Lec 1:

web.mit.edu/18.06

Fund^l prob of lin alg is to solve a sys of lin eqs

n lin. eqs, n unknowns:

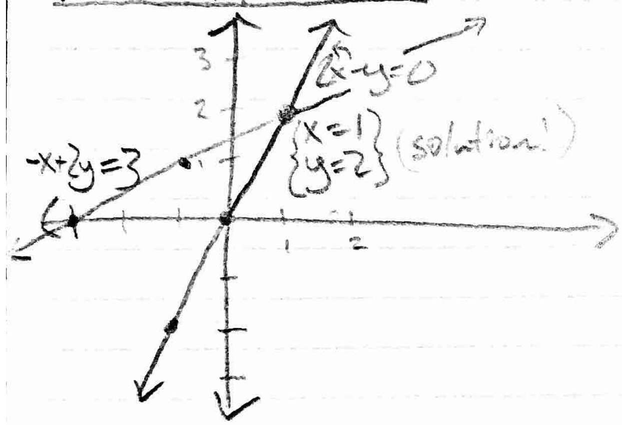
- Row picture,
- column picture
- matrix form

$$2x - y = 0$$
$$-x + 2y = 3$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A \quad x = b$$

ROW PICTURE



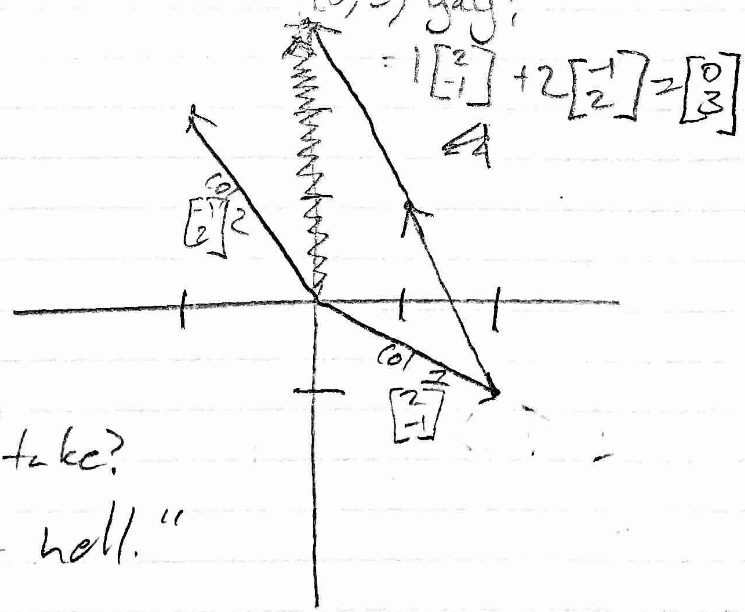
COL. PICTURE

$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Find the right linear combination!

"Multiply by some numbers and add"

"What combination should I take? Why not the right combination? What the hell."



With any x, y, can get any RHS!

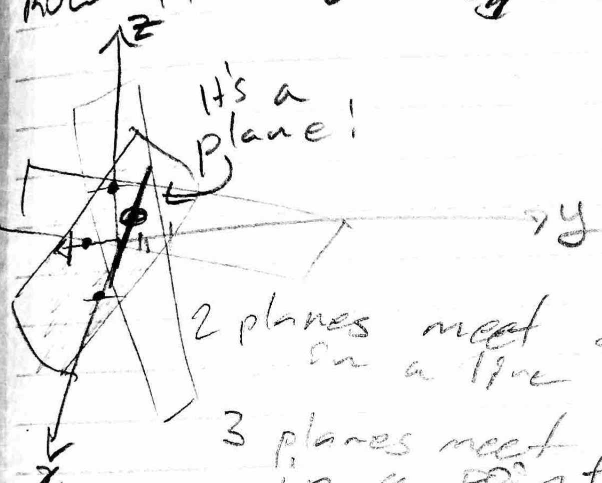
Lec 1, pg 2

$$
\begin{aligned}
2x - y &= 0 \\
-x + 2y - z &= -1 \\
-3y + 4z &= 4
\end{aligned}
$$

MATRIX FORM

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

ROW PICTURE



2 planes meet in a line ~~unless~~

3 planes meet in a point → unless they're special

Ugh! Too hard to draw!

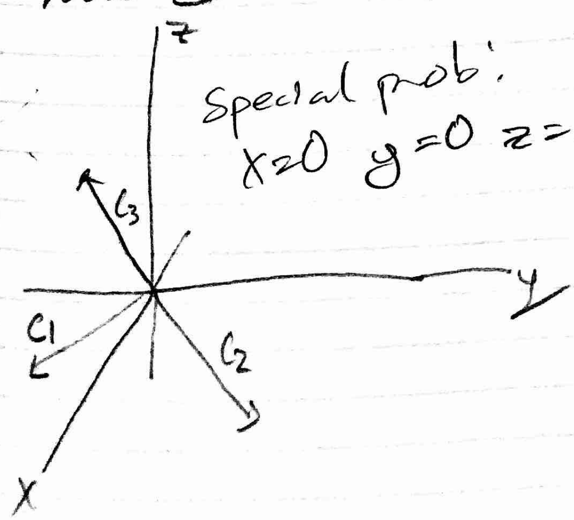
COLUMN PICTURE

$$x\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Now 3D vectors!

