Lec 14: Orthogonality

2017-11-27

Orthogonal = perpendicular

$x^Ty = 0 \quad (\vec{x}\cdot\vec{y} = 0)$

$\|x\|^2 + \|y\|^2 = \|x+y\|^2$ if $x^Ty = 0$

$x^Tx + y^Ty = (x+y)^T(x+y)$

$x^Tx + y^Ty = x^Tx + y^Ty + x^Ty + y^Tx$

$x^Ty + y^Tx = 0 \quad (x^Ty = y^Tx)$

$2x^Ty = 0$

$\boxed{x^Ty = 0}$

"The one thing about math is you're supposed to follow the rules."

"& The floor... is not a great subspace MIT only built it so-so."

Two subspaces are orthogonal if every vector in one is ortho. to every vector in the other.

Row space is ortho to nullspace. $x \in \mathbb{Z}$

$A_1^T x = 0$

$A_2^T x = 0$

$A_3^T x = 0$

$(c_1 row_1 + c_2 row_2 + \ldots)^T x = 0 \quad \text{Q(-)}$

Column space is ortho. to left nullspace.

Nullspace & row space are ortho. complements in $\mathbb{R}^n$.

A^TA: square, symmetric

Claim: $A^TA\hat{x} = A^Tb$: $\hat{x}$ has the best soln.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} x = b$$

$$A^TA = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

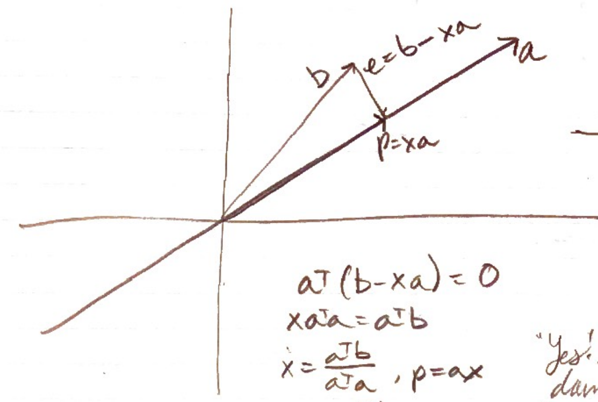
$N(A^TA) = N(A)$

rank A^TA = rank A

A^TA is invertible iff A has ind. cols.

Lec 15: Projections

2017-11-12



$a^T(b - xa) = 0$

$xa^Ta = a^Tb$

$x = \frac{a^Tb}{a^Ta}, \; p = ax$

$p = \frac{a^Tb}{a^Ta}$

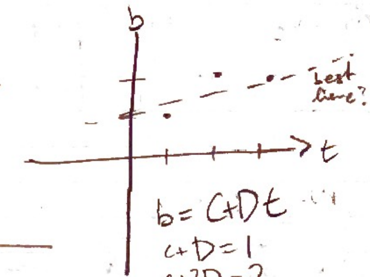
Projection matrix $P = \frac{aa^T}{a^Ta}$

rank $P = 1$

$\boxed{P^T = P, \; P^2 = P}$

Instead of $Ax = b$ solve $Ax = p$

p is proj of b onto $C(A)$



$b = C + Dt$

$C + D = 1$

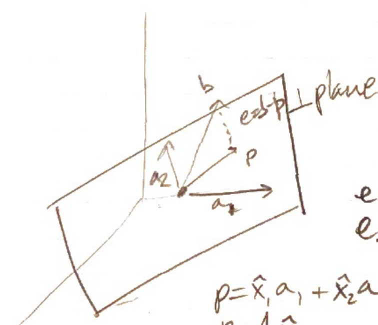
$C + 2D = 2$

$C + 3D = 2$

"You'll just feel, OK, the damn thing came out right!"

"Eight A's in a row is quite obscure, but... you see that it works?"

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$



$e \in N(A^T)$

$e \perp C(A)$

$p = \hat{x}_1 a_1 + \hat{x}_2 a_2 + \ldots$

$p = A\hat{x}$

$b - A\hat{x}$ is perp. to plane

$a_1^T(b - A\hat{x}) = 0$

$a_2^T(b - A\hat{x}) = 0$

$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} = A^T$

$A^T(b - A\hat{x}) = 0$

$A^TA\hat{x} = A^Tb$

$\hat{x} = (A^TA)^{-1}A^Tb$

$p = A\hat{x} = A(A^TA)^{-1}A^T b$

$P = A(A^TA)^{-1}A^T \leftarrow$ Proj. matrix

Note that if A is square then $P = I$.

Psets 14~15 2017-11-28

(14.1)
$$\begin{aligned} x_1 - x_2 \quad &= 1 \\ x_2 - x_3 &= 1 \\ -x_1 \quad + x_3 &= -1 \\ \hline 0 &= 1 \end{aligned} \qquad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

(14.2) $\boxed{r \perp n,\ c \perp l}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad c = xa,\ d = xb$, $b = ya$

$A = a\begin{bmatrix} 1 & y \\ x & xy \end{bmatrix}$ ~~Best B(C(A))=~~ basis for $C(A)$: $\begin{bmatrix} 1 \\ x \end{bmatrix}$

" " $C(A^T)$: $\begin{bmatrix} 1 \\ y \end{bmatrix}$

" " $N(A)$: $\begin{bmatrix} -y \\ 1 \end{bmatrix}$

" " $N(A^T)$: $\begin{bmatrix} -x \\ 1 \end{bmatrix}$ Q.E.D.

$\boxed{\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}}$

(15.1) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$A \quad \vec{x} \approx \vec{b}$

$A^T A \vec{x} = A^T \vec{b}$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$I\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $P \neq I$. Not square.

$\boxed{\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$

$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

P is square.

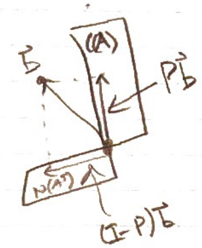
(15.2)
$$\begin{aligned} &(I-P)^2 \\ &= (I-P)(I-P) \\ &= (I-P)I - (I-P)P \\ &= I^2 - PI - IP + P^2 \\ &= I - 2P + P \\ &= I - P, \quad \text{Q.E.D.} \end{aligned}$$

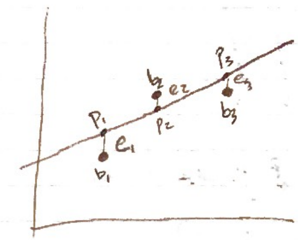
Nullspace.

