Machine Learning 1.13: Latent Variables

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- This lecture: We don't.

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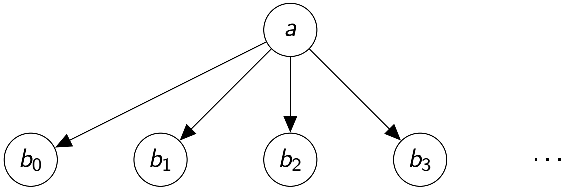
- This lecture: We don't.

- Structure often unknown, incomplete, or a reasonable guess.
- Don't have all latent RVs when training (Often hypothetical RVs).

Aside: Plate Notation



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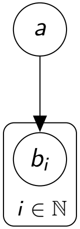

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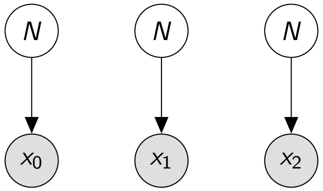
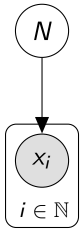
- **Plate notation** solves this.
- Note indices.
- Can cover multiple RVs.

One Model

- Shared parameters can be made explicit using plates:





(e.g. Gaussian distribution N over x)

- Much easier for complex models.
- Important for fully Bayesian.

Examples

Two examples:

- Latent Semantic Analysis (recommender system).
- Gaussian Mixture Model (density estimation).

Example I: Recommender System

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- Estimates what you will like/dislike.

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- Two approaches:
  - Content-based filtering:
    Each product has features. Match user to products with features similar to previously liked (e.g. per user logistic regression model).
  - Collaborative filtering:
    Identify users with similar likes. Assume they have similar opinions for products where one has given an opinion and the other has not (e.g. clustering users).

  Hybrids often used in practise.

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- Has multiple equivalent graphical models!
- **Not probabilistic** (though easy to fix).

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  ∴ Robert likes Star Wars.
- With multiple attributes:
  e.g. Susan likes action films but really hates fantasy. Star Wars is a fantasy action film.
  ∴ Susan dislikes Star Wars a little bit.

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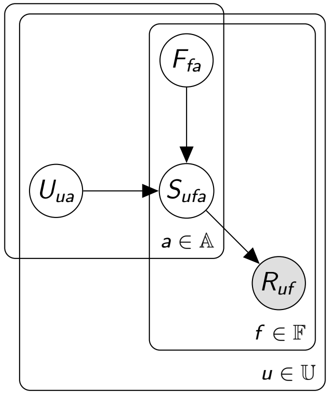
- **Unknown** $\therefore$ Inferred from data.
- Unlikely to have a human interpretation. Does this matter?

Graphical Model



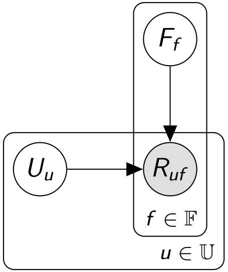
- a – Index of an attribute, $\in \mathbb{A}$.
- u – Index of a user, $\in \mathbb{U}$.
- f – Index of a film, $\in \mathbb{F}$.

- F_{fa} – Strength of attribute a in film f.
- U_{ua} – How much user u likes attribute a.

- S_{ufa} – Score: Effect of attribute a for user u on film f (multiplication).
- R_{uf} – Rating given by user u of film f (addition).

- Training data: **Only** R_{uf}, extremely sparse.

Simplified



- You usually simplify by not showing the attributes. (the arrows are now vectors of attribute weights)
- This is also the standard PCA graphical model.

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In practise:
Too much data, so use gradient descent to minimise error instead.

Results

From the Netflix challenge:

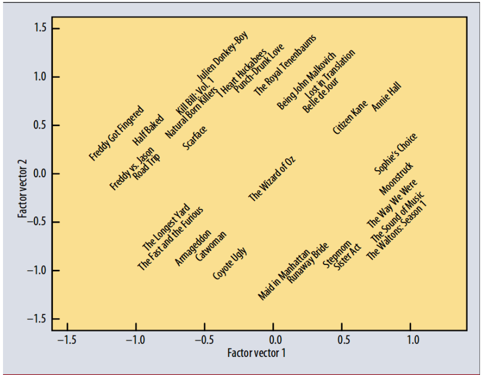

Figure 3. The first two vectors from a matrix decomposition of the Netflix Prize data. Selected movies are placed at the appropriate spot based on their factor vectors in two dimensions. The plot reveals distinct genres, including clusters of movies with strong female leads, fraternity humor, and quirky independent films.

