A Sparse Signal Representation and Reconstruction Perspective for Direction-of-Arrival Estimation

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***Abstract*—Direction-of-Arrival (DoA) estimation is an important research branch in array signal processing, which aims to find and distinguish the incoming directions of multiple propagating electromagnetic waves. The input signals to the DoA estimation algorithms are generally collected by sensor or antenna array. In this project, we study how to cast the DoA estimation problem into a sparse signal representation and reconstruction problem, providing a completely different perspective to understand the source localization problem. A mixed $\ell_{2,1}$-norm optimization framework is proposed and we enforce sparsity by imposing penalty on the ℓ_1-norm of the source signal vector. A singular value decomposition (SVD) based approach is proposed to handle the complexity issues caused by a large number of snapshots. A spatial spectrum is derived through solving the mixed optimization problem, which can be solved efficiently by reformulating it into a standard second-order cone (SOC) optimization problem. We use the MATLAB convex optimization toolbox CVX to solve the SOCP problem and compare the performance of this new approach and the traditional Multiple Signal Classification (MUSIC) algorithm. The simulation results show that this new approach has much higher spectrum resolution than the MUSIC algorithm and is more effective in dealing with correlated sources.**

I. Introduction

Direction-of-Arrival estimation, or equivalently, source localization, has a wide range of application scenarios including effective beamforming in smart antennas, acoustic localization and target tracking systems [1]- [3]. Traditional DoA algorithms can be roughly classified into three general categories: 1) Non-parametric spatial spectrum based algorithms like CAPON [4], which develops a spatial spectrum by designing filter banks, 2) Subspace based algorithms like MUSIC [5] and 3) Maximum likelihood (ML) based algorithms like deterministic maximum likelihood (DML) method and stochastic maximum likelihood (SML) [1]. All these aforementioned DoA algorithms rely on the statistical properties of the source signals, and most of them performs well under the assumption that these source signals are independent of each other. However, the performance degrades when dealing with correlated sources. Another drawback is that sometimes extra information is required to implement the DoA estimation. For example, MUSIC algorithm requires the knowledge of the number of remote sources in order to work well.

Sparse signal representation and reconstruction based DoA estimation approach provides a brand new way of understanding the source localization problem and has many potential advantages [6]- [8]. For example, most of the sparse signal reconstruction based DoA estimation approaches do not require the knowledge the the actual number of remote sources, i.e., they can estimate both the number of the signal sources and the exact direction of each source. The most direct way to enforce sparsity is to minimize the ℓ_0-norm of the signal vector, i.e., the number of non-zero entries of the vector. However, the computational complexity grows exponentially with the dimension of the signal vector because we have to consider all possible numbers of non-zero entries, which makes the ℓ_0-norm minimization problem intractable given a large signal vector with large dimension. To address this issue, Jeffs *et al.* [9] use a ℓ_p-norm with $p \leq 1$ to approximate the ℓ_0-norm.

In this project[1], we investigate the development of the sparse representation and reconstruction based DoA estimation by reviewing several seminal papers. Then we follow and re-evaluate the ℓ_1-SVD based DoA estimation methodology presented in [10], in which the authors use ℓ_2-norm (i.e., least square) to penalize noise and use ℓ_1-norm as an approximation of the ℓ_0-norm to enforce sparsity of the signal vector. More specifically, under the signal model $\mathbf{y}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{n}(t)$, the goal is to represent $\mathbf{x}(t)$ in a sparse form and utilize the sparse nature to reconstruct it given that we have single or multiple samples of $\mathbf{y}(t)$, which are referred to as snapshots. In practical application scenarios, in order to mitigate the effect of noise and improve the accuracy of estimation, a large number of snapshots will be collected at various sampling time points. For example, 200 snapshots will be common for many applications. However, the complexity increase as the number of snapshots grows and one important fact we have observed is that, there may be redundancy in the snapshot data when its number is very large, which implies that a tradeoff between data size and computation complexity must be considered. [10] propose to use SVD of the snapshot data matrix instead of the original matrix. As long as the number of sources is relatively small, the rank of the data matrix will not be too large, which is easier to handle. Doing SVD reduces the redundancy of snapshots data while keeps all the useful information we want, and the experiment results validate its effectiveness. Simulations are conducted with a focus on the performance

[1]This project aims to expand my personal knowledge of sparse signal representation and reconstruction based DoA estimation approaches. The core part of the project includes: 1) The basics of sparse signal representation DoA eestimation problem and 2) Re-evaluation of the methodology and experiments of [10]. No innovation is involved in this report.

comparison of the ℓ_1-SVD approach and MUSIC algorithm, which enjoys better performance than other traditional DoA estimation algorithms.

