A Focused Sequent Calculus System of Skew Monoidal Closed Categories

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1 Introduction

Recent discoveries on skew monoidal categories [4] [3] [2] (check references in previous papers, at least include papers cited by previous three papers) provide us a good reasons to study their corresponding proof systems. In previous work [7] [5] [6] [8], sequent calculus proof system for skew monoidal categories, prounital closed catgories, partial normal monoidal categories, and skew symmetric monoidal categories are presented respectively. However, there is still lack of a proof analysis on skew monoidal closed categories. Therefore, in this paper we give sound and complete sequent calculus system with skew monoidal closed categories. Moreover, we use focusing strategy from [1] to solve the coherence problem of skew monoidal closed categories.

Interestingly, because of having two connectives in our sequent calculus system, the forcusing strategy becomes subtle and involved. It cannot just divide derivations into invertible part and non-inertible part then fix their order. We discovered that it is no harm to arrange invertible rules in a fixed order, but in non-invertible rules, bad thing happens. The naive focused sequent calculus system cannont prove some provable sequents in original sequent calculus. We will see this involved focused sequent calculus system in (?) section.

This paper will go through in this order, in section two, we see the Hilbert style calculus of skew monoidal closed categories and its proof equivalences.

In third section, a cut-free sequent caculus of skew monoidal closed categories is presented. The key feature is that we have a special formula called stoup at the leftmost position in antecedent of a given sequent. In the same time, for any left rules, it can only be applied to the stoup formula. We will also see preliminarily relationship with Hilbert style calculus.

Next section, we will see a focused sequent calculus system and its connection with original sequent calulus system. We will see that all derivations in a same equivalence class under proof conversion relation, they will correspond to a unique derivation in focused system.

2 Skew monoidal closed categories

A *skew monoidal closed category*[3] is a category $\mathcal{C}$ with a unit I, a pair of adjoint functors $A \otimes - \dashv - \multimap B$, and six natural transformations

$$\lambda_A : \mathsf{I} \otimes A \Longrightarrow A \qquad \rho_A : A \Longrightarrow A \otimes \mathsf{I} \qquad \alpha_{A,B,C} : (A \otimes B) \otimes C \Longrightarrow A \otimes (B \otimes C)$$

$$i_A : \mathsf{I} \multimap A \Longrightarrow A \qquad j_A : \mathsf{I} \Longrightarrow A \multimap A \qquad \mathsf{L}_{A,B,C} : B \multimap C \Longrightarrow (A \multimap B) \multimap (A \multimap C)$$

satisfying the laws

Laws of skew-monoidal closed categoris:

$$\frac{}{A \Longrightarrow A}\,\mathsf{id} \qquad \frac{A \Longrightarrow B \quad B \Longrightarrow C}{A \Longrightarrow C}\,\mathsf{comp}$$

$$\frac{A \Longrightarrow C \quad B \Longrightarrow D}{A \otimes B \Longrightarrow C \otimes D}\,\otimes \qquad \frac{C \Longrightarrow A \quad B \Longrightarrow D}{A \multimap B \Longrightarrow C \multimap D}\,\multimap$$

$$\frac{}{\mathsf{I} \otimes A \Longrightarrow A}\,\lambda \qquad \frac{}{A \Longrightarrow A \otimes \mathsf{I}}\,\rho \qquad \frac{}{(A \otimes B) \otimes C \Longrightarrow A \otimes (B \otimes C)}\,\alpha$$

$$\frac{}{\mathsf{I} \multimap A \Longrightarrow A}\,i \qquad \frac{}{\mathsf{I} \Longrightarrow A \multimap A}\,j \qquad \frac{}{B \multimap C \Longrightarrow (A \multimap B) \multimap (A \multimap C)}\,\mathsf{L}$$

$$\frac{}{(A \multimap B) \otimes A \Longrightarrow B}\,\epsilon_{A,B} \qquad \frac{}{A \Longrightarrow B \multimap (A \otimes B)}\,\eta_{A,B}$$

$$\frac{A \otimes B \Longrightarrow C}{A \Longrightarrow (B \multimap C)}\,\mathsf{adj}_1 \qquad \frac{A \Longrightarrow (B \multimap C)}{A \otimes B \Longrightarrow C}\,\mathsf{adj}_2$$

Notice that we do not need all of these rules. For example, we can derive $i, j, \mathsf{L}, \epsilon$, and adj_1 from $\lambda, \rho, \alpha, \eta$, and adj_2, vice versa.