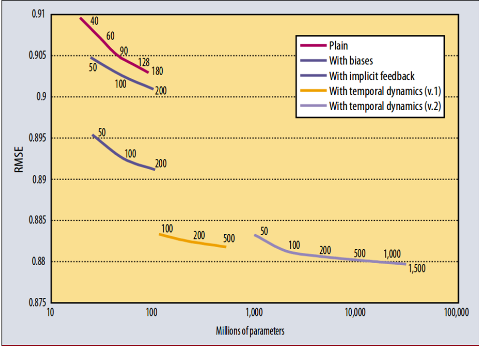

Figure 4. Matrix factorization models' accuracy. The plots show the root-mean-square error of each of four individual factor models (lower is better). Accuracy improves when the factor model's dimensionality (denoted by numbers on the charts) increases. In addition, the more refined factor models, whose descriptions involve more distinct sets of parameters, are more accurate. For comparison, the Netflix system achieves RMSE = 0.9514 on the same dataset, while the grand prize's required accuracy is RMSE = 0.8563.

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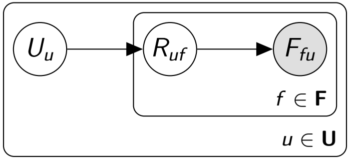
Normal graphical model – weird conditional structure for this problem:

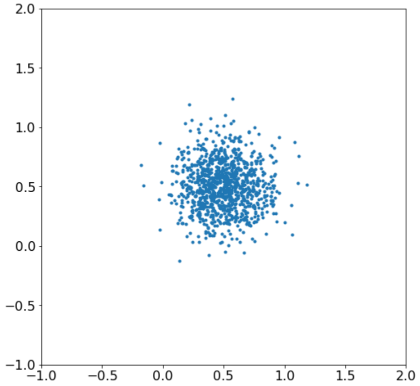


(user $\rightarrow$ document, rating $\rightarrow$ topic, film $\rightarrow$ word)

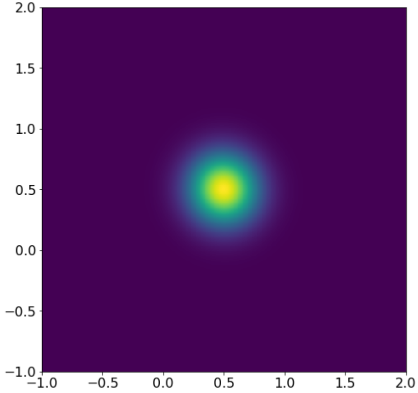
Example II: Density Estimation

- Density estimation:
  - Input: Samples, x, drawn from an unknown probability distribution D.
  - Output: An estimate of D.

Input:



Output: (Gaussian)



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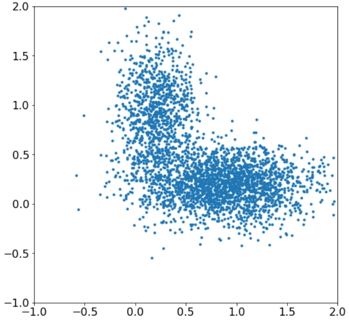
- Uses:
    - Visualisation.
    - Clustering.
    - Regression/classification via Bayes rule.
    - Abnormality detection.
    - Part of larger models, e.g. voice recognition.

