A Focused Sequent Calculus System of Skew Monoidal Closed Categories

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1 Introduction

Recent discoveries on skew monoidal categories [4] [3] [2] (check references in previous papers, at least include papers cited by previous three papers) provide us a good reasons to study their corresponding proof systems. In previous work [7] [5] [6] [8], sequent calculus proof system for skew monoidal categories, prounital closed catgories, partial normal monoidal categories, and skew symmetric monoidal categories are presented respectively. However, there is still lack of a proof analysis on skew monoidal closed categories. Therefore, in this paper we give sound and complete sequent calculus system with skew monoidal closed categories. Moreover, we use focusing strategy from [1] to solve the coherence problem of skew monoidal closed categories.

Interestingly, because of having two connectives in our sequent calculus system, the forcusing strategy becomes subtle and involved. It cannot just divide derivations into invertible part and non-inertible part then fix their order. We discovered that it is no harm to arrange invertible rules in a fixed order, but in non-invertible rules, bad thing happens. The naive focused sequent calculus system cannont prove some provable sequents in original sequent calculus. We will see this involved focused sequent calculus system in (?) section.

This paper will go through in this order, in section two, we see the Hilbert style calculus of skew monoidal closed categories and its proof equivalences.

In third section, a cut-free sequent caculus of skew monoidal closed categories is presented. The key feature is that we have a special formula called stoup at the leftmost position in antecedent of a given sequent. In the same time, for any left rules, it can only be applied to the stoup formula. We will also see preliminarily relationship with Hilbert style calculus.

Next section, we will see a focused sequent calculus system and its connection with original sequent calulus system. We will see that all derivations in a same equivalence class under proof conversion relation, they will correspond to a unique derivation in focused system.

2 Skew monoidal closed categories

A *skew monoidal closed category*[3] is a category $\mathcal{C}$ with a unit I, two bifunctors $- \otimes - : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ and $- \multimap - : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ satisfy $\otimes \dashv \multimap$, and six natural transformations

$$\lambda_A : \mathsf{I} \otimes A \Longrightarrow A \qquad \rho_A : A \Longrightarrow A \otimes \mathsf{I} \qquad \alpha_{A,B,C} : (A \otimes B) \otimes C \Longrightarrow A \otimes (B \otimes C)$$

$$i_A : \mathsf{I} \multimap A \Longrightarrow A \qquad j_A : \mathsf{I} \Longrightarrow A \multimap A \qquad \mathsf{L}_{A,B,C} : B \multimap C \Longrightarrow (A \multimap B) \multimap (A \multimap C)$$

Give a set At which contains countably infinite atomic formulae, we can generate a free skew monoidal closed category according to folloing rules:

$$\frac{}{A \Longrightarrow A}\,\mathsf{id} \qquad \frac{A \Longrightarrow B \quad B \Longrightarrow C}{A \Longrightarrow C}\,\mathsf{comp}$$

$$\frac{A \Longrightarrow C \quad B \Longrightarrow D}{A \otimes B \Longrightarrow C \otimes D}\,\otimes \qquad \frac{C \Longrightarrow A \quad B \Longrightarrow D}{A \multimap B \Longrightarrow C \multimap D}\,\multimap$$

$$\frac{}{\mathsf{I} \otimes A \Longrightarrow A}\,\lambda \qquad \frac{}{A \Longrightarrow A \otimes \mathsf{I}}\,\rho \qquad \frac{}{(A \otimes B) \otimes C \Longrightarrow A \otimes (B \otimes C)}\,\alpha$$

$$\frac{}{\mathsf{I} \multimap A \Longrightarrow A}\,i \qquad \frac{}{\mathsf{I} \Longrightarrow A \multimap A}\,j \qquad \frac{}{B \multimap C \Longrightarrow (A \multimap B) \multimap (A \multimap C)}\,\mathsf{L}$$

$$\frac{}{(A \multimap B) \otimes A \Longrightarrow B}\,\epsilon_{A,B} \qquad \frac{}{A \Longrightarrow B \multimap (A \otimes B)}\,\eta_{A,B}$$

$$\frac{A \otimes B \Longrightarrow C}{A \Longrightarrow (B \multimap C)}\,\mathsf{adj}_1 \qquad \frac{A \Longrightarrow (B \multimap C)}{A \otimes B \Longrightarrow C}\,\mathsf{adj}_2$$

