Direct Solver used in NHN2022

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August 29, 2022

1 Discretization of Equation (S3)

Equation (S3) in NH2018 reads

$$\frac{\partial}{\partial \mu}\left[\frac{1}{2\Omega\mu}\frac{\partial}{\partial \mu}(u_{\mathrm{REF}}\cos\phi)\right]+\frac{2\Omega a^2 H\mu}{R(1-\mu^2)}e^{z/H}\frac{\partial}{\partial z}\left[\frac{e^{(\kappa-1)z/H}}{\frac{\partial\tilde{\theta}}{\partial z}}\frac{\partial}{\partial z}u_{\mathrm{REF}}\cos\phi\right]=-a\frac{\partial}{\partial\mu}\left(\frac{q_{\mathrm{REF}}}{2\Omega\mu}\right) \tag{1}$$

where $\mu \equiv \sin\phi$. Changing the variable from μ to $\sin\phi$ and with the substitutions $\tilde{u} \equiv u_{\mathrm{REF}}\cos\phi$ and $\tilde{q} \equiv \frac{q_{\mathrm{REF}}}{\sin\phi}$, equation (1) becomes

$$\frac{\partial}{\partial\phi}\left[\frac{1}{\sin\phi\cos\phi}\frac{\partial\tilde{u}}{\partial\phi}\right]+\frac{4\Omega^2 a^2 H\sin\phi}{R\cos\phi}e^{z/H}\frac{\partial}{\partial z}\left[\frac{e^{(\kappa-1)z/H}}{\frac{\partial\tilde{\theta}}{\partial z}}\frac{\partial\tilde{u}}{\partial z}\right]=-a\frac{\partial\tilde{q}}{\partial\phi} \tag{2}$$

Discretizing each term in equation (2) on the uniform ϕ (index j) and z (index k) grids:

$$\begin{aligned}
\frac{\partial}{\partial\phi}\left[\frac{1}{\sin\phi\cos\phi}\frac{\partial\tilde{u}}{\partial\phi}\right] &\approx \frac{1}{(\Delta\phi)^2}\left[\frac{\tilde{u}_{j+1,k}-\tilde{u}_{j,k}}{\sin\phi_{j+1/2}\cos\phi_{j+1/2}}-\frac{\tilde{u}_{j,k}-\tilde{u}_{j-1,k}}{\sin\phi_{j-1/2}\cos\phi_{j-1/2}}\right],\\
\frac{\partial}{\partial z}\left[\frac{e^{(\kappa-1)z/H}}{\frac{\partial\tilde{\theta}}{\partial z}}\frac{\partial}{\partial z}\tilde{u}\right] &\approx \frac{1}{(\Delta z)^2}\left[\frac{e^{\frac{(\kappa-1)z_{k+1/2}}{H}}(\tilde{u}_{j,k+1}-\tilde{u}_{j,k})}{\frac{\partial\tilde{\theta}}{\partial z}_{k+1/2}}-\frac{e^{\frac{(\kappa-1)z_{k-1/2}}{H}}(\tilde{u}_{j,k}-\tilde{u}_{j,k-1})}{\frac{\partial\tilde{\theta}}{\partial z}_{k-1/2}}\right],\\
a\frac{\partial\tilde{q}}{\partial\phi} &\approx -\frac{a}{2\Delta\phi}\left(\tilde{q}_{j+1}-\tilde{q}_{j-1}\right)
\end{aligned}$$

would turn (2) to the following form:

$$a_{j,k}\tilde{u}_{j+1,k}+b_{j,k}\tilde{u}_{j-1,k}+c_{j,k}\tilde{u}_{j,k+1}+d_{j,k}\tilde{u}_{j,k-1}-e_{j,k}\tilde{u}_{j,k}=f_{j,k} \tag{3}$$

where

$$a_{j,k}=\frac{1}{\sin\phi_{j+1/2}\cos\phi_{j+1/2}} \tag{4}$$

$$b_{j,k}=\frac{1}{\sin\phi_{j-1/2}\cos\phi_{j-1/2}} \tag{5}$$

$$c_{j,k}=\frac{4\Omega^2 a^2 H\sin\phi_j}{R\cos\phi_j}e^{z_k/H}\frac{(\Delta\phi)^2}{(\Delta z)^2}\frac{e^{\frac{(\kappa-1)z_{k+1/2}}{H}}}{\frac{\partial\tilde{\theta}}{\partial z}_{k+1/2}} \tag{6}$$

