HW6 报告

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1 报告要求

1.1 要求 1 给出计算过程证明

因为只需要输出各项系数乘以 2^{n-1} 后的结果，则不妨直接对整个式子乘以 2^{n-1}，则有式1：

$$\begin{aligned} 2^{n-1}x^n &= 2^{n-1}c_nT_n(x) + 2^{n-1}c_{n-2}T_{n-2} + 2^{n-1}c_{n-4}T_{n-4} + \cdots \\ &= a_{n,n}T_n(x) + a_{n,n-2}T_{n-2}(x) + a_{n,n-4}T_{n-4}(x) + \cdots \end{aligned} \tag{1}$$

同时已知 Chebyshev 多项式的递推公式：

$$2x \cdot T_0(x) = 2 \cdot T_1(x) \tag{2a}$$

$$2x \cdot T_n(x) = T_{n+1}(x) + T_{n-1}(x) \quad (n \geq 1) \tag{2b}$$

两边同时乘以 2x，并采用递推公式化简得：

$$\begin{aligned} 2^n x^{n+1} &= 2x \cdot 2^{n-1}x^n \\ &= 2x \cdot (2^{n-1}c_nT_n(x) + 2^{n-1}c_{n-2}T_{n-2} + 2^{n-1}c_{n-4}T_{n-4} + \cdots) \\ &= 2^{n-1}c_n(T_{n+1}(x) + T_{n-1}(x)) + 2^{n-1}c_{n-2}(T_{n-1}(x) + T_{n-3}(x)) + \cdots \\ &= 2^{n-1}c_nT_{n+1}(x) + 2^{n-1}(c_n + c_{n-2})T_{n-1}(x) + 2^{n-1}(c_{n-2} + c_{n-4})T_{n-3}(x) + \cdots \\ &= a_{n,n}T_{n+1}(x) + (a_{n,n} + a_{n,n-2})T_{n-1}(x) + (a_{n,n-2} + a_{n,n-4})T_{n-3}(x) + \cdots \\ &= a_{n+1,n+1}T_{n+1}(x) + a_{n+1,n-1}T_{n-1}(x) + a_{n+1,n-3}T_{n-3}(x) + \cdots \end{aligned}$$

故综上分析，系数有初始值 $a_{1,0} = 0$，$a_{1,1} = 1$，且有类金字塔形的递推形式：

$$\begin{aligned} a_{n+1,n+1} &= a_{n,n} \\ a_{n+1,k} &= a_{n,k+1} + a_{n,k-1} \quad (2 \leq k \leq n) \\ a_{n+1,1} &= a_{n,2} + 2a_{n,0} \\ a_{n+1,0} &= a_{n,1} \end{aligned} \tag{3}$$

则系数的计算过程可使用如下图1的递推方法得到

```
int ans[n + 1][n + 1];
for (int i = 0; i <= n; i++){
    for (int j = 0; j <= n; j++){
        ans[i][j] = 0;
    }
}
ans[1][1] = 1;

for (int i = 2; i <= n; i++){
    ans[i][0] = ans[i - 1][1];
    ans[i][1] = ans[i - 1][2] + ans[i - 1][0] * 2;
    for (int j = 2; j < i; j++){
        ans[i][j] = ans[i - 1][j + 1] + ans[i - 1][j - 1];
    }
    ans[i][i] = ans[i - 1][i - 1];
}
```

Figure 1: 计算过程

1.2 要求 2 给出 n=6,7,11,12,15 的结果

根据如上算法，在计算 n=15 的过程中可一次性得到所有结果：

n = 2: 1 1

n = 3: 1 3

n = 4: 1 4 3

n = 5: 1 5 10

n = 6: 1 6 15 10

n = 7: 1 7 21 35

n = 8: 1 8 28 56 35

n = 9: 1 9 36 84 126

n = 10: 1 10 45 120 210 126

n = 11: 1 11 55 165 330 462

n = 12: 1 12 66 220 495 792 462

n = 13: 1 13 78 286 715 1287 1716

n = 14: 1 14 91 364 1001 2002 3003 1716

n = 15: 1 15 105 455 1365 3003 5005 6435