Lec 16 2017-11-29

If $\vec{b} \in C(A)$, $P\vec{b} = \vec{b}$

If $\vec{b} \perp C(A)$, $P\vec{b} = 0$





$C + D = 1$

$C + 2D = 2$

$C + 3D = 2$

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$A \quad \vec{x} = \vec{b}$

Minimize $\|e\|^2 = \|A\vec{x} - \vec{b}\|^2$

$= (C+D-1)^2 + (C+2D-2)^2 + (C+3D-3)^2$

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$

$A^TA\hat{x} = A^T\vec{b}$

$\begin{bmatrix} 3 & 6 & 5 \\ 6 & 14 & 11 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 5/3 \\ 0 & 2 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 1/2 \end{bmatrix}$

Best line: $y = \frac{2}{3} + \frac{1}{2}t$

$\hat{x} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$, $\vec{p} = \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix}$, $\vec{e} = \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$

Suppose $A^TAx = 0$.

$x^TA^TAx = 0$

$(Ax)^TAx = 0$

$Ax = 0$

$x = 0$ Q.E.D.

Pset 16 2017-11-29

18.1 $(-1, 7),\ (1, 7),\ (2, 21)$

$C - D = 7$
$C + D = 7$
$C + 2D = 21$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$$

$A^TA\vec{x} = A^Tb$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}\left[\begin{array}{cc|c} 1 & -1 & 7 \\ 1 & 1 & 7 \\ 1 & 2 & 21 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 2 & 35 \\ 2 & 6 & 42 \end{array}\right]$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\vec{x} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & -7 \\ 1 & 3 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 21 \\ 0 & 7 & 28 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & 4 \end{bmatrix}$$

$\boxed{b = 9 + 4t}$

$\vec{x} = (A^TA)^{-1}A^Tb$
$A\vec{x} = A(A^TA)^{-1}A^T\vec{b}$
$= Pb$

16.2
$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix} = \vec{p}$$

$$\begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} - \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3/7 & -1/7 \\ -1/7 & 3/14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4/7 & -5/14 \\ 2/7 & 1/14 \\ 1/7 & 2/7 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \cancel{\begin{bmatrix} -3 & 5 & -7 \\ 13 & -3 & 20 \\ 21 & 7 & 35 \end{bmatrix}}$$

$$\frac{1}{14}\begin{bmatrix} 5 & 3 & 6 \\ 3 & 13 & -2 \\ 6 & -2 & 10 \end{bmatrix}$$

Pset 16 contd 2017-11-29

16.3
$$\frac{1}{14}\begin{bmatrix} 5 & 3 & 6 \\ 3 & 13 & -2 \\ 6 & -2 & 10 \end{bmatrix}\begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix} = \Big[$$

Lec 17 2017-11-30

Orthonormal vectors

$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

made-up word

"Orthonormal matrix": matrix Q with O.N. columns

"Orthogonal matrix": O.N. matrix, square

real word

$Q^T = Q^{-1}$ if $\exists Q^{-1}$

$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ "adhemar matrix": ⊥Q made of ±1

Orthonormal vectors are indep.

Q has ON cols ← what Q means

$P = Q(Q^TQ)^{-1}Q^T = QQ^T$ {= I if Q is square}

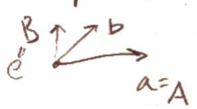
$A^TA\hat{x} = A^Tb$

$Q^TQ\hat{x} = Q^Tb$

$\hat{x} = Q^Tb$ don't need inversion!

Gram-Schmidt Orthogonalization

indep. vectors a, b → orthogonal $\vec{A}, \vec{B}$ → orthonormal q_1, q_2



$B = b - \frac{\vec{A}^Tb}{\vec{A}^T\vec{A}}\vec{A} = b - proj_A$

$C = c - \frac{\vec{A}^Tc}{\vec{A}^T\vec{A}}\vec{A} - \frac{\vec{B}^Tc}{\vec{B}^T\vec{B}}B$

Lec 17 contd 2017-11-30

$a = A = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $b = \begin{bmatrix}1\\0\\2\end{bmatrix}$, $B = \begin{bmatrix}1\\0\\2\end{bmatrix} - \frac{3}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$

$Q = [q_1 \; q_2] = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$

$A = QR$ R is upper-tri.

Rect 17

2017-11-30

(17.1)

$Qx = 0$

$Q^TQx = Q^T0$

$x = 0$ Q.E.D.

(17.2) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$A = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

$B = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{A^Tb}{A^TA} a = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{-1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix}$

$q_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \\ 0 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ -1/\sqrt{6} \\ -1/\sqrt{6} \\ \sqrt{2}/\sqrt{3} \end{bmatrix} \rightarrow Q = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{3} & 1 & 0 \\ -\sqrt{3} & 1 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

$C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \frac{A^Tc}{A^TA} A - \frac{B^Tc}{B^TB} B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - 0\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{-1}{2}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

$[1 \ 1 \ 1 \ 1]x = 0$

$\{A, B, C\}$ and $\{a, b, c\}$ are independent.

$\text{rank}([1 \cdots 1]) = 1 \Rightarrow \dim N([1 \cdots 1]) = 3$

$\{A, B, C, a, b, c\} \subset N([1 \cdots 1])$

$\therefore \{A, B, C\}$ and $\{a, b, c\}$ are bases. QED

Lec 18

Determinants

~~written~~ written $|A|$ or $\det A$

Only for square matrices

① $\det I = 1$ $\qquad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

② when you exchange rows, you ~~get~~ reverse the sign. $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

③ ⓐ $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

ⓑ $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

If two rows are equal:

can exchange them

$|A| = |A'|$

$|A| = -|A|$

$|A| = 0$

$\det A = \det U = \det R$

$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l\begin{vmatrix} a & b \\ a & b \end{vmatrix}$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Row of zeros $\Rightarrow \det A = 0$

$\det U = \begin{vmatrix} d_1 & \cdots & \cdots \\ 0 & d_2 & \cdots \\ 0 & 0 & d_3 \\ 0 & 0 & 0 & d_n \end{vmatrix} = (-1)^{\#\text{ of rexes}}(d_1)(d_2)(d_3)\cdots(d_n)$

$\det A = 0$ iff A is singular.

$\det(AB) = (\det A)(\det B)$ $\qquad$ *note $\det(A+B)$ is not $\det A + \det B$

$\det A^{-1} = \frac{1}{\det A}$

$\det(A^T) = \det A$

Pset 18

18.1

If the entries of every row sum to 0, then ∃ a lin. comb of rows that equals $(0 \dots 0)$. Then $\det A = 0$.

Subtracting I from A gives a matrix whose rows add to 0.

Yes.

