

A Focused Sequent Calculus System of Skew Monoidal Closed Categories

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1 Introduction

Recent discoveries on skew monoidal categories [4] [3] [2] (check references in previous papers, at least include papers cited by previous three papers) provide us a good reasons to study their corresponding proof systems. In previous work [7] [5] [6] [8], sequent calculus proof system for skew monoidal categories, prounital closed categories, partial normal monoidal categories, and skew symmetric monoidal categories are presented respectively. However, there is still lack of a proof analysis on skew monoidal closed categories. Therefore, in this paper we give sound and complete sequent calculus system with skew monoidal closed categories. Moreover, we use focusing strategy from [1] to solve the coherence problem of skew monoidal closed categories.

Interestingly, because of having two connectives in our sequent calculus system, the focusing strategy becomes subtle and involved. It cannot just divide derivations into invertible part and non-invertible part then fix their order. We discovered that it is no harm to arrange invertible rules in a fixed order, but in non-invertible rules, bad thing happens. The naive focused sequent calculus system cannot prove some provable sequents in original sequent calculus. We will see this involved focused sequent calculus system in (?) section.

This paper will go through in this order, in section two, we see the Hilbert style calculus of skew monoidal closed categories and its proof equivalences.

In third section, a cut-free sequent calculus of skew monoidal closed categories is presented. The key feature is that we have a special formula called stoup at the leftmost position in antecedent of a given sequent. In the same time, for any left rules, it can only be applied to the stoup formula. We will also see preliminarily relationship with Hilbert style calculus.

Next section, we will see a focused sequent calculus system and its connection with original sequent calculus system. We will see that all derivations in a same equivalence class under proof conversion relation, they will correspond to a unique derivation in focused system.

2 Skew monoidal closed categories

A *skew monoidal closed category*[3] is a category \mathcal{C} with a unit I , two bifunctors $- \otimes - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and $- \multimap - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ satisfy $\otimes \dashv \multimap$, and six natural transformations

$$\begin{aligned} \lambda_A : I \otimes A &\Rightarrow A & \rho_A : A &\Rightarrow A \otimes I & \alpha_{A,B,C} : (A \otimes B) \otimes C &\Rightarrow A \otimes (B \otimes C) \\ i_A : I \multimap A &\Rightarrow A & j_A : I &\Rightarrow A \multimap A & L_{A,B,C} : B \multimap C &\Rightarrow (A \multimap B) \multimap (A \multimap C) \end{aligned}$$

Give a set At which contains countably infinite atomic formulae, we can generate a free skew monoidal closed category according to following rules:

$$\begin{aligned} & \frac{}{A \Rightarrow A} \text{id} \quad \frac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C} \text{comp} \\ & \frac{A \Rightarrow C \quad B \Rightarrow D}{A \otimes B \Rightarrow C \otimes D} \otimes \quad \frac{C \Rightarrow A \quad B \Rightarrow D}{A \multimap B \Rightarrow C \multimap D} \multimap \\ & \frac{}{I \otimes A \Rightarrow A} \lambda \quad \frac{}{A \Rightarrow A \otimes I} \rho \quad \frac{}{(A \otimes B) \otimes C \Rightarrow A \otimes (B \otimes C)} \alpha \\ & \frac{}{I \multimap A \Rightarrow A} i \quad \frac{}{I \Rightarrow A \multimap A} j \quad \frac{}{B \multimap C \Rightarrow (A \multimap B) \multimap (A \multimap C)} L \\ & \frac{}{(A \multimap B) \otimes A \Rightarrow B} \epsilon_{A,B} \quad \frac{}{A \Rightarrow B \multimap (A \otimes B)} \eta_{A,B} \end{aligned}$$

$$\frac{A \otimes B \Longrightarrow C}{A \Longrightarrow (B \multimap C)} \text{adj}_1 \quad \frac{A \Longrightarrow (B \multimap C)}{A \otimes B \Longrightarrow C} \text{adj}_2$$