Remark 2.0.1 We do not need all of these rules to generate whole skew monoidal closed category. For example, suppose we have $\lambda, \rho, \alpha, \epsilon$, and adj_1, then i, j, L, η, and adj_2 are admissible.

$$\begin{aligned}
i &=_{\mathsf{def}} \mathsf{comp}(\rho, (\mathsf{adj}_2(\mathsf{id}_{\mathsf{I} \multimap A}))) \\
j &=_{\mathsf{def}} \mathsf{adj}_1(\lambda_A) \\
\mathsf{L} &=_{\mathsf{def}} \mathsf{adj}_1(\mathsf{adj}_1(\mathsf{comp}(\alpha, (\mathsf{comp}(\mathsf{id}_{B \multimap C} \otimes \epsilon_{A,B}), \epsilon_{B,C})))) \\
\eta &=_{\mathsf{def}} \mathsf{adj}_1(\mathsf{id}_{A \otimes B}) \\
\mathsf{adj}_2 &=_{\mathsf{def}} \mathsf{comp}((f \otimes \mathsf{id}_B), \epsilon_{B,C}) && \text{given } f : A \Longrightarrow B \multimap C
\end{aligned}$$

3 Sequent calculus for skew monoidal closed categories

In this section, we introduce sequent calculus system for skew monoidal closed categories. Following the settings from [7], [5], and [6], the sequent $S \mid \Gamma \vdash C$ in our calculus system splits in three parts. First S is called a stoup formula where we can apply to any left rule. Γ is a list of formulae where we can us pass to move stoup formula into context and no other rules can be applied. C is a formula just like in any single succeedent sequent calculus sense. Seqeunt calculus system of skew monoidal closed categries:

$$\frac{A \mid \Gamma \vdash C}{- \mid A, \Gamma \vdash C}\,\mathsf{pass} \qquad \frac{- \mid \Gamma \vdash C}{\mathsf{I} \mid \Gamma \vdash C}\,\mathsf{IL} \qquad \frac{A \mid B, \Gamma \vdash C}{A \otimes B \mid \Gamma \vdash C}\,\otimes\mathsf{L}$$

$$\frac{S \mid \Gamma \vdash A \quad - \mid \Delta \vdash B}{S \mid \Gamma, \Delta \vdash A \otimes B}\,\otimes\mathsf{R} \qquad \frac{S \mid \Gamma, A \vdash B}{S \mid \Gamma \vdash A \multimap B}\,\multimap\mathsf{R}$$

$$\frac{}{A \mid \ \vdash A}\,\mathsf{ax} \qquad \frac{}{- \mid \ \vdash \mathsf{I}}\,\mathsf{IR} \qquad \frac{- \mid \Gamma \vdash A \quad B \mid \Delta \vdash C}{A \multimap B \mid \Gamma, \Delta \vdash C}\,\multimap\mathsf{L}$$

If we interpret morphism $A \Longrightarrow C$ as a sequent $A \mid \ \vdash C$, then we can see the natural transformations are derivable in this seqent calculus. For example, natural transformations λ, ρ, and α(subscripts are omitted when there is no amubiguity) are derivable:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A \mid \ \vdash A}\,\mathsf{ax}}{- \mid A \vdash A}\,\mathsf{pass}}{\mathsf{I} \mid A \vdash A}\,\mathsf{IL}}{\mathsf{I} \otimes A \mid \ \vdash A}\,\otimes\mathsf{L} \qquad \cfrac{\cfrac{}{A \mid \ \vdash A}\,\mathsf{ax} \qquad \cfrac{}{- \mid \ \vdash \mathsf{I}}\,\mathsf{IR}}{A \mid \ \vdash A \otimes \mathsf{I}}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A \mid \ \vdash A}\,\mathsf{ax} \qquad \cfrac{\cfrac{\cfrac{}{B \mid \ \vdash B}\,\mathsf{ax} \qquad \cfrac{\cfrac{}{C \mid \ \vdash C}\,\mathsf{ax}}{- \mid C \vdash C}\,\mathsf{pass}}{B \mid C \vdash B \otimes C}\,\otimes\mathsf{R}}{- \mid B, C \vdash B \otimes C}\,\otimes\mathsf{L}}{A \mid B, C \vdash A \otimes (B \otimes C)}\,\otimes\mathsf{R}}{A \otimes B \mid C \vdash A \otimes (B \otimes C)}\,\otimes\mathsf{L}}{(A \otimes B) \otimes C \vdash \ \vdash A \otimes (B \otimes C)}\,\otimes\mathsf{L}$$