18.2

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \quad \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}} \quad \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{bmatrix}} \quad = (a-b)(c-a) \begin{vmatrix} \cancel{1} & \cancel{a} & \cancel{a} \\ 0 & \cancel{1} & \cancel{b+a} \\ \cancel{0} & \cancel{1} & c+a \end{vmatrix}$$

$$\cancel{\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{bmatrix}} \quad \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$$

$$= \cancel{(a-b)(c-a)} \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & \end{vmatrix}$$

$$= (a-b)(c-b)(c-a)$$

Lec 19

2017-12-06

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$= \underset{\text{sing}}{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}} + \underset{ad}{\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}} + \underset{-bc}{\begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}} + \underset{\text{sing}}{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}$$

$$= ad - bc$$

$$\det A = \sum \pm a_{1\alpha} a_{2\beta} c_{3\gamma} \dots a_{n\omega}$$

cofactor:

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix}$$

$$\det A = a_{11} C_{11} + a_{12} C_{12} + \dots + a_{1n} C_{1n}$$

Strang 2020

"Onward to cofactors."

"Oh, hell. Oh, I shouldn't have said 'hell', because it's all right."

Lec 20

2017-12-06

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & \vdots \\ C_{n1} & \cdots & & C_{nn} \end{bmatrix}^T$$

cofactor matrix

$Ax = b$

$x = A^{-1}b = \frac{1}{\det A} C^T b$

$x_1 = \frac{\det B_1}{\det A}$ ← Cramer's Rule

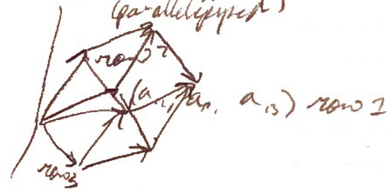
$x_2 = \frac{\det B_2}{\det A}$ MUCH slower than elimination!!

$B_1 = \begin{bmatrix} b & \begin{matrix} n-1 \\ \text{cols of} \\ A \end{matrix} \end{bmatrix}$

B_j: a with col j replaced by b

Mathematics for the Million – Bell

det A = ±volume of a box (parallelepiped)



If orthogonal: $Q^TQ = I$

$|Q|^2 = 1$

$Q = \pm 1$

$A_{\text{simplex}} = \frac{1}{n!} \det A$

Lec 21

2017-12-?

Eigenvalues, Eigenvectors

☐ proj. math for vec. sp?

☐ ~~tr~~ trace

$\det[A - \lambda I] = 0$

Eigenvector: x s.t. $Ax = \lambda x$

Eigenvalue: λ

If A is singular, $\lambda = 0$ is eval.

If A is a proj matrix:

$Px = x$ iff x in the plane or $x \perp$ plane

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda = 1$

$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\lambda = -1$

"A repeated λ is the source of all trouble in 18.06."

$\sum a_{ii}$ is called the trace

sum of evals is ~~called~~ trace.

$Ax = \lambda x$

$(A - \lambda I)x = 0$

$\therefore A - \lambda I$ is singular

$\therefore \det(A - \lambda I)$ is 0.

Find λ first.

trace $= \lambda_1 + \lambda_2$

det $= \lambda_1 \lambda_2$

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$\lambda_1 = 4, \lambda_2 = 2$

In 2x2 case: $\lambda^2 - t\lambda + d = 0$ (t = trace, d = det)

If I add nI to A, λs increase by λ and xs don't change.

$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Lec 21 2017-12-09

Antisymmetric matrices have ~~tme~~ imaginary evals.

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rightarrow i, -i$

$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \rightarrow \det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda)$

$\lambda_1 = 3, \lambda_2 = 3$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0 \rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, NO 2nd evec

Lec 22 2017-12-09

$S^{-1}AS = \Lambda$

□ diag → mat
□ ~~mat~~ → diag

Suppose A has n indep evecs

$AS = A\begin{bmatrix} x_1 & x_2 & x_3 \dots \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 \dots \end{bmatrix}$

$= S\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}$

$AS = S\Lambda$

If n indep evecs:

$S^{-1}AS = \Lambda$

$A = S\Lambda S^{-1}$

A ~~as~~ is a matrix times a diag times the inverse

~~If~~ If $Ax = \lambda x$,

$A^2 x = \lambda^2 x \Rightarrow$ x are same, λ are squared

$A^2 = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1}$

$A^k = S\Lambda^k S^{-1}$

$A^k \rightarrow 0$ as $k \rightarrow \infty$ if all $|\lambda_i| < 1$

$u_{k+1} = Au_k$

start with u_0

$u_k = A^k u_0$

~~Formally~~

$u_0 = c_1 x_1 + c_2 x_2 + \dots = Sc$

$Au_0 = c_1\lambda_1 x_1 + c_2\lambda_2 x_2 + \dots$

$A^{100} u_0 = c_1 \lambda_1^{100} x_1 + \dots$

$= A^{100} Sc$

$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\begin{bmatrix} \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$

$x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

Fibonacci

$F_{k+2} = F_{k+1} + F_k$ $\quad u_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} u_k$

$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \lambda^2 - \lambda - 1 = 0$

$\lambda = \frac{1 \pm \sqrt{1+4}}{2}$

$\lambda = \frac{1 \pm \sqrt{5}}{2} \Rightarrow \lambda_1 = \frac{1}{2}(1+\sqrt{5}) \approx 1.618\dots$

$\lambda_2 = \frac{1}{2}(1-\sqrt{5}) \approx -.618$

$F_{100} \approx c_{100}\left(\frac{1+\sqrt{5}}{2}\right)^{100}$

Psets 19-20

2017-12-09

□ cofactor matrix

19.1

$$\cancel{\det}\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

Cofactors are easiest.

19.2

$\det P_n = 1$

$\det(P_n') = \det P_n + \begin{vmatrix} \text{top rows} \\ 0 0 \dots 1 \end{vmatrix}$

$\det P_n^{-1} = \det P_n - \det P_{n-1}$

$= 0$

QED

20.1

$$C = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 1 & 1 \\ -6 & 3 & 0 \end{bmatrix}$$

$$AC^T = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$C_{1,3} = 0.$

20.2

$$\det \begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}$$

$= \rho^2((\cos^2\phi\ \sin\phi)(\cos^2\theta + \sin^2\theta) + (\sin^2\phi\sin\phi)(\cos^2\theta + \sin^2\theta))$

$= \rho^2\sin^2\phi$

Pset 21

□ Evals

□ Evecs

21.1 (a) 3

(b) $B^TB = S\Lambda S^{-1} S^{-1^T}\Lambda S^T = S\Lambda(S^TS)^{-1}\Lambda S^T$