Remark 2.0.1 We do not need all of these rules to generate whole skew monoidal closed category. For example, suppose we have $\lambda, \rho, \alpha, \epsilon$, and adj_1 , then i, j, \mathbf{L}, η , and adj_2 are admissible.

$$\begin{aligned} i &=_{\text{df}} \text{comp}(\rho, (\text{adj}_2(\text{id}_{\mathbf{I} \multimap A}))) \\ j &=_{\text{df}} \text{adj}_1(\lambda_A) \\ \mathbf{L} &=_{\text{df}} \text{adj}_1(\text{adj}_1(\text{comp}(\alpha, (\text{comp}(\text{id}_{B \multimap C} \otimes \epsilon_{A,B}), \epsilon_{B,C})))) \\ \eta &=_{\text{df}} \text{adj}_1(\text{id}_{A \otimes B}) \\ \text{adj}_2 &=_{\text{df}} \text{comp}((f \otimes \text{id}_B), \epsilon_{B,C}) \quad \text{given } f : A \Longrightarrow B \multimap C \end{aligned}$$

3 Sequent calculus for skew monoidal closed categories

In this section, we introduce sequent calculus system for skew monoidal closed categories. Following the settings from [7], [5], and [6], the sequent $S \mid \Gamma \vdash C$ in our calculus system splits in three parts. First S is called a stoup formula where we can apply to any left rule. Γ is a list of formulae where we can use pass to move stoup formula into context and no other rules can be applied. C is a formula just like in any single succedent sequent calculus sense. Sequent calculus system of skew monoidal closed categories:

$$\begin{aligned} &\frac{A \mid \Gamma \vdash C}{- \mid A, \Gamma \vdash C} \text{pass} \quad \frac{- \mid \Gamma \vdash C}{\mathbf{I} \mid \Gamma \vdash C} \text{IL} \quad \frac{A \mid B, \Gamma \vdash C}{A \otimes B \mid \Gamma \vdash C} \otimes \text{L} \\ &\frac{S \mid \Gamma \vdash A \quad - \mid \Delta \vdash B}{S \mid \Gamma, \Delta \vdash A \otimes B} \otimes \text{R} \quad \frac{S \mid \Gamma, A \vdash B}{S \mid \Gamma \vdash A \multimap B} \multimap \text{R} \\ &\frac{}{A \mid \vdash A} \text{ax} \quad \frac{}{- \mid \vdash \mathbf{I}} \text{IR} \quad \frac{- \mid \Gamma \vdash A \quad B \mid \Delta \vdash C}{A \multimap B \mid \Gamma, \Delta \vdash C} \multimap \text{L} \end{aligned}$$

If we interpret morphism $A \Longrightarrow C$ as a sequent $A \mid \vdash C$, then we can see the natural transformations are derivable in this sequent calculus. For example, natural transformations λ, ρ , and α (subscripts are omitted when there is no ambiguity) are derivable:

$$\begin{aligned} &\frac{}{A \mid \vdash A} \text{ax} \quad \frac{}{- \mid A \vdash A} \text{pass} \quad \frac{}{\mathbf{I} \mid A \vdash A} \text{IL} \quad \frac{}{A \mid \vdash A} \text{ax} \quad \frac{}{- \mid \vdash \mathbf{I}} \text{IR} \\ &\frac{}{\mathbf{I} \otimes A \mid \vdash A} \otimes \text{L} \quad \frac{}{A \mid \vdash A \otimes \mathbf{I}} \otimes \text{R} \\ &\frac{}{A \mid \vdash A} \text{ax} \quad \frac{}{B \mid \vdash B} \text{ax} \quad \frac{}{C \mid \vdash C} \text{ax} \quad \frac{}{- \mid C \vdash C} \text{pass} \\ &\frac{}{B \mid C \vdash B \otimes C} \otimes \text{R} \quad \frac{}{- \mid B, C \vdash B \otimes C} \otimes \text{L} \\ &\frac{}{A \mid B, C \vdash A \otimes (B \otimes C)} \otimes \text{L} \quad \frac{}{A \otimes B \mid C \vdash A \otimes (B \otimes C)} \otimes \text{L} \\ &\frac{}{(A \otimes B) \otimes C \vdash A \otimes (B \otimes C)} \otimes \text{L} \end{aligned}$$