The rest of this report is organized as follows. In Section II, we present the formulation of the DoA estimation problem in the sparse signal representation and reconstruction framework, and give a brief introduction to the MUSIC algorithm for comprehensiveness. The ℓ_1-SVD based approach and simulation results are presented in Section III and IV respectively. Finally, we conclude this report and discuss some issues related to this work in Section V.

II. Sparse Signal Reconstruction Based DoA Estimation

A. General Signal Model of DoA Estimation

The goal of the DoA estimation is to determine the incoming directions of electromagnetic waves that reach the antenna or sensor array. The available information is the arrangement of the antenna array (i.e., linear or non-linear, 2-dimensional or 3-dimensional, etc), the receiving gain of each antenna and the measurements of the received signals (snapshots) at the antenna array. For ease of discussion, we assume that all possible signal sources are narrowband and farfield waves, and linear antenna arrangement is used. The advantage is that narrowband signal can be effectively represented by its central frequency component and farfield guarantees that the the waves emitted from any one of the sources will impinge on all the antennas with the same arrival direction (plane wave).

Consider a set of K sources and M antennas, as shown in Figure 1. The spacing of two consecutive antennas is d, usually chosen to be half wavelength, i.e., $d = \frac{\lambda}{2}$. Correspondingly, the difference in travelling time between two antennas is equal to $\Delta t = \frac{\Delta}{c} = \frac{d \sin\theta_k}{c}$, in which c is the speed of light. Without loss of generality, the signal of the k-th ($k = 1, 2, \cdots, K$) source can be written as

$$s_k(t) = f_k(t) e^{j\omega_k t} \tag{1}$$

In eq.(1), $f_k(t)$ is the slow-changing signal envelope and $e^{j\omega_k t}$ is the narrowband high-frequency carrier with a central frequency of ω_k. If we use the first antenna as a reference point, i.e., assume the signal wave $s_k(t)$ arrivals at the first antenna at time $t = 0$, then the received signal wave at the m-th antenna will be

$$\begin{aligned} s_k^m(t) &= s_k(t-(m-1)\Delta t) \\ &= f_k(t-(m-1)\Delta t) e^{j\omega_k (t-(m-1)\Delta t} \\ &\approx f_k(t) e^{j\omega_k t} e^{-j\omega_k (m-1)\Delta t} \\ &= s_k(t) e^{-j\omega_k (m-1)\Delta t} \\ &= s_k(t) e^{-j(m-1)\frac{2\pi d \sin\theta_k}{\lambda}} \\ &= s_k(t) a_m(\theta_k) \end{aligned} \tag{2}$$

As a result, the overall received signal from all sources at the m-th ($m = 1, 2, \cdots, M$) antenna is

$$y_m(t) = \textstyle\sum_{k=1}^{K} a_m(\theta_k) s_k^m(t) + n_m(t) \tag{3}$$

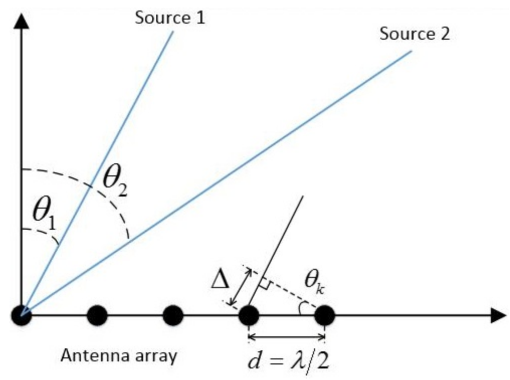


Fig. 1. Signal model of antenna array and signal sources

We can further write the received signals in vector form:

$$\mathbf{y}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \tag{4}$$

In eq.(4), $\mathbf{y}(t)$ and $\mathbf{n}(t)$ is the received signal vector and measurement noise vector with their m-th element being $y_m(t)$ and $n_m(t)$ respectively. $\mathbf{s}(t)$ is the source signal vector with its k-th element being $s_k(t)$. $\mathbf{A} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_K)]$ is the steering matrix with its columns $\mathbf{a}(\theta_k)$ being a steering vector associated with the direction θ_k. For uniform linear array (ULA), $\mathbf{A}$ is a Vandermond matrix with independent columns. Denote $\phi_k = \frac{2\pi \sin\theta_k}{\lambda}$, then the steering vector can be written as

$$\mathbf{a}(\theta_k) = \begin{bmatrix} 1 \\ e^{-j\phi_k} \\ \vdots \\ e^{-j(M-1)\phi_k} \end{bmatrix} \tag{5}$$

