A Categorification of Group Cohomology

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Picard Categories

- Groupoid
- Symmetric monoidal
- Group-like:
 - For all X, there is a Y such that $X \otimes Y \cong I \cong Y \otimes X$
- Essential data:
 - $\pi_0(\mathscr{C}) = Obj(\mathscr{C}) / \cong$
 - $\pi_1(\mathscr{C}) = \mathscr{C}(I,I)$
 - $K: \pi_0(\mathscr{C}) \to \pi_1(\mathscr{C}),$ $X \mapsto \beta_{X,X} \in \mathscr{C}(X \otimes X, X \otimes X) \cong \pi_1(\mathscr{C})$

Examples

- Pic(R) := R-**Mod** $_{inv}^{\cong}$, for $R \in \mathbf{CRing}$
 - Note: $\pi_0(Pic(R)) = pic(R)$
- $\Pi_1 X$ for $X \in \Omega^3$ **Top**
- Z, "Super Integers"

•
$$Obj(\mathfrak{Z}) = \mathbb{Z}, \ \ \mathfrak{Z}(n,m) \cong \begin{cases} \mathbb{Z}/2, & \text{if } n = m \\ 0, & \text{else} \end{cases}$$

- Call $\mathcal{Z}(n,n) = \{\pm 1_n\}$
- $(\beta: n+m \to m+n) = (-1_{n+m})^{nm}$

Free Picard Categories

Theorem (H)

The forgetful functor $U : \mathbf{Pic} \to \mathbf{Grpd}$ has a left biadjoint given by

$$\mathcal{Z}[_]: \mathbf{Grpd} \to \mathbf{Pic}.$$

Specifically: For $\mathscr{G} \in \mathbf{Grpd}$ and $\mathscr{A} \in \mathbf{Pic}$,

$$\mathsf{Pic}(\mathcal{Z}[\mathscr{G}],\mathscr{A}) \simeq \mathsf{Grpd}(\mathscr{G},\mathscr{A})$$

as Picard categories, pseudonatural in $\mathscr G$ and $\mathscr A$.

Tensoring over **Grpd**

Corollary (H)

Pic is tensored over Grpd

Specifically: For $\mathscr{G} \in \mathbf{Grpd}$ and $\mathscr{A} \in \mathbf{Pic}$, there exists $\mathscr{A}[\mathscr{G}] \in \mathbf{Pic}$ so that for all $\mathscr{B} \in \mathbf{Pic}$,

$$\mathsf{Pic}(\mathscr{A}[\mathscr{G}],\mathscr{B}) \simeq \mathsf{Grpd}(\mathscr{G},\mathsf{Pic}(\mathscr{A},\mathscr{B}))$$

pseudonaturally.

Proposition (H)

Pic is cotensored over Grpd

Picard Cohomology

Definition: Category of Modules over $\mathscr{G} \in \mathbf{Pic}$

For $\mathscr{G} \in \mathbf{Pic}$, \mathscr{G} -**Mod** := PsFunk $(\Sigma\mathscr{G}, \mathbf{Pic})$

Definition: Picard Cohomology

For $\mathscr{G} \in \mathbf{Pic}$ and $\mathscr{M} \in \mathscr{G}\mathbf{-Mod}$, define

$$\mathscr{H}^n(\mathscr{G};\mathscr{M}) := \mathbb{R}^nig(\mathscr{G}\operatorname{\mathsf{-Mod}}ig(\mathcal{Z}_{\mathsf{triv}},_ig)ig)(\mathscr{M})$$

i.e.

$$\mathscr{H}^{n}\left(\mathscr{G};\mathscr{M}\right)=\mathsf{Ext}_{\mathscr{G}\text{-}\mathsf{Mod}}^{n}\left(\mathcal{Z}_{\mathsf{triv}},\mathscr{M}\right)$$

i.e.

$$\mathscr{H}^n(\mathscr{G};\mathscr{M}) = \mathbb{L}^nig(\mathscr{G} ext{-}\mathsf{Mod}\,(\,_\,,\mathscr{M})\,ig)(\mathfrak{Z}_\mathsf{triv})$$

A first computation: $\mathcal{H}^m(\mathbb{Z}/n;\mathbb{Z})$

Proposition (H)

The chain complex

$$\ldots \xrightarrow{N} \mathfrak{Z}[\mathbb{Z}/n] \xrightarrow{x-1} \mathfrak{Z}[\mathbb{Z}/n] \xrightarrow{N} \mathfrak{Z}[\mathbb{Z}/n] \xrightarrow{x-1} \mathfrak{Z}[\mathbb{Z}/n] \xrightarrow{0} 0 \xrightarrow{0} \ldots$$

provides a projective resolution of

$$\dots \xrightarrow{0} 0 \xrightarrow{0} \mathcal{Z}_{triv} \xrightarrow{0} 0 \xrightarrow{0} \dots$$

Lemma (H)

Applying \mathbb{Z}/n -**Mod** $(_, \mathcal{Z}_{triv})$ to the above yields

$$0 \xrightarrow{0} 0 \xrightarrow{0} \mathcal{Z} \xrightarrow{0} \mathcal{Z} \xrightarrow{n} \mathcal{Z} \xrightarrow{0} \mathcal{Z} \xrightarrow{n} \dots$$

A first computation: $\mathscr{H}^m(\mathbb{Z}/n;\mathbb{Z})$

Theorem (H)

For n even,

$$\mathcal{H}^{m}\left(\mathbb{Z}/n;\mathbb{Z}
ight) = egin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & m=0 \ \mathbb{Z}/2 \oplus \Sigma(\mathbb{Z} imes \mathbb{Z}/2) & m=1 \ \mathbb{Z}/n \oplus \mathbb{Z}/2 & m>1 \ ilde{s} \ even \ \mathbb{Z}/2 \oplus \Sigma(\mathbb{Z}/n imes \mathbb{Z}/2) & m>1 \ ilde{s} \ odd \end{cases}$$