Thanks to the invertibility of $\otimes L$ and $\multimap R$ [7] [5], two adjoint rules are also admissible in sequent calculus system:

$$\frac{\frac{f}{A \otimes B \mid \vdash C} \otimes L^{-1}}{A \mid \vdash B \multimap C} \multimap R \quad \frac{\frac{f}{A \mid \vdash B \multimap C} \multimap R^{-1}}{A \otimes B \mid \vdash C} \otimes L$$

Therefore, we can summarize two admissible rules in SMCC system:

$$\frac{f}{A \otimes B \mid \vdash C} \text{adj}_1 \quad \frac{f}{A \mid \vdash B \multimap C} \text{adj}_2$$

Another important thing is that we have to ensure this sequent calculus system could not prove the inverse of any natural transformations in section 1. Here is one example, ρ^{-1} is not derivable in our system:

$$\frac{??}{\frac{A \mid I \vdash A}{A \otimes I \mid \vdash A} \otimes L}$$

We interpret ρ^{-1} into $A \otimes I \mid \vdash A$, then according to bottom-up proof search strategy, we first apply $\otimes L$ but we get stuck immediately. Therefore, ρ^{-1} is not derivable in our sequent calculus. Other cases are similar.

Next we see cut-freeness of this sequent system:

Theorem 3.0.1 Two cut rules **scut** and **ccut**

$$\frac{S \mid \Gamma \vdash A \quad A \mid \Delta \vdash C}{S \mid \Gamma, \Delta \vdash C} \text{scut} \quad \frac{- \mid \Gamma \vdash A \quad S \mid \Delta_0, A, \Delta_1 \vdash C}{S \mid \Delta_0, \Gamma, \Delta_1 \vdash C} \text{ccut}$$

are admissible in sequent calculus system for skew monoidal closed categories.

Proof : According to previous works [7] and [5] we know two cut rules, **scut** and **ccut** are admissible in each system separately. As our new sequent calculus system is a union of two previous systems, we can just check the \otimes and \multimap interaction cases.

Dealing with **scut** first, same as proof strategy in previous papers, we proof by induction on left premise of **scut** rule.

1. First case is **scut**(($\otimes L$) f), g) where $f : A' \mid B', \Gamma \vdash A, g : A \mid \Delta \vdash C$, then we do subinduction on g to obtain two subcases:
 - a. $g = \multimap R g'$, then we let $\text{scut}((\otimes L f), (\multimap R g')) =_{\text{df}} \multimap R(\text{scut}((\otimes L f), g'))$.
 - b. $g = \multimap L g'$, then we let $\text{scut}((\otimes L f), (\multimap R g')) =_{\text{df}} \otimes L(\text{scut}(f, (\multimap R g')))$
2. Other cases are similar.

Next we see the cut-freeness of **ccut** rule. Similarly, we prove it by induction on the second premise.

1. **ccut**($f, (\multimap L(g, h))$) where $f : - \mid \Gamma \vdash A, g : - \mid \Delta_1 \vdash A', h : B' \mid \Delta_2 \vdash C$, then we do subinduction on f :

- a. The only possibility is $A = A' \otimes B'$ and $f = \otimes R(f_1, f_2)$ where $f_1 : - \mid \Gamma_1 \vdash A'$, $f_2 : - \mid \Gamma_2 \vdash B'$, then depending on $A' \otimes B'$ in Δ_1 or Δ_2 , we let $\text{ccut}(f, (\multimap L(g, h))) =_{\text{def}} \multimap L(\text{ccut}(f, g), h)$ or $\multimap L(g, (\text{ccut}(f, h)))$, respectively.
2. $\text{ccut}(f, (\otimes R(g, h)))$ where $f : - \mid \Gamma \vdash A$, $g : S \mid \Delta_1 \vdash A'$, $h : - \mid \Delta_2 \vdash B'$ is similar as above.
3. $\text{ccut}(f, (\otimes Lg))$ and $\text{ccut}(f, (\multimap Rh))$ cases are similar, we permute ccut up, where $g : A' \mid B', \Delta_0, A, \Delta_1 \vdash C$, $h : S \mid \Delta_0, A, \Delta_1, A' \vdash B'$.