B. Multiple Signal Classification (MUSIC) Algorithm

MUSIC algorithm is a traditional DoA estimation method which enjoys excellent performance. The MUSIC algorithm is based on the concept of signal subspace and noise subspace. Denote $\mathbf{R}_y = \frac{1}{n}\sum_{i=1}^{n} \mathbf{y}_i \mathbf{y}_i^\dagger$ as the sample (snapshot) covariance matrix[2], where $\mathbf{y}_i$ is the i-th ($i = 1, 2, \cdots, n$) snapshot. Assume that the eigenvalues of $\mathbf{R}_y$ are sorted in accordance with size, which is

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M > 0 \tag{6}$$

in which the first K eigenvalues correspond to signal $\mathbf{s}(t)$ and the remaining $M-K$ eigenvalues correspond to noise $\mathbf{n}(t)$. MUSIC algorithm obtains a so-called noise matrix

$$\mathbf{E}_N = [\mathbf{v}_{K+1}, \mathbf{v}_{K+1}, \cdots, \mathbf{v}_M] \tag{7}$$

where $\mathbf{v}_i$, $i = K+1, K+2, \cdots M$ is the i-th eigenvector of sample covariance matrix $\mathbf{R}_y$. The spatial spectrum of MUSIC algorithm is defined as

$$\mathrm{P}(\theta) = \frac{1}{\|\mathbf{E}_N \mathbf{a}(\theta)\|_2^2} \tag{8}$$

[2]Here we use $\dagger$ to represent conjugate transpose.

in which $\mathbf{a}(\theta) = \left[1, e^{j\frac{2\pi d\sin(\theta)}{\lambda}}, \cdots, e^{j(M-1)\frac{2\pi d\sin(\theta)}{\lambda}}\right]^\dagger$ is the steering vector associated with θ. Since the steering vector $\mathbf{a}(\theta)$ associated with the actual source direction $\theta = \theta_k$ is orthogonal with the noise matrix $\mathbf{E}_N$, the square ℓ_2-norm $\|\mathbf{E}_N\mathbf{a}(\theta)\|_2^2$ will be very small thus producing sharp peaks at the actual source directions. By looking at these peaks of the spectrum the DoA estimate can be obtained.

C. The Concept of Sparse Signal Representation

The general form of a sparse representation problem is that given $\mathbf{y} \in \mathbb{C}^M$ and $\mathbf{A} \in \mathbb{C}^{M\times N}$ where $M < N$, we want to find the most sparse solution $\mathbf{x} \in \mathbb{C}^N$ such that $\mathbf{y} = \mathbf{A}\mathbf{x}$. The sparse property of the solution is crucial since without this assumption, the equation will have infinitely many solutions. The most direct way to measure sparsity is the ℓ_0-norm which is denoted by $\|\mathbf{x}\|_0$. Thus, this leads to an optimization problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x}\|_0 \\ \text{s.t.} \quad & \mathbf{y} = \mathbf{A}\mathbf{x} \end{aligned} \tag{9}$$

Unfortunately, due to the high complexity nature of the combinatorial optimization problems, (9) is almost intractable when the dimension of the variable $\mathbf{x}$ is relatively large. A direct way to handle this is to use ℓ_1-norm, which is differentiable given that $\mathbf{x} \succeq 0$, to approximate the ℓ_0-norm. The sparse representation problem with the presence of noise has the form of $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$, where $\mathbf{n}$ is the random noise vector. To find the solutions of $\mathbf{x}$ with sparsity, we can formulate the problem into a mixed ℓ_2-norm and ℓ_1-norm optimization problem

$$\min_{\mathbf{x}} \quad \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_1 \tag{10}$$

in which λ is a regularization factor that enforces sparsity on the solutions, while the least square term minimizes the error caused by noise.

