XX50215 Statistics for Data Science

Problems 5 - Solutions

1. A random point (X,Y) is distributed uniformly on the square with vertices (1,1), (1,-1), (-1, 1), (-1,-1). That is the joint pdf f(x,y) = 0.25 on the square. Determine the probabilities of the following events.

   a. $X^2 + Y^2 <= 1$
   b. 2X –Y > 0
   c. |X + Y| < 2

> a. The circle $x^2+y^2 <= 1$ has area π so $P(X^2 + Y^2 <= 1) = \pi/4$
> b. The area below the line y=2x is half the area of the square so P(2X –Y > 0) = 2/4 = 1/2.
> c. As X and Y are constrained to +/-1 their sum must be less than 2. Therefore, P(|X + Y| < 2) = 1

2. I leave for work between 8AM and 8:30AM and takes between 40 and 50 minutes to get there. Assuming that departure time and journey length are independent and each is uniformly distributed what is the probability that I will arrive before 9AM?

> We need two random variables. We can use X for departure time and Y for journey time. We will express X as the number of minutes after 8AM so X ~= uniform(0,30) and Y ~= uniform(40,50). The joint pdf is 1/300 on the rectangle (0,30) x (40,50).
>
> P(arrive before 9AM) = P(X+Y < 60) = $\int_{40}^{50}\int_{0}^{60-y} 1/300\, dxdy = \frac{1}{2}$
>
> You may find it helpful to sketch the rectangle on a graph and draw the line x+y < 60 through it.

3. If a stick is broken at random into three pieces, what is the probability that the pieces can be put together in a triangle?

> One interpretation of "a stick is broken at random into three pieces" is this. Suppose the length of the stick is 1. Let X and Y denote the two points where the stick is broken. Let X and Y both have uniform$(0,1)$ distributions, and assume X and Y are independent. Then the joint distribution of X and Y is uniform on the unit square. In order for the three pieces to form a triangle, the sum of the lengths of any two pieces must be greater than the length of the third. This will be true if and only if the length of each piece is less than $1/2$. To calculate the probability of this, we need to identify the sample points (x,y) such that the length of each piece is less than $1/2$. If $y > x$, this will be true if $x < 1/2$, $y - x < 1/2$ and $1 - y < 1/2$. These three inequalities define the triangle with vertices $(0,1/2)$, $(1/2,1/2)$ and $(1/2,1)$. (Draw a graph of this set.) Because of the uniform distribution, the probability that (X,Y) falls in the triangle is the area of the triangle, which is $1/8$. Similarly, if $x > y$, each piece will have length less than $1/2$ if $y < 1/2$, $x - y < 1/2$ and $1 - x < 1/2$. These three inequalities define the triangle with vertices $(1/2,0)$, $(1/2,1/2)$ and $(1,1/2)$. The probability that (X,Y) is in this triangle is also $1/8$. So the probability that the pieces form a triangle is $1/8 + 1/8 = 1/4$.

4. Suppose that the random variable Y has a binomial distribution with n trials and success probability X, where n is a constant and X is a uniform(0,1) random variable. Find EY and Var Y.

$$\mathrm{E}Y = \mathrm{E}\{\mathrm{E}(Y|X)\} = \mathrm{E}nX = \frac{n}{2}.$$

$$\mathrm{Var}Y = \mathrm{Var}\left(\mathrm{E}(Y|X)\right) + \mathrm{E}\left(\mathrm{Var}(Y|X)\right) = \mathrm{Var}(nX) + \mathrm{E}nX(1-X) = \frac{n^2}{12} + \frac{n}{6}.$$

5. Show that any random variable is uncorrelated with a constant.

Let a be a constant. Cov(a, X) = E(aX) – EaEX = aEX – aEX = 0.

6. How many terms are in the expansion of $(x_1 + x_2 + x_3)^4$?

There is one term for each ordered triple (b1, b2, b3) with b1+b2+b3 = 4. One way to count these triples is to represent them as collections of 2 bars and 4 stars. E.g. (1,3,0) would be represented by *|***| The number of such collections is $\binom{4+2}{4} = 15$.

There are 15 terms in the expansion.