4 Focusing

Before getting into focused sequent calculus system, we need to define new proof equivalences in original sequent calculus system. In [7] and [5], Uustalu et al. provided proof equivalences between \otimes only and \multimap only systems respectively. Again, similar as cut elimination proof above, we have to give \otimes and \multimap interaction cases here:

For $f : - \mid \Gamma, A \vdash C$, IL :

$$\frac{\frac{f}{- \mid \Gamma, A \vdash C} \multimap R}{\frac{- \mid \Gamma \vdash A \multimap C}{\text{IL}}} \multimap R \stackrel{=}{=} \frac{\frac{f}{- \mid \Gamma, A \vdash C} \text{IL}}{\frac{\text{I} \mid \Gamma, A \vdash C}{\text{IL}}} \multimap R$$

For $f : A \mid B, \Gamma, C \vdash D$:

$$\frac{\frac{f}{A \mid B, \Gamma, C \vdash D} \multimap R}{\frac{A \mid B, \Gamma \vdash C \multimap D}{A \otimes B \mid \Gamma \vdash C \multimap D} \otimes L} \multimap R \stackrel{=}{=} \frac{\frac{f}{A \mid B, \Gamma, C \vdash D} \otimes L}{\frac{A \otimes B \mid \Gamma, C \vdash D}{A \otimes B \mid \Gamma \vdash C \multimap D} \multimap R} \otimes L$$

For $f : - \mid \Gamma \vdash A$, $g : B \mid \Delta \vdash C$, and $h : - \mid \Delta' \vdash D$:

$$\frac{\frac{f}{- \mid \Gamma \vdash A} \quad \frac{\frac{g}{B \mid \Delta \vdash C} \quad \frac{h}{- \mid \Delta' \vdash D}}{B \mid \Delta, \Delta' \vdash C \otimes D} \otimes R}{\frac{A \multimap B \mid \Gamma, \Delta, \Delta' \vdash C \otimes D}{\multimap L}} \multimap L \stackrel{=}{=} \frac{\frac{f}{- \mid \Gamma \vdash A} \quad \frac{g}{B \mid \Delta \vdash C}}{\frac{A \multimap B \mid \Gamma, \Delta \vdash C}{\multimap L}} \multimap L \quad \frac{h}{- \mid \Delta' \vdash D} \otimes R$$

The focused sequent calculus here has four phases with special annotations on \vdash . We explain the intuition of this focused system from bottom-up proof search perspective. Every proof in focused calculus is a tag-free sequent ended at phase RI .

$$\frac{S \mid \Gamma, A \vdash_{\text{RI}}^x B}{S \mid \Gamma \vdash_{\text{RI}}^x A \multimap B} \multimap R \quad \frac{S \mid \Gamma \vdash_{\text{LI}}^x P}{S \mid \Gamma \vdash_{\text{RI}}^x P} \text{LI2RI}$$

