Computer Animation and Games I

CM50244

2D and 3D Transformations

Overview

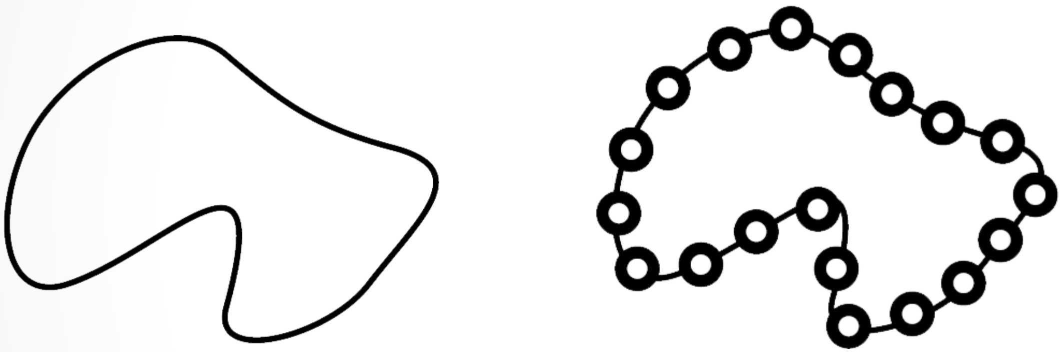
- 2D Transformations
  - 2x2 Matrix Transforms
  - Rotation, Scale, Stretch, Shear, Translation
  - Homogeneous coordinates
- Compound transformations
  - Rotation about an arbitrary point
- 3D Transformations

Overview

- **2D Transformations**
  - **2x2 Matrix Transforms**
  - **Rotation, Scale, Stretch, Shear, Translation**
  - **Homogeneous coordinates**
- Compound transformations
  - Rotation about an arbitrary point
- 3D Transformations

Recall: Procedural Rigging

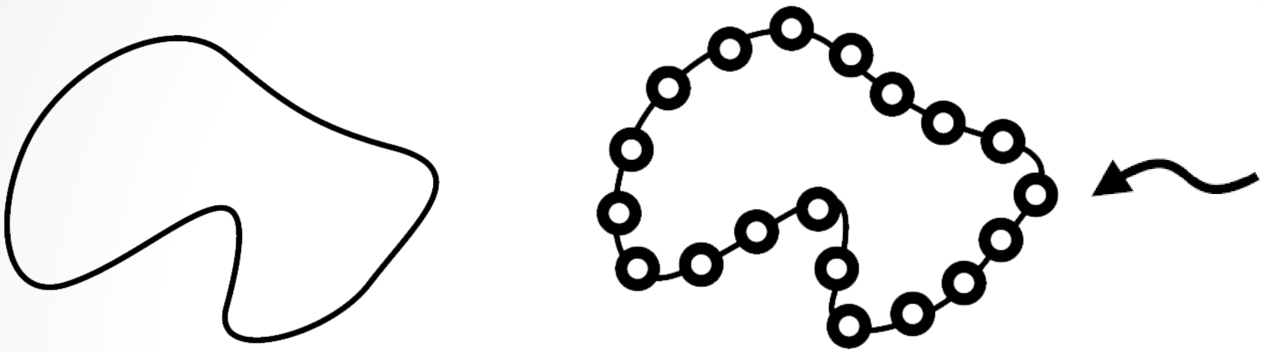
- Apply function to points specifying the shape



$$\mathbf{p}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$\mathbf{p}' = f(\mathbf{p})$$

2D Matrix Transforms



$$\mathbf{p}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

- We can transform the shape using Matrix operations e.g.