$\det B^TB = 0$

(c) ~~no~~ no

~~B2n~~

$\cancel{SA}$ $SAS^{-1} = I$

$SA = SI$

$A = I$

(d) $(S\Lambda^2S^{-1} + I)^{-1}$

$(S\Lambda^2S^{-1} + SIS^{-1})^{-1}$

$(S(\Lambda^2+I)S^{-1})^{-1}$

$S(\Lambda^2+I)^{-1}S^{-1}$

1, 2, 3

$\{1, \frac{1}{2}, \frac{1}{3}\}$

21.2

A: $\{1, 4, 6\}$

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & -\lambda \end{vmatrix} = 2\lambda^2 - \lambda^3 + 3\lambda - 6$$

$\lambda^3 - 2\lambda^2 - 3\lambda + 6$

$(\lambda-2)(\lambda^2-3)$

B: $\{2, \pm\sqrt{3}\}$

$$\begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix}$$

$\alpha = \frac{\lambda}{2}$

$= 8((1-\alpha)^3 + 2 - 3(1-\alpha)) = 0$

$\alpha^3 - 3\alpha^2 + \cancel{3\alpha} + \cancel{1} \cancel{2} - \cancel{3\alpha} + \cancel{1} = 0$

$\alpha^3 - 3\alpha^2 = 0$

$(\alpha^2)(\alpha - 3) = 0$

C: $\{0, 6\}$

Pset 22

2017-12-09

22.1 $(4-\lambda)(2-\lambda)=0$

$\lambda \in \{2, 4\}$

$$\begin{bmatrix}2 & 0\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0\\1 & -2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}$$

$$S \in \left\{\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix} * \begin{bmatrix}x & 0\\0 & y\end{bmatrix} \;\middle|\; x, y \in \mathbb{C}\right\}$$

22.2

$$\begin{bmatrix}.6 & .9\\.4 & .1\end{bmatrix} = \frac{1}{10}\begin{bmatrix}6 & 9\\4 & 1\end{bmatrix}$$

$\alpha = 10\lambda$

$(6-\alpha)(1-\alpha)-36=0$

$\alpha^2-7\alpha-30=(\alpha-10)(\alpha+3)=0 \Rightarrow \lambda \in \{1, -.3\}$

$$\Lambda = \begin{bmatrix}1 & 0\\0 & -.3\end{bmatrix}$$

$$S = \begin{bmatrix}9 & 1\\4 & -1\end{bmatrix}$$

Sec 23

2017-12-11

1st-order const coeff ODEs system

$$\frac{du_1}{dt} = -u_1 + 2u_2$$
$$\frac{du_2}{dt} = u_1 - 2u_2$$

$$A = \begin{bmatrix}-1 & 2\\1 & -2\end{bmatrix} \qquad \frac{du}{dt} = Au$$

$\lambda = 0, -3$

$\lambda_1 = 0$

$\lambda_2 = -3$

set $u = Sv$

$x = \begin{bmatrix}2\\1\end{bmatrix}$ $Ax_1 = 0x_1$ $\qquad S\frac{dv}{dt} = ASv$

$x_2 = \begin{bmatrix}1\\-1\end{bmatrix}$ $Ax_2 = -3x_2$

$$u(t) = c_1 e^{\lambda_1 t}x_1 + c_2 e^{\lambda_2 t}x_2$$

$$\begin{bmatrix}u_1\\u_2\end{bmatrix} = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2 e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix}$$

$Sc = u(0)$

if $c_1 = c_2 = \frac{1}{3}$:

$$\frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix} + \frac{1}{3}e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix}$$

1) Stability: if all $Re(\lambda) < 0$
2) Steady state if $\lambda_1 = 0$ and other $\lambda_i \le 0$
3) blows up if any $\lambda > 0$

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \dots + \frac{(At)^n}{n!} + \dots$$

$$= SS^{-1} + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}}{2}t^2 + \dots + \frac{S\Lambda^n S^{-1}}{n!}t^n + \dots$$

$$= S\left(I + \Lambda t + \frac{\Lambda^2 t^2}{2} + \dots + \frac{\Lambda^n t^n}{n!} + \dots\right)S^{-1}$$

$$= S e^{\Lambda t} S^{-1}$$ (assuming A is diagonalizable)

$$e^{\Lambda t} = \begin{bmatrix}e^{\lambda_1 t} & & 0\\ & \ddots & \\ 0 & & e^{\lambda_n t}\end{bmatrix}$$

Lec 23 2017-12-11

$y'' + by' + ky = 0$

$u = \begin{bmatrix} y' \\ y \end{bmatrix}$

$u' = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix}$

$\frac{du}{dt} = Au$

$\int \cancel{du} = \cancel{\int Au\,dt}$

$\cancel{A\int u\,dt}$

$u = e^{At}c$

$u = Se^{\Lambda t}S^{-1}$

Lec 24 2017-12-1[illegible]

$A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$

Markov matrix

~~all powers~~:

① all entries ≥ 0

② all cols sum to 1

(also holds for all A^n)

Goes to a steady state iff $0 \leq \lambda_i \leq 1 \ \forall i$

Any Markov matrix has $\lambda_1 = 1$ because:

Consider $A - I$

All cols add to zero $\therefore [1\ 1\ 1] \in N(A^T) \therefore$ rows are dep.

$\therefore$ singular

$\therefore \det = 0$

$\therefore \det A = 1$ QED

λ of $A = \lambda$ of A^T

"This is the first time in the history of [LA] that an eigenvector has had

$\begin{bmatrix} u_{CA} \\ u_{MA} \end{bmatrix}_{t=k+1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} u_{CA} \\ u_{MA} \end{bmatrix}_{t=k}$

$\begin{bmatrix} u_{CA} \\ u_{MA} \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$ Populations after 100 yrs?

$\lambda_1 = 1, \ \lambda_2 = .7$

$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Projection w orthonormal

expand $q_1, \ldots q_n$

$v = x_1 q_1 + x_2 q_2 + \ldots$

$q_1^T v = x_1$

Lec 24

2017-1012

Fourier Series

$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos(2x) + \dots$

vectors

$v^T w = v_1 w_1 + \dots + v_n w_n$

functions

$f^T g = \int_0^{2\pi} f(x) g(x)\, dx$

$a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x \, dx$

Pset 23

23.1

$\frac{du}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} u$

$\begin{vmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{vmatrix} = -\lambda^3 + \lambda(a^2+b^2+c^2) = 0$