D. Sparse Signal Representation and Reconstruction Based DoA Estimation

In this section we introduce the overcomplete representation of the steering matrix and source signal vector with respect to all possible source directions. Consider a set of possible source directions $\Theta = \{\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_{N_\theta}\}$, where N_θ is the number of potential source directions which is usually much greater than the actual number of sources K. Then we construct an overcomplete steering matrix $\mathbf{A}^{\text{sparse}}$ in terms of Θ with its columns being steering vectors associated with the directions in Θ, $\mathbf{A}^{\text{sparse}} = [\mathbf{a}(\hat{\theta}_1), \mathbf{a}(\hat{\theta}_2), \cdots, \mathbf{a}(\hat{\theta}_{N_\theta})]$. Then we represent the source signal $\mathbf{s}^{\text{sparse}}(t)$ by an $\theta_{N_\theta} \times 1$ vector, in which the n-th entry is equal to $s_k(t)$ if source k is in the direction of $\hat{\theta_n}$ for some k and equal to zero if there is no source coming from the direction of $\hat{\theta_n}$. It is easy to see that $\mathbf{s}^{\text{sparse}}(t)$ is sparse. As a result, the problem becomes $\mathbf{y}(t) = \mathbf{A}^{\text{sparse}}\mathbf{s}^{\text{sparse}}(t) + \mathbf{n}(t)$. The corresponding optimization problem is

$$\min_{\mathbf{s}^{\text{sparse}}(t),\, \lambda} \quad \|\mathbf{y} - \mathbf{A}^{\text{sparse}}\mathbf{s}^{\text{sparse}}(t)\|_2^2 + \lambda\|\mathbf{s}^{\text{sparse}}(t)\|_1 \tag{11}$$

The optimal solution of the above optimization problem yields a spatial spectrum with a resolution of $\hat{\theta_n}$, and the peaks of the spectrum give the estimate of the DoA. This is the basic methodology of sparse signal representation and reconstruction based DoA estimation approach under the assumption that only one snapshot is available. However, to improve the accuracy of the estimation, multiple snapshots are collected in practical scenarios. One problem related to a large number of snapshots is still computational complexity, which renders real-time DoA applications impossible. To address this issue, the authors of [10] proposed to use the singular value decomposition (SVD) of the snapshot matrix to reduce complexity. We refer to this approach as ℓ_1-SVD method, which we will present in the following section.

III. ℓ_1-SVD Based DoA Estimation Approach

A. ℓ_1-SVD Approach

In this section, we deal with the DoA estimation problem with multiple snapshots. Under the multiple snapshots setting, we have $\mathbf{y}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t)$, $t \in \{t_1, t_2, \cdots, t_T\}$, where T is the total number of snapshots. Denote $\mathbf{Y} = [\mathbf{y}(t_1), \mathbf{y}(t_2), \cdots, \mathbf{y}(t_T)]$, $\mathbf{S} = [\mathbf{s}(t_1), \mathbf{s}(t_2), \cdots, \mathbf{s}(t_T)]$ and $\mathbf{N} = [\mathbf{n}(t_1), \mathbf{n}(t_2), \cdots, \mathbf{n}(t_T)]$, we have

$$\mathbf{Y} = \mathbf{A}\mathbf{S} + \mathbf{N} \tag{12}$$

The signal model with multiple snapshots (12) is significantly different from that with a single snapshot (4) since $\mathbf{S}$ spreads both temporally ($t = t_1, t_2, \cdots, t_T$) and spatially ($m = 1, 2, \cdots, M$) while $\mathbf{s}(t)$ spread only in spatial dimension. However, sparsity only needs to be enforced in the spatial dimension of the source signal matrix $\mathbf{S}$, and there is no sparsity property in the temporal dimension. This authors came up with a way to enforce sparsity temporally as follows: 1) Compute the ℓ_2-norm of all snapshots of a specific spatial index of $\mathbf{S}$, i.e., $s_i^{(\ell_2)} = \|(s_i(t_1), s_i(t_2), \cdots, s_i(t_T))\|_2$ and then 2) penalize the ℓ_1-norm of $\mathbf{s}^{(\ell_2)} = [s_1^{(\ell_2)}, s_2^{(\ell_2)}, \cdots, s_{N_\theta}^{(\ell_2)}]$. Therefore, the optimization problem becomes