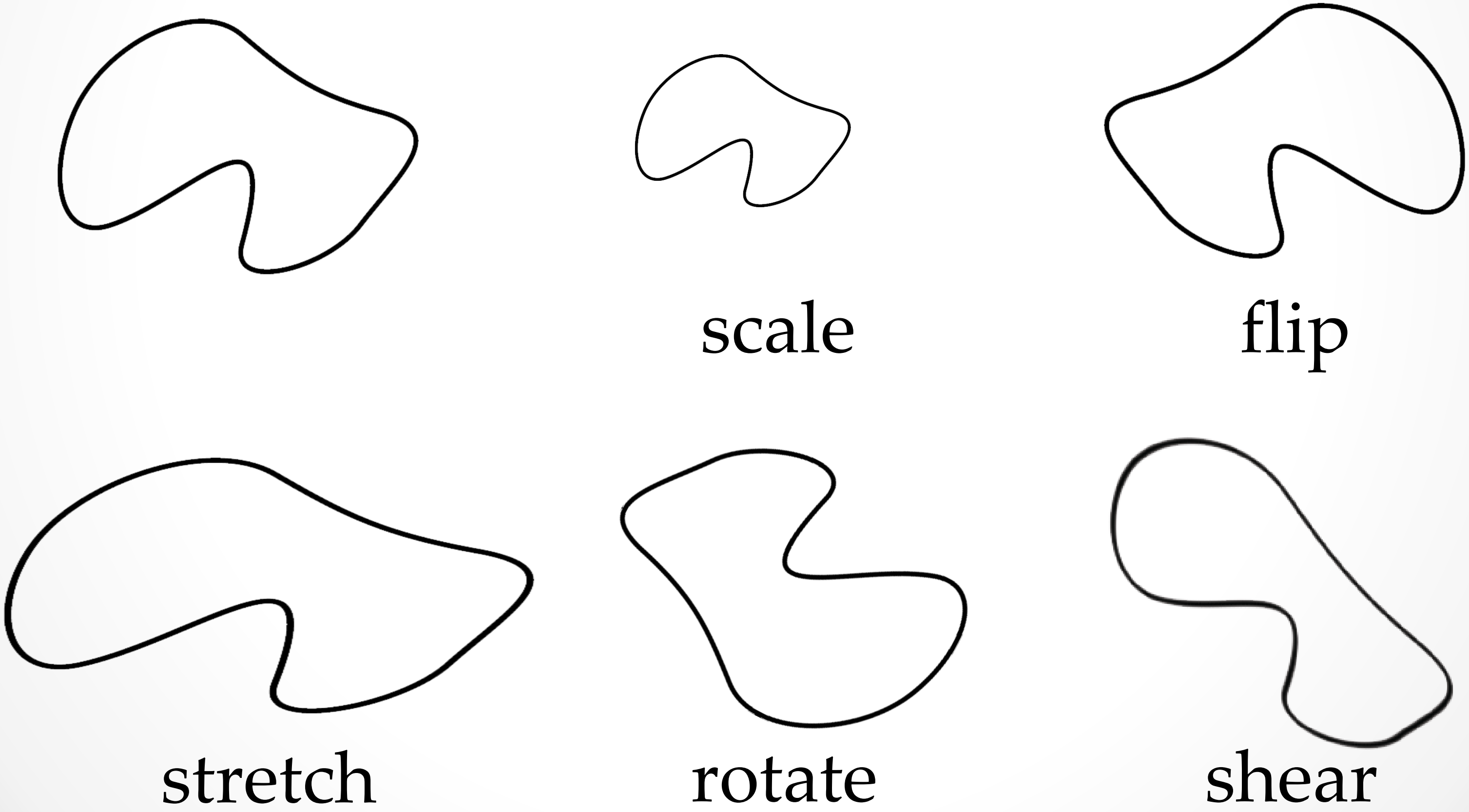
$$\mathbf{p}' = \mathbf{M}\mathbf{p}$$

- **M** is a 2X2 matrix, 4 degrees of freedom

2D Matrix Transforms

- What transformations can be represented by:

$$\mathbf{p}' = \mathbf{M}\mathbf{p}$$



2D Matrix Transforms

- Scale

$$x' = \lambda x,\ \ y' = \lambda y \quad \Rightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Stretch