Special prob!

$x=0 \quad y=0 \quad z=1 \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



Lec 1, pg 3

Can I solve $Ax = b \ \exists\, b$?

Do lin combos of cols fill 3D space?

For this A, yes!
A is non-singular, invertible

If cols lie in same plane (line), then their LCs lie in that plane (line).

Mult matrix by vector: It's columns again!

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

Pset 1

(1.1) $0w_1 + 0w_2 + 0w_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

w_i are independent!

They lie on a line.

(1.2) $\begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$

(1.3) $\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \times \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$ True

lec 2 pg 1

$x + 2y + z = 2$
$3x + 8y + z = 12$
$4y + z = 2$
$Ax = b$

Mult & subtract (Elimination)

1st pivot

$$\begin{matrix} \boxed{1} & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{matrix}$$

(2,1)

$$\begin{matrix} \boxed{1} & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{matrix}$$

If ya got trouble, exchange rows

$\downarrow (3,1)$ (nothing b/c already 0)

Zero pivots are bad! → not invertible

2nd pivot

$$\begin{matrix} 1 & 2 & 1 & 2 \\ 0 & \boxed{2} & -2 & 6 \\ 0 & 4 & 1 & 2 \end{matrix}$$

$\downarrow (3,2)$

In upper-triangular matrix, product of pivots is det

$$\begin{matrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{matrix}$$ U (upper-triangular) 3rd pivot

$$\begin{matrix} 2 \\ 6 \\ -10 \end{matrix}$$ ← c

$x + 2y + z = 2 \rightarrow x = 2$
$2y - 2z = 6 \rightarrow y = 1$
$5z = -10 \rightarrow z = -2$

Back substitution

Elimination w/ Matrices

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

Permutation matrix exchange rows 1 and 2

$$P_{12} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{21} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

$$E_{32} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$E_{32}(E_{21}(A)) = U$

$(E_{32} E_{21}) A = U$

Matrix mult. is associative; Not commutative

"You're got to keep the matrices in their Gauss-given order here."

Pset 2

(2.1)

$$\begin{matrix} 2 & 3 & 5 \\ 6 & 15 & 12 \end{matrix}$$

$$\begin{matrix} 2 & 3 & 5 \\ 0 & 6 & -3 \end{matrix}$$

$$\begin{matrix} 2 & 0 & 13/2 \\ 0 & 6 & -3 \end{matrix}$$

$$\boxed{\begin{aligned} x &= 13/4 \\ y &= -1/2 \end{aligned}}$$

(2.2)

$$\overset{E}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

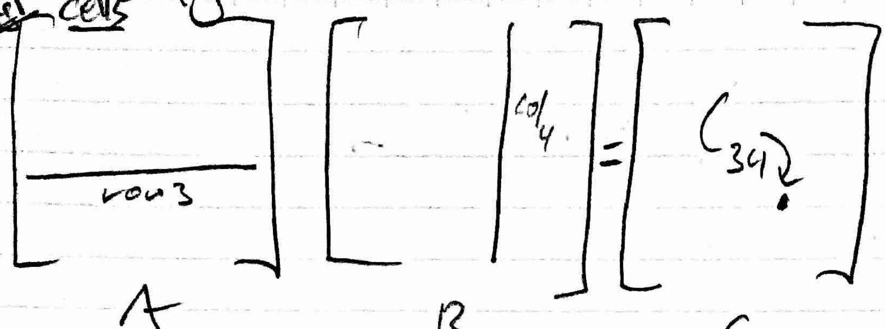
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} M$$

lec 3 pg 1

way1 cells



mxn nxp mxp

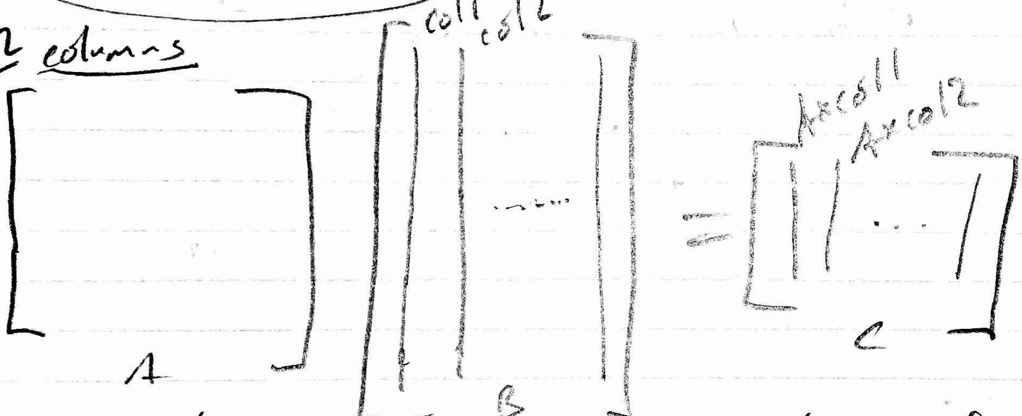
$$C_{34} = (\text{row } 3 \text{ of } A) \cdot (\text{col } 4 \text{ of } B)$$

