Computer Animation and Games I

CM50244

Today's Lectures

- 3D Rotations
- Interpolation
- Skeleton-based Animation

3D Rotations

Recap: 2D Transformations

- Matrix transformations in 2D (e.g. Rotation, Scale, Stretch, Shear)

$$\begin{bmatrix} p'_1 \\ p'_2 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

- If we use Homogeneous Coordinates, translation can be represented as matrix multiplication

Euclidean

$$\begin{bmatrix} p'_1 \\ p'_2 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} m_{13} \\ m_{23} \end{bmatrix}$$

homogenous

$$\begin{bmatrix} p'_1 \\ p'_2 \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix}$$

Compound Transformations

- Homogeneous transformations are easily concatenated

$$\mathbf{p}' = \mathbf{T}_3\mathbf{T}_2\mathbf{T}_1\mathbf{p}$$

- Note that the order of operations matters, in general

$$\mathbf{T}_1\mathbf{T}_2 \neq \mathbf{T}_2\mathbf{T}_1$$

3D Affine Transformations

3D

$$\begin{bmatrix} p'_1 \\ p'_2 \\ p'_3 \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix}$$

12 degrees of freedom

Recall 2D

$$\begin{bmatrix} p'_1 \\ p'_2 \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix}$$

6 degrees of freedom

Overview

- 3D Rotation About x/y/z Axis
- 3D Arbitrary Rotation Representations
  - Euler angles
  - Axis-angle
  - Quaternions

Overview

- **3D Rotation About x/y/z Axis**
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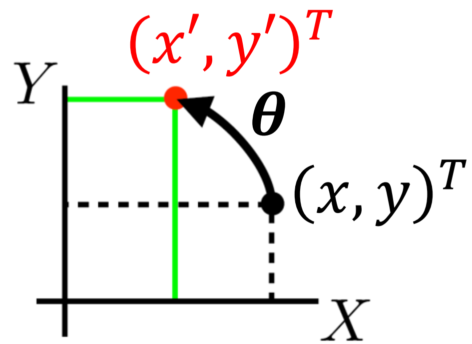
Recap: 2D Rotation



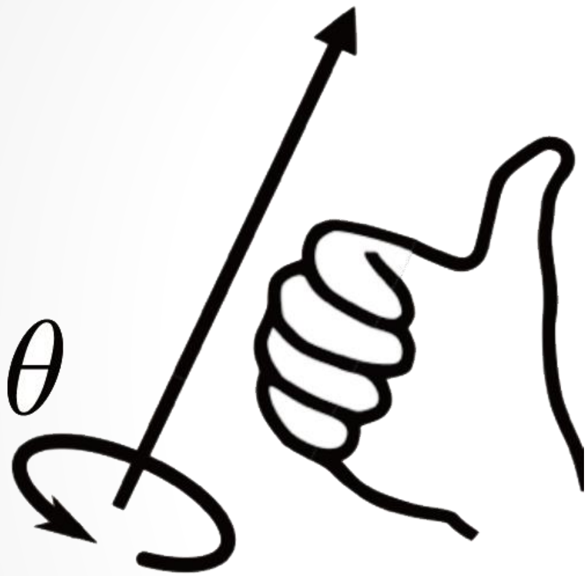
Euclidean

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

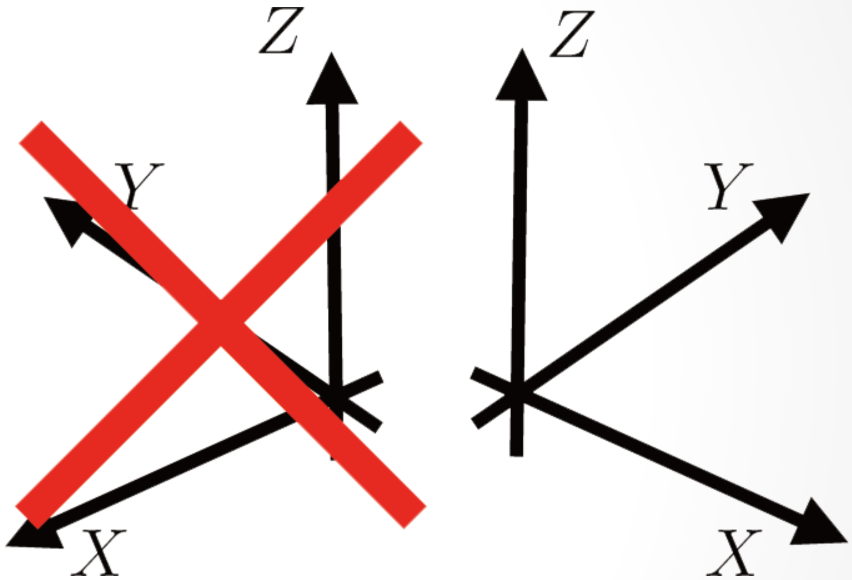
homogenous

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3D Rotation Conventions



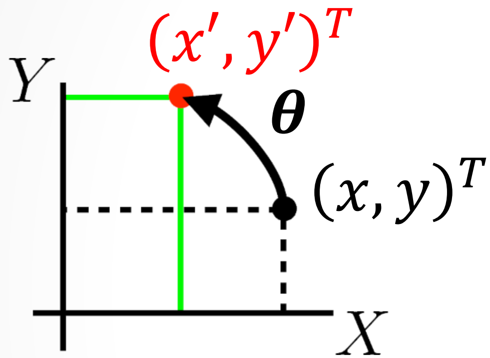
Right-Hand Screw Rule:
defines positive rotation