$$\begin{array}{c}
\frac{- \mid \Gamma \vdash_{\text{LI}}^x P}{\text{I} \mid \Gamma \vdash_{\text{LI}}^x P} \text{IL} \quad \frac{A \mid B, \Gamma \vdash_{\text{LI}}^x P}{A \otimes B \mid \Gamma \vdash_{\text{LI}}^x P} \otimes \text{L} \quad \frac{T \mid \Gamma \vdash_{\text{P}}^x P}{T \mid \Gamma \vdash_{\text{LI}}^x P} \text{P2LI} \\
\\
\frac{A \mid \Gamma \vdash_{\text{LI}}^x P}{- \mid A, \Gamma \vdash_{\text{P}}^x P} \text{pass} \quad \frac{T \mid \Gamma \vdash_{\text{F}}^x P}{T \mid \Gamma \vdash_{\text{P}}^x P} \text{F2P} \\
\\
\frac{T \mid \Gamma^\bullet \vdash_{\text{RI}}^\bullet A \quad - \mid \Delta \vdash_{\text{RI}} B}{T \mid \Gamma, \Delta \vdash_{\text{F}}^x A \otimes B} \otimes \text{R} \\
\\
\frac{- \mid \Gamma^\circ \vdash_{\text{RI}} A \quad B \mid \Delta \vdash_{\text{LI}} P \quad \text{if } x = \bullet, \text{ then } \exists F^\circ \in \Gamma}{A \multimap B \mid \Gamma, \Delta \vdash_{\text{F}}^x P} \multimap \text{L} \\
\\
\frac{}{X \mid \vdash_{\text{F}} X} \text{ax} \quad \frac{}{- \mid \vdash_{\text{F}} \text{I}} \text{IR}
\end{array}$$

- In RI, only right invertible rules can be applied, in particular, $\multimap \text{R}$. Therefore, we decompose C until it becomes a positive P where $P \neq A \multimap B$, then move to phase LI.
- Next, we destruct S by left invertible rules $\otimes \text{L}$ and IL until we obtain an irreducible stoup S' where S' is not in $A' \otimes B'$.
- Then we move to passivation phase P, where only **pass** can be applied to sequents. Notice that for each decomposition of **pass**, we go back to phase LI to do left invertible rules again because the passivated formula A could be $A' \otimes B'$ or I .
- In phase F, we have four rules, $\otimes \text{R}$, $\multimap \text{L}$, **ax**, and **IR**. Here we have special restrictions on $\otimes \text{L}$ and $\multimap \text{L}$. In general, we prefer to decompose \multimap in stoup prior then \otimes in conclusion. However, if there are some formula are packed into conclusion, then whole bottom-up proof strategy fails. We will see comparasion between involed and naive focused system later. For rule $\otimes \text{R}$, we let its left and right premises back to phase RI because A and B could be negative formulas. Especially, the left premise becomes a tagged sequent and formulae in Γ become tagged formulae. Notice that $\otimes \text{R}$ is the only rule produce tags in its premise, so it let us specify the origin of formulae in context easily. In a $\multimap \text{L}$ application, if conclusion sequent is tagged, then at least exists a formula F is white in the context of left premise. Remembering that we prefer $\multimap \text{L}$ prior to $\otimes \text{R}$, so if there is any $\otimes \text{R}$ before $\multimap \text{L}$, then we have to ensure that we cannot do $\multimap \text{L}$ first. The white tag is a guardian for that because context in left premise of $\otimes \text{R}$ application are tagged. Therefore, any white formula F in left presmie of $\multimap \text{L}$ applicaton means that it is impossible to obtain same sequeunt by decomposing \multimap in stoup first. In conclusion, under this tag setting, we can eliminate redundant non-determinism in our focused system.

$$\begin{array}{c}
\frac{\frac{\frac{\frac{}{ax}}{Z \mid \vdash_4 Z}}{Z \mid \vdash_F Z}}{Z \mid \vdash_2 Z}}{- \mid Z \vdash_2 Z} \text{pass} \\
\frac{\frac{Y \mid X \vdash_{RI} Y}{Y \mid \vdash_{RI} X \multimap Y} \multimap R}{- \mid Z \vdash_{RI} Z} \otimes R \\
\frac{\frac{\frac{Y \mid Z \vdash_4 (X \multimap Y) \otimes Z}{Y \mid Z \vdash_F (X \multimap Y) \otimes Z} \text{4to3}}{Y \mid Z \vdash_2 (X \multimap Y) \otimes Z} \text{3to2}}{- \mid \vdash_{RI} X} \multimap L \\
\frac{X \multimap Y \mid Z \vdash_F (X \multimap Y) \otimes Z}{X \multimap Y \mid Z \vdash_2 (X \multimap Y) \otimes Z} \text{3to2} \\
\frac{X \multimap Y \mid Z \vdash_2 (X \multimap Y) \otimes Z}{X \multimap Y \mid Z \vdash_{RI} (X \multimap Y) \otimes Z} \text{2to1}
\end{array}$$