$$= a_{31}b_{14} + a_{32}b_{24} + \ldots = \sum_{k=1}^{n} a_{3k}b_{k4}$$

Iff square, same size

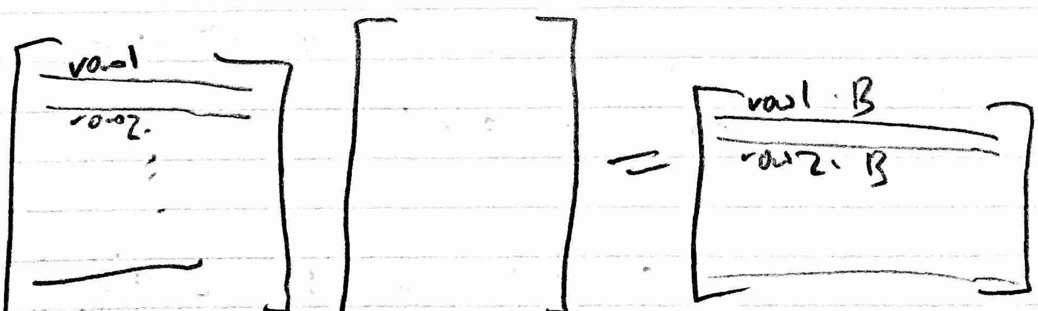
cols x rows

way2 columns



columns of C are combinations of columns of A.

way3 rows



lec 3 pg 2

way 4 rows & cols

$$\begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1 & 6\end{bmatrix} = \begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix}$$ → "minimal," as

$$AB = \sum_i col_i \times row_i$$

$$\begin{bmatrix}2 & 7\\3 & 8\\4 & 9\end{bmatrix}\begin{bmatrix}1 & 6\\0 & 0\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1 & 6\end{bmatrix} + \begin{bmatrix}7\\8\\9\end{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix}$$

Block multiplication

Blocky rows × blocky cols

$$\underbrace{\begin{bmatrix}A_1 & A_2\\A_3 & A_4\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}B_1 & B_2\\B_3 & B_4\end{bmatrix}}_{B} = \underbrace{\begin{bmatrix}A_1B_1 + A_2B_3 & \\ & \end{bmatrix}}_{C}$$

INVERSES!

if $\exists A^{-1}$

$A^{-1}A = I = AA^{-1}$

For square matrices, left inv = right inv

Invertible, nonsingular → $\exists A^{-1}$

Singular case:

eg: $\begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}$ Not invertible, because $[1\ 0]$ isn't in it's row space.

Because $Ax = 0$ for some x

$\exists A^{-1}\begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} A^{-1} \rightarrow \begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ ⇒ contradiction! ∴ A^{-1} DNE.

Lec 3 pg 3

$$\begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{bmatrix}a&\\b&\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$A \quad A^{-1}$

Like solving several systems (2 here)

$A \times$ col j of A^{-1} = col j of I

Gauss-Jordan (solve 2 eqns at once)

$$\begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \rightarrow \left[\begin{array}{cc|cc}1&3&1&0\\2&7&0&1\end{array}\right]$$

$$\begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} \qquad A \quad I$$

"And I'm gonna do whatever Gauss wants, right?"

$$\left[\begin{array}{cc|cc}1&3&1&0\\0&1&-2&1\end{array}\right]$$ Gauss is done Jordan keep going

$$\left[\begin{array}{cc|cc}1&0&7&-3\\0&1&-2&1\end{array}\right] \qquad \begin{bmatrix}1&3\\2&7\end{bmatrix}\begin{bmatrix}7&-3\\-2&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$ yay!

$I \quad A^{-1}$

Why?

$\begin{bmatrix}A & I\end{bmatrix}$ By block mult

$E\begin{bmatrix}A & I\end{bmatrix}=\begin{bmatrix}E\times A & E\times I\end{bmatrix}$

But $E \times A = I \therefore E = A^{-1}$

$= \begin{bmatrix}I & A^{-1}\end{bmatrix}$

Pset 3

3.1

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$+$

$$AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 12 \\ 20 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 12 \\ 23 & 24 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 12 \\ 23 & 24 \end{bmatrix}$$ yay!

3.2

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & b-ac & 1 & -a & -ac \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad U^{-1} = \begin{bmatrix} 1 & -a & -b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

Lec 4 p1 2017-09-13

$AA^{-1} = I = A^{-1}A$

$ABB^{-1}A^{-1} = I = B^{-1}A^{-1}AB$ put on socks then shoes
take off shoes then socks

$AA^{-1} = I$

$(A^{-1})^T A^T = I$ A^T = A transpose

$\therefore (A^T)^{-1} = (A^{-1})^T$

$A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$

$E_{21}A = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ Lower tri * upper tri
1 on diag 2 on diag

$A = L\ U$

$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$

$A = L\ D\ U$

$E_{32}E_{31}E_{21}A = U$

$A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U$
$= L\ U$

$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}}\underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix} = E$ $EA = U$

invs $\underset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\underset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L$ $A = LU$

No interference
just keep a record

If no row exchanges, multipliers go directly into L.