Thanks to the invertibility of $\otimes\mathsf{L}$ and $\multimap\mathsf{R}$ [7] [5], two adjoint rules are also admissible in sequent calculus system:

$$\dfrac{\dfrac{\overset{f}{A \otimes B \mid \ \vdash C}}{A \mid B \vdash C}\ \otimes\mathsf{L}^{-1}}{A \mid \ \vdash B \multimap C}\ \multimap\mathsf{R} \qquad \dfrac{\dfrac{\overset{f}{A \mid \ \ B \multimap C}}{A \mid B \vdash C}\ \multimap\mathsf{R}^{-1}}{A \otimes B \mid \ \vdash C}\ \otimes\mathsf{L}$$

Therefore, we can summarize two admissible rules in SMCC system:

$$\dfrac{\overset{f}{A \otimes B \mid \ \vdash C}}{A \mid \ \vdash B \multimap C}\ \mathsf{adj}_1 \qquad \dfrac{\overset{f}{A \mid \ \vdash B \multimap C}}{A \otimes B \mid \ \vdash C}\ \mathsf{adj}_2$$

Another important thing is that we have to ensure this sequent calculus system could not prove the inverse of any natural transformations in section 1. Here is one example, ρ^{-1} is not derivable in our system:

$$\dfrac{\overset{??}{A \mid \mathsf{I} \vdash A}}{A \otimes \mathsf{I} \mid \ \vdash A}\ \otimes\mathsf{L}$$

We interpret ρ^{-1} into $A \otimes \mathsf{I} \mid \ \vdash A$, then according to bottom-up proof search strategy, we first apply $\otimes\mathsf{L}$ but we get stuck immediately. Therfore, ρ^{-1} is not derivable in our sequent calculus. Other cases are similar.

Next we see cut-freeness of this sequent system:

Theorem 3.0.1 Two cut rules scut and ccut

$$\dfrac{S \mid \Gamma \vdash A \qquad A \mid \Delta \vdash C}{S \mid \Gamma, \Delta \vdash C}\ \mathsf{scut} \qquad \dfrac{- \mid \Gamma \vdash A \qquad S \mid \Delta_0, A, \Delta_1 \vdash C}{S \mid \Delta_0, \Gamma, \Delta_1 \vdash C}\ \mathsf{ccut}$$

are admissible in sequent calculus system for skew monoidal closed categories.

Proof : According to previous works [7] and [5] we know two cut rules, scut and ccut are admissible in each system seperately. As our new sequent calculus system is a union of two previous systems, we can just check the $\otimes$ and $\multimap$ interaction cases.

Dealing with scut first, same as proof strategy in previous papers, we proof by induction on left premise of scut rule.

1. First case is $\mathsf{scut}((\otimes\mathsf{L} f), g)$ where $f : A' \mid B', \Gamma \vdash A, g : A \mid \Delta \vdash C$, then we do subinduction on g to obtain two subcases:

   a. $g = \multimap \mathsf{R} g'$, then we let $\mathsf{scut}((\otimes\mathsf{L} f), (\multimap \mathsf{R} g')) =_{\mathsf{def}} \multimap \mathsf{R}(\mathsf{scut}((\otimes\mathsf{L} f), g'))$.

   b. $g = \multimap \mathsf{L} g'$, then we let $\mathsf{scut}((\otimes\mathsf{L} f), (\multimap \mathsf{R} g')) =_{\mathsf{def}} \otimes\mathsf{L}(\mathsf{scut}(f, (\multimap \mathsf{R} g')))$

2. Other cases are similar.

Next we see the cut-freeness of ccut rule. Similarly, we prove it by induction on the second premise.