This counter example occurs when we insist to decompose \multimap in stoup prior than \otimes in succedent. Therefore we induce $\multimap \text{L}^\otimes$ rule to solve this situation:

$\frac{\frac{X \mid \vdash_4 X}{X \mid \vdash_F X} \text{ 4to3} \quad \frac{X \mid \vdash_2 X}{- \mid X \vdash_2 X} \text{ 3to2} \quad \frac{- \mid X \vdash_2 X}{- \mid X \vdash_{RI} X} \text{ 2to1}$	$\frac{\frac{Y \mid \vdash_4 Y}{Y \mid \vdash_F Y} \text{ 4to3} \quad \frac{Y \mid \vdash_2 Y}{Y \mid \vdash_{RI} Y} \text{ 3to2} \quad \frac{Y \mid Z \vdash_4 Y \otimes Z}{Y \mid Z \vdash_F Y \otimes Z} \text{ 2to1} \quad \frac{Y \mid Z \vdash_2 Y \otimes Z}{- \mid L^\otimes}$	$\frac{\frac{Z \mid \vdash_4 Z}{Z \mid \vdash_F Z} \text{ 4to3} \quad \frac{Z \mid \vdash_2 Z}{- \mid Z \vdash_2 Z} \text{ 3to2} \quad \frac{- \mid Z \vdash_2 Z}{- \mid Z \vdash_{RI} Z} \text{ 2to1} \quad \frac{- \mid Z \vdash_{RI} Z}{\otimes R}$
$\frac{\frac{X \multimap Y \mid Z \vdash_F (X \multimap Y) \otimes Z}{X \multimap Y \mid Z \vdash_2 (X \multimap Y) \otimes Z} \text{ 3to2} \quad \frac{X \multimap Y \mid Z \vdash_2 (X \multimap Y) \otimes Z}{X \multimap Y \mid Z \vdash_{RI} (X \multimap Y) \otimes Z} \text{ 2to1}$		

We are going to show that this focused sequent calculus is equivalent with the original system which means that:

Theorem 4.0.1 1. For any derivation $f : S \mid \Gamma \vdash_{\text{RI}} C$, there is a derivation $\text{emb } f : S \mid \Gamma \vdash C$.
2. For any derivation $f : S \mid \Gamma \vdash C$, there is a derivation $\text{focus } f : S \mid \Gamma \vdash_{\text{RI}} C$.

Proof : 1. First theorem is straightforward, we just erase labels in a focused derivation then it would become a derivation in original sequent calculus system.

2. By induction on f .

5 Soundness and completeness

Theorem 5.0.1 For any derivation $f : S \mid \Gamma \vdash C$, there is a derivation sound $f : \llbracket S \mid \Gamma \rrbracket_a \Longrightarrow C$.

Proof : We prove soundness by induction on f . Our interpretation $\llbracket - \rrbracket$ is same as the definition in [7] so we only have to deal with two \multimap cases. However, when proving soundness of \multimap L rule, we need

to use two admissible adjoint rules repeatedly. Therefore we need to overload $\llbracket - \rrbracket$ then make it corresponds to the interpretation in [5]. Then we can obtain two repeated application version of adjoint rules Gadj_1 and Gadj_2 (we use $\llbracket - \rrbracket_a$ and $\llbracket - \rrbracket_s$ to indicate the interpretation): $\frac{\llbracket S \mid \Gamma \rrbracket_a \Rightarrow A}{S \Rightarrow \llbracket \Gamma \mid C \rrbracket_s} \text{Gadj}_1$

and $\frac{A \Rightarrow \llbracket \Gamma \mid C \rrbracket_s}{\llbracket A \mid \Gamma \rrbracket_a \Rightarrow C} \text{Gadj}_2$