$$\min_{\mathbf{S},\, \lambda} \quad \|\mathbf{Y} - \mathbf{A}\mathbf{S}\|_F^2 + \lambda\|\mathbf{s}^{(\ell_2)}\|_1 \tag{13}$$

in which the Frobenius norm $\|\cdot\|_F$ for a matrix $\mathbf{D} = [d_{ij}]_{M\times N}$ is defined as $\|\mathbf{D}\|_F = \sqrt{\sum_{i=1}^M \sum_{j=1}^N |d_{ij}|^2}$. Denote the optimal solution of (13) as $\mathbf{s}^{(\ell_2)*} = [s_1^{(\ell_2)*}, s_2^{(\ell_2)*}, \cdots, s_{N_\theta}^{(\ell_2)*}]$, then the spatial spectrum is defined as

$$\mathrm{P}^{sparse}(\hat{\theta}_i) = s_i^{(\ell_2)*}, \quad i = 1, 2, \cdots, N_\theta \tag{14}$$

In practical scenarios, t_T can be in the order of magnitude of several thousands, which imposes high complexity in solving (14), especially when N_θ is large too, i.e., when higher resolution is expected.

To release the computation pressure, take the SVD of the snapshot matrix $\mathbf{Y}$, we get

$$\mathbf{Y} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^\dagger \tag{15}$$

Then the matrix $\mathbf{Y}_{SVD} = \mathbf{U\Lambda D}_K$, of dimension $M \times K$, keeps the most of the signal power. $\mathbf{D}_K = [\mathbf{I}_K\ \mathbf{O}_{(T-K)\times K}]^\dagger$ is a $T \times K$ matrix, and $\mathbf{I}_K$ is the $K \times K$ identity matrix and $\mathbf{O}_{(T-K)\times}$ is a $(T-K)\times K$ matrix whose entries are all zero. Also, denote $\mathbf{S}_{SVD} = \mathbf{SVD}_K$, and $\mathbf{N}_{SVD} = \mathbf{NVD}_K$, we obtain

$$\mathbf{Y}_{SVD} = \mathbf{AS}_{SVD} + \mathbf{N}_{SVD} \tag{16}$$

which can be equivalently expressed column by column:

$$\mathbf{y}^{SVD}(k) = \mathbf{As}^{SVD}(k) + \mathbf{n}^{SVD}(k),\ k = 1, 2, \cdots, K \tag{17}$$

Similarly, to enforce sparsity of the source signal matrix, define $\tilde{s}_i^{(\ell_2)} = \|(s_i^{SVD}(1), s_i^{SVD}(2), \cdots, s_i^{SVD}(K))\|_2$, then (13) can be reformulated as

$$\min_{\mathbf{S}_{SVD},\lambda} \quad \|\mathbf{Y}_{SVD} - \mathbf{AS}_{SVD}\|_F^2 + \lambda\|\tilde{\mathbf{s}}^{(\ell_2)}\|_1 \tag{18}$$

The corresponding spatial spectrum is

$$\mathrm{P}_{SVD}^{sparse}(\hat{\theta}_i) = \tilde{s}_i^{(\ell_2)*}, \quad i = 1, 2, \cdots, N_\theta \tag{19}$$

In (19) we are assuming that all the actual DoA lie within the presumed set of possible directions $\Theta = \{\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_{N_\theta}\}$. If the actual DoA, θ, of a source doesn't belong to Θ, then the algorithm gives the nearest direction estimate $\hat{\theta}_i$ of θ such that $|\hat{\theta}_i - \theta|$ is minimized. The difference in angle of two consecutive directions, i.e., $\hat{\theta}_{j+1} - \hat{\theta}_j$ is called the resolution of the set Θ. For example, if we choose $\Theta = \{0°, 1°, \cdots, 180°, \}$, then the resolution is $1°$. We can use higher resolution to enhance the accuracy of the DoA estimation, but that will increase the computation load. A proper resolution of Θ should be in accordance with the particular problem setting.