1. $\mathsf{ccut}(f, (\multimap \mathsf{L}(g, h)))$ where $f : - \mid \Gamma \vdash A$, $g : - \mid \Delta_1 \vdash A'$, $h : B' \mid \Delta_2 \vdash C$, then we do subinduciton on f:

a. The only possibility is $A = A' \otimes B'$ and $f = \otimes\mathsf{R}(f_1, f_2)$ where $f_1 : - \mid \Gamma_1 \vdash A'$, $f_2 : - \mid \Gamma_2 \vdash B'$, then depending on $A' \otimes B'$ in Δ_1 or Δ_2, we let $\mathsf{ccut}(f, (\multimap \mathsf{L}(g, h))) =_{\mathsf{def}} \multimap \mathsf{L}(\mathsf{ccut}(f, g), h)$ or $\multimap \mathsf{L}(g, (\mathsf{ccut}(f, h)))$, respectively.

2. $\mathsf{ccut}(f, (\otimes\mathsf{R}(g, h)))$ where $f : - \mid \Gamma \vdash A$, $g : S \mid \Delta_1 \vdash A'$, $h : - \mid \Delta_2 \vdash B'$ is similar as above.

3. $\mathsf{ccut}(f, (\otimes\mathsf{L}g))$ and $\mathsf{ccut}(f, (\multimap \mathsf{R}h))$ cases are similar, we permute ccut up, where $g : A' \mid B', \Delta_0, A, \Delta_1 \vdash C$, $h : S \mid \Delta_0, A, \Delta_1, A' \vdash B'$.

4 Focusing

Before getting into focused sequent calculus system, we need to define new proof equivalences in original sequent calculus system. In [7] and [5], Uustalu et al. provided proof equivalences between $\otimes$ only and $\multimap$ only systems respectively. Again, similar as cut elimination proof above, we have to give $\otimes$ and $\multimap$ interaction cases here:

For $f : - \mid \Gamma, A \vdash C$, IL:

$$\dfrac{\dfrac{\dfrac{f}{- \mid \Gamma, A \vdash C}}{- \mid \Gamma \vdash A \multimap C}\multimap\mathsf{R}}{\mathsf{I} \mid \Gamma \vdash A \multimap C}\mathsf{IL} \quad \doteq \quad \dfrac{\dfrac{\dfrac{f}{- \mid \Gamma, A \vdash C}}{\mathsf{I} \mid \Gamma, A \vdash C}\mathsf{IL}}{\mathsf{I} \mid \Gamma \vdash A \multimap C}\multimap\mathsf{R}$$

For $f : A \mid B, \Gamma, C \vdash D$:

$$\dfrac{\dfrac{\dfrac{f}{A \mid B, \Gamma, C \vdash D}}{A \mid B, \Gamma \vdash C \multimap D}\multimap\mathsf{R}}{A \otimes B \mid \Gamma \vdash C \multimap D}\otimes\mathsf{L} \quad \doteq \quad \dfrac{\dfrac{\dfrac{f}{A \mid B, \Gamma, C \vdash D}}{A \otimes B \mid \Gamma, C \vdash D}\otimes\mathsf{L}}{A \otimes B \mid \Gamma \vdash C \multimap D}\multimap\mathsf{R}$$

For $f : - \mid \Gamma \vdash A$, $g : B \mid \Delta \vdash C$, and $h : - \mid \Delta' \vdash D$:

$$\dfrac{\dfrac{f}{- \mid \Gamma \vdash A} \quad \dfrac{\dfrac{g}{B \mid \Delta \vdash C} \quad \dfrac{h}{- \mid \Delta' \vdash D}}{B \mid \Delta, \Delta' \vdash C \otimes D}\otimes\mathsf{R}}{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash C \otimes D}\multimap\mathsf{L}$$

$$\doteq$$

$$\dfrac{\dfrac{\dfrac{f}{- \mid \Gamma \vdash A} \quad \dfrac{g}{B \mid \Delta \vdash C}}{A \multimap B \mid \Gamma, \Delta \vdash C}\multimap\mathsf{L} \quad \dfrac{h}{- \mid \Delta' \vdash D}}{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash C \otimes D}\otimes\mathsf{R}$$