$\lambda(\lambda^2 - (a^2+b^2+c^2)) = 0$

$\lambda \in \{0, \pm\sqrt{a^2+b^2+c^2}\}$ $\quad \sqrt{a^2+b^2+c^2} = s$

$x_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\begin{bmatrix} -s & c & -b \\ -c & -s & a \\ b & -a & -s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & \frac{-c}{s} & \frac{b}{s} & 0 \\ 1 & \frac{s}{c} & \frac{-a}{c} & 0 \\ 1 & \frac{-a}{b} & \frac{-s}{b} & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & \frac{-c}{s} & \frac{b}{s} & 0 \\ 0 & \frac{s^2+c^2}{cs} & \frac{-a^2+b^2}{cs} & 0 \\ 0 & \frac{c^2-a^2}{bs} & \frac{-s^2+b^2}{bs} & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & \frac{-c}{s} & \frac{b}{s} \\ 0 & 1 & \frac{-a^2+b^2}{s^2+c^2} \\ 0 & 1 & \frac{-s^2+b^2}{c^2-a^2} \end{bmatrix}$ Bleh

$\begin{bmatrix} 1 & 0 \\ & \\ & \end{bmatrix}$

$\|u(t)\|^2 = u_1^2 + u_2^2 + u_3^2$

$\frac{d}{dt}(\|u(t)\|^2) = 2(u_1' u_1 + u_2' u_2 + u_3' u_3)$

$2(cu_1u_2 - bu_1u_3 + au_2u_3 - cu_1u_2 + bu_1u_3 - au_2u_3)$

$= 0$

$\|u\|$ is not changing.

$u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $a^2+b^2+c^2 = k$ for some k.

$e^{At} = \begin{bmatrix} e^t & \frac{1}{2}e^{3t} - \frac{1}{2}e^t \\ 0 & e^{3t} \end{bmatrix}$

$\frac{d}{dt}(e^{At}) = \begin{bmatrix} e^t & \frac{3}{2}e^{3t} - \frac{1}{2}e^t \\ 0 & 3e^{3t} \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^t & \frac{1}{2}e^{3t} - \frac{1}{2}e^t \\ 0 & e^{3t} \end{bmatrix}$

$= \begin{bmatrix} e^t & \frac{3}{2}e^{3t} - \frac{1}{2}e^t \\ 0 & 2e^{3t} \end{bmatrix}$

23.2

$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0$ ~~$\to (\lambda-2)^2 = 2 \to \lambda = 2 \pm \sqrt{2}$~~

$\lambda \in \{1, 3\}$

$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \to x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \to x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$. $e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$

$\begin{bmatrix} e^t & e^{3t} \\ 0 & 2e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$

Pset 24 2017-12-12

24.1 $\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix} \to$ ⓐ $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

ⓑ -1 is an eigenvalue. Also ~~#~~ $\det A < 0$ but the trace > 0.

ⓒ If both eigenvalues were negative, the trace would be negative, but ~~the~~ its not.

24.2 ~~ⓐ~~ $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ invertible, orthogonal, permutation, Markov

$B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ projection, Markov, diagonalizable

$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \lambda^3 + 2 - 3\lambda = 0$ $\quad \lambda - 1 - \lambda$

$\lambda^3 - 3\lambda + 2 = 0$

$$\begin{array}{r} \lambda^2 + \lambda - 2 \\ \lambda - 1 \,\big)\, \lambda^3 \quad -3\lambda + 2 \\ -\lambda^3 + \lambda^2 \\ \hline \lambda^2 - 3\lambda \\ -\lambda^2 + \lambda \\ \hline -2\lambda + 2 \\ +2\lambda - 2 \\ \hline \end{array}$$

$\frac{-1 \pm \sqrt{1+9}}{}$ $\quad \frac{-1 \pm 3}{2}$ $1, -2$

$\lambda \in \{1, 1, -2\}$

~~$A = LU$~~, $A = QR$, $S\Lambda S^{-1}$

None for B $\quad Q\Lambda Q^T$

24.3 $A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ .2 & .3 & .5 \end{bmatrix}$

$\begin{vmatrix} .7-\lambda & .1 & .2 \\ .1 & .6-\lambda & .3 \\ .2 & .3 & .5-\lambda \end{vmatrix}$

Lec 25 2017-12-23 Dcnorm

Symmetric matrices (real)

$A = A^T$

① λ_i are real

② $\vec{x}_i$ are orthogonal (or can be chosen perpendicular) $\to$ can be orthonormal!

Usually $A = S\Lambda S^{-1}$

If $A = A^T$, $S\Lambda S^{-1} = (S\Lambda S^{-1})^T = (S^{-1})^T \Lambda^T (S)^T \Rightarrow S^{-1} = S^T \to S$ is orthonormal

$A = Q\Lambda Q^T$ $\leftarrow$ "Spectral Thm". (Spectrum = set of λ_i)

"Principal Axis Thm" in mech.

$Ax = \lambda x \to \bar{A}\bar{x} = \bar{\lambda}\bar{x}$ $\quad$ b/c $\bar{a}\bar{b} = \overline{ab}\ \forall a, b \in \mathbb{C}$

A is real so

$S\Lambda S^{-1} = (S\Lambda S^{-1})^T = \bar{S}^T \Lambda^T S^T$

$A\bar{x} = \bar{\lambda}\bar{x}$

$\bar{x}^T A^T = \bar{x}^T \bar{\lambda}$

A is symm, so

$\bar{x}^T A = \bar{x}^T \bar{\lambda}$

mul both sides by x:

$\bar{x}^T A x = \bar{x}^T \bar{\lambda} x$

Lmul both sides of original by $\bar{x}^T$

$\bar{x}^T A x = \bar{x}^T \lambda x$

$\lambda \bar{x}^T x = \bar{\lambda} \bar{x}^T x$, and $\bar{x}^T x \neq 0$, so $\quad \to = \bar{x}_1 x_1 + \bar{x}_2 x_2 + \ldots$

$\lambda = \bar{\lambda}$, so $\lambda \in \mathbb{R}$.

$(a-ib)(a+ib)$

For complex matrices, want $\bar{A}^T = A$. $\quad a^2 + b^2 > 0$

$A = Q\Lambda Q^T$

Every symm matrix is the sum of perp. proj. matrices

For symm matrices:

~~So~~ # of pivots = # pos. λ_s

Positive definite matrices are great!

$A^T = A$ and all $\lambda_i > 0$? ~~test all~~

$\Leftrightarrow$ all pivots > 0?

