilberger. The Sequent Calculus of Skew Monoidal Categories. *Electronic Notes in Theoretical Computer Science*, 341:345–370, December 2018.
- [8] Niccolò Veltri. Coherence via Focusing for Symmetric Skew Monoidal Categories. In Alexandra Silva, Renata Wassermann, and Ruy de Queiroz, editors, *Logic, Language, Information, and Computation*, Lecture Notes in Computer Science, pages 184–200, Cham, 2021. Springer International Publishing.