Our focused system is with 4 phases. From bottom-up proof search, we decompose $\multimap \mathsf{R}$ until $\multimap$ is not the principal connective in succedent. In phase 2, we destruct the formula in stoup until it is not in the form of $A \otimes B$. Next we proceed to phase 3 to remove $\multimap$ in stoup. Lastly in phase 4, we deal with $\otimes$ in succedent and close the proof tree. Notice that we have a non-intuitive rule $\multimap \mathsf{L}^{\otimes}$. We need this rule to keep soundness for our focused system. Without the rule, we would face a counter example:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}??\\ - \mid \ \vdash_{\mathsf{RI}} X\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}??\\ Y \mid X \vdash_{\mathsf{RI}} Y\end{matrix}}{Y \mid \ \vdash_{\mathsf{RI}} X \multimap Y}\multimap \mathsf{R} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Z \mid \ \vdash_4 Z}\mathsf{ax}}{Z \mid \ \vdash_{\mathsf{F}} Z}\mathsf{4to3}}{Z \mid \ \vdash_2 Z}\mathsf{3to2}}{- \mid Z \vdash_2 Z}\mathsf{pass}}{- \mid Z \vdash_{\mathsf{RI}} Z}\mathsf{2to1}}{Y \mid Z \vdash_4 (X \multimap Y) \otimes Z}\otimes\mathsf{R}}{Y \mid Z \vdash_{\mathsf{F}} (X \multimap Y) \otimes Z}\mathsf{4to3}}{Y \mid Z \vdash_2 (X \multimap Y) \otimes Z}\mathsf{3to2}}{X \multimap Y \mid Z \vdash_{\mathsf{F}} (X \multimap Y) \otimes Z}\multimap \mathsf{L}}{X \multimap Y \mid Z \vdash_2 (X \multimap Y) \otimes Z}\mathsf{3to2}}{X \multimap Y \mid Z \vdash_{\mathsf{RI}} (X \multimap Y) \otimes Z}\mathsf{2to1}$$

This counter example occurs when we insist to decompose $\multimap$ in stoup prior than $\otimes$ in succedent. Therefore we induce $\multimap \mathsf{L}^{\otimes}$ rule to solve this situation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X \mid \ \vdash_4 X}\mathsf{ax}}{X \mid \ \vdash_{\mathsf{F}} X}\mathsf{4to3}}{X \mid \ \vdash_2 X}\mathsf{3to2}}{- \mid X \vdash_2 X}\mathsf{pass}}{- \mid X \vdash_{\mathsf{RI}} X}\mathsf{2to1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Y \mid \ \vdash_4 Y}\mathsf{ax}}{Y \mid \ \vdash_{\mathsf{F}} Y}\mathsf{4to3}}{Y \mid \ \vdash_2 Y}\mathsf{3to2}}{Y \mid \ \vdash_{\mathsf{RI}} Y}\mathsf{2to1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Z \mid \ \vdash_4 Z}\mathsf{ax}}{Z \mid \ \vdash_{\mathsf{F}} Z}\mathsf{4to3}}{Z \mid \ \vdash_2 Z}\mathsf{3to2}}{- \mid Z \vdash_2 Z}\mathsf{pass}}{- \mid Z \vdash_{\mathsf{RI}} Z}\mathsf{2to1}}{Y \mid Z \vdash_4 Y \otimes Z}\otimes\mathsf{R}}{Y \mid Z \vdash_{\mathsf{F}} Y \otimes Z}\mathsf{4to3}}{Y \mid Z \vdash_2 Y \otimes Z}\mathsf{3to2}}{X \multimap Y \mid Z \vdash_{\mathsf{F}} (X \multimap Y) \otimes Z}\multimap \mathsf{L}^{\otimes}}{X \multimap Y \mid Z \vdash_2 (X \multimap Y) \otimes Z}\mathsf{3to2}}{X \multimap Y \mid Z \vdash_{\mathsf{RI}} (X \multimap Y) \otimes Z}\mathsf{2to1}$$

Focused system:

$$\text{Phase } \mathsf{RI}\text{:} \quad \dfrac{S \mid \Gamma, A \vdash^x_{\mathsf{RI}} B}{S \mid \Gamma \vdash^x_{\mathsf{RI}} A \multimap B}\multimap \mathsf{R} \qquad \dfrac{S \mid \Gamma \vdash^x_{\mathsf{LI}} P}{S \mid \Gamma \vdash_{\mathsf{RI}} P}\mathsf{LI2RI}$$

$$\text{Phase } \mathsf{LI}\text{:} \quad \dfrac{- \mid \Gamma \vdash^x_{\mathsf{LI}} P}{\mathsf{I} \mid \Gamma \vdash^x_{\mathsf{LI}} P}\mathsf{IL} \qquad \dfrac{A \mid B, \Gamma \vdash^x_{\mathsf{LI}} P}{A \otimes B \mid \Gamma \vdash^x_{\mathsf{LI}} P}\otimes\mathsf{L}$$