$\Leftrightarrow$ All subdeterminants are positive

Lec 26 2017-12-26

~~z~~

$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ ~~le~~ length in $\mathbb{C}^n$

Length ~~z^2~~ $= \bar{z}^T z$

$z^H z$ ~~H~~ H → ~~Hermi~~ ber-me-shon

inner product: $y^H x$ (named after "ermeet") Hermite

~~Symmetric:~~ $\begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$

Hermitian: $\bar{A}^H = A$

— real λ, $\perp x$ ~~Ex~~

$A = Q\Lambda Q^{-1} = Q\Lambda Q^H$ $Q^H Q = I \rightarrow Q$ is unitary

$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$ $q_i^H q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Fourier Matrices:

$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} \end{bmatrix}$ BUT $w^n = 1$, $w = e^{i2\pi/n}$

$F_n = e^{i\frac{2\pi}{n}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & n-1 \\ 0 & 2 & 4 & & 2(n-1) \\ \vdots & \vdots & & & \\ 0 & n-1 & (n-1)2 & \cdots & (n-1)^2 \end{bmatrix}}$

$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$

$\begin{bmatrix} F_{64} \end{bmatrix} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \begin{bmatrix} P_{64} \end{bmatrix}$ $D = \begin{bmatrix} 1 & & & \\ & w & & \\ & & w^2 & \\ & & & \ddots & w^{31} \end{bmatrix}$

~~Ps~~ Pset 26 2017-~~12~~-26

$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$F_4 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_2 & \\ & F_2 \end{bmatrix} P$

$P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $D_4 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

$F_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} I & D_4 \\ I & -D_4 \end{bmatrix} \begin{bmatrix} F_2 & 0 \\ 0 & F_2 \end{bmatrix}$ $F_1 = [1]$

$= \begin{bmatrix} F_2 & D_4 F_2 \\ F_2 & -D_4 F_2 \end{bmatrix}$ $P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix}$

$\begin{bmatrix} A & \\ & B \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}$

$= \begin{bmatrix} AC & AD \\ BC & BD \end{bmatrix}$

1	1	-1	1
2	2	3	2
3	3	-2	5
4	4	4	6
5	5	-5	3
6	6	7	4
7	7	-6	7
8	8	8	8
	P_2	P_4	P_8

25 | 26

26

26

Lec 27 2017.12.28

Positive Definite Matrices

If $A^T = A$, then the following are equiv:

1. all $\lambda_i > 0$
2. all subdets > 0
3. all pivots > 0
4. $x^T A x > 0$ for any vector $x \neq 0$

$\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$ is positive semi-definite ≥ 0 instead of > 0.

$x^T A x = f(x_1, x_2) = 2x_1^2 + 12x_1x_2 + 18x_2^2$
$ax^2 + 2bxy + cy^2$

can get dets, pivots from

Minimum: $f_{x_i} = 0$, $f_{x_i x_j} > 0$

$\frac{d^2}{dx_i dx_j}(x^T A x)$

$\frac{d}{dx}(Ax) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$

$\lim_{h \to 0} \frac{Ax + Ah - Ax}{h}$

$[a_{11}x_1 + a_1$

$A = Q \Lambda Q^T$

Q gives directions of axes
Λ gives magnitudes

Lec 28 2017-12-29

Similar matrices

$A^T A$ is pos def for any rectangular A

If A is pos def, A^{-1} is pos def $(Q\Lambda Q^T)^{-1} = Q\Lambda^{-1}Q^T$

If A, B pos def, $A+B$ is pos def! $x^T(A+B)x = x^TAx + x^TBx > 0$

$\rightarrow x^T A^T A x = (Ax)^T(Ax) \geq 0$ so at least pos s-def
pos def iff rank $= n$

(n×n matrices)
A and B are similar if $\exists M$ s.t. $B = M^{-1}AM$
$S^{-1}AS = \Lambda \rightarrow A$ is similar to Λ

matrices w/ same S?

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$\underset{M^{-1}}{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}} \underset{M}{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}$

~~Similar matrices:~~

- $B = M^{-1}AM$
- Same λ_i $\longrightarrow$ $Ax = \lambda x$
- Same det
- Same trace
- Same # of x_i

$AMM^{-1}x = \lambda x$
$M^{-1}AMM^{-1}x = M^{-1}\lambda x$
$B(M^{-1}x) = \lambda(M^{-1}x)$
$Bx' = \lambda x'$

"I think of these matrices as people by this point in 18.06"

BAD CASE:

$\lambda_1 = \lambda_2 = 4$

$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4I$ is in its own fam

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \Rightarrow$ big fam (Jordan form)

$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$ One $J_i \Leftrightarrow$ One x_i

Every square A is sim to a Jordan matrix J;

Good case: $J = \Lambda$

$J = \begin{bmatrix} \boxed{J_1} & & \\ & \ddots & \\ & & \boxed{J_n} \end{bmatrix}$ One block per x_i

proof?

Don't need to compute!!

Dec 29 2017-12-29

Singular Value Decomposition (SVD)

$A = U\Sigma V^T$ (U, V orthogonal, Σ diagonal)

SVD: an $\perp$n basis for rowspace → $\perp$n basis for col space

nullspace → zeros on diag of Σ

$\perp$n = orthonormal

$A[v_1 v_2 \dots v_r] = [u_1 u_2 \dots]\Sigma$

$A[v_1, v_2 \dots v_r] = [u_1, u_2 \dots u_r]\begin{bmatrix}\sigma_1 & & \\ & \sigma_2 & 0 \\ 0 & & \sigma_r\end{bmatrix}$

$A \quad V = U \quad \Sigma$

$A = [u \dots]\Sigma[v \dots]$

$A^TA = V\Sigma^TU^TU\Sigma V^T = V\Sigma^T\Sigma V^T$

$= V\begin{bmatrix}\sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \sigma_n^2\end{bmatrix}V^T$

evecs, evals of A^TA

$AA^T = U\Sigma V^TV\Sigma U^T = U\begin{bmatrix}\sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \sigma_n^2\end{bmatrix}U^T$

$A\begin{bmatrix}4 & 4 \\ -3 & 3\end{bmatrix} \Rightarrow A^TA = \begin{bmatrix}4 & -3 \\ 4 & 3\end{bmatrix}\begin{bmatrix}4 & 4 \\ -3 & 3\end{bmatrix} = \begin{bmatrix}25 & 7 \\ 7 & 25\end{bmatrix} \Rightarrow \begin{bmatrix}\sqrt{2}/2 \\ \sqrt{2}/2\end{bmatrix}, 32$

$\begin{bmatrix}\sqrt{2}/2 \\ -\sqrt{2}/2\end{bmatrix}, 18$

$A = \underset{U}{\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}}\underset{\Sigma}{\begin{bmatrix}\sqrt{32} & 0 \\ 0 & \sqrt{18}\end{bmatrix}}\underset{V^T}{\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}}$

$AA^T = \begin{bmatrix}4 & 4 \\ -3 & 3\end{bmatrix}\begin{bmatrix}4 & -3 \\ 4 & 3\end{bmatrix} = \begin{bmatrix}32 & 0 \\ 0 & 18\end{bmatrix} \Rightarrow = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}32 & 0 \\ 0 & 18\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$