Gaussian Mixture Model

- Instead of one Gaussian distribution describe data with several.

$$P(x|\alpha, \mu, \Sigma) = \sum_{z=1}^{K} \alpha_z \mathcal{N}(x|\mu_z, \Sigma_z)$$

Input:



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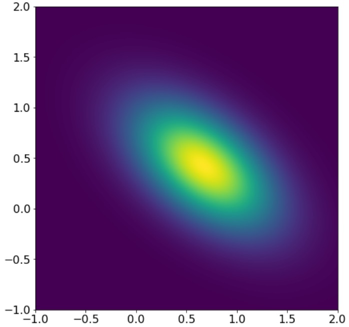
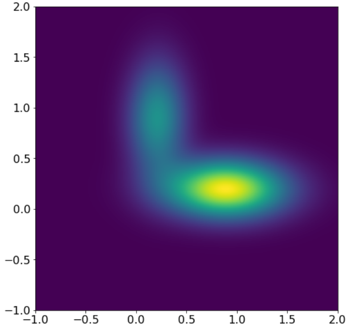


GMM with 2 components:



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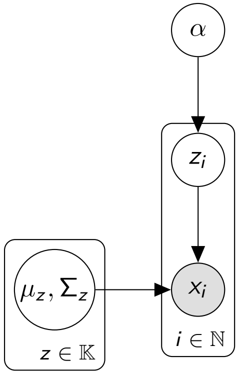

- i – Index of exemplar, $\in \mathbb{N}$.
- z – Index of component, $\in \mathbb{K}$.
- μ_z, Σ_z – Mean and covariance for component z.
- α – Categorical distribution over component membership.
- z_i – Which component feature vector i belongs to.
- x_i – Feature vector for exemplar i.

Fitting to Data

- No analytical solution.
- Belief propagation problematic.

- New algorithm: **Expectation Maximisation**

Expectation Maximisation

- Finds maximum likelihood and maximum a posteriori solutions.
- Similar to gradient descent:
  - Start from random parameters.
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  - **Missing values**: Latent variables associated with data. (Often called latent variables!)
  - **Observed variables**: The data.

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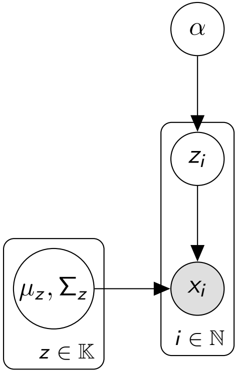
- Two steps:
    - **E-step**: Calculate **likelihood function** of missing values with model parameters fixed. (Often referred to as calculating **expectation**. It isn't.)
    - **M-Step**: **Maximise** model parameters given expectation of missing values.

  Alternate between until convergence.

EM for GMM



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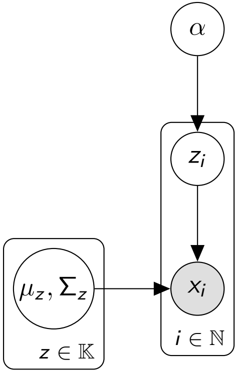

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- Missing values: z
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- Likelihood of z_i:
  Categorical distribution, $z_i \sim \texttt{Cat}(w_i)$
  $\sum_{z=1}^{K} w_{iz} = 1$

Initialisation

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- Much better:
  1. Run k-means with K centres.
  2. Set $\alpha \propto$ number assigned to each centre.
  3. Fit μ and Σ to points assigned to each centre.
- This is close to the goal $\therefore$ less iterations required to converge.
- There are clever k-means initialisations too.

E-Step

Set (for all i, z):

$$w_{iz} \propto \alpha_z \mathcal{N}(x_i | \mu_z, \Sigma_z)$$

(remember to normalise, so $\sum_{z=1}^{K} w_{iz} = 1$)

M-Step

Set (for all z):

$$\alpha_z \propto T_z,\, T_z = \sum_{i=1}^{N} w_{iz}$$

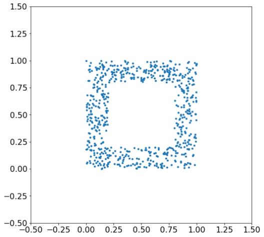
(remember to normalise, so $\sum_{z=1}^{K} \alpha_z = 1$)

$$\mu_z = \frac{1}{T_z} \sum_{i=1}^{N} w_{iz} x_i$$

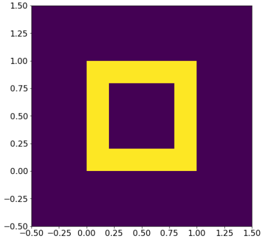
$$\Sigma_z = \frac{1}{T_z} \sum_{i=1}^{N} w_{iz} (x_i - \mu_z)(x_i - \mu_z)^T$$