B. SOCP Transformation of the ℓ_1-SVD Problem

It is shown in [10] that the optimization problem (18) can be reformulated as a standard second-order cone (SOC) optimization problem, which then can be efficiently solved by general SOC programming software. The SOCP is as follows

$$\begin{aligned} \min \quad & p + \lambda q \\ \text{s.t.} \quad & \|(\mathbf{z}_1^\dagger, \mathbf{z}_2^\dagger, \cdots, \mathbf{z}_K^\dagger)\|_2^2 \le p \\ & \mathbf{1}^\dagger \mathbf{r} \le q \\ & \hat{s}_i^{(\ell_2)} \le r_i,\ \forall i = 1, 2, \cdots, N_\theta \end{aligned} \tag{20}$$

in which $\mathbf{z}_k = \mathbf{y}^{SVD}(k) - \mathbf{As}^{SVD}(k)$, $k = 1, 2, \cdots, K$ and $\mathbf{1}$ is a $N_\theta \times 1$ vector whose entries are all equal to one. To clearly show the connection to a standard SOCP, we modify (20) as

$$\begin{aligned} \min \quad & p + \lambda q \\ \text{s.t.} \quad & \|\widetilde{\mathbf{A}}\mathbf{s} + \mathbf{y}\|_2^2 \le p \\ & \mathbf{1}^\dagger \mathbf{r} \le q \\ & \|\widetilde{\mathbf{M}}_i \mathbf{s}\|_2 \le r_i,\ \forall i = 1, 2, \cdots, N_\theta \end{aligned} \tag{21}$$

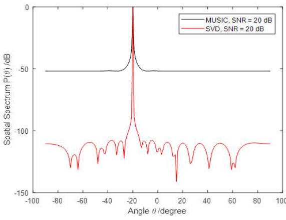
where the optimization variables are $p, q, \mathbf{s}$ and $\mathbf{r}$ respectively. The parameter in (21) are denoted as follows: $\widetilde{\mathbf{A}} = \mathrm{diag}(-\mathbf{A}, -\mathbf{A}, \cdots, -\mathbf{A})$ is a $KM \times KN_\theta$ matrix with its K sub-blocks all being the steering matrix $\mathbf{A}$ associated with $\Theta = \{\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_{N_\theta}\}$. $\widetilde{\mathbf{M}}_i = diag(\mathbf{M}_i, \mathbf{M}_i, \cdots, \mathbf{M}_i)$ is a $KN_\theta \times KN_\theta$ diagonal matrix with its sub-blocks $\mathbf{M}_i$ being a matrix of which the i-th diagonal element is one and all other elements are zero. $\mathbf{y} = [\mathbf{y}^{SVD,\dagger}(1), \mathbf{y}^{SVD,\dagger}(2), \cdots, \mathbf{y}^{SVD,\dagger}(K)]^\dagger$ and the optimization variable $\mathbf{s} = [\mathbf{s}^{SVD,\dagger}(1), \mathbf{s}^{SVD,\dagger}(2), \cdots, \mathbf{s}^{SVD,\dagger}(K)]^\dagger$, $\mathbf{r} = [r_1, r_2, \cdots, r_{N_\theta}]$. Since the objective function is linear and all constraints are expressed in the form of ℓ_2-norm and linear inequality, this is a standard SOCP optimization problem which can be effectively solved using MATLAB CVX.

IV. Simulation Result

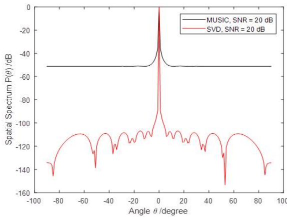
In this section, we present the simulation results of the ℓ_1-SVD DoA estimation approach and compare to the traditional MUSIC algorithm. The spatial spectrum of these two algorithms are given in (8) and (19) respectively. Here we choose $\Theta = \{0°, 1°, \cdots, 180°\}$, so the direction resolution is $1°$. We consider a uniform linear array (ULA) with $M = 10$ antennas and a spacing of $d = \frac{\lambda}{2}$. Without loss of generality, we assume the signal envelope to be $f(t) = e^{j\omega t}$, where ω is frequency belonging to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We also assume the noise to be additive white Gaussian noise (AWGN) with a unit variance. Various number of independent or correlated sources are considered in the simulation. We use MATLAB CVX optimization toolbox to solve the SOCP problem. The choice of the penalty factor λ has huge impact on the performance of the sparse representation based approach. Due to time limit, we do not study how to derive a criteria to decide the optimal value of λ, which is also related to the distribution of the noise. Simple heuristic search was used in the simulation to find the value of λ that yields a fairly good DoA estimation performance.

