苏王捷 23000 11075

2.1 考虑增广矩阵 $\begin{bmatrix} 0 & 2 & 1 & 5 \\ 1 & 1 & 0 & 3 \\ 2 & 3 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 2 & 0 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & 1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 2 & 0 \\ 0 & -\frac{1}{2} & -1 & 3 \\ 0 & 2 & 1 & 5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & 3 & 2 & 0 \\ 0 & 2 & 1 & 5 \\ 0 & -\frac{1}{2} & -1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 2 & 0 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & -\frac{3}{4} & \frac{17}{4} \end{bmatrix} \Rightarrow x_3 = -\frac{17}{3}, \quad x_2 = \frac{16}{3}, \quad x_1 = -\frac{7}{3}$

2.2 由对称正定矩阵得 $\forall x \in \mathbb{R}^n$ 且 $x \neq 0$ 有 $x^TAx > 0 \Rightarrow \|x\|_A = \sqrt{x^TAx} > 0$

且当 $x=0$ 时 $\Leftrightarrow x^TAx = 0 \Leftrightarrow \|x\|_A = 0 \Rightarrow$ 满足正定性

$\forall \alpha \in \mathbb{R}$ 和 $x \in \mathbb{R}^n$, $\|\alpha x\|_A = \sqrt{(\alpha x)^T A (\alpha x)} = \sqrt{\alpha^2 x^TAx} = |\alpha| \|x\|_A \Rightarrow$ 满足齐次性

$\forall x, y \in \mathbb{R}^n$ 令 $A = LL^T \Rightarrow \|x\|_A = \|L^Tx\|$ 则

$$\|x+y\|_A = \|L^T(x+y)\| = \|L^Tx + L^Ty\| \le \|L^Tx\| + \|L^Ty\| = \|x\|_A + \|y\|_A$$

$\Rightarrow$ 满足三角不等式

综上 $\|\cdot\|_A$ 是一个向量范数

2.3 易知 $\|B\|_\infty = \max\limits_{1 \le i \le n} \{1+n-i\} = n$ 令 $D = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & \cdots & & 0 \end{bmatrix}_{n \times n}$ 则有 $D^2 = \begin{bmatrix} 0 & 0 & 1 & \\ & \ddots & \ddots & \ddots \\ & & \ddots & 0 \\ & & & 0 \end{bmatrix}_{n \times n}$

$\Rightarrow D^k = 0 \ (k \ge n)$ 则 $B = I - D - D^2 - \cdots - D^{n-1}$

$\Rightarrow B^{-1} = I + D + 2D^2 + 4D^3 + \cdots + 2^{n-2}D^{n-1}$

则 $\|B^{-1}\|_\infty = 1+1+2+4+\cdots+2^{n-2} = 2^{n-1}$

$\Rightarrow \mathrm{Cond}(B)_\infty = \|B\|_\infty \|B^{-1}\|_\infty = n \cdot 2^{n-1}$ 证毕

2.4 1. 已知 $n=0$ 时 $H_0 = H_0^T$, $H_0H_0^T = [1] = 2^0 I_{2^0}$ 满足条件

设当 $n=k$ 时 $H_k = H_k^T$, $H_kH_k^T = 2^k I_{2^k}$

当 $n=k+1$ 时 $H_{k+1} = \begin{bmatrix} H_k & H_k \\ H_k & -H_k \end{bmatrix} \Rightarrow H_{k+1}^T = \begin{bmatrix} H_k^T & H_k^T \\ H_k^T & -H_k^T \end{bmatrix} = \begin{bmatrix} H_k & H_k \\ H_k & -H_k \end{bmatrix} = H_{k+1}$ 成立

$$H_{k+1}H_{k+1}^T = H_{k+1}^2 = \begin{bmatrix} 2H_k^2 & 0 \\ 0 & 2H_k^2 \end{bmatrix} = \begin{bmatrix} 2^{k+1}I_{2^k} & 0 \\ 0 & 2^{k+1}I_{2^k} \end{bmatrix}$$

$$= 2^{k+1}\begin{bmatrix} I_{2^k} & 0 \\ 0 & I_{2^k} \end{bmatrix} = 2^{k+1} I_{2^k+2^k} = 2^{k+1} I_{2^{k+1}}$$ 成立

综上 有 $H_n = H_n^T$, $H_nH_n^T = 2^n I_{2^n}$

2. $H_0 = L_0 D_0 L_0^T = [1] \Rightarrow L_0 = [1], D_0 = [1]$

$$H_n = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix} = \begin{bmatrix} L_{n-1}D_{n-1}L_{n-1}^T & L_{n-1}D_{n-1}L_{n-1}^T \\ L_{n-1}D_{n-1}L_{n-1}^T & -L_{n-1}D_{n-1}L_{n-1}^T \end{bmatrix}$$

$$= \begin{bmatrix} L_{n-1} & 0 \\ L_{n-1} & L_{n-1} \end{bmatrix} \begin{bmatrix} D_{n-1} & 0 \\ 0 & -2D_{n-1} \end{bmatrix} \begin{bmatrix} L_{n-1}^T & L_{n-1}^T \\ 0 & L_{n-1}^T \end{bmatrix} = L_n D_n L_n^T$$

$$\therefore L_n = \begin{bmatrix} L_{n-1} & \\ L_{n-1} & L_{n-1} \end{bmatrix}, \quad D_n = \begin{bmatrix} D_{n-1} & 0 \\ 0 & -2D_{n-1} \end{bmatrix}, \quad D_0 = [1]$$