1 Skew monoidal closed categories

A *skew monoidal closed category* $\mathcal{C}$ is a category $\mathcal{C}$ with a unit I, adjoint functors $X \otimes - : \mathcal{C} \to \mathcal{C}$ and $- \multimap X : \mathcal{C}^{op} \to \mathcal{C}(X \otimes - \dashv - \multimap X)$, natural transformations $\lambda, \rho, \alpha, i, j, \mathsf{L}$ satisfying all Mac Lane axioms.

We can generate a free skew monoidal closed category over a set At by following rules:

$$\frac{}{A \Longrightarrow A}\,\mathsf{id} \qquad \frac{A \Longrightarrow B \quad B \Longrightarrow C}{A \Longrightarrow C}\,\mathsf{comp}$$

$$\frac{A \Longrightarrow C \quad B \Longrightarrow D}{A \otimes B \Longrightarrow C \otimes D}\,\otimes \qquad \frac{C \Longrightarrow A \quad B \Longrightarrow D}{A \multimap B \Longrightarrow C \multimap D}\,\multimap$$

$$\frac{}{\mathsf{I} \otimes A \Longrightarrow A}\,\lambda \qquad \frac{}{A \Longrightarrow A \otimes \mathsf{I}}\,\rho \qquad \frac{}{(A \otimes B) \otimes C \Longrightarrow A \otimes (B \otimes C)}\,\alpha$$

$$\frac{}{\mathsf{I} \multimap A \Longrightarrow A}\,i \qquad \frac{}{\mathsf{I} \Longrightarrow A \multimap A}\,j \qquad \frac{}{B \multimap C \Longrightarrow (A \multimap B) \multimap (A \multimap C)}\,\mathsf{L}$$

$$\frac{}{(A \multimap B) \otimes A \Longrightarrow B}\,\epsilon_{A,B} \qquad \frac{}{A \Longrightarrow B \multimap (A \otimes B)}\,\eta_{A,B}$$

$$\frac{A \otimes B \Longrightarrow C}{A \Longrightarrow (B \multimap C)}\,\mathsf{adj}_1 \qquad \frac{A \Longrightarrow (B \multimap C)}{A \otimes B \Longrightarrow C}\,\mathsf{adj}_2$$

Notice that we do not need all of these rules. For example, we can derive $i, j, \mathsf{L}, \epsilon$, and adj_1 from $\lambda, \rho, \alpha, \eta$, and adj_2, vice versa.

2 Sequent calculus for skew monoidal closed categories

Seqeunt calculus system of skew monoidal closed categries(SMCC):

$$\frac{A \mid \Gamma \longrightarrow C}{- \mid A, \Gamma \longrightarrow C}\,\mathsf{pass} \qquad \frac{- \mid \Gamma \longrightarrow C}{\mathsf{I} \mid \Gamma \longrightarrow C}\,\mathsf{IL} \qquad \frac{A \mid B, \Gamma \longrightarrow C}{A \otimes B \mid \Gamma \longrightarrow C}\,\otimes\mathsf{L}$$

$$\frac{S \mid \Gamma \longrightarrow A \quad - \mid \Delta \longrightarrow B}{S \mid \Gamma, \Delta \longrightarrow A \otimes B}\,\otimes\mathsf{R} \qquad \frac{S \mid \Gamma, A \longrightarrow B}{S \mid \Gamma \longrightarrow A \multimap B}\,\multimap\mathsf{R}$$

$$\frac{}{A \mid \quad \longrightarrow A}\,\mathsf{ax} \qquad \frac{}{- \mid \quad \longrightarrow \mathsf{I}}\,\mathsf{IR} \qquad \frac{- \mid \Gamma \longrightarrow A \quad B \mid \Delta \longrightarrow C}{A \multimap B \mid \Gamma, \Delta \longrightarrow C}\,\multimap\mathsf{L}$$

$$\frac{S \mid \Gamma \vdash A \qquad A \mid \Delta \vdash C}{S \mid \Gamma, \Delta \vdash C}\,\mathsf{scut}$$

$$\frac{- \mid \Gamma \vdash A \qquad S \mid \Delta_0, A, \Delta_1 \vdash C}{S \mid \Delta_0, \Gamma, \Delta_1 \vdash C}\,\mathsf{ccut}$$

3 Focusing

Proof equivelences:

For $f : - \mid \Gamma, A \vdash C$, IL:
$$\dfrac{\dfrac{\overset{f}{- \mid \Gamma, A \vdash C}}{- \mid \Gamma \vdash A \multimap C}\multimap\mathsf{R}}{\mathsf{I} \mid \Gamma \vdash A \multimap C}\mathsf{IL} \doteq \dfrac{\dfrac{\overset{f}{- \mid \Gamma, A \vdash C}}{\mathsf{I} \mid \Gamma, A \vdash C}\mathsf{IL}}{\mathsf{I} \mid \Gamma \vdash A \multimap C}\multimap\mathsf{R}$$