$$\text{Phase } \mathsf{LI}\text{:} \quad \dfrac{T \mid \Gamma \vdash^x_{\mathsf{P}} P}{T \mid \Gamma \vdash^x_{\mathsf{LI}} P}\mathsf{P2LI}$$

$$\text{Phase } \mathsf{P}\text{:} \quad \dfrac{A \mid \Gamma \vdash^x_{\mathsf{LI}} P}{- \mid A, \Gamma \vdash^x_{\mathsf{P}} P}\mathsf{pass}$$

$$\text{Phase } \mathsf{F}\text{:} \quad \dfrac{- \mid \Gamma_0 \vdash_{\mathsf{RI}} A \qquad B \mid \Gamma_1 \vdash^x_{\mathsf{LI}} P \qquad B \neq T \text{ or } C \neq A' \otimes B'}{A \multimap B \mid \Gamma_0, \Gamma_1 \vdash_{\mathsf{F}} C}\multimap \mathsf{L}$$

$$\text{Phase } \mathsf{F}\text{:} \quad \dfrac{T \mid \Gamma \vdash_{\mathsf{RI}} A \qquad - \mid \Delta \vdash_{\mathsf{RI}} B}{T \mid \Gamma, \Delta \vdash_{\mathsf{F}} A \otimes B}\otimes\mathsf{R} \qquad \dfrac{}{A \mid \ \vdash_{\mathsf{F}} A}\mathsf{ax}$$

$$\text{Phase } \mathsf{F}\text{:} \quad \frac{}{- \mid \ \vdash_{\mathsf{F}} \mathsf{I}} \ \mathsf{IR}$$

T in stoup means that T is irreducible(empty or atom). P in conclusion means that P is positive(not in $A \multimap B$). Notice that our interpretation $[\![-]\!]_s$ is same as the definition in [5].

We are going to show that this focused sequent calculus is equivalent with the original system which means that:

Theorem 4.0.1 1. For any derivation $f : S \mid \Gamma \vdash_{\mathsf{RI}} C$, there is a derivation $\mathsf{emb}\ f : S \mid \Gamma \vdash C$.

2. For any derivation $f : S \mid \Gamma \vdash C$, there is a derivation $\mathsf{focus}\ f : S \mid \Gamma \vdash_{\mathsf{RI}} C$.

Proof : 1. First theorem is straightforward, we just erase labels in a focused derivation then it would become a derivation in original sequent calculus system.

2. By induction on f.

5 Soundness and completeness

Theorem 5.0.1 For any derivation $f : S \mid \Gamma \vdash C$, there is a derivation $\mathsf{sound}\ f : [\![S \mid \Gamma]\!]_a \Longrightarrow C$.

Proof : We prove soundness by induction on f. Our interpretation $[\![-]\!]$ is same as the definition in [7] so we only have to deal with two $\multimap$ cases. However, when proving soundness of $\multimap \mathsf{L}$ rule, we need to use two admissible adjoint rules repeatedly. Therefore we need to overload $[\![-]\!]$ then make it corresponds to the interpretation in [5]. Then we can obtain two repeated application version of adjoint rules $\mathsf{Gadj_1}$ and $\mathsf{Gadj_2}$(we use $[\![-]\!]_a$ and $[\![-]\!]_s$ to indicate the interpretation): $\dfrac{[\![S \mid \Gamma]\!]_a \Longrightarrow A}{S \Longrightarrow [\![\Gamma \mid C]\!]_s}\ \mathsf{Gadj_1}$ and $\dfrac{A \Longrightarrow [\![\Gamma \mid C]\!]_s}{[\![A \mid \Gamma]\!]_a \Longrightarrow C}\ \mathsf{Gadj_2}$