$A = \begin{bmatrix}4 & 3 \\ 8 & 6\end{bmatrix}$ AA^T $\begin{bmatrix}4 & 3 \\ 8 & 6\end{bmatrix}\begin{bmatrix}4 & 8 \\ 3 & 6\end{bmatrix} = \begin{bmatrix}25 & 50 \\ 50 & 100\end{bmatrix}$

$A^TA = \begin{bmatrix}4 & 8 \\ 3 & 6\end{bmatrix}\begin{bmatrix}4 & 3 \\ 8 & 6\end{bmatrix} = \begin{bmatrix}80 & 60 \\ 60 & 45\end{bmatrix} \Rightarrow 0, 125$

$A = \frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2 \\ 2 & -1\end{bmatrix}\begin{bmatrix}\sqrt{125} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}.8 & .6 \\ .6 & -.8\end{bmatrix}$

$v_1 \dots v_r$: orthonormal basis for row space

$u_1 \dots u_r$: " " " column space

$v_{r+1} \dots v_n$: " " " nullspace

$u_{r+1} \dots u_m$: " " " left nullspace

Pset 25 2017-12-29

25.1

$Ax = \lambda x \Rightarrow x^TAx = \lambda x^Tx \Rightarrow \lambda = \frac{x^TAx}{x^Tx}$???

$\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}i \\ 1\end{bmatrix} = i\begin{bmatrix}i \\ 1\end{bmatrix}$

$\underset{x^T}{[i \;\; 1]}\underset{A}{\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}}\underset{x}{\begin{bmatrix}i \\ 1\end{bmatrix}} = \underset{\lambda}{i}\,\underset{x^T}{[i \;\; 1]}\underset{x}{\begin{bmatrix}i \\ 1\end{bmatrix}}$

x^Tx can be 0.

25.2

~~$x^TAx = 0,\ x^TBx = 0 \Rightarrow x^TABx$~~

~~$Q_1\Lambda_1Q_1^TQ_2\Lambda_2Q_2^T$~~

$(Q\Lambda Q^T)^{-1} = Q\Lambda^{-1}Q^T$

$P\Lambda_1P^TQ\Lambda_2Q^T$

$PP^T = I,\ QQ^T = I$

$PQQ^TP^T = I$

$(PQ)(PQ)^T = I$

~~$x^TAx = 0,\ x^TBx = 0$~~

~~$Q\Lambda_1Q^TR\Lambda_2R^T$~~

~~x^TAxx^TB~~

$x^TA\,Bx$

$x^TAx > 0$

(a) NO

$\begin{bmatrix}a & b \\ b & c\end{bmatrix}\begin{bmatrix}d & e \\ e & f\end{bmatrix} = \begin{bmatrix}ad+be & ae+bf \\ bd+ce & cf+be\end{bmatrix}$

$\begin{bmatrix}2 & 1 \\ 1 & 2\end{bmatrix}\begin{bmatrix}4 & 1 \\ 1 & 5\end{bmatrix} = \begin{bmatrix}9 & 7 \\ 6 & 11\end{bmatrix}$ not symm, let alone P.D.

(b) yes $QQ^T = I$ $RR^T = I$

$QR(QR)^T = QRR^TQ^T = QQ^T = I$

(c) $e^{sA}e^{tA} = e^{(s+t)A}$, $(e^{tA})^{-1} = e^{-tA}$ → YES

(d) $\det(AB) = \det A \det B = 1 \cdot 1 = 1$

$\det(A^{-1}) = (\det A)^{-1} = 1^{-1} = 1$

Psets 27, 28

2017-12-29

27.1

$ABx = \lambda x$

$x^TB^TA^TBx = \lambda x^TBx$

$B^TA^TB = \lambda B$

$ABx = \lambda x$

$Bx = \lambda A^{-1}x$

$x^TBx = \lambda x^TA^{-1}x$

$\frac{x^TBx}{x^TAx} = \lambda > 0$ QED

27.2

$x^2 + 12xy + 9y^2 = (x+6y)^2 - 27y^2$

sometimes +, sometimes −

28.1

$$M = \begin{bmatrix} i & k & & \\ a & b & c & d \\ j & l & & \\ e & f & g & h \end{bmatrix} = \begin{bmatrix} b & c & m & n \\ a & b & c & d \\ f & g & p & q \\ e & f & g & h \end{bmatrix}$$

$$JM = \begin{bmatrix} a & b & c & d \\ & & & \\ e & f & g & h \\ & & & \end{bmatrix} = \begin{bmatrix} b & c \\ a & b \\ f & g \\ e & f \end{bmatrix} = MK$$

$b^2ph + bcfq + bdg^2 - bdpf - bcgh - b^2gq - caph - c^2ge$

$$\left[\begin{array}{cccc|cccc} b & c & m & n & 1 & 0 & 0 & 0 \\ a & b & c & d & 0 & 1 & 0 & 0 \\ f & g & p & q & 0 & 0 & 1 & 0 \\ e & f & g & h & 0 & 0 & 0 & 1 \end{array}\right]$$

28.2

(a) $B = MAM^{-1} \to B^2 = M^2A^2M^{-2}$ QED

(b) $\underset{A}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{B}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$ not sim.

$A^2 = B^2$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) $\underset{A}{\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}}, \underset{B}{\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}}$ $MAM^{-1} = B$ They have the same eigenvalues

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$$

(d) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ can't be decomposed

(e) $\begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} A \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = B$, $\begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}^2 = I$

Lec 29

2017-12-29

29.1

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $A^TA = V\Sigma^2V^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$a+b=3, ab=1$

$a + \frac{1}{a} = 3$ $a^2+1 = 3a$

$a^2 - 3a + 1 = 0$ $a = \frac{3 \pm \sqrt{9-4}}{2}$ $\sqrt{\frac{3+\sqrt{5}}{2}}$ $\frac{(1+\sqrt{5})^2}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$

$\frac{(\sqrt{5}-1)^2}{4} = \frac{3-\sqrt{5}}{2}$

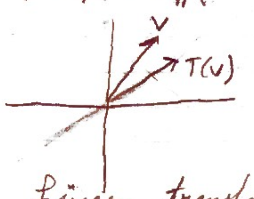
29.2

$A^TA = \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \sigma_n^2 \end{bmatrix} = \Sigma$; $U, V = I$

Lec. 30

2017-12-30

$T: \mathbb{R}^2 \to \mathbb{R}^2$



Given bases:
1st col: $T(v_1) = a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m$

$T = \frac{d}{dx}()$ Input: $c_1 + c_2x + c_3x^2$
Basis: $1, x, x^2$
Output: $c_2 + 2c_3x$
basis: $1, x$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} v = \frac{dv}{dx}$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix}$

Linear transformations

$T(v+w) = T(v) + T(w)$
$T(cv) = cT(v)$
$T(cv+dw) = cT(v) + dT(w)$
$\to T(0) = 0$

Proj. is a LT
shift is not a LT
Length is not a LT
Rotation is LT

$T(v) = Av$ for any matrix A (rect.)