For $f : A \mid B, \Gamma, C \vdash D$:
$$\dfrac{\dfrac{\overset{f}{A \mid B, \Gamma, C \vdash D}}{A \mid B, \Gamma \vdash C \multimap D}\multimap\mathsf{R}}{A \otimes B \mid \Gamma \vdash C \multimap D}\otimes\mathsf{L} \doteq \dfrac{\dfrac{\overset{f}{A \mid B, \Gamma, C \vdash D}}{A \otimes B \mid \Gamma, C \vdash D}\otimes\mathsf{L}}{A \otimes B \mid \Gamma \vdash C \multimap D}\multimap\mathsf{R}$$

For $f : - \mid \Gamma \vdash A$, $g : B \mid \Delta \vdash C$, and $h : - \mid \Delta' \vdash D$:
$$\dfrac{\overset{f}{- \mid \Gamma \vdash A} \qquad \dfrac{\overset{g}{B \mid \Delta \vdash C} \qquad \overset{h}{- \mid \Delta' \vdash D}}{B \mid \Delta, \Delta' \vdash C \otimes D}\otimes\mathsf{R}}{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash C \otimes D}\multimap\mathsf{L} \doteq \dfrac{\dfrac{\overset{f}{- \mid \Gamma \vdash A} \qquad \overset{g}{B \mid \Delta \vdash C}}{A \multimap B \mid \Gamma, \Delta \vdash C}\multimap\mathsf{L} \qquad \overset{h}{- \mid \Delta' \vdash D}}{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash C \otimes D}\otimes\mathsf{R}$$

A counter example occurs when we insist to decompose $\multimap$ in stoup prior than $\otimes$ in succedent.
$$\dfrac{\dfrac{\dfrac{\overset{??}{- \mid \ \vdash_1 X} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{??}{Y \mid X \vdash_1 Y}}{Y \mid \ \vdash_1 X \multimap Y}\multimap\mathsf{R} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Z \mid \ \vdash_4 Z}\mathsf{ax}}{Z \mid \ \vdash_3 Z}\mathsf{4to3}}{Z \mid \ \vdash_2 Z}\mathsf{3to2}}{- \mid Z \vdash_2 Z}\mathsf{pass}}{- \mid Z \vdash_1 Z}\mathsf{2to1}}{Y \mid Z \vdash_4 (X \multimap Y) \otimes Z}\otimes\mathsf{R}}{Y \mid Z \vdash_3 (X \multimap Y) \otimes Z}\mathsf{4to3}}{Y \mid Z \vdash_2 (X \multimap Y) \otimes Z}\mathsf{3to2}}{X \multimap Y \mid Z \vdash_3 (X \multimap Y) \otimes Z}\multimap\mathsf{L}}{X \multimap Y \mid Z \vdash_2 (X \multimap Y) \otimes Z}\mathsf{3to2}}{X \multimap Y \mid Z \vdash_1 (X \multimap Y) \otimes Z}\mathsf{2to1}$$

Focused system:

$$\text{Phase 1: } \dfrac{S \mid \Gamma, A \vdash_1 B}{S \mid \Gamma \vdash_1 A \multimap B}\multimap\mathsf{R} \qquad \dfrac{S \mid \Gamma \vdash_2 C \qquad C \neq A \multimap B}{S \mid \Gamma \vdash_1 C}\mathsf{2to1}$$

$$\text{Phase 2: } \dfrac{- \mid \Gamma \vdash_2 C}{\mathsf{I} \mid \Gamma \vdash_2 C}\mathsf{IL} \qquad \dfrac{A \mid B, \Gamma \vdash_2 C}{A \otimes B \mid \Gamma \vdash_2 C}\otimes\mathsf{L}$$

$$\text{Phase 2: } \dfrac{A \mid \Gamma \vdash_2 C}{- \mid A, \Gamma \vdash_2 C}\mathsf{pass} \qquad \dfrac{S \mid \Gamma \vdash_3 C \qquad S \neq A \otimes B \qquad C \neq A \multimap B}{S \mid \Gamma \vdash_2 C}\mathsf{3to2}$$

$$\text{Phase 3: } \dfrac{- \mid \Gamma_0 \vdash_1 A \qquad B \mid \Gamma_1 \vdash_2 C \qquad B \neq T \text{ or } C \neq A' \otimes B'}{A \multimap B \mid \Gamma_0, \Gamma_1 \vdash_3 C}\multimap\mathsf{L}$$

$$\text{Phase 3: } \dfrac{S \mid \Gamma \vdash_1 A \qquad - \mid \Delta \vdash_1 B}{S \mid \Gamma, \Delta \vdash_3 A \otimes B}\otimes\mathsf{R} \qquad \dfrac{}{A \mid \ \vdash_3 A}\mathsf{ax}$$

$$\text{Phase 4: } \dfrac{}{- \mid \ \vdash_3 \mathsf{I}}\mathsf{IR}$$