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3.1 $A=\begin{bmatrix}10 & -2 & -1\\ -2 & 10 & -1\\ -1 & -2 & 5\end{bmatrix} \Rightarrow \begin{cases} x_1 = 0.2x_2+0.1x_3+0.3\\ x_2=0.2x_1+0.1x_3+1.5\\ x_3=0.2x_1+0.4x_2+2\end{cases}$

① Jacobi 法 $\Rightarrow \begin{cases} x_1^{(k+1)} = 0.2x_2^{(k)}+0.1x_3^{(k)}+0.3\\ x_2^{(k+1)}=0.2x_1^{(k)}+0.1x_3^{(k)}+1.5\\ x_3^{(k+1)}=0.2x_1^{(k)}+0.4x_2^{(k)}+2\end{cases}$

$x^{(1)}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0.3\\1.5\\2\end{pmatrix}$, $x^{(2)}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0.2\times1.5+0.1\times2+0.3\\0.2\times0.3+0.1\times2+1.5\\0.2\times0.3+0.4\times1.5+2\end{pmatrix}=\begin{pmatrix}0.8\\1.76\\2.66\end{pmatrix}$

② Gauss-Seidal 法 $\Rightarrow \begin{cases} x_1^{(k+1)} = 0.2x_2^{(k)}+0.1x_3^{(k)}+0.3\\ x_2^{(k+1)}=0.2x_1^{(k+1)}+0.1x_3^{(k)}+1.5\\ x_3^{(k+1)}=0.2x_1^{(k+1)}+0.4x_2^{(k+1)}+2\end{cases}$

$x^{(1)}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0.3\\0.2\times0.3+1.5\\0.2\times0.3+0.4\times1.56+2\end{pmatrix}=\begin{pmatrix}0.3\\1.56\\2.684\end{pmatrix}$, $x^{(2)}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0.8804\\1.94448\\2.953872\end{pmatrix}$

3.2 $\Rightarrow \begin{cases} x_1 = -\dfrac{a_{12}}{a_{11}}x_2+\dfrac{b_1}{a_{11}}\\ x_2=-\dfrac{a_{21}}{a_{22}}x_1+\dfrac{b_2}{a_{22}}\end{cases}$ 则 $M=\begin{bmatrix}0 & -\dfrac{a_{12}}{a_{11}}\\ -\dfrac{a_{21}}{a_{22}} & 0\end{bmatrix}$

$|M-\lambda I|=0 \Leftrightarrow \lambda^2=\dfrac{a_{12}a_{21}}{a_{11}a_{22}}$ 由定理知 Jacobi 法收敛的条件是

$\rho(M)<1 \Rightarrow$ 即 $\lambda^2<1 \Rightarrow \left|\dfrac{a_{12}a_{21}}{a_{11}a_{22}}\right|<1$ 证毕

3.3 $A=\begin{bmatrix}4 & 3 & 0\\ 3 & 4 & -1\\ 0 & -1 & 4\end{bmatrix}$

① 最速下降法 $r^{(0)}=b-Ax^{(0)}=\begin{bmatrix}24\\30\\-24\end{bmatrix}\Rightarrow p^{(0)}=r^{(0)}=\begin{bmatrix}24\\30\\-24\end{bmatrix}$

$\alpha^{(0)}=\dfrac{\langle r^{(0)},r^{(0)}\rangle}{B(r^{(0)},r^{(0)})}=\dfrac{2052}{13968}=0.1469$, $x^{(1)}=x^{(0)}+\alpha^{(0)}p^{(0)}=\begin{bmatrix}3.5256\\4.407\\-3.5256\end{bmatrix}$

$r^{(1)}=b-Ax^{(1)}=\begin{bmatrix}-3.325\\-1.732\\-5.489\end{bmatrix}=p^{(1)}$, $\alpha^{(1)}=\dfrac{\langle r^{(1)},r^{(1)}\rangle}{B(r^{(1)},r^{(1)})}=0.2298$