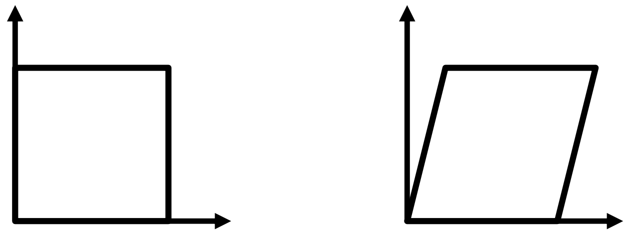
$$x' = \alpha x,\ \ y' = \beta y \quad \Rightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Matrix Transforms

- Flip (across y-axis)

$$x' = -x, \;\; y' = y \quad \Rightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

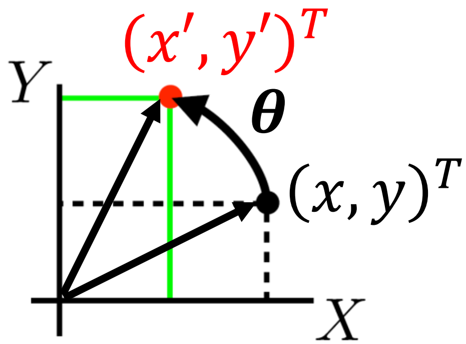
- Shear (Skew)



$$x' = x + ky, \;\; y' = y \quad \Rightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Matrix Transforms

- Rotation



$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

What about translation?

- Translation is vector addition

$$\begin{bmatrix} p'_1 \\ p'_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

- Can we represent translation as a linear operation? (matrix multiplication)

Homogeneous Coordinates

- Yes, if we add a "1" to the vector

$$\begin{bmatrix} p'_1 \\ p'_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & t_1 \\ a_{21} & a_{22} & t_2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix}$$

- These are called homogeneous coordinates

$$\begin{bmatrix} p'_1 \\ p'_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_1 \\ a_{21} & a_{22} & t_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix}$$

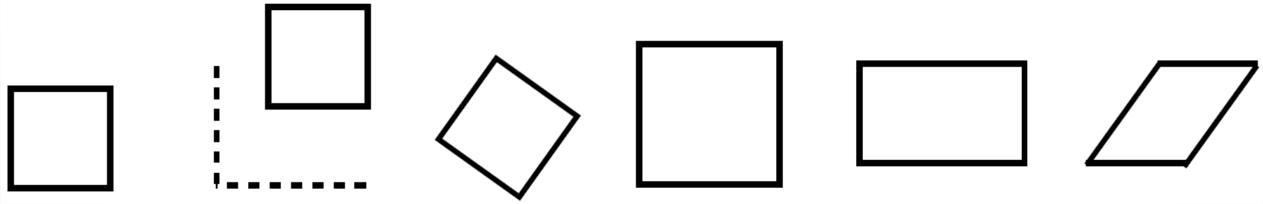
Translation is now a linear operation...

Affine Transformations

- What range of transformations can we represent using

$$\begin{bmatrix} p'_1 \\ p'_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_1 \\ a_{21} & a_{22} & t_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix}$$

- 6 degrees of freedom: translation (2), rotation, scale, stretch, shear



Examples

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix}$$

Translate

Overview

- 2D Transformations
  - 2x2 Matrix Transforms
  - Rotation, Scale, Stretch, Shear, Translation
  - Homogeneous coordinates
- **Compound transformations**
  - **Rotation about an arbitrary point**
- 3D Transformations

Compound Transformations

- Translate: $\mathbf{T}$

$$\mathbf{p}' = \mathbf{T}\mathbf{p}$$

- Scale: $\mathbf{S}$

$$\mathbf{p}'' = \mathbf{S}\mathbf{p}'$$

- Matrix multiplication is associative, we have

$$\begin{aligned}\mathbf{p}'' &= \mathbf{S}\mathbf{p}' \\ &= \mathbf{S}(\mathbf{T}\mathbf{p}) \\ &= (\mathbf{S}\mathbf{T})\mathbf{p}\end{aligned}$$

