XX50215 Statistics for Data Science

Problems 1 - Solutions

1. Consider the vector $v = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$ in R^3

   a. Determine the equation of the line L through the origin and parallel to *v*.

   b. Consider the vector $w = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ in R^3 and find $proj_L(w)$.

Solution:

a. $L = \{tv : t \in R\}$

b. $proj_L(w) = \frac{1}{||v||2}(w.v)v = \frac{41}{49}\begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$ (Orthogonal projection on to the line L)

$||v||^2 = 6^2 + 2^2 + 3^2 = 36 + 4 + 9 = 49$

6*3 + 2*4 + 5*2 = 18 + 8+ 15 = 41

2. Is the following matrix invertible?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

Justify your answer. If A^{-1} exists, find it.

Solution:

Can we row reduce A to the identity matrix? We were allowed to do with the rows the same things we are allowed to do with the individual equations in a system of equations:

1. Multiply any row by a non-zero constant.

2. Add a constant multiple of any row to any other row.

3. Interchange any two rows.

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow[(III)+(II)^*]{} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 4 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow[2(III)^*+(I)]{} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow[-2(III)+(II)^*]{} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[(1/3)(II)+(I)^*,(III)^*]{} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow[-(1/3)(II)^*,(1/2)((III)^*,(I)^*]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Yes, so it is invertible. Now perform these row operations on the identify matrix to obtain A^{-1}

$$A^{-1} = \begin{bmatrix} 1/6 & 1/6 & -1/2 \\ 2/3 & -1/3 & 1 \\ 1/6 & 1/6 & 1/2 \end{bmatrix}$$

3. Consider the matrix,

$$A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}$$

a. Find its eigenvalues and eigenvectors.

Write the vector $u(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ as a combination of those eigenvectors.

b. Solve the equation $\frac{du}{dt} = Au$ starting with the same vector u(0) at time t = 0.

Solution:

a. $\det(A - \lambda I) = \lambda^2 + 4$.

Therefore eigenvalues are 2i, -2i.

Two associated eigenvectors are $[1\ -2i]^T$, $[1\ 2i]^T$.

u(0) is the sum of these two vectors.

b. Add factors of $e^{\lambda t}$ to each term.

$$u(t) = e^{2it}\begin{bmatrix} 1 \\ -2i \end{bmatrix} + e^{-2it}\begin{bmatrix} 1 \\ 2i \end{bmatrix}$$