$x^{(2)}=x^{(1)}+\alpha^{(1)}p^{(1)}=\begin{bmatrix}2.762\\4.009\\-4.787\end{bmatrix}$, $r^{(2)}=b-Ax^{(2)}=\begin{bmatrix}0.925\\0.891\\-0.841\end{bmatrix}=p^{(2)}$

$\alpha^{(2)}=\dfrac{\langle r^{(2)},r^{(2)}\rangle}{B(r^{(2)},r^{(2)})}=0.1485$, $x^{(3)}=x^{(2)}+\alpha^{(2)}p^{(2)}=\begin{bmatrix}2.899\\4.141\\-4.912\end{bmatrix}$

② 共轭梯度法 $r^{(0)}=d^{(0)}=\begin{bmatrix}24\\30\\-24\end{bmatrix}$, $\alpha^{(0)}=\dfrac{\langle r^{(0)},d^{(0)}\rangle}{B(d^{(0)},d^{(0)})}=0.1469$

$x^{(1)}=x^{(0)}+\alpha^{(0)}d^{(0)}=\begin{bmatrix}3.5256\\4.407\\-3.5256\end{bmatrix}$, $r^{(1)}=b-Ax^{(1)}=\begin{bmatrix}-3.325\\-1.732\\-5.489\end{bmatrix}$

$\beta^{(0)}=-\dfrac{B(d^{(0)},r^{(1)})}{B(d^{(0)},d^{(0)})}=0.0215$, $d^{(1)}=r^{(1)}+\beta^{(0)}d^{(0)}=\begin{bmatrix}-2.808\\-1.086\\-6.006\end{bmatrix}$

$$\alpha^{(1)}=\frac{\langle r^{(1)},d^{(1)}\rangle}{B(d^{(1)},d^{(1)})}=0.2378,\quad x^{(2)}=x^{(1)}+\alpha^{(1)}d^{(1)}=\begin{bmatrix}-2.858\\4.149\\-4.954\end{bmatrix}$$

③ 与精确解 $x^*=\begin{bmatrix}3\\4\\-5\end{bmatrix}$ 对比

最速下降法 残差 $r=\begin{bmatrix}-0.101\\0.141\\0.088\end{bmatrix}$

共轭梯度法 残差 $r=\begin{bmatrix}-0.142\\0.149\\0.046\end{bmatrix}$

3.4 $B=D^{-1}(D-A)=D^{-1}(L+U)=D^{-1}L+D^{-1}U$

则 $\tilde{x}^{(k)}=D^{-1}L\tilde{x}^{(k)}+D^{-1}Ux^{(k)}+D^{-1}b$ (式 1)

$x^{(k+1)}=D^{-1}L\tilde{x}^{(k)}+D^{-1}Ux^{(k+1)}+D^{-1}b$ (式 2)

$\Rightarrow D\tilde{x}^{(k)}=L\tilde{x}^{(k)}+Ux^{(k)}+b \Rightarrow (D-L)\tilde{x}^{(k)}=Ux^{(k)}+b$

$\Rightarrow \tilde{x}^{(k)}=(D-L)^{-1}Ux^{(k)}+(D-L)^{-1}b$

$\Rightarrow Dx^{(k+1)}=L\tilde{x}^{(k)}+Ux^{(k+1)}+b \Rightarrow (D-U)x^{(k+1)}=L\tilde{x}^{(k)}+b$

$\Rightarrow x^{(k+1)}=(D-U)^{-1}L\tilde{x}^{(k)}+(D-U)^{-1}b$

$=(D-U)^{-1}L(D-L)^{-1}Ux^{(k)}+(D-U)^{-1}[L(D-L)^{-1}+I]b$

$=Mx^{(k)}+f$

则 $M=(D-U)^{-1}L(D-L)^{-1}U$

$f=(D-U)^{-1}[L(D-L)^{-1}+I]b=(D-U)^{-1}[L(D-L)^{-1}+(D-L)(D-L)^{-1}]b$

$=(D-U)^{-1}(L+D-L)(D-L)^{-1}b=(D-U)^{-1}D(D-L)^{-1}b$

$=(D-U)^{-1}D(D-L)^{-1}Dg$

证毕