In the first part of the simulation, we have $T = 100$ snapshots and only one source coming from various directions. The signal-to-noise ratio (SNR) is set to be 20 dB. Consider a source signal with a envelope signal frequency $\frac{\pi}{3}$ and a DoA of $-20°, -0°, 50°$ and $75°$ respectively. Figure 2 shows the corresponding spatial spectra of these two algorithms. It can be seen that in this case, both algorithms yield the correct source direction estimate. However, for the same SNR, ℓ_1-SVD has much sharper peaks than MUSIC, implying higher spectrum resolution which is a highly desired property. Another advantage of the ℓ_1-SVD approach is that is doesn't require the knowledge of the number of actual sources but MUSIC does, as we have seen in the derivation of the algorithm.

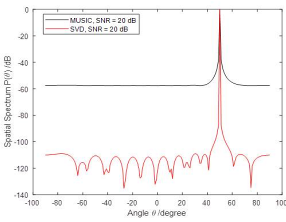
For the next step, we compared the performance of these two algorithms under two closely correlated source signals. The number of snapshots is $T = 100$ and SNR=20 dB. The signal envelope frequencies are both $\frac{\pi}{3}$. We evaluate the performance under a source DoA of $[-30°, 30°], [-10°, 10°], [-5°, 5°]$ and $[-4°, 4°]$ respectively. The simulation results is shown in Figure 3. ℓ_1-SVD still enjoys sharper peaks than MUSIC. In the first three source DoA settings, in which the two sources are fairly spread



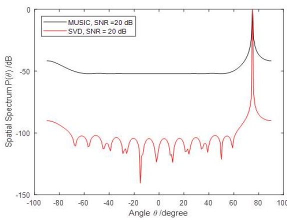
(a) Source direction=$-20°$



(b) Source direction=$0°$

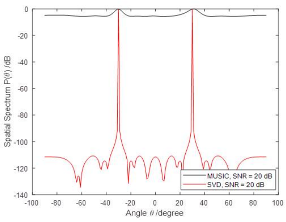


(c) Source direction=$50°$

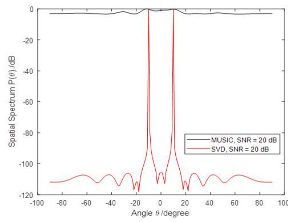


(d) Source direction=$75°$

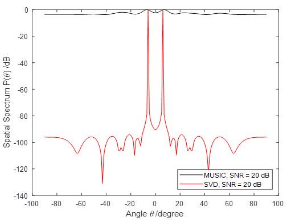
Fig. 2. Single DoA estimate of ℓ_1-SVD and MUSIC. In this case, both algorithms yield the correct direction estimate. However, ℓ_1-SVD has higher resolution than MUSIC due to the shaper peaks at the actual source direction.



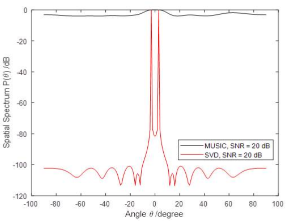
(a) DoA=$-30°, 30°$, SNR=20 dB



(b) DoA=$-10°, 10°$, SNR=20 dB



(c) DoA=$-5°, 5°$, SNR=20 dB

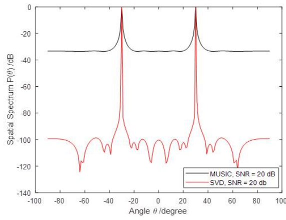


(d) DoA=$-4°, 4°$, SNR=20 dB

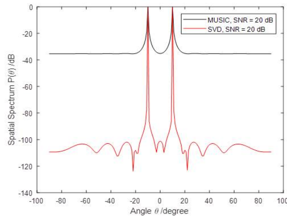
Fig. 3. DoA estimation of two perfectly correlated sources. ℓ_1-SVD has sharper and distinguishable peaks while MUSIC merges the two peaks when the two sources are close to each other, leading to resolution degradation.

apart, both algorithms tell the correct direction estimate despite MUSIC having less sharper peaks. However, when the sources are very close to each other, say, only a $8°$ difference, MUSIC is unable to give the correct direction because the two peaks are merged together whole ℓ_1-SVD still works pretty well. This simulation exposed MUSIC algorithm's incapability in dealing with correlated sources.