$$d_{j,k}=\frac{4\Omega^2 a^2 H\sin\phi_j}{R\cos\phi_j}e^{z_k/H}\frac{(\Delta\phi)^2}{(\Delta z)^2}\frac{e^{\frac{(\kappa-1)z_{k-1/2}}{H}}}{\frac{\partial\tilde{\theta}}{\partial z}_{k-1/2}} \tag{7}$$

$$e_{j,k}=a_{j,k}+b_{j,k}+c_{j,k}+d_{j,k} \tag{8}$$

$$f_{j,k}=-\frac{a\Delta\phi}{2}(\tilde{q}_{j+1,k}-\tilde{q}_{j-1,k}). \tag{9}$$

2 Boundary conditions

The boundary conditions are listed in Equations (S14)-(S16) in the Supplementary materials of NHN22. Equation (S14) gives:

$$\tilde{u}_{j_{max},k} = 0, \tag{10}$$

$$\tilde{u}_{j,1} = 0. \tag{11}$$

Equation (S15) gives:

$$\tilde{u}_{j,k_{max}} = \tilde{u}_{j,k_{max-1}} - \frac{\Delta z R \cos\phi_j e^{-\kappa z_{max}/H}}{2\Omega a H \sin\phi_j} \frac{\theta_{j+1,k_{max}} - \theta_{j-1,k_{max}}}{2\Delta\phi}. \tag{12}$$

Equation (S16) gives:

$$\tilde{u}_{5,k} = (K - 2\pi\Omega a^2 \cos^2\phi_5)/2\pi a \tag{13}$$

where K is the Kelvin's circulation at 5°N equivalent latitude, which is evaluated as the surface integral of absolute vorticity over the domain where QGPV is greater than q_{REF} at 5°N.

3 Direct solver for poisson equation

Define $(j_{max} - 2)$-element vectors:

$$p_k = \begin{bmatrix} U_{2,k} \\ U_{3,k} \\ \dots \\ U_{j_{max}-1,k} \\ U_{j_{max}-1,k} \end{bmatrix}, \tag{14}$$

and

$$r_k = \begin{bmatrix} F_{2,k} - B_{2,k}U_{1,k} \\ F_{3,k} \\ \dots \\ F_{j_{max}-2,k} \\ F_{j_{max}-1,k} - A_{j_{max}-1,k}U_{j_{max}} \end{bmatrix}, \tag{15}$$

such that (3) can be written as

$$Q_k p_k + C_k p_{k+1} + D_k p_{k-1} = r_k \tag{16}$$

where

$$[Q_k]_{i,j} = \begin{cases} -E_{j+1,k} \text{ for } i = j \\ A_{j,k} \text{ for } i+1 = j \\ B_{j+2,k} \text{ for } i-1 = j \end{cases}, \text{ where } i,j \in [1, j_{max} - 2] \tag{17}$$

$$[C_k]_{i,j} = c_{j+1,k} \tag{18}$$

$$[D_k]_{i,j} = c_{j+1,k} \tag{19}$$

Let

$$p_{k+1} = S_k p_k + T_k. \tag{20}$$

Substitute (20) into (16) yields

$$\begin{aligned} &Q_k p_k + C_k(S_k p_k + T_k) + D_k p_{k-1} = r_k \\ &p_k = -(Q_k + C_k S_k)^{-1} D_k p_{k-1} + (Q_k + C_k S_k)^{-1}(r_k - C_k T_k) \end{aligned} \tag{21}$$

Comparing (21) with (20), we get

$$S_{k-1} = -(Q_k + C_k S_k)^{-1} D_k \tag{22}$$

and

$$T_{k-1} = (Q_k + C_k S_k)^{-1}(r_k - C_k T_k). \tag{23}$$

From upper boundary condition (12), we have

$$S_{k_{max}-1} = I, T_{k_{max}-1} = -t, \tag{24}$$

where

$$
\begin{aligned}
t_j &= \frac{\Delta z R \cos\phi_j e^{-\kappa z_{max}/H}}{2\Omega a H \sin\phi_j} \frac{\theta_{j+1,k_{max}} - \theta_{j-1,k_{max}}}{2\Delta\phi}, \\
t_1 &= t_{j_{max}} = 0.
\end{aligned} \tag{25}
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Using (22) and (23), one can determine all S_k and T_k from k_{max} down to $k = 1$. Finally, starting from the lower boundary condition (11),

$$p_1 = 0.$$

One can use (20) to determine all p_k.