We can obtain the soundness proof of $\multimap \mathsf{L}$ immediately. For any $f : - \mid \Gamma \vdash A$ and $g : B \mid \Delta \vdash C$:

$$\dfrac{\dfrac{\dfrac{\text{Lemma 4.3 in [7]}}{[\![A \multimap B \mid \Gamma]\!]_a \Longrightarrow A \multimap B \otimes [\![- \mid \Gamma]\!]_a} \qquad \dfrac{\dfrac{\dfrac{\mathsf{sound} f}{[\![- \mid \Gamma]\!]_a \Longrightarrow A} \quad \dfrac{\dfrac{\mathsf{sound} g}{[\![B \mid \Delta]\!]_a \Longrightarrow C}}{B \Longrightarrow [\![\Delta \mid C]\!]_s}\ \mathsf{Gadj_1}}{A \multimap B \Longrightarrow [\![- \mid \Gamma]\!]_a \multimap [\![\Delta \mid C]\!]_s}\ \multimap}{[\![A \multimap B \otimes [\![- \mid \Gamma]\!]_a \Longrightarrow [\![\Delta \mid C]\!]_s}\ \mathsf{adj_2}}{[\![A \multimap B \mid \Gamma]\!]_a \Longrightarrow [\![\Delta \mid C]\!]_s}\ \mathsf{comp}}{[\![A \multimap B \mid \Gamma, \Delta]\!]_a \Longrightarrow C}\ \mathsf{Gadj_2}$$

Theorem 5.0.2 (Weak completeness) For any derivation $f : A \Longrightarrow C$, there is a derivation complt $f : A \mid \ \vdash C$.

Proof : In previous studies from Uustalu et al. ([7] and [5]), main cases of completenes are proved. Therefore, we only have to show new cases in skew monoidal closed categories. In particular, new cases are unit, counit, and two adjoint rules.

- (Counit)Case $f = \epsilon : (A \multimap B) \otimes A \Longrightarrow B$. We define:

$$\mathsf{complt}\left(\frac{}{(A \multimap B) \otimes A \Longrightarrow B}\ \epsilon\right) =_{\text{df}} \dfrac{\dfrac{\dfrac{\dfrac{}{A \mid \ \vdash A}\ \mathsf{ax}}{- \mid A \vdash A}\ \mathsf{pass} \qquad \dfrac{}{B \mid \ \vdash B}\ \mathsf{ax}}{A \multimap B \mid A \vdash B}\ \multimap \mathsf{L}}{(A \multimap B) \otimes A \mid \ \vdash B}\ \otimes\mathsf{L}$$

- (Unit)Case $f = \eta : A \Longrightarrow B \multimap (A \otimes B)$. We define:

$$\mathsf{complt}\left(\frac{}{A \Longrightarrow B \multimap (A \otimes B)}\,\eta\right) \quad =_{\mathrm{df}} \quad \dfrac{\dfrac{}{A \mid \ \vdash A}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{B \mid \ \vdash B}\,\mathsf{ax}}{- \mid B \vdash B}\,\mathsf{pass}}{\dfrac{A \mid B \vdash A \otimes B}{A \mid \ \vdash B \multimap (A \otimes B)}\multimap\mathsf{R}}\,\otimes\mathsf{R}$$

Two adjoint rules need invertibility of $\otimes\mathsf{L}$ and $\multimap \mathsf{R}$ which are proved respectively in [7] and [5].

- Case $f = \mathsf{adj}_1\ g$ where $g : A \otimes B \Longrightarrow C$. We define:

$$\mathsf{complt}\left(\dfrac{\begin{array}{c}\vdots\, g \\ A \otimes B \Longrightarrow C\end{array}}{A \Longrightarrow B \multimap C}\,\mathsf{adj}_1\right) \quad =_{\mathrm{df}} \quad \dfrac{\dfrac{\begin{array}{c}\vdots\, \mathsf{complt}\ g \\ A \otimes B \mid \ \vdash C\end{array}}{A \mid B \vdash C}\,\otimes\mathsf{L}^{-1}}{A \mid \ \vdash B \multimap C}\multimap\mathsf{R}$$

- Case $f = \mathsf{adj}_2\ g$ where $g : A \Longrightarrow B \multimap C$. We define:

$$\mathsf{complt}\left(\dfrac{\begin{array}{c}\vdots\, g \\ A \Longrightarrow B \multimap C\end{array}}{A \otimes B \Longrightarrow C}\,\mathsf{adj}_2\right) \quad =_{\mathrm{df}} \quad \dfrac{\dfrac{\begin{array}{c}\vdots\, \mathsf{complt}\ g \\ A \mid \ \vdash B \multimap C\end{array}}{A \mid B \vdash C}\multimap\mathsf{R}^{-1}}{A \otimes B \mid \ \vdash C}\,\otimes\mathsf{L}$$

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