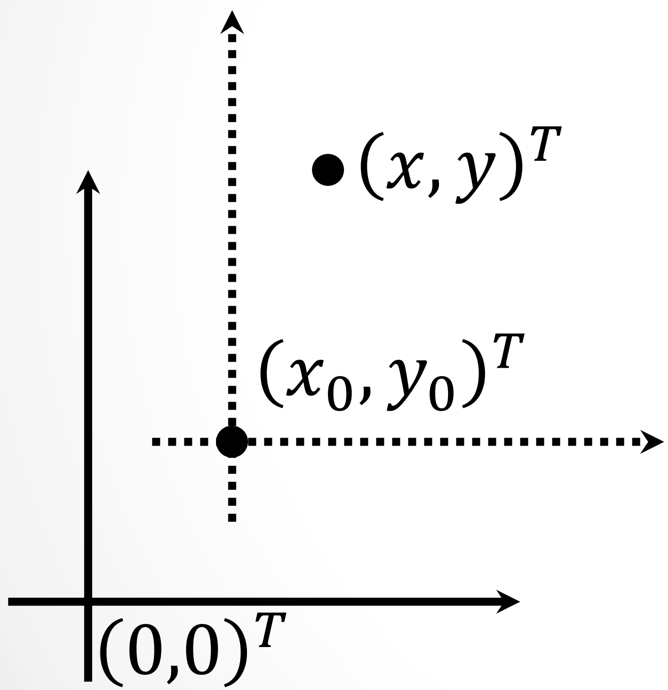
- The compound transformation $\mathbf{M}$ is

$$\mathbf{M} = \mathbf{S}\mathbf{T}$$

$$\mathbf{p}'' = \mathbf{M}\mathbf{p}$$

Rotation about a Point

- How can we rotate an object about a point?



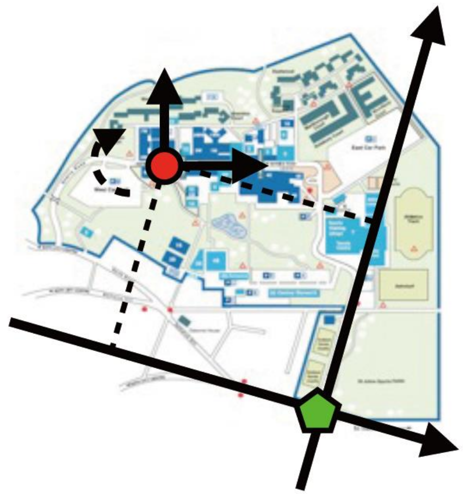
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

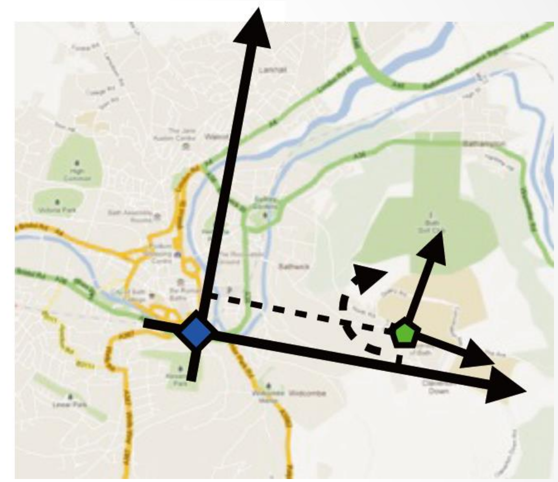
Local/Global Coordinates

- Where are you right now?

row 3, seat 4



8W2.1



Univ. of Bath

$$\mathbf{p}_r = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

(room coordinate)

$$\mathbf{p}_c = \mathbf{R}_1 \mathbf{p}_r + \mathbf{t}_1$$

$$\tilde{\mathbf{p}}_c = \mathbf{T}_1 \tilde{\mathbf{p}}_r$$

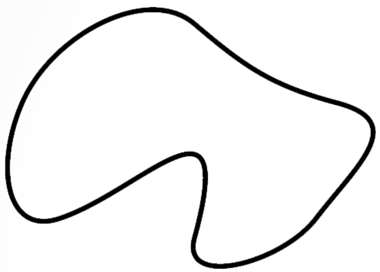
(campus coordinate)