How many ops on $n \times n$ matrix A?
$n = 100$ how many steps?

$$\begin{bmatrix} & & \\ & 100 \times 100 & \\ & & \end{bmatrix} \to \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & & \\ 0 & 99 \times 99 & \\ 0 & & \end{bmatrix} \quad \text{cost} \approx 100^2$$

A

1 op. is (mult, subtract)

"It's not that I don't like 18.01 but 18.06 is better."

"Calculus is like sums except it's continuous."

$$\begin{bmatrix} 12 & & \\ 0 & 12 & \\ 0 & 0 & 98^2 \\ 0 & 0 & \end{bmatrix} \quad \text{cost} \approx 99^2$$

total cost $A \to LU \approx n^2 + (n-1)^2 + \ldots + 2^2 + 1^2 \approx \boxed{\frac{1}{3} n^3}$

total cost of $B \approx n^2$

B is cheap, A is expensive. Can do lots of B once have $A = LU$

3×3 perms: 6 of them it's a group! $P^{-1} = P^T$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$4 \times 4 \to 24$ P's

Pset 4 2017-09-17

4.1

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -2 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -2 & 1 & 0 \\ 0 & -6 & 1 & -2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 4 & -3 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}$$

$$L = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}}$$

Pset 4 p2 2017-09-17

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

iff $a \neq 0$, $a \neq b$, $b \neq c$, $c \neq d$

Lec 5 p1 2017-09-22

Permutations

P: execute row exchanges

$A = LU = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{bmatrix} \ddots & \\ 0 & \ddots \end{bmatrix}$

only if $P = I$ w/ row ex

$PA = LU$ ⟵ any invertible A. For almost all A, don't need P

rows in right order

$P = I$ w/ [reordered] rows.

$\exists$ n! of P poss

$\exists P^{-1}$

$P^{-1} = P^T$

$P^T P = I$

there are more of them!

$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$

$(A^T)_{ij} = A_{ji}$

Symmetric matrices:

$A^T = A$

$\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$

$R^T R$ is always symmetric $\forall$ rectangular R

$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & \cdot & \cdot \\ 7 & \cdot & \cdot \end{bmatrix}$

Logic?

$(R^T R)^T = R^T (R^T)^T = R^T R$ QED

Lec 5 p2 2017-09-22

VECTOR SPACES & SUBSPACES

$\mathbb{R}^2$ = all 2D vectors

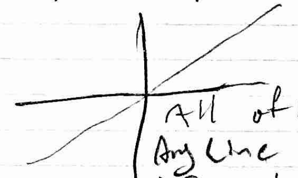
$\mathbb{R}^3$ = all 3D vectors

$\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is in $\mathbb{R}^3$

$\mathbb{R}^n$ = all nD vectors

must be closed under $*$ by scalar & $+$ ~~vecs~~

subspace: space inside other space.



NOT $\mathbb{Z}^2$ b/c not closed under $*_n$

All of $\mathbb{R}^2$

Any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in L$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ only $\leftarrow Z$

For $\mathbb{R}^3$: $\mathbb{R}^3$

plane thr origin

line thr origin

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

A

$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ ⟶ col ~~sp~~space $C(A)$ is lin combs of $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$

Pset 5

5.1

(a)
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

5.2

(a)
$$\begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix} \quad \boxed{10}$$

(b)
$$\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} \quad \boxed{6}$$

5.3

(a) True.
$$(A+B)^T = A^T + B^T = (A+B)$$

(b) True
$$A^T = -A$$
$$B^T = -B$$
$$(A+B)^T = A^T + B^T = -(A+B)$$

(c) False.
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & 5 \\ 6 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Lec 6 2017-10-02

- Vector space: can add any 2 vectors in space & mult by numbers
  - Linear combs of members are members.
  - Ex: $\mathbb{R}^3$

- Subspaces.
  - Vector space entirely contained in another
  - a plane through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is a subspace of $\mathbb{R}^3$
  - " line " " " " " "

- Given 2 subsp, plane P & line L
  - Is $P \cup L$ a subspace?
    - NO.
  - Is $P \cap L$ a subspace?
    - YES! True for any subsp S, T
    - If $V \in S$ & $V \in T$, $W \in S$ & $W \in T$
      lin combs $\in S$ and lin combs $\in T$
      $\therefore$ lin combs $\in S \cap T$
      $\therefore$ $S \cap T$ is a s.s. QED

COLUMN SPACE

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \qquad \underline{4} \times 3$$

Vectors in $\mathbb{R}^4$

~~C(A)~~ $C(A)$ is column space of A

All lin combs of columns

$$a\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + b\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + c\begin{bmatrix}2\\3\\4\\5\end{bmatrix}$$

4 equations in 3 unknowns

$$\underset{A}{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}$$

Column space is answer to ~~set of all solutions~~

For what b can this be solved?

Can be solved exactly when $b \in C(A)$

2017-10-02

Lec 6 p2

$C\left(\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\\ 3 & 1 & 4\\ 4 & 1 & 5\end{bmatrix}\right)$ is a 2D subspace of $\mathbb{R}^4$.