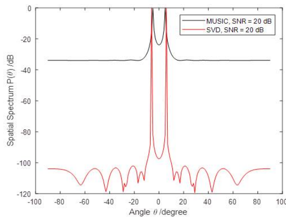
We evaluate another case in which the two source signals are less correlated. We set the envelope frequency to be $\frac{\pi}{3}$ and $\frac{\pi}{3.01}$. The source DoA is still $[-30°, 30°], [-10°, 10°], [-5°, 5°]$ and $[-4°, 4°]$ respectively. The simulation result is shown in Figure 4. We can see in



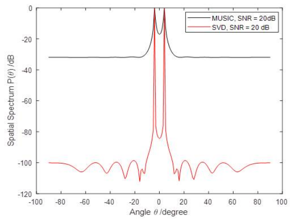
(a) DoA=$-30°, 30°$, SNR=20 dB



(b) DoA=$-10°, 10°$, SNR=20 dB



(c) DoA=$-5°, 5°$, SNR=20 dB



(d) DoA=$-4°, 4°$, SNR=20 dB

Fig. 4. DoA estimation of two less correlated sources. ℓ_1-SVD outperforms MUSIC. However, MUSIC algorithm performs slightly better than the perfectly correlated case.

this case, ℓ_2-SVD still outperforms MUSIC and the difference is that, since the two sources are less correlated due to the envelope frequency difference, MUSIC algorithm performs slightly better than the perfectly correlated case. From the overall simulation results we can reach the conclusion that under the same SNR, ℓ_1-SVD algorithm has much higher spectrum resolution and are effective in dealing with correlated sources. Since SVD is done, only a few columns of the snapshots are remained, so the computation complexity can be reduced significantly.

V. Conslusion

In this project, we studied how to cast the traditional DoA estimation problem into the sparse signal representation and reconstruction based framework. For the purpose of comparison, we briefly introduced the methodology of the traditional MUSIC algorithm. To handle the computation complexity of the problem when a large number of snapshots are available, ℓ_1-SVD method was proposed to reduce the snaps shot matrix to only several columns, which makes the problem tractable while keeps all the useful information of the data. We next formulated the mixed $\ell_2, 1$-norm optimization as a standard SOCP problem and used MATLAB CVX toolbox to sovle the SOCP problem. Simulation results showed that ℓ_1-SVD approach has much higher spectrum resolution than MUSIC and is more effective in dealing with correlated sources. In addition, ℓ_1-norm approach does not require the knowledge the number of actual sources, which is considered another advantage over MUSIC algorithm. One interesting research direction is to decide the criteria to find the best penalty factor instead of using heuristic search, which adds extra computation load and does not guarantee a global optima.

REFERENCES

[1] Krim, Hamid, and Mats Viberg. "Two decades of array signal processing research: the parametric approach." IEEE signal processing magazine 13.4 (1996): 67-94.

[2] Johnson, Don H., and Dan E. Dudgeon. Array signal processing: concepts and techniques. Englewood Cliffs: PTR Prentice Hall, 1993.

[3] Stoica, Petre. Introduction to spectral analysis. Prentice hall, 1997.

[4] Capon, Jack. "High-resolution frequency-wavenumber spectrum analysis." Proceedings of the IEEE 57.8 (1969): 1408-1418.

[5] Schmidt, Ralph. "Multiple emitter location and signal parameter estimation." IEEE transactions on antennas and propagation 34.3 (1986): 276-280.

[6] Gorodnitsky, Irina F., and Bhaskar D. Rao. "Sparse signal reconstruction from limited data using FOCUSS: A re-weighted minimum norm algorithm." IEEE Transactions on signal processing 45.3 (1997): 600-616.

[7] Fuchs, J-J. "Linear programming in spectral estimation. Application to array processing." Acoustics, Speech, and Signal Processing, 1996. ICASSP-96. Conference Proceedings., 1996 IEEE International Conference on. Vol. 6. IEEE, 1996.

[8] Fuchs, J-J. "On the application of the global matched filter to DOA estimation with uniform circular arrays." IEEE Transactions on Signal Processing 49.4 (2001): 702-709.

[9] Jeffs, Brian D. "Sparse inverse solution methods for signal and image processing applications." Acoustics, Speech and Signal Processing, 1998. Proceedings of the 1998 IEEE International Conference on. Vol. 3. IEEE, 1998.

[10] Malioutov, Dmitry, Mujdat Cetin, and Alan S. Willsky. "A sparse signal reconstruction perspective for source localization with sensor arrays." IEEE transactions on signal processing 53.8 (2005): 3010-3022.