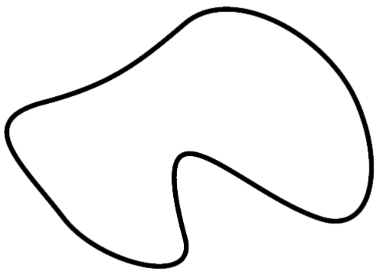
$$\begin{aligned} \mathbf{p}_b &= \mathbf{R}_2 \mathbf{p}_c + \mathbf{t}_2 \\ &= \mathbf{R}_2 (\mathbf{R}_1 \mathbf{p}_r + \mathbf{t}_1) + \mathbf{t}_2 \end{aligned}$$

$$\boxed{\tilde{\mathbf{p}}_b = \mathbf{T}_2 \mathbf{T}_1 \tilde{\mathbf{p}}_r}$$

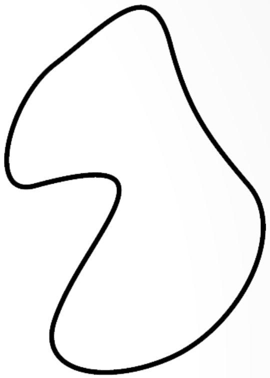
Order Matters!

- Note that for matrices $\mathbf{AB} \neq \mathbf{BA}$ in general

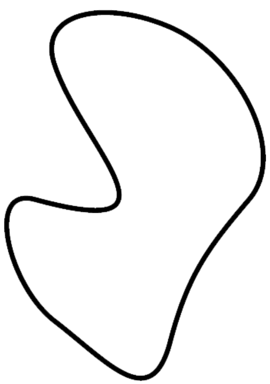




Flip Vertical



Rotate 90° CW



Rotate 90° CW

Flip Vertical

Overview

- 2D Transformations
  - 2x2 Matrix Transforms
  - Rotation, Scale, Stretch, Shear, Translation
  - Homogeneous coordinates
- Compound transformations
  - Rotation about an arbitrary point
- **3D Transformations**

Recap: 2D Transformations

- Matrix transformations in 2D (e.g. Rotation, Scale, Stretch, Shear)

$$\begin{bmatrix} p'_1 \\ p'_2 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

- If we use Homogeneous Coordinates, translation can be represented as matrix multiplication

Euclidean

$$\begin{bmatrix} p'_1 \\ p'_2 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} m_{13} \\ m_{23} \end{bmatrix}$$

homogenous

$$\begin{bmatrix} p'_1 \\ p'_2 \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix}$$

Homogeneous Coordinates in 3D

- As before, we can add a "1" to the last coordinate to make translation a linear operation (matrix multiplication):

$$\begin{bmatrix} p'_1 \\ p'_2 \\ p'_3 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} p'_1 \\ p'_2 \\ p'_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix}$$

3D Affine Transformations

3D
$$\begin{bmatrix} p_1' \\ p_2' \\ p_3' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix}$$

12 degrees of freedom

Recall 2D
$$\begin{bmatrix} p_1' \\ p_2' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix}$$