3 Sequent calculus for skew monoidal closed categories

Seqeunt calculus system of skew monoidal closed categries(SMCC):

$$\frac{A \mid \Gamma \vdash C}{- \mid A, \Gamma \vdash C}\,\mathsf{pass} \qquad \frac{- \mid \Gamma \vdash C}{\mathsf{I} \mid \Gamma \vdash C}\,\mathsf{IL} \qquad \frac{A \mid B, \Gamma \vdash C}{A \otimes B \mid \Gamma \vdash C}\,\otimes\mathsf{L}$$

$$\frac{S \mid \Gamma \vdash A \quad - \mid \Delta \vdash B}{S \mid \Gamma, \Delta \vdash A \otimes B}\,\otimes\mathsf{R} \qquad \frac{S \mid \Gamma, A \vdash B}{S \mid \Gamma \vdash A \multimap B}\,\multimap\mathsf{R}$$

$$\frac{}{A \mid \ \vdash A}\,\mathsf{ax} \qquad \frac{}{- \mid \ \vdash \mathsf{I}}\,\mathsf{IR} \qquad \frac{- \mid \Gamma \vdash A \quad B \mid \Delta \vdash C}{A \multimap B \mid \Gamma, \Delta \vdash C}\,\multimap\mathsf{L}$$

If we interpret morphism $A \Longrightarrow C$ as a sequent $A \mid \ \vdash C$, then we can see the natural transformations are derivable in this seqent calculus. For example, natural transformations λ, ρ, and α(subscripts are omitted when there is no amubiguity) are derivable:

$$\frac{\dfrac{\dfrac{\dfrac{}{A \mid \ \vdash A}\,\mathsf{ax}}{- \mid A \vdash A}\,\mathsf{pass}}{\mathsf{I} \mid A \vdash A}\,\mathsf{IL}}{\mathsf{I} \otimes A \mid \ \vdash A}\,\otimes\mathsf{L} \qquad \frac{\dfrac{}{A \mid \ \vdash A}\,\mathsf{ax} \quad \dfrac{}{- \mid \ \vdash \mathsf{I}}\,\mathsf{IR}}{A \mid \ \vdash A \otimes \mathsf{I}}$$

$$\frac{\dfrac{\dfrac{\dfrac{}{A \mid \ \vdash A}\,\mathsf{ax} \quad \dfrac{\dfrac{\dfrac{}{B \mid \ \vdash B}\,\mathsf{ax} \quad \dfrac{\dfrac{}{C \mid \ \vdash C}\,\mathsf{ax}}{- \mid C \vdash C}\,\mathsf{pass}}{B \mid C \vdash B \otimes C}\,\otimes\mathsf{R}}{- \mid B, C \vdash B \otimes C}\,\otimes\mathsf{L}}{A \mid B, C \vdash A \otimes (B \otimes C)}\,\otimes\mathsf{R}}{A \otimes B \mid C \vdash A \otimes (B \otimes C)}\,\otimes\mathsf{L}}{(A \otimes B) \otimes C \vdash \ \vdash A \otimes (B \otimes C)}\,\otimes\mathsf{L}$$

Thanks for the invertibility of $\otimes\mathsf{L}$ and $\multimap\mathsf{R}$ [7] [5], two adjoint rules are also interpretable in sequent calculus system:

$$\frac{\dfrac{\overset{f}{A \otimes B \mid \ \vdash C}}{A \mid B \vdash C}\,\otimes\mathsf{L}^{-1}}{A \mid \ \vdash B \multimap C}\,\multimap\mathsf{R} \qquad \frac{\dfrac{\overset{f}{A \mid \quad B \multimap C}}{A \mid B \vdash C}\,\multimap\mathsf{R}^{-1}}{A \otimes B \mid \ \vdash C}\,\otimes\mathsf{L}$$

Therefore, we can summarize two admissible rules in SMCC system:

$$\frac{\overset{f}{A \otimes B \mid \ \vdash C}}{A \mid \ \vdash B \multimap C}\,\mathsf{adj}_1 \qquad \frac{\overset{f}{A \mid \ \vdash B \multimap C}}{A \otimes B \mid \ \vdash C}\,\mathsf{adj}_2$$

