Question 1. Let $X_1, \ldots, X_n$ be a random sample from a population depending on an unknown parameter $\theta > 0$. Which of the following quantities is a statistic.

A. $T = \sum_{i=1}^{n}(X_i - \theta)^2$.

B. $T = \sum_{i=1}^{n} X_i/\theta$.

C. $T = \max\{X_1, \ldots, X_n\}$. Does not depend on θ.

D. $T = \theta$.

Question 2. Consider the population distributed as Bernoulli(θ) where θ denotes the probability of observing the value 1 from a randomly chosen individual. What is the parameter space in this case?

A. $\{0, 1\}$.

B. $[0, 1]$. All possible values for θ.

C. $\{0, 1, 2, \ldots, n\}$ where n is the sample size.

D. $(0, \infty)$.

Question 3. Let $X_1, \ldots, X_n$ be a random sample from a population depending on an unknown parameter θ and let T be an estimator for θ such that $\text{Bias}_\theta(T) < 0$. Which of the following is true about the mean squared error, $\text{MSE}_\theta(T)$, of T?

A. $\text{MSE}_\theta(T) = (\text{Bias}_\theta(T))^2 + \text{Var}(T)$.

B. $\text{MSE}_\theta(T) = (\text{Bias}_\theta(T))^2 - \text{Var}(T)$.

C. $\text{MSE}_\theta(T) = \text{Var}(T) - (\text{Bias}_\theta(T))^2$.

D. $\text{MSE}_\theta(T) = \text{Var}(T)$.

Question 4. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} f(x|\theta)$ where

$$f(x|\theta) = \theta x^{\theta - 1},\ x \in (0, 1),\ \theta > 0.$$

Which of the following is the method of moments estimator for θ.

A. $\bar{x}$.

B. $\dfrac{\bar{x}}{1 - \bar{x}}$.

C. $2\bar{x}$.

D. None of the above.

$$E(X) = \int_0^1 x \cdot \theta \cdot x^{\theta-1} dx = \theta \cdot \int_0^1 x^\theta dx = \frac{\theta}{\theta+1}\left[x\right]_0^1 = \frac{\theta}{\theta+1}$$

Set $\frac{\theta}{\theta+1} = \bar{x} \Rightarrow \theta = \bar{x} \cdot (\theta+1) \Rightarrow (1-\bar{x})\theta = \bar{x} \Rightarrow \theta = \frac{\bar{x}}{1-\bar{x}}$

Question 5. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} f(x|\theta)$ where

$$f(x|\theta) = \theta x^{\theta-1}, \; x \in (0,1), \theta > 0.$$

Which of the following is the maximum likelihood estimator for θ.

A. $-\log \bar{x}$.

B. $-\dfrac{n}{\sum_{i=1}^{n} \log x_i}$.

C. The same as the method of moments estimator.

D. Does not exist in closed form.

$l(\theta) = n \log\theta + (\theta-1)\sum \log x_i$

$l'(\theta) = \frac{n}{\theta} + \sum \log x_i = 0$

$\Rightarrow \theta = -\frac{n}{\sum \log x_i}$

Question 6. Let $X \sim \mathrm{U}(\theta, \theta+1)$. A single observation $x = 2$ is made. Which of the following is a 95% confidence interval for θ?

A. $[0.05, 1.95]$.

B. $[1.05, 1.95]$.

C. $[1.025, 1.975]$.

D. $[1.025, 2.975]$.

$Y = X - \theta \sim U(0,1)$

$P(0.025 < Y < 0.975) = 0.95$

$P(0.025 < X - \theta < 0.975) = 0.95$

$-0.975 < \theta - X < -0.025$

$X - 0.975 < \theta < X - 0.025 = (1.025, 1.975)$

Question 7. Let $X \sim f(x|\theta)$ where

$$f(x|\theta) = \theta x^{\theta-1}, \; x \in (0,1), \; \theta > 0.$$

We wish to test

$$\mathrm{H}_0\colon \theta = 1 \text{ v.s } \mathrm{H}_1\colon \theta = 2$$

based on a single observation $X = x$, and decided to reject H_0 if $x > 0.5$. Which of the following corresponds to the probability of Type I Error for this rule?

A. 0.25.

B. 0.5.

C. 0.75

D. 0.95.

$P(X > 0.5) = \int_{0.5}^{1} 1 \cdot x^{1-1}\, dx = \int_{0.5}^{1} dx = 0.5$

Question 8. Let $X \sim f(x|\theta)$ where

$$f(x|\theta) = \theta x^{\theta-1}, \ x \in (0,1), \ \theta > 0.$$

We wish to test

$$\mathrm{H}_0\colon\ \theta = 1 \text{ v.s } \mathrm{H}_1\colon\ \theta = 2$$

based on a single observation $X = x$, and decided to reject H_0 if $x > 0.5$. Which of the following corresponds to the probability of Type II Error for this rule?

A. 0.25.

$$P(X<0.5|\theta=2) = \int_0^{0.5} 2x^{2-1}\,dx = [x^2]_0^{0.5} = 0.25$$

B. 0.5.

C. 0.75

D. 0.95.

Question 9. In Bayesian inference, which of the following is **not** needed for inference?

A. The prior distribution.

B. The observed data.

C. The likelihood function.

D. All of the above are needed for inference.

Question 10. Let $X \sim f(x|\theta)$ where

$$f(x|\theta) = \theta^x(1-\theta), \ x = 0, 1, 2, \ldots, \ \theta \in (0,1).$$

Assume a Beta(α, β) prior for θ, i.e.

$$\pi(\theta) = \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}.$$

Which of the following corresponds to the posterior distribution for θ given a single observation $X = x$?

A. Beta$(x+\alpha, 1+\beta)$.

B. Beta$(1+\beta, x+\alpha)$.

C. Beta$(x+\alpha-1, \beta)$.

D. Beta$(\beta, x+\alpha-1)$.

$$f(x|\theta) \cdot \pi(\theta) = \frac{1}{B(\alpha,\beta)}\theta^{x+\alpha-1}(1-\theta)^{1+\beta-1}$$

which is of the form

$$\pi(\theta|x) = C \cdot \theta^{A-1}(1-\theta)^{B-1}$$

$$\text{Beta}(x+\alpha, 1+\beta)$$