<u>NULL SPACE</u>

All solutions x ~~forma~~ to $Ax = 0$.

$$\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\\ 3 & 1 & 4\\ 4 & 1 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$A \quad x = b$

$N(A)$: 3 columns $\downarrow$ subsp of $\mathbb{R}^3$

Contains $\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}$

$N(A) = c\begin{bmatrix}1\\1\\-1\end{bmatrix}$

"Actually, we've used it but without proving it, but that's OK. We only live so long—just skip that proof."

Line in $\mathbb{R}^3$

If $Av = 0$ & $Aw = 0$, then
$A(v+w) = 0$
b/c $Av + Aw = 0$

Solutions to

$$\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\\ 3 & 1 & 4\\ 4 & 1 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

NOT a space

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}, \ldots$

it's a line $\notin \begin{bmatrix}0\\0\\0\end{bmatrix}$!

02 ~~Lec.~~ Lec. 7 p1 2017-10-24

Can use elimination to get null space:

$$Ax = 0$$
$$EAx = E0$$
$$\text{rref}(A)x = 0$$

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

(row) echelon form

rank of A
= # of pivots in echelon form
= 2

$$\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \text{pivot} & \text{free} & \text{pivot} & \text{free} \end{array}$$

Can give any values to free vars.

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$
$$2x_3 + 4x_4 = 0$$

1 (for x_2), 0 (for x_4)

$$x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

"special" solutions b/c gave 0 & 1 to free vars.

$r = 2$
$n - r = 4 - 2 = 2$ ← # of special sols.

2017-10-24

Lec 7 p2

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF: get 0_s above and below pivots; pivots $= 1$

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Zero rows below

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

I in pivot rows & cols!

$$\begin{array}{c|c} & \text{Pivots} \quad\quad \text{Free cols} \\ \begin{array}{c} \text{Pivots} \\ \\ \text{Zeros} \end{array} & \begin{bmatrix} \underbrace{I}_{r\text{ cols}} & \begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix} \\ 0 & \underbrace{0}_{n-r\text{ cols}} \end{bmatrix} \end{array}$$

r rows, $m-r$ rows

N is nullspace mat.

$RN = 0$

$$\begin{bmatrix} -F \\ I \end{bmatrix} \qquad Rx = 0$$

$x_{pivot} = -F x_{free} \Rightarrow x_{pivot} = -F$, $x_{free} = I$

$\Uparrow$

$x_{pivot} + F x_{free} = 0$

$\Uparrow$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$$

$\Uparrow$

$$\begin{bmatrix} I & F \\ 0 \ldots & \end{bmatrix} x = 0$$

$\Uparrow$

$rref(A)\, x = 0$

$\Uparrow$

$Ax = 0$

-24 Lec 7 p3 2017-10-24

$$~~A = \begin{bmatrix} 2 & 0 & 3 & -9 & 6 \\ 9 & -5 & 0 & 4 & 8 \end{bmatrix}~~$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$$

$$\underset{\text{sub}}{\Rightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix}$$

$$\underset{\text{Rex}}{\Rightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

$$\underset{\text{sub}}{\Rightarrow} \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \text{PC} \quad \text{PC} \quad \text{free} \end{array}$$

$\text{rank}(A) = \text{rank}(A^T)$

$r = 2$

2 special sols

$x_1 + 2x_2 + 3x_3 = 0$
$2x_2 + 2x_3 = 0$

$$x = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\boxed{x = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}$$

sub

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

plus E1

$$\begin{array}{c} \text{I} \qquad \text{F} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \text{zeros} \end{array}$$

$$\begin{bmatrix} F \\ I \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

2014-10-29

Pset 7

7.1

(a) $A = \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 2 & -2 & 11 & -3 \end{bmatrix}$

$$\downarrow$$

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 0 & -12 & -3 & -21 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$\downarrow$$

$$\boxed{\begin{bmatrix} 1 & 0 & \frac{23}{4} & \frac{29}{4} \\ 0 & 1 & \frac{1}{4} & \frac{7}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} = R}$$

(b) $\boxed{2}$

(c) $N = \begin{bmatrix} -\frac{23}{4} & -\frac{29}{4} \\ \frac{1}{4} & -\frac{7}{4} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\frac{23}{4} \\ \frac{1}{4} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -\frac{29}{4} \\ -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix}$

7.2

$A_1, A_2 \ni \text{rank}(A_1 B) = 1, \text{rank}(A_2 B) = 0$

$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\cancel{A_1 [\,0\,2\,0\,0\,]}$ $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Lec 8: $Ax = b$ 2017-10-24

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

Goal: completely solve $Ax = b$

$Ax = b$

Augmented matrix $= [A \ b]$

$\Downarrow$

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

Solvability ~~Condition on b:~~

$Ax = b$ iff $b \in C(A)$

iff a combo of rows of A gives a zero row, same combo of entries of b gives 0.