According to previous works [7] and [5] we know two cut rules, scut and ccut are admissible in this system(interaction cases are left so I have to write them down).

$$\frac{S \mid \Gamma \vdash A \qquad A \mid \Delta \vdash C}{S \mid \Gamma, \Delta \vdash C}\;\mathsf{scut}$$

$$\frac{- \mid \Gamma \vdash A \qquad S \mid \Delta_0, A, \Delta_1 \vdash C}{S \mid \Delta_0, \Gamma, \Delta_1 \vdash C}\;\mathsf{ccut}$$

4 Focusing

Before getting into focused sequent calculus system, we need to define proof equivalences in original sequent calculus system. In previous works [7] and [5] from Uustalu et al. , they provided proof equivalences between $\otimes$ only and $\multimap$ only systems respectively. Therefore, there are only $\otimes$ and $\multimap$ interaction cases are left:

For $f : - \mid \Gamma, A \vdash C$, IL:

$$\frac{\dfrac{\dfrac{f}{- \mid \Gamma, A \vdash C}}{- \mid \Gamma \vdash A \multimap C}\;{\multimap}\mathsf{R}}{\mathsf{I} \mid \Gamma \vdash A \multimap C}\;\mathsf{IL} \;\circeq\; \frac{\dfrac{\dfrac{f}{- \mid \Gamma, A \vdash C}}{\mathsf{I} \mid \Gamma, A \vdash C}\;\mathsf{IL}}{\mathsf{I} \mid \Gamma \vdash A \multimap C}\;{\multimap}\mathsf{R}$$

For $f : A \mid B, \Gamma, C \vdash D$:

$$\frac{\dfrac{\dfrac{f}{A \mid B, \Gamma, C \vdash D}}{A \mid B, \Gamma \vdash C \multimap D}\;{\multimap}\mathsf{R}}{A \otimes B \mid \Gamma \vdash C \multimap D}\;{\otimes}\mathsf{L} \;\circeq\; \frac{\dfrac{\dfrac{f}{A \mid B, \Gamma, C \vdash D}}{A \otimes B \mid \Gamma, C \vdash D}\;{\otimes}\mathsf{L}}{A \otimes B \mid \Gamma \vdash C \multimap D}\;{\multimap}\mathsf{R}$$

For $f : - \mid \Gamma \vdash A$, $g : B \mid \Delta \vdash C$, and $h : - \mid \Delta' \vdash D$:

$$\frac{\dfrac{f}{- \mid \Gamma \vdash A} \qquad \dfrac{\dfrac{g}{B \mid \Delta \vdash C} \qquad \dfrac{h}{- \mid \Delta' \vdash D}}{B \mid \Delta, \Delta' \vdash C \otimes D}\;{\otimes}\mathsf{R}}{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash C \otimes D}\;{\multimap}\mathsf{L}$$

$$\circeq$$

$$\frac{\dfrac{\dfrac{f}{- \mid \Gamma \vdash A} \qquad \dfrac{g}{B \mid \Delta \vdash C}}{A \multimap B \mid \Gamma, \Delta \vdash C}\;{\multimap}\mathsf{L} \qquad \dfrac{h}{- \mid \Delta' \vdash D}}{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash C \otimes D}\;{\otimes}\mathsf{R}$$

Our focused system is with 4 phases. From bottom-up proof search, we decompose $\multimap \mathsf{R}$ until $\multimap$ is not the principal connective in succedent. In phase 2, we destruct the formula in stoup until it is not in the form of $A \otimes B$. Next we proceed to phase 3 to remove $\multimap$ in stoup. Lastly in phase 4, we deal with $\otimes$ in succedent and close the proof tree. Notice that we have a non-intuitive rule $\multimap \mathsf{L}^{\otimes}$. We need this rule to keep soundness for our focused system. Without the rule, we would face a counter example:

$$
\dfrac{\begin{array}{c}??\\ - \mid\ \ \vdash_{\mathsf{RI}} X\end{array} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}??\\ Y \mid X \vdash_{\mathsf{RI}} Y\end{array}}{Y \mid\ \ \vdash_{\mathsf{RI}} X \multimap Y}\multimap\mathsf{R} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Z \mid\ \ \vdash_4 Z}\mathsf{ax}}{Z \mid\ \ \vdash_{\mathsf{F}} Z}\mathsf{4to3}}{Z \mid\ \ \vdash_2 Z}\mathsf{3to2}}{- \mid Z \vdash_2 Z}\mathsf{pass}}{- \mid Z \vdash_{\mathsf{RI}} Z}\mathsf{2to1}}{Y \mid Z \vdash_4 (X \multimap Y) \otimes Z}\otimes\mathsf{R}}{Y \mid Z \vdash_{\mathsf{F}} (X \multimap Y) \otimes Z}\mathsf{4to3}}{Y \mid Z \vdash_2 (X \multimap Y) \otimes Z}\mathsf{3to2}}{\dfrac{X \multimap Y \mid Z \vdash_{\mathsf{F}} (X \multimap Y) \otimes Z}{\dfrac{X \multimap Y \mid Z \vdash_2 (X \multimap Y) \otimes Z}{X \multimap Y \mid Z \vdash_{\mathsf{RI}} (X \multimap Y) \otimes Z}\mathsf{2to1}}\mathsf{3to2}}\multimap\mathsf{L}
$$

This counter example occurs when we insist to decompose $\multimap$ in stoup prior than $\otimes$ in succedent. Therefore we induce $\multimap \mathsf{L}^{\otimes}$ rule to solve this situation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X \mid\ \ \vdash_4 X}\mathsf{ax}}{X \mid\ \ \vdash_{\mathsf{F}} X}\mathsf{4to3}}{X \mid\ \ \vdash_2 X}\mathsf{3to2}}{- \mid X \vdash_2 X}\mathsf{pass}}{- \mid X \vdash_{\mathsf{RI}} X}\mathsf{2to1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Y \mid\ \ \vdash_4 Y}\mathsf{ax}}{Y \mid\ \ \vdash_{\mathsf{F}} Y}\mathsf{4to3}}{Y \mid\ \ \vdash_2 Y}\mathsf{3to2}}{Y \mid\ \ \vdash_{\mathsf{RI}} Y}\mathsf{2to1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Z \mid\ \ \vdash_4 Z}\mathsf{ax}}{Z \mid\ \ \vdash_{\mathsf{F}} Z}\mathsf{4to3}}{Z \mid\ \ \vdash_2 Z}\mathsf{3to2}}{- \mid Z \vdash_2 Z}\mathsf{pass}}{- \mid Z \vdash_{\mathsf{RI}} Z}\mathsf{2to1}}{Y \mid Z \vdash_4 Y \otimes Z}\otimes\mathsf{R}}{Y \mid Z \vdash_{\mathsf{F}} Y \otimes Z}\mathsf{4to3}}{Y \mid Z \vdash_2 Y \otimes Z}\mathsf{3to2}}{\dfrac{X \multimap Y \mid Z \vdash_{\mathsf{F}} (X \multimap Y) \otimes Z}{\dfrac{X \multimap Y \mid Z \vdash_2 (X \multimap Y) \otimes Z}{X \multimap Y \mid Z \vdash_{\mathsf{RI}} (X \multimap Y) \otimes Z}\mathsf{2to1}}\mathsf{3to2}}\multimap\mathsf{L}^{\otimes}
$$

Focused system:

$$
\text{Phase } \mathsf{RI}\text{:} \quad \dfrac{S \mid \Gamma, A \vdash^{x}_{\mathsf{RI}} B}{S \mid \Gamma \vdash^{x}_{\mathsf{RI}} A \multimap B}\multimap\mathsf{R} \qquad \dfrac{S \mid \Gamma \vdash^{x}_{\mathsf{LI}} P}{S \mid \Gamma \vdash_{\mathsf{RI}} P}\mathsf{LI2RI}
$$

$$
\text{Phase } \mathsf{LI}\text{:} \quad \dfrac{- \mid \Gamma \vdash^{x}_{\mathsf{LI}} P}{\mathsf{I} \mid \Gamma \vdash^{x}_{\mathsf{LI}} P}\mathsf{IL} \qquad \dfrac{A \mid B, \Gamma \vdash^{x}_{\mathsf{LI}} P}{A \otimes B \mid \Gamma \vdash^{x}_{\mathsf{LI}} P}\otimes\mathsf{L}
$$