6 degrees of freedom

Compound Transformations

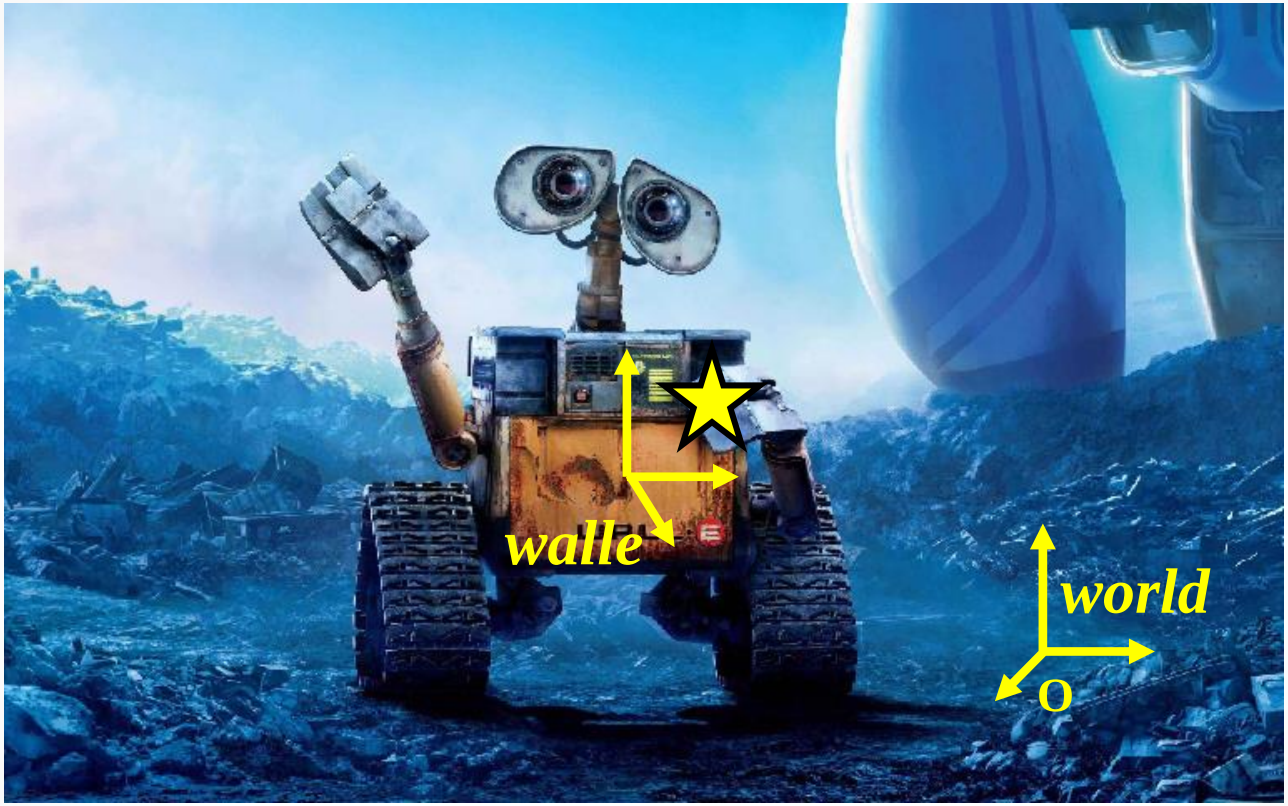
- Homogeneous transformations in 3D are easily concatenated

$$\mathbf{p}' = \mathbf{T}_3\mathbf{T}_2\mathbf{T}_1\mathbf{p}$$

- Note that the order of transformations matters, in general

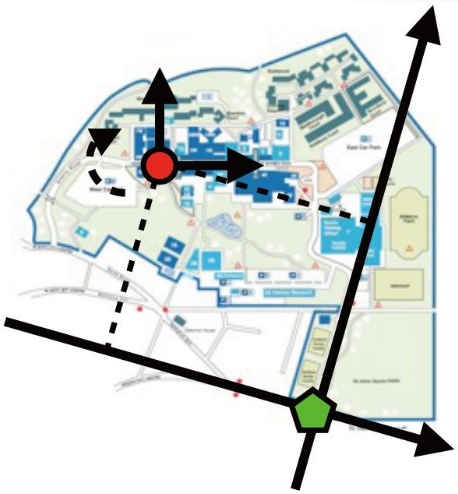
$$\mathbf{T}_1\mathbf{T}_2 \neq \mathbf{T}_2\mathbf{T}_1$$

3D Local/Global Coordinates



Recap: 2D Local/Global Coordinates

- Where are you right now?