To find complete solution:

① $X_{particular}$: set all free vars to 0 (one way)

solve $Ax = b$ for pivots

$x_2 = 0$, $x_4 = 0$

$x_3 = 3/2$, $x_1 = -2$

$$\begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

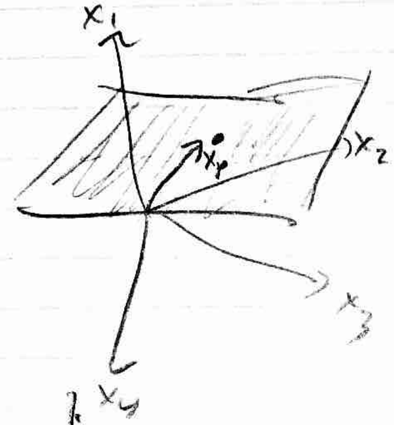
② $X_{nullspace}$: add any vector in null(A)

because:

$A(x_p + x_n) = Ax_p + Ax_n = b + 0 = b$

 lec 8, 2

$$x_{complete} = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}}_{\substack{x_p \\ \text{one guy}}} + \underbrace{c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\substack{x_n \\ \text{whole subspace}}}$$



"So I have to draw a four-dimensional picture on this MIT cheap blackboard."

<u>In general:</u>

$m \times n$ matrix of rank r (know $r \le m$, $r \le n$)

"full rank": r is as big as can be

↳ full column rank: $r = n$

→ no free vars

→ $N(A) = \{\text{zero vector}\}$

→ sol. to $Ax = b$: $x = x_p$ if $\exists\, x_p$.

(0 or 1 solns)

• if all cols. independent

$$\text{rref}\left(\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solvable only if b is a lin comb of cols, then 1 sol (x_p)

full row rank: $r = m$

↳ m pivots, every row has a pivot

Can solve for all b

$n - r = n - m$ free vars

$$\begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & \text{"F"} \\ 0 & 1 & \end{bmatrix}$$

$r = m = n$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

exactly the invertible matrices!

$rref(A) = I$

$N(A) = \{0\}$

$\forall b,\ \exists x \ni Ax = b$

Full rank (r&c) $r=m=n$	Full col rank $r=n<m$	Full row rank $r=m<n$	not full rank $r<m,\ r<n$
$R=I$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = \begin{bmatrix} I & F \end{bmatrix}$	$R = \begin{bmatrix} I & F \\ 0 & \end{bmatrix}$
1 sol to $Ax=b$	(0-1 solns)	(∞ solns b/c of N)	(0 or ∞ solns)

Lec 9 2017-10-25

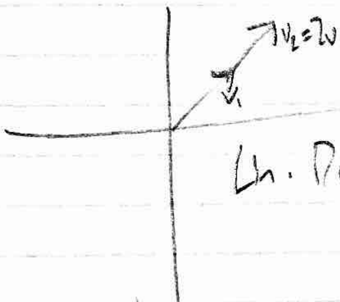
Linear independence (of vectors)
Spanning a space
Basis for a subspace

- Suppose A is $m \times n$, $m<n$
  Then $\exists$ nonzero sols to $Ax=0$
  Because there are free vars (≥ 1), can give nonzero vals to those.

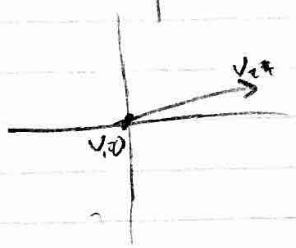
~~Vec indep~~

__Linear independence__

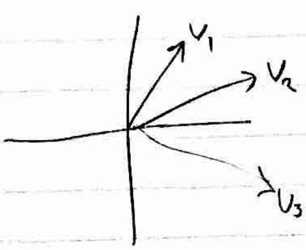
Vectors $x_1, x_2, \ldots x_n$ independent if no linear combination
$$c_1x_1 + c_2x_2 + \ldots + c_nx_n = 0$$
except when $c_i = 0$



Lin. Dep. b/c $2v_1 - v_2 = 0$



$3071 v_1 + 0 v_2 = 0$



Lin dep. A
$$\begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \end{array} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

No free vars → Cols. are independent if $N(A) = \{0\}$

Yes free vars → dependent if $Ac = 0$ $\exists c \neq 0$

Lec 9 p2 2014-10-25

1 SPANNING A SPACE

- Vectors $v_1, \dots v_\ell$ span a space if the space consists of all the LCs of $v_1, \dots v_\ell$

~~to span means~~

BASIS

- A basis for a space is a sequence of vectors $v_1, v_2, \dots, v_d$ such that:
  1. they are independant.
  2. they span the space.
- Basis tells all about a subspace.

Ex: $\mathbb{R}^3$

- $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ is one basis.
  - Ind b/c $N\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right) = \{0\}$
- Another basis: $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$ doesn't work

- To test, put in a matrix, see if you get free vars.

- Given a space, every basis has the same # of vectors, called the dimension.

- $\begin{array}{cccc} & & & \\ \end{array}$ $\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix}$ rank (A) = # pivot columns = dimension of $C(A)$

  in col space

- If you have dim vectors, all ind., must be a basis.

