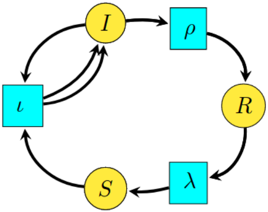
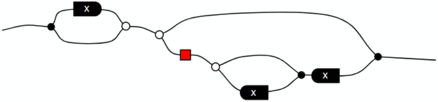
Hypergraph categories as cospan algebras

Brendan Fong, with David Spivak

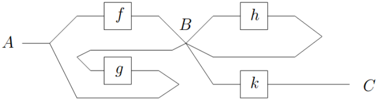
Category Theory Octoberfest 2018
City College New York
27 October 2018



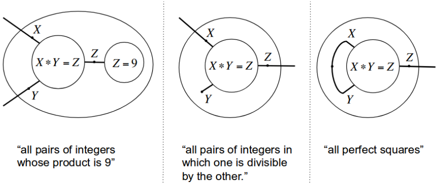
Baez, Pollard: *A compositional framework for reaction networks*



Bonchi, Sobocinski, Zanasi: *A categorical semantics of signal flow graphs*



Rosebrugh, Sabadini, Walters: *Calculating colimits compositionally*



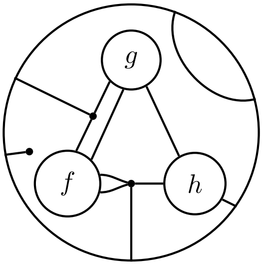
Spivak: *The operad of wiring diagrams*

Outline

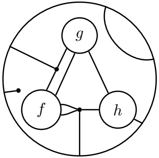
I. Hypergraph categories
II. Cospan algebras
III. The equivalence

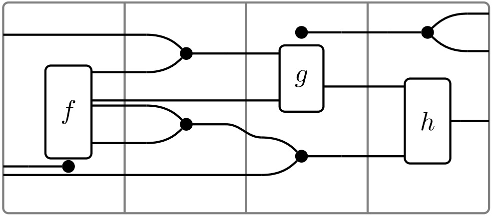
I. Hypergraph categories

Abstractly, how do we construct this?



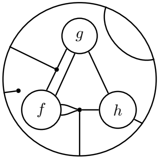
. . . as structured monoidal category

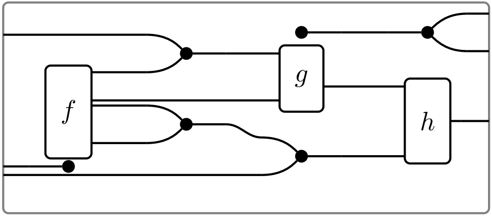




$$(1 \otimes f \otimes -\!\bullet \otimes 1); (\succ\!\!\bullet\!- \otimes 1 \otimes \succ\!\!\bullet\!- \otimes 1); (\bullet\!- \otimes g \otimes \succ\!\!\bullet\!-); (-\!\bullet\!\!\prec \otimes h).$$

. . . as structured monoidal category





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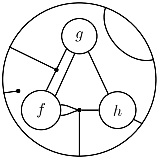
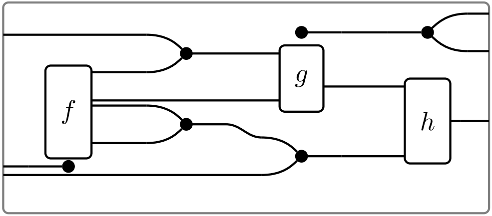
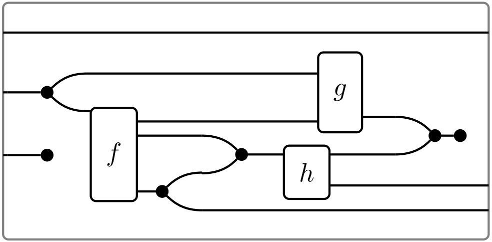
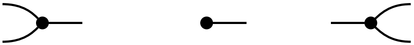