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects over y axis

understand LTs by looking at matrix

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$\begin{bmatrix} \vdots & & \vdots \\ \vdots & & \vdots \end{bmatrix} v$

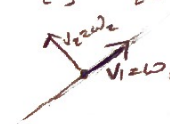
A (2x3)

To understand a LT, need only understand basis.

coordinates off basis:
basis $\{v_1, v_2, v_3\}$
$\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

$\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$T: \mathbb{R}^n \to \mathbb{R}^m$
Choose basis $v_1, \dots v_n$ for inputs
Choose basis $w_1, \dots w_m$ for outputs.



$v = c_1v_1 + c_2v_2$
$T(v) = c_1v_1$
$T(v) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} v$

An eigenvector basis gives a diagonal matrix Λ

Lec 31 - Change of Basis

2017-12-30

"that [video signal] wd go to the bottom of Blac..."

512 (box) 512: i^{th} pixel, $0 \le x \le 255$

$n = (512)^2$

JPEG - Joint Photographic Experts Group

Standard basis: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \end{bmatrix} \dots$

Any LT has similar A in all bases

Better basis: $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \vdots \\ -1 \end{bmatrix}$ whatever

Fourier basis:

$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \vdots \\ \omega^{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \vdots \\ \omega^{2n-2} \end{bmatrix} \dots$

Break down into 8x8 blocks

64-dimensional

$\underbrace{\qquad\qquad}_{64 \text{ vectors}}$

Drop out all the vectors with really small values

"So, Washington had a file of 30 million murderers, cheaters on quizzes, that sort of stuff..."

Wavelets basis: ($\mathbb{R}^8$)

$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \vdots \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \dots$

orthogonal!

$W = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & \\ \vdots & \vdots & \vdots & & -1 & \text{etc.} \\ & -1 & -1 & & & \\ & & & 0 & & \end{bmatrix}$

$p = Wc$ decoding
$c = W^{-1}p$ encoding

W^{-1} is a diag. $\cdot W^T$

Good basis:
1. easy to multiply
 - FFT or FWT (fast wavelet T)
2. a few vectors give enough info

W = new basis vectors in terms of old

$x = Wc$ ← in new basis

Psets 30, 31 2017-12-30

30.1 a) Yes.

b) $T(x, y) = (2x, 2y)$ $\quad \sqrt{(2x)^2+(2y)^2} = 2\sqrt{x^2+y^2}$

$\sqrt{(2x)^T(2x)} = \sqrt{4x^Tx} = 2\sqrt{x^Tx}$

c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ cartesian and $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ polar.

30.2 $T(x, y) = (\sqrt{x}, \sin(y))$

31.1 The dot product $v_1^T v_2$ of any two vectors v_1 and v_2 is $\underbrace{a*1 + a*1 + \dots}_{n \text{ times}} \underbrace{+ a*-1 + a*-1 + \dots}_{n \text{ times}} + \underbrace{a*0 + a*0 \dots}_{\text{several times}} + \underbrace{a*0 + \dots}_{\text{several times}} = 0$

where $a = \pm 1$

31.2 $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\}$

The first basis is probably better for all three

Lec 32 - Left, Right and pseudo-inverse 2017-12-30

$m = n = r$:

(full rank)

$AA^{-1} = I = A^{-1}A$

$r = n$:

(full col rank)

nullspace $= \{0\}$

row space $=$ everything

[I columns

(A^TA) is invertible (full rank)

$(A^TA)^{-1}A^T$ is the "left inverse" A^{-1}_{left}

$A^{-1}_{left} A = I$

$r = m$:

full row rank

$A^{-1}_{right} = A^T(AA^T)^{-1}$

"I see you guys are in a happy mood - I don't know if that means 18.06 is ending or the quiz was good, but..."

"I shouldn't say anything bad about calculus, but I will ~ " : ~22:00

"statisticians are like least-squares doing least squares"

there are other L, R I happy! They're inverses! (but these always are our favorites)

If you put them in the wrong order, you get $A(A^TA)^{-1}A^T$ or $A^T(AA^T)^{-1}A$ (projection matrices)

A is a bijection from the row space to the col space;

Pseudo-inverse denoted A^+. For x in row space:

$A^+(Ax) = x$ For $x \neq y$: suppose $Ax = Ay$. Then $A(x-y) = 0$

Then $x-y \in N(A)$, but $x-y \in C(A^T)$, so $(x-y) = 0$, $\therefore x = y$, but $x \neq y$, so by contra. $Ax \neq Ay$

$A = U\Sigma V^T$ $\quad \Sigma = \begin{bmatrix} \sigma_1, \sigma_2 & 0 \\ 0 & 0 \end{bmatrix}_{m \times n} \rightarrow \Sigma^+ = \begin{bmatrix} 1/\sigma_1, 1/\sigma_2 & 0 \\ 0 & 0 \end{bmatrix}_{n \times m}$ $\quad \Sigma\Sigma^+ = \begin{bmatrix} 1, 1 & 0 \\ 0 & 0 \end{bmatrix}_{m \times m}$

$A^+ = V\Sigma^+U^T$ $\quad \Sigma^+\Sigma = \begin{bmatrix} 1, 1 & 0 \\ 0 & 0 \end{bmatrix}_{n \times n}$

Pset 32 2017-12-30

32.1

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A_r^{-1} = A^T(AA^T)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix}$$

32.2 Only a pseudoinverse, as $m=n>r$.

$$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}. \quad AA^T = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 25 & 50 \\ 50 & 100 \end{bmatrix} = \underbrace{\tfrac{1}{\sqrt5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} 125 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma^2}\underbrace{\begin{bmatrix} \; \end{bmatrix}}_{U^T}$$

$$A^TA = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix} = \underbrace{\begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}}_{V}\underbrace{\begin{bmatrix} 125 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma^2}\underbrace{\begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}}_{V^T}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix}$$

$$A = \begin{bmatrix} 1/\sqrt5 & 2/\sqrt5 \\ 2/\sqrt5 & -1/\sqrt5 \end{bmatrix}\begin{bmatrix} 5\sqrt5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}$$

$$A^+ = \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}\begin{bmatrix} \sqrt5/25 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt5 & 2/\sqrt5 \\ 2/\sqrt5 & -1/\sqrt5 \end{bmatrix}$$

$$= \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}\begin{bmatrix} 1/25 & 2/25 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4/125 & 6/125 \\ 3/125 & -8/125 \end{bmatrix}$$