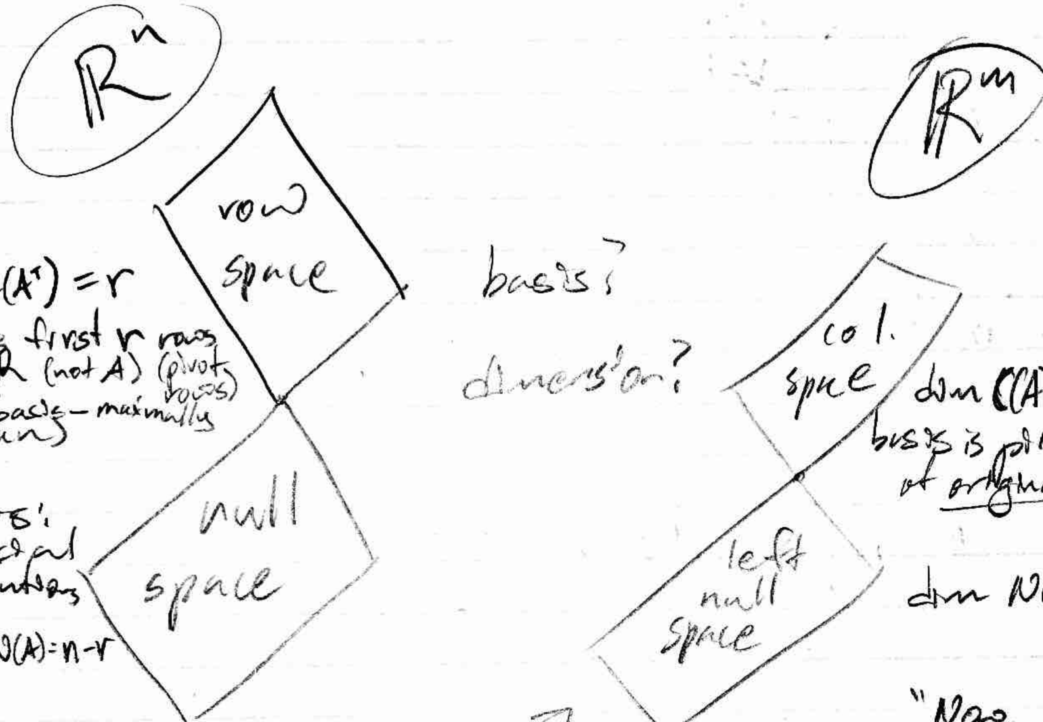
$\dim C(A) = r$

$\dim N(A) = \#\text{free vars} = n-r$

Lec 10 p1 2017-10-25

4 SUBSPACES

- Column space $C(A) \to$ in $\mathbb{R}^m$
- Null space $N(A) \xrightarrow{\text{mean}} \to$ in $\mathbb{R}^n$
- Row space $C(A^T) \to$ in $\mathbb{R}^n$
  - All LCs of rows of A
  - Column space of A^T
- Null space of the transpose $N(A^T) \to$ in $\mathbb{R}^m$
  "Left null space of A"



dim $C(A^T) = r$
basis is first r rows of R (not A) (pivot rows)
("best" basis - maximally clean)

basis: special solutions
dim $N(A) = n - r$

dim $C(A) = r$
basis is pivot columns of original A (not R)

dim $N(A^T) = m - r$

"Now, what's the dimension of this poor, misbegotten fourth subspace?"

$y^T A = [0 \dots]$
$A^T y = 0$

$$EA = R \quad \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis is rows of E that correspond to 0 rows

New vector space: M = {all 3x3 matrices!}

Subspaces:

All upper tri matrices
All symmetric matrices
All diagonal matrices $\to$ dim = 3, basis = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Pset 8 2017-10-25

8.1 (a) The complete solution is x_p plus a L.C. of x_n.

(b) If the dimension of $N(A)$ is nonzero, $Ax=b$ has inf'ly many non particular solutions.

(c) If A is invertible the null space contains 0.

8.2

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$x_n = c\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad x_p = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$x = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + c\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

8.3 $N(A) = N(C)$

$A^{-1}b = C^{-1}b = x \quad \forall b$

$$\begin{bmatrix} A^{-1} \end{bmatrix} \begin{bmatrix} b \end{bmatrix}$$

~~$A^{-1}b$~~

for $b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $x =$ first col of A = first col. of C

for $b = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ $x =$ 2nd col A = 2nd col C

$\vdots$

for $b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ $x =$ last col A = last col C

since all cols of A = of C

$\therefore A = C$ QED.

25 Pset 9 2017-10-25

9.1

$V_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, V_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\{V_1, V_2, V_3\} \quad (3)}$$

9.2

$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = 0$

$\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\boxed{\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}}$

$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = 0$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \to \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = 0$

$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = 0$

$$\boxed{\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

Pset 10 2017-10-25

(10.1)

(a) $r<m$, $r\leq n$ because $\dim C(A) < m$ ~~the column space's dimension~~

(b) $r<n$, $r\leq m$

$\therefore$ there are free columns

$\therefore N(A^T) \neq \{0\}$

(10.2)

Column space.