$$
\text{Phase } \mathsf{LI}\text{:} \quad \dfrac{T \mid \Gamma \vdash^{x}_{\mathsf{P}} P}{T \mid \Gamma \vdash^{x}_{\mathsf{LI}} P}\mathsf{P2LI}
$$

$$
\text{Phase } \mathsf{P}\text{:} \quad \dfrac{A \mid \Gamma \vdash^{x}_{\mathsf{LI}} P}{- \mid A, \Gamma \vdash^{x}_{\mathsf{P}} P}\mathsf{pass}
$$

$$
\text{Phase } \mathsf{F}\text{:} \quad \dfrac{- \mid \Gamma_0 \vdash_{\mathsf{RI}} A \qquad B \mid \Gamma_1 \vdash^{x}_{\mathsf{LI}} P \qquad B \neq T \text{ or } C \neq A' \otimes B'}{A \multimap B \mid \Gamma_0, \Gamma_1 \vdash_{\mathsf{F}} C}\multimap\mathsf{L}
$$

$$
\text{Phase } \mathsf{F}\text{:} \quad \dfrac{T \mid \Gamma \vdash_{\mathsf{RI}} A \qquad - \mid \Delta \vdash_{\mathsf{RI}} B}{T \mid \Gamma, \Delta \vdash_{\mathsf{F}} A \otimes B}\otimes\mathsf{R} \qquad \dfrac{}{A \mid\ \ \vdash_{\mathsf{F}} A}\mathsf{ax}
$$

$$\text{Phase } \mathsf{F}\text{:}\quad \frac{}{- \mid \ \vdash_{\mathsf{F}} \mathsf{I}}\ \mathsf{IR}$$

T in stoup means that T is irreducible(empty or atom). P in conclusion means that P is positive(not in $A \multimap B$). Notice that our interpretation $[\![-]\!]_s$ is same as the definition in [5].

We are going to show that this focused sequent calculus is equivalent with the original system which means that:

Theorem 4.0.1

1. For any derivation $f : S \mid \Gamma \vdash_{\mathsf{RI}} C$, there is a derivation $\mathsf{emb}\ f : S \mid \Gamma \vdash C$.
2. For any derivation $f : S \mid \Gamma \vdash C$, there is a derivation $\mathsf{focus}\ f : S \mid \Gamma \vdash_{\mathsf{RI}} C$.

Proof :

1. First theorem is straightforward, we just erase labels in a focused derivation then it would become a derivation in original sequent calculus system.
2. By induction on f.

5 Soundness and completeness

Theorem 5.0.1 For any derivation $f : S \mid \Gamma \vdash C$, there is a derivation $\mathsf{sound}\ f : [\![S \mid \Gamma]\!]_a \Longrightarrow C$.

Proof : We prove soundness by induction on f. Our interpretation $[\![-]\!]$ is same as the definition in [7] so we only have to deal with two $\multimap$ cases. However, when proving soundness of $\multimap\mathsf{L}$ rule, we need to use two admissible adjoint rules repeatedly. Therefore we need to overload $[\![-]\!]$ then make it corresponds to the interpretation in [5]. Then we can obtain two repeated application version of adjoint rules Gadj_1 and Gadj_2(we use $[\![-]\!]_a$ and $[\![-]\!]_s$ to indicate the interpretation): $\dfrac{[\![S \mid \Gamma]\!]_a \Longrightarrow A}{S \Longrightarrow [\![\Gamma \mid C]\!]_s}\ \mathsf{Gadj}_1$ and $\dfrac{A \Longrightarrow [\![\Gamma \mid C]\!]_s}{[\![A \mid \Gamma]\!]_a \Longrightarrow C}\ \mathsf{Gadj}_2$

