苏王捷 2300011075

8.1 ① $x_{n+1} = 1 + 1/x_n^2 \Rightarrow \varphi(x) = 1 + 1/x^2$，$\varphi'(x) = -\frac{2}{x^3}$

在 $x \in [1.4, 1.55]$ 中，$1.4 \le \varphi(x) \le 1.55$，$|\varphi'(x)| \le 0.73 < 1$

则在 $x_0 = 1.5$ 附近迭代格式收敛

② $x_{n+1} = \sqrt{1/(x_n - 1)} \Rightarrow \varphi(x) = \sqrt{1/(x-1)}$，$\varphi'(x) = -\frac{1}{2(\sqrt{x-1})^3}$

有 $|\varphi'(x_0)| = \sqrt{2} > 1 \Rightarrow$ 不满足充分条件

则在 $x_0 = 1.5$ 附近迭代格式不收敛

③ $x_{n+1} = \sqrt[3]{1 + x_n^2} \Rightarrow \varphi(x) = \sqrt[3]{1 + x^2}$，$\varphi'(x) = \frac{1}{3}(1+x^2)^{-\frac{2}{3}} 2x = \frac{2x}{3}(1+x^2)^{-\frac{2}{3}}$

在 $x \in [1.4, 1.6]$ 中，$1.4 \le \varphi(x) \le 1.6$，$|\varphi'(x)| \le 0.45 < 1$

则在 $x_0 = 1.5$ 附近迭代格式收敛

8.2 $f_1 = x^2 + y^2 - 5$，$f_2 = (x+1)y - 3x - 1$

$$\left[\frac{\partial f_i}{\partial x_j}\right] = \begin{bmatrix} 2x & 2y \\ y-3 & x+1 \end{bmatrix} \qquad (x_0, y_0) = (1, 1)$$

$$\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \Delta x_0 \\ \Delta y_0 \end{bmatrix} = \begin{bmatrix} -(1+1-5) \\ -(2-3-1) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} \Delta x_0 \\ \Delta y_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{5}{4} \end{bmatrix}$$

$\Rightarrow (x_1, y_1) = (\frac{5}{4}, \frac{9}{4})$

$$\begin{bmatrix} \frac{5}{2} & \frac{9}{2} \\ -\frac{3}{4} & \frac{9}{4} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \end{bmatrix} = \begin{bmatrix} -(\frac{25}{16} + \frac{81}{16} - 5) \\ -(\frac{9}{4} \times \frac{9}{4} - \frac{15}{4} - 1) \end{bmatrix} = \begin{bmatrix} -\frac{13}{8} \\ -\frac{5}{16} \end{bmatrix} \Rightarrow \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ -\frac{2}{9} \end{bmatrix}$$

$\Rightarrow (x_2, y_2) = (1, \frac{73}{36})$

8.3 对收敛点 $x^*$ 有 $f(x^*) = 0$，$g(x^*) = x^* - \frac{f(x^*)}{f'(x^*)} - \frac{f''(x^*)}{2f'(x^*)}\left[\frac{f(x^*)}{f'(x^*)}\right]^2 = x^*$

$$g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} - \frac{f'''(x)f'(x) - (f''(x))^2}{2(f'(x))^2}\left[\frac{f(x)}{f'(x)}\right]^2 - \frac{f''(x)}{f'(x)} \frac{f(x)}{f(x)} \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}$$

$$= \frac{f^2(x)[3f''^2(x) - f'(x)f'''(x)]}{2(f'(x))^4}$$ 有 $g'(x^*) = 0$

$$g''(x^*) = f^2(x^*)\left[\frac{3f''^2(x) - f'(x)f'''(x)}{2[f'(x)]^4}\right]' + 2f(x^*)f'(x^*)\frac{3f''^2(x) - f'(x)f'''(x)}{2[f'(x)]^4} = 0$$

$$g'''(x^*) = f^2(x^*)\left[\frac{3f''^2(x) - f'(x)f'''(x)}{2[f'(x)]^4}\right]'' + 4f(x^*)f'(x^*)\left(\frac{3f''^2(x) - f'(x)f'''(x)}{2[f'(x)]^4}\right)' +$$

$$2[f'(x^*)]^2 \frac{3f''^2(x) - f'(x)f'''(x)}{2[f'(x)]^4} = \frac{3[f''(x^*)]^2 - f'(x^*)f'''(x^*)}{[f'(x^*)]^2} \ne 0$$

则 $p = 3 \Rightarrow$ 该方法三阶收敛