Nullspace.

lec 11

2017-10-26

Todo: new spaces
~~R~~ rank 1 matrices
small world graphs

NEW SPACES

M: all 3x3
- Symmetric 3x3
- Upper-tri 3x3

Basis for M: dim=9

$$\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}, \ldots$$

$$\ldots \quad \ldots \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix}$$

Symmetric matrices: S

dim=9

Basis:

$$\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix}$$

$$\begin{bmatrix} 0&1&0\\1&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&1\\0&1&0 \end{bmatrix}.$$

Upper-tri matrices: U

dim=6

$$\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix} \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}$$

$$\begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0\\0&0&1\\0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix}$$

Diagonal matrices; $S \cap U$

dim=3

$$\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix}$$

$S \cup U$ is not a subspace

~~$S \cup U$~~

$S+U$: all LCs of members of S & of U

= any element of S + any element of U

= all 3x3 = M

$$\dim S + \dim U = \dim(S \cap U) + \dim(S+U)$$

$$6+6 = 3+9$$

$$\frac{d^2y}{dx^2} + y = 0$$

$y = \cos x, \sin x, e^{ix}, \ldots$

(sort of a null spaces)

Complete solution:

$y = c_1 \cos x + c_2 \sin x$

vector space!

Basis: $\{\sin x, \cos x\}$

$\dim\left(\substack{\text{solution}\\\text{space}}\right) = 2$

"Five minutes of 18.06 is enough to take care ~~of~~ of 18.03."

Lec 11 p2

2017-10-26

Rank 1 matrices:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

Basis for $C(A^T)$: $\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

Basis for $C(A)$: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$r = 1$

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$$

Every $r=1$ matrix has the form uv^T

"Bldg blocks": every rank r matrix is the sum of r rank-1 matrices.

~~$R(A) + R(B) \geq R(A+B)$~~

$R(A+B) \leq R(A) + R(B)$

$\mathbb{R}^4$: $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

$S =$ all v in $\mathbb{R}^4$

with $v_1 + v_2 + v_3 + v_4 = 0$

$\dim = 3$

$S = N(\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix})$

$r = 1 \rightarrow \dim(N(A)) = n - r = 3$

Basis: special solutions

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

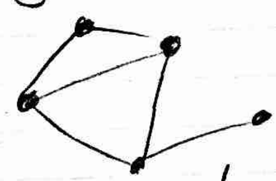
$C(A^T) = c\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$C(A) = c[1]$

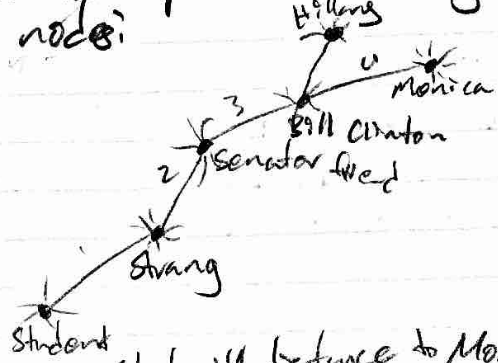
$N(A^T) = \{0\}$

Small world graphs:

A graph is a bunch of nodes & edges connecting them



How far apart are any two nodes:



Student's distance to Monica is 4

"All your Clinton distances immediately drop down to three by taking linear algebra. That's, they are added bonus for taking linear algebra."

2017-10-26

Lec 12 p1

Applications!

- Graphs + Networks
- Incidence Matrices
- Kirchoff's Laws

$e = Ax$

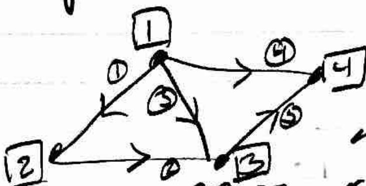
$y = Ce$

$A^T y = f$

$x_1 \dots x_n$ = pots @ nodes

$A^T C A x = f$

Graph: nodes & edges



directions are arbitrary (for now)

rows

m = 5 edges (1 for each edge)

n = 4 (# of nodes)

Incidence Matrix:

$$\begin{array}{c c c c c c}
 & \text{node } 1 & 2 & 3 & 4 & \\
A = & \left[\begin{matrix} -1 \\ 0 \\ -1 \\ -1 \\ 0 \end{matrix}\right. & \begin{matrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{matrix}\right] & \begin{matrix} \text{edge } 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}
\end{array}$$

Loops correspond to L.D. rows

Really big matrix but mostly zeros

Solve $Ax = 0$

$$Ax = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

x = potentials of nodes

Ax = Differences in potential across edges

$$x = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \dim N(A) = 1$$

Means you ~~can't~~ can only calculate potentials to a constant.

$r = 3$

5×3 u $r=3$

$$R=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\end{bmatrix} \quad N(R)=\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

$$\begin{bmatrix}R\\0\end{bmatrix} \quad \begin{bmatrix}u&u\\u&0\end{bmatrix}\to\begin{bmatrix}u&u\\0&-u\end{bmatrix}\to\begin{bmatrix}u&0\\0&-u\end{bmatrix}$$

$$Ax=\begin{bmatrix}2\\4\\2\end{bmatrix} \quad x=\begin{bmatrix}2\\0\\0\end{bmatrix}+c\begin{bmatrix}1\\1\\0\end{bmatrix}+d\begin{bmatrix}0\\0\\1\end{bmatrix} \quad \to\begin{bmatrix}u&0\\0&u\end{bmatrix}$$

$\downarrow$

3×3 $A=\begin{bmatrix}1&-1&0\\2&-2&0\\1&-1&0\end{bmatrix}$ $N(CD)=N(D)$ if C is invertible

$r=1$

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$