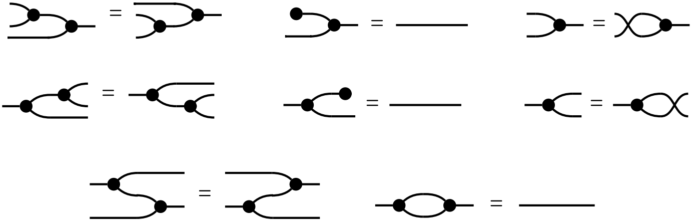


A **special commutative Frobenius monoid** on X is



$\mu: X \otimes X \to X$ $\qquad$ $\eta: I \to X$ $\qquad$ $\delta: X \to X \otimes X$ $\qquad$ $\epsilon: X \to I$

obeying



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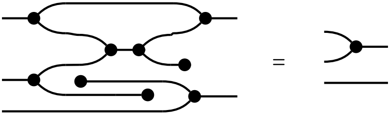
$\mu\colon X \otimes X \to X$ $\qquad \eta\colon I \to X$ $\qquad \delta\colon X \to X \otimes X$ $\qquad \epsilon\colon X \to I$

obeying *the spider theorem*



A **hypergraph category** is a symmetric monoidal category in which each object X is equipped with a Frobenius structure in a way compatible with the monoidal product.

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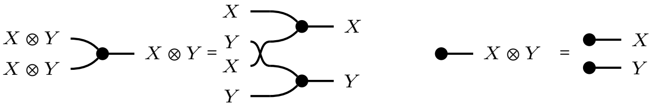
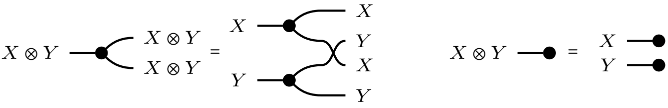
This means that the Frobenius structure on I is $(\rho_I^{-1}, \mathrm{id}_I, \rho_I, \mathrm{id}_I)$ and for all X, Y, the Frobenius structure on $X \otimes Y$ is





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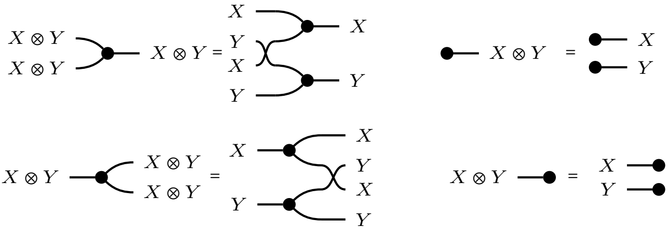


A **hypergraph functor** is a strong symmetric monoidal functor (F, φ) such that if $(\mu_X, \eta_X, \delta_X, \epsilon_X)$ is the Frobenius structure on X, then $\left(\varphi_{X,X}; F\mu_X,\ \ \varphi_I; F\eta_X,\ \ F\delta_X; \varphi^{-1}_{X,X},\ \ F\epsilon_X; \varphi_I^{-1}\right)$ is the Frobenius structure on FX.

Let $\mathbf{Hyp}$ be the 2-category with

objects: hypergraph categories

morphisms: hypergraph functors

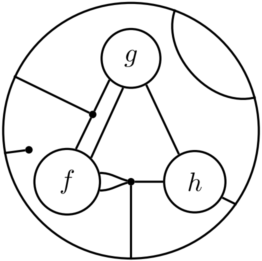
2-morphisms: monoidal natural transformations.

Let $\mathbf{Hyp}_{\mathtt{OF}}$ be the full sub-2-category of objectwise-free hypergraph categories.

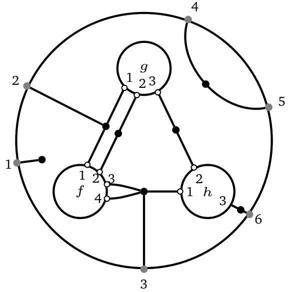
Theorem (Coherence for hypergraph categories)
$\mathbf{Hyp}_{\mathtt{OF}}$ *and* $\mathbf{Hyp}$ *are 2-equivalent.*

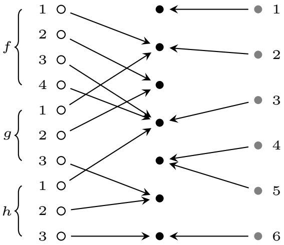
II. Cospan algebras

Abstractly, how do we construct this?



…as operad algebra





$$A \longrightarrow N \longleftarrow B$$

Define $\mathbf{Cospan}_\Lambda = \coprod_{\lambda \in \Lambda} \mathbf{Cospan}(\mathbf{FinSet})$.

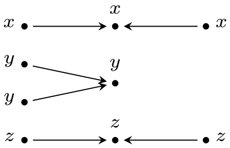
$\mathbf{Cospan}_\Lambda$ is the symmetric monoidal category with

objects: Λ-typed finite sets $t\colon X \to \Lambda$.

morphisms: cospans over Λ.