For n odd,

$$\mathscr{H}^m\left(\mathbb{Z}/n;\mathbb{Z}
ight) = egin{cases} \mathbb{Z} & m=0 \ \Sigma(\mathbb{Z}) & m=1 \ \mathbb{Z}/n & m>1 \ is \ even \ \Sigma(\mathbb{Z}/n) & m>1 \ is \ odd \end{cases}$$

A second computation: $\mathscr{H}^m(\mathbb{Z}/n;\mathbb{Z})$

Lemma (H)

Applying $\mathbb{Z}/n\text{-}\mathbf{Mod}\left(_,\mathbb{Z}_{triv}\right)$ to the previous resolution yields

$$0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \dots$$

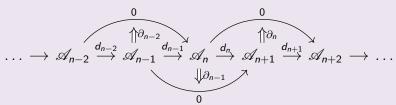
Theorem (H)

For all n,

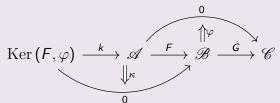
$$\mathscr{H}^m\left(\mathbb{Z}/n;\mathbb{Z}
ight) = egin{cases} \mathbb{Z} & m=0 \ \Sigma(\mathbb{Z}) & m=1 \ \mathbb{Z}/n & m>1 \ is \ even \ \Sigma(\mathbb{Z}/n) & m>1 \ is \ odd \end{cases}$$

Chain complexes of Picard categories

Chain complex of Picard categories



Compute cohomology using relative kernel and cokernel



• If $\mathcal{H}^n(\mathscr{A}_{\bullet}) \simeq 0$, call \mathscr{A}_{\bullet} relative exact at \mathscr{A}_n

\mathscr{G} -modules vs. Picard Categories

Theorem (H)

Let \mathscr{A}, \mathscr{B} , and \mathscr{D} be bicategories, and assume that \mathscr{A} has all bicategorical limits (dually, colimits) of shape \mathscr{D} . Then for any $F: \mathscr{D} \to \mathsf{PsFunk}(\mathscr{B}, \mathscr{A})$, $\operatorname{Lim} F$ exists and all its data are computed objectwise. Dually, the same holds for $\operatorname{Colim} F$.

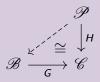
Proposition

Any biadjunction $F \dashv G$ between bicategories \mathscr{A} and \mathscr{B} yields for any bicategory \mathscr{D} a biadjunction $F^{\mathscr{D}} \dashv G^{\mathscr{D}}$ between $PsFunk(\mathscr{D},\mathscr{A})$ and $PsFunk(\mathscr{D},\mathscr{B})$.

Projective Picard categories

Definition

 $\mathscr{P} \in \mathbf{Pic}$ is *projective* if for all



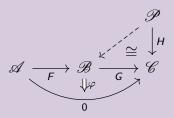
with G essentially surjective, such a lift exists.

- Problems:
 - No homological rephrasing
 - $\mathscr P$ is projective if and only if $\mathscr P=\mathfrak Z^{\oplus\kappa}=\mathfrak Z[\mathscr G]$ for discrete $\mathscr G$

Relative projective Picard categories

Definition (H)

 $\mathscr{P} \in \mathbf{Pic}$ is *relative projective* if for all



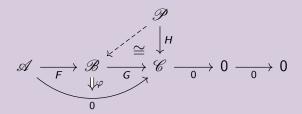
with G essentially surjective and φ -full, such a lift exists.

ullet " φ -full" is a generalization of full

Relative projective Picard categories

Definition, rephrased

 $\mathscr{P} \in \mathbf{Pic}$ is *relative projective* if for all



with row relative exact at \mathscr{C} , such a lift exists.

Relative projective Picard categories

Theorem (H)

 $\mathscr{P} \in \mathbf{Pic}$ is relative projective if and only if $\mathbf{Pic}(\mathscr{P}, \underline{\hspace{0.1cm}})$ is relative exact.

• But not all free Picard categories are relative projective!

Proposition (H)

 $\mathbb{Z}[\mathcal{G}]$ is relative projective if and only if for all $G \in \mathcal{G}$, End(G) is free.

Proposition (H)

For free $A \in \mathbf{Ab}$, $\Sigma A \in \mathbf{Pic}$ is relative projective but not free.

Thank you \sim !

References



J. C. Baez and A. D. Lauda, *Higher-dimensional algebra. v: 2-groups.*, Theory and Applications of Categories [electronic only] **12** (2004), 423–491.



A. del Río, J. Martínez-Moreno, and E. Vitale, *Chain complexes of symmetric categorical groups.*, J. Pure Appl. Algebra **196** (2005), no. 2-3, 279–312. doi:10.1016/j.jpaa.2004.08.029



M. Dupont, Abelian Categories in dimension 2., Ph.D. thesis, Université catholique de Louvain, 2008. arXiv:0809.1760



N. Gurski, N. Johnson, and A. M. Osorno, *Star product on Picard Categories.*, 2018, personal correspondence.



N. Johnson and A. M. Osorno, *Modeling stable one-types.*, Theory Appl. Categ. **26** (2012), 520–537.



M. Kapranov, Supergeometry in mathematics and physics., 2015. arXiv:1512.07042



T. Pirashvili, On Abelian 2-categories and derived 2-functors., 2010. arXiv:1007 4138



______, Projective and injective symmetric categorical groups and duality., Proc. Am. Math. Soc. **143** (2015), no. 3, 1315–1323. doi:10.1090/S0002-9939-2014-12354-9