Demonstration

Input:



Ground Truth:



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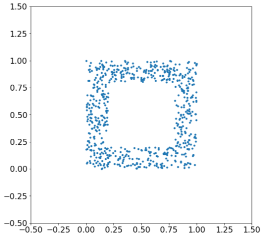

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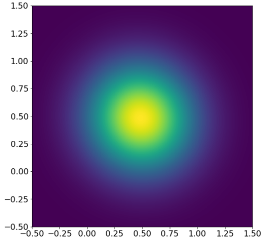

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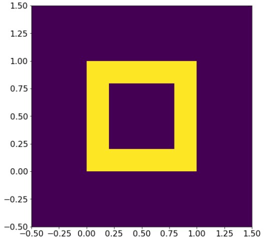

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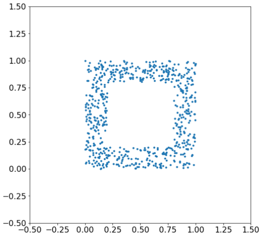

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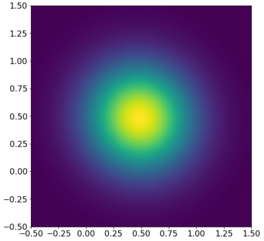

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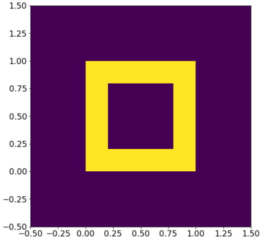
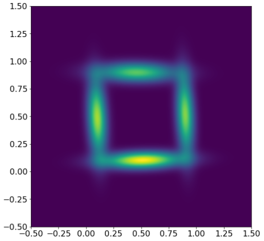


GMM with 4 components:



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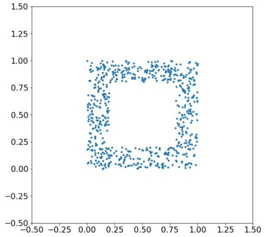

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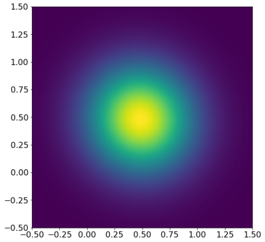

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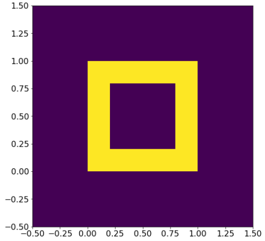

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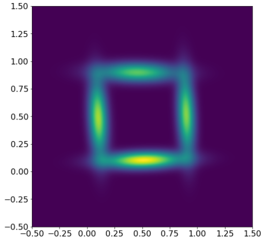
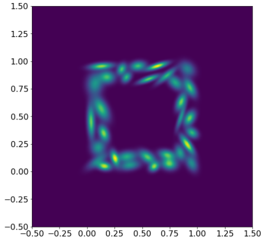


GMM with 32 components:



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- BIC is crude – poor assumptions. Many other options.
- Non-parametric techniques, e.g. Dirichlet process prior ideal.

Summary

- Real problems are not neat.
- Plate notation.

- Latent Semantic Analysis.
- Gaussian Mixture Model.

Further Reading

- Paper describing the use of LSA on the Netflix challenge:
  "*Matrix Factorization Techniques for Recommender Systems*"
  by Koren, Bell & Volinsky (2009) (source of results images).

- Original EM paper, if you're so inclined:
  "*Maximum Likelihood from Incomplete Data via the EM Algorithm*",
  by Dempster, Laird, & Rubin (1977).
  (Special cases were discovered before, has mistake in convergence proof)

- If you sensibly decide to ignore the above:
  Chapter 9 of "*Pattern Recognition and Machine Learning*" by Bishop.