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[dr, "t"'] & N \arrow[d, "s"] & Y \arrow[l, "f_2"'] \arrow[dl, "u"] \\ & \Lambda & \end{tikzcd}$$

monoidal product: disjoint union $\oplus$



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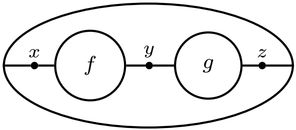

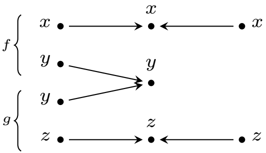

Let **CospanAlg** be the category with

objects: lax symmetric monoidal functors

$$\Lambda \qquad\qquad A\colon (\mathbf{Cospan}_\Lambda, \oplus) \longrightarrow (\mathbf{Set}, \times)$$

morphisms: monoidal natural transformations

$$\begin{array}{c} \Lambda \\ {\scriptstyle f}\big\downarrow \\ \mathsf{List}(\Lambda') \end{array} \qquad\qquad \begin{tikzcd} \mathbf{Cospan}_\Lambda \arrow[d, "\mathbf{Cospan}_f"'] \arrow[dr, bend left, "A"] \arrow[dr, phantom, "\Downarrow\alpha"] & \\ \mathbf{Cospan}_{\Lambda'} \arrow[r, bend right, "A'"'] & \mathbf{Set} \end{tikzcd}$$

III. The equivalence

Theorem

$\mathbf{Hyp}_{\mathtt{OF}}$ *and* $\mathbf{CospanAlg}$ *are (1-)equivalent.*

Proof sketch:

1. Work over Λ.
2. Frobenius monoids define cospan algebra.
3. Cospan algebras define homsets of hypergraph categories.

Lemma
There is a Grothendieck fibration $\mathsf{Gens}\colon \mathbf{Hyp}_{\mathtt{OF}} \to \mathbf{Set}_{\mathsf{List}}$ sending an objectwise-free hypergraph category to its set of generating objects.

This implies

$$\mathbf{Hyp}_{\mathtt{OF}} \cong \int^{\Lambda \in \mathbf{Set}_{\mathsf{List}}} \mathbf{Hyp}_{\mathtt{OF}(\Lambda)}$$

Note also

$$\mathbf{CospanAlg} = \int^{\Lambda \in \mathbf{Set}_{\mathsf{List}}} \mathbf{Lax}(\mathbf{Cospan}_{\Lambda}, \mathbf{Set})$$

2. Frobenius defines cospan algebras

Lemma
$\mathbf{Cospan}_\Lambda$ is the free hypergraph category over Λ (ie. with objects generated by Λ). That is, there is an adjunction

$$\begin{tikzcd} \mathbf{Set}_{\mathsf{List}} \arrow[r, shift left, "\mathbf{Cospan}\_"] \arrow[r, phantom, "\perp"] & \mathbf{Hyp}_{\mathtt{OF}} \arrow[l, shift left, "\mathsf{Gens}"] \end{tikzcd}$$

Given a hypergraph category $\mathcal{H}$ over Λ, we can construct a cospan algebra

$$A_{\mathcal{H}}\colon \mathbf{Cospan}_\Lambda \xrightarrow{\mathsf{Frob}} \mathcal{H} \xrightarrow{\mathcal{H}(I,-)} \mathbf{Set}.$$

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3. Cospans define hypergraph structure

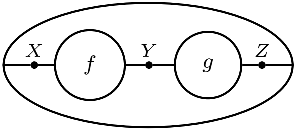
Lemma
Hypergraph categories are self dual compact closed.

Given a cospan algebra A over Λ, we may define a hypergraph category $\mathcal{H}_A$ over Λ with homsets

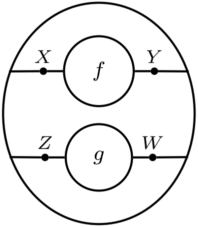
$$\mathcal{H}_A(X, Y) = A(X \oplus Y).$$

3. Cospans define hypergraph structure

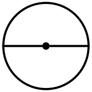
The remaining structure is defined by certain cospans.



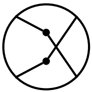
composition



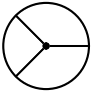
monoidal product



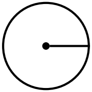
identity



braiding



(co)multiplication



(co)unit

Theorem (Coherence for hypergraph categories)
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$\mathbf{Hyp}_{\mathtt{OF}}$ and $\mathbf{CospanAlg}$ are (1-)equivalent.