row 3, seat 4

8W2.1

$$\mathbf{p}_r = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

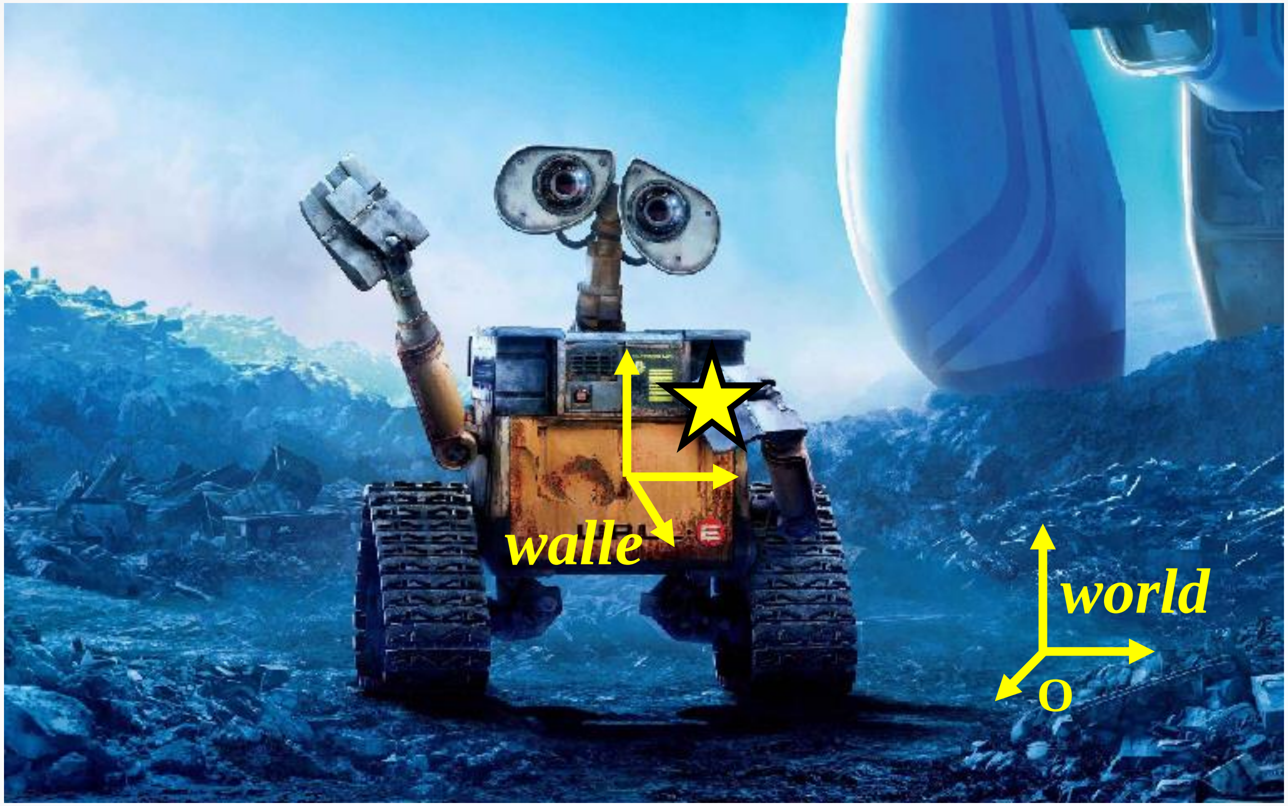
$$\mathbf{p}_c = \mathbf{R}_1 \mathbf{p}_r + \mathbf{t}_1$$

$$\tilde{\mathbf{p}}_c = \mathbf{T}_1 \tilde{\mathbf{p}}_r$$

(room coordinate)

(campus coordinate)