We can obtain the soundness proof of $\multimap\mathsf{L}$ immediately. For any $f : - \mid \Gamma \vdash A$ and $g : B \mid \Delta \vdash C$:

$$\dfrac{\dfrac{\dfrac{\text{Lemma 4.3 in [7]}}{[\![A \multimap B \mid \Gamma]\!]_a \Longrightarrow A \multimap B \otimes [\![- \mid \Gamma]\!]_a} \quad \dfrac{\dfrac{\dfrac{\overset{\mathsf{sound}f}{[\![- \mid \Gamma]\!]_a \Longrightarrow A} \quad \dfrac{\overset{\mathsf{sound}g}{[\![B \mid \Delta]\!]_a \Longrightarrow C}}{B \Longrightarrow [\![\Delta \mid C]\!]_s}\ \mathsf{Gadj}_1}{A \multimap B \Longrightarrow [\![- \mid \Gamma]\!]_a \multimap [\![\Delta \mid C]\!]_s}\ \multimap}{[\![A \multimap B \otimes [\![- \mid \Gamma]\!]_a \Longrightarrow [\![\Delta \mid C]\!]_s}\ \mathsf{adj}_2}{}}{[\![A \multimap B \mid \Gamma]\!]_a \Longrightarrow [\![\Delta \mid C]\!]_s}\ \mathsf{comp}}{[\![A \multimap B \mid \Gamma, \Delta]\!]_a \Longrightarrow C}\ \mathsf{Gadj}_2$$

Theorem 5.0.2 (Weak completeness) For any derivation $f : A \Longrightarrow C$, there is a derivation $\mathsf{complt}\ f : A \mid \ \vdash C$.

Proof : In previous studies from Uustalu et al. ([7] and [5]), main cases of completenes are proved. Therefore, we only have to show new cases in skew monoidal closed categories. In particular, new cases are unit, counit, and two adjoint rules.

- (Counit)Case $f = \epsilon : (A \multimap B) \otimes A \Longrightarrow B$. We define:

$$\mathsf{complt}\left(\frac{}{(A \multimap B) \otimes A \Longrightarrow B}\ \epsilon\right) \quad =_{\mathrm{df}} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A \mid \ \vdash A}\ \mathsf{ax}}{- \mid A \vdash A}\ \mathsf{pass} \quad \dfrac{}{B \mid \ \vdash B}\ \mathsf{ax}}{A \multimap B \mid A \vdash B}\ \multimap\mathsf{L}}{(A \multimap B) \otimes A \mid \ \vdash B}\ \otimes\mathsf{L}$$

- (Unit)Case $f = \eta : A \Longrightarrow B \multimap (A \otimes B)$. We define:

$$\mathsf{complt}\left(\frac{}{A \Longrightarrow B \multimap (A \otimes B)}\,\eta\right) \quad =_{\mathrm{df}} \quad \dfrac{\dfrac{}{A \mid\ \vdash A}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{B \mid\ \vdash B}\,\mathsf{ax}}{- \mid B \vdash B}\,\mathsf{pass}}{\dfrac{A \mid B \vdash A \otimes B}{A \mid\ \vdash B \multimap (A \otimes B)}\,\multimap \mathsf{R}}\,\otimes\mathsf{R}$$

Two adjoint rules need invertibility of $\otimes\mathsf{L}$ and $\multimap \mathsf{R}$ which are proved respectively in [7] and [5].

- Case $f = \mathsf{adj}_1\ g$ where $g : A \otimes B \Longrightarrow C$. We define:

$$\mathsf{complt}\left(\dfrac{\begin{array}{c}\vdots\ g\\ A \otimes B \Longrightarrow C\end{array}}{A \Longrightarrow B \multimap C}\,\mathsf{adj}_1\right) \quad =_{\mathrm{df}} \quad \dfrac{\dfrac{\begin{array}{c}\vdots\ \mathsf{complt}\ g\\ A \otimes B \mid\ \vdash C\end{array}}{A \mid B \vdash C}\,\otimes\mathsf{L}^{-1}}{A \mid\ \vdash B \multimap C}\,\multimap \mathsf{R}$$

- Case $f = \mathsf{adj}_2\ g$ where $g : A \Longrightarrow B \multimap C$. We define:

$$\mathsf{complt}\left(\dfrac{\begin{array}{c}\vdots\ g\\ A \Longrightarrow B \multimap C\end{array}}{A \otimes B \Longrightarrow C}\,\mathsf{adj}_2\right) \quad =_{\mathrm{df}} \quad \dfrac{\dfrac{\begin{array}{c}\vdots\ \mathsf{complt}\ g\\ A \mid\ \vdash B \multimap C\end{array}}{A \mid B \vdash C}\,\multimap \mathsf{R}^{-1}}{A \otimes B \mid\ \vdash C}\,\otimes\mathsf{L}$$

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