We can obtain the soundness proof of $\multimap \text{L}$ immediately. For any $f : - \mid \Gamma \vdash A$ and $g : B \mid \Delta \vdash C$:

$$\frac{\frac{\text{Lemma 4.3 in [7]}}{\llbracket A \multimap B \mid \Gamma \rrbracket_a \Rightarrow A \multimap B \otimes \llbracket - \mid \Gamma \rrbracket_a} \quad \frac{\frac{\frac{\text{sound } f}{\llbracket - \mid \Gamma \rrbracket_a \Rightarrow A} \quad \frac{\frac{\text{sound } g}{\llbracket B \mid \Delta \rrbracket_a \Rightarrow C} \text{Gadj}_1}{B \Rightarrow \llbracket \Delta \mid C \rrbracket_s} \multimap}{A \multimap B \Rightarrow \llbracket - \mid \Gamma \rrbracket_a \multimap \llbracket \Delta \mid C \rrbracket_s} \text{adj}_2}{\llbracket A \multimap B \otimes \llbracket - \mid \Gamma \rrbracket_a \rrbracket_s \Rightarrow \llbracket \Delta \mid C \rrbracket_s} \text{comp}}{\llbracket A \multimap B \mid \Gamma, \Delta \rrbracket_a \Rightarrow C} \text{Gadj}_2$$

Theorem 5.0.2 (Weak completeness) For any derivation $f : A \Rightarrow C$, there is a derivation $\text{complt } f : A \mid \vdash C$.

Proof : In previous studies from Uustalu et al. ([7] and [5]), main cases of completeness are proved. Therefore, we only have to show new cases in skew monoidal closed categories. In particular, new cases are unit, counit, and two adjoint rules.

- (Counit)Case $f = \epsilon : (A \multimap B) \otimes A \Rightarrow B$. We define:

$$\text{complt} \left(\frac{}{(A \multimap B) \otimes A \Rightarrow B} \epsilon \right) =_{\text{df}} \frac{\frac{\frac{A \mid \vdash A}{- \mid A \vdash A} \text{ax} \quad \frac{B \mid \vdash B}{- \mid B \vdash B} \text{ax}}{A \multimap B \mid A \vdash B} \text{pass}}{(A \multimap B) \otimes A \mid \vdash B} \otimes \text{L}$$

- (Unit)Case $f = \eta : A \Rightarrow B \multimap (A \otimes B)$. We define:

$$\text{complt} \left(\frac{}{A \Rightarrow B \multimap (A \otimes B)} \eta \right) =_{\text{df}} \frac{\frac{\frac{A \mid \vdash A}{- \mid A \vdash A} \text{ax} \quad \frac{\frac{B \mid \vdash B}{- \mid B \vdash B} \text{ax}}{- \mid B \vdash B} \text{pass}}{A \mid B \vdash A \otimes B} \otimes \text{R}}{A \mid \vdash B \multimap (A \otimes B)} \multimap \text{R}$$

Two adjoint rules need invertibility of $\otimes \text{L}$ and $\multimap \text{R}$ which are proved respectively in [7] and [5].

- Case $f = \text{adj}_1 g$ where $g : A \otimes B \Rightarrow C$. We define:

$$\text{complt} \left(\frac{\vdots g}{\frac{A \otimes B \Rightarrow C}{A \Rightarrow B \multimap C} \text{adj}_1} \right) =_{\text{df}} \frac{\frac{\vdots \text{complt } g}{A \otimes B \mid \vdash C} \otimes \text{L}^{-1}}{A \mid \vdash B \multimap C} \multimap \text{R}$$

- Case $f = \text{adj}_2 g$ where $g : A \Longrightarrow B \multimap C$. We define:

$$\text{complt} \left(\frac{\vdots g}{\frac{A \Longrightarrow B \multimap C}{A \otimes B \Longrightarrow C} \text{adj}_2} \right) =_{\text{df}} \frac{\frac{\vdots \text{complt } g}{A \mid \vdash B \multimap C} \multimap R^{-1}}{\frac{A \mid B \vdash C}{A \otimes B \mid \vdash C} \otimes L}$$

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