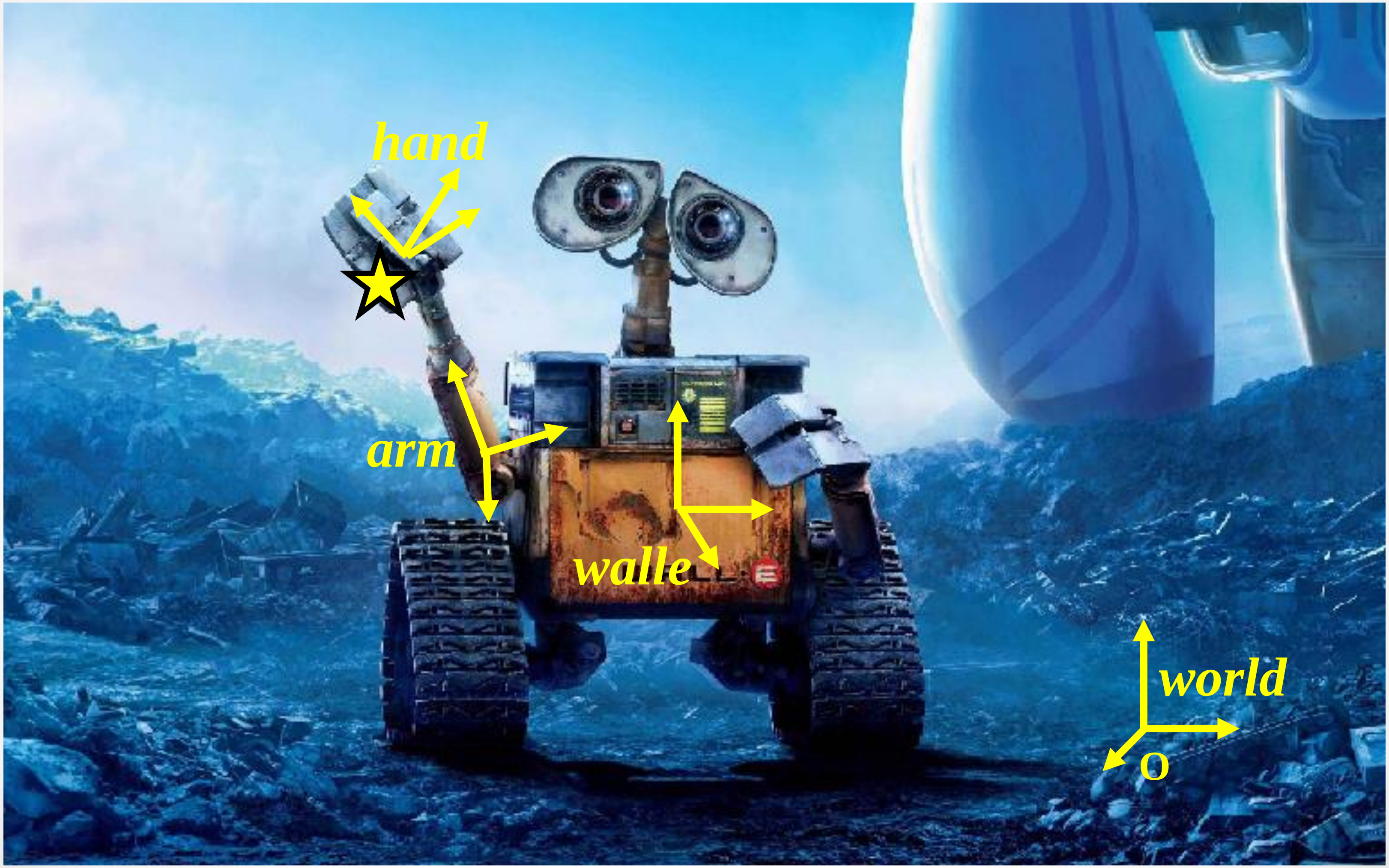
3D Local/Global Coordinates



$$\mathbf{p}_{world} = \mathbf{R}_1 \; \mathbf{p}_{walle} + \mathbf{t}_1$$

$$\tilde{\mathbf{p}}_{world} = \mathbf{T}_1 \; \tilde{\mathbf{p}}_{walle}$$

Coordinate Transformations



Coordinate Transformations

- Hand coordinates to arm coordinates:

$$\tilde{\mathbf{p}}_{arm} = \mathbf{T}_3\, \tilde{\mathbf{p}}_{hand}$$

- Arm coordinates to Wall-e coordinates:

$$\tilde{\mathbf{p}}_{walle} = \mathbf{T}_2\, \tilde{\mathbf{p}}_{arm}$$

- Wall-e coordinates to world coordinates:

$$\tilde{\mathbf{p}}_{world} = \mathbf{T}_1\, \tilde{\mathbf{p}}_{walle}$$

- Hand coordinates to world coordinates:

$$\tilde{\mathbf{p}}_{world} = \mathbf{T}_1 \mathbf{T}_2 \mathbf{T}_3\, \tilde{\mathbf{p}}_{hand}$$