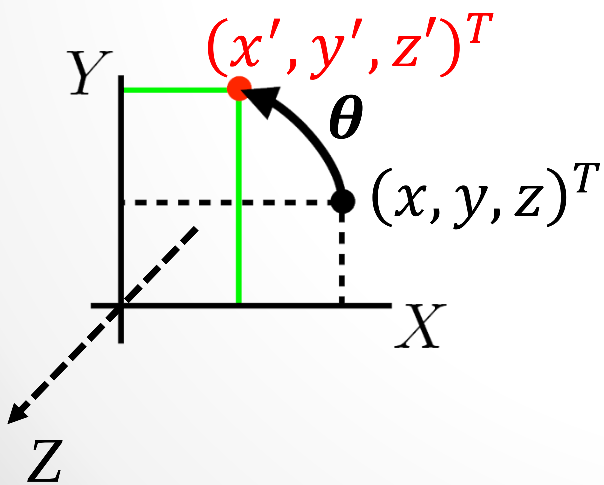
Right-Handed
Coordinate Systems

3D Rotations

- Find matrix for a positive rotation of θ about the Z axis



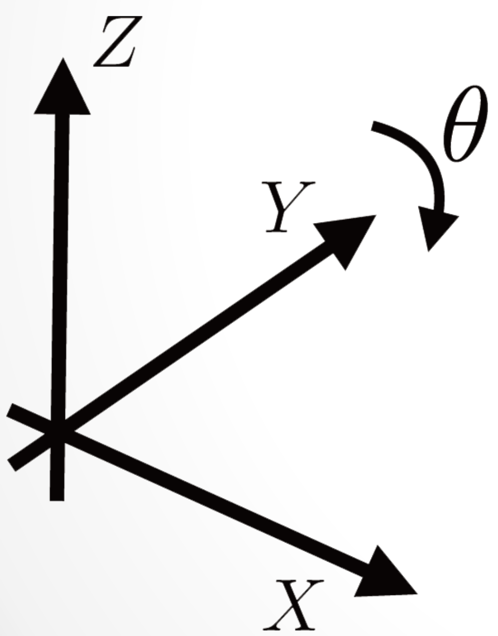
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

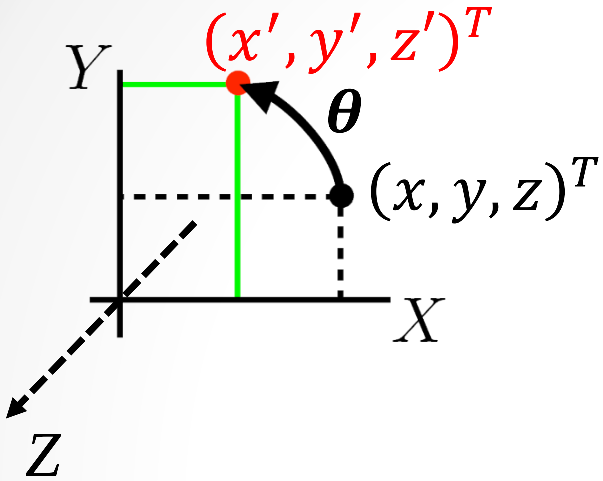
3D Rotations

- Find matrix for a positive rotation of θ about the Y axis



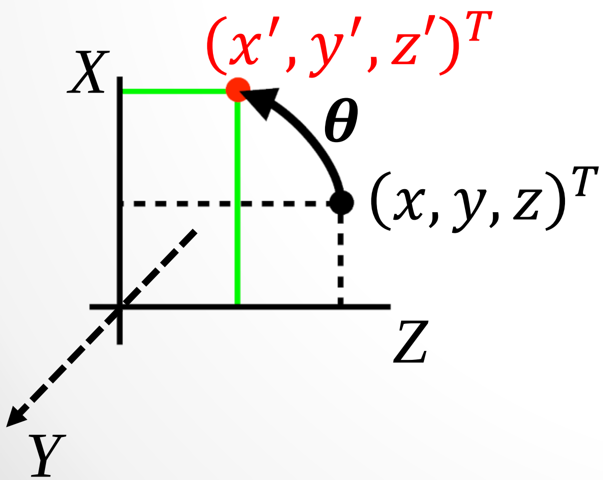
$$\mathbf{R}_y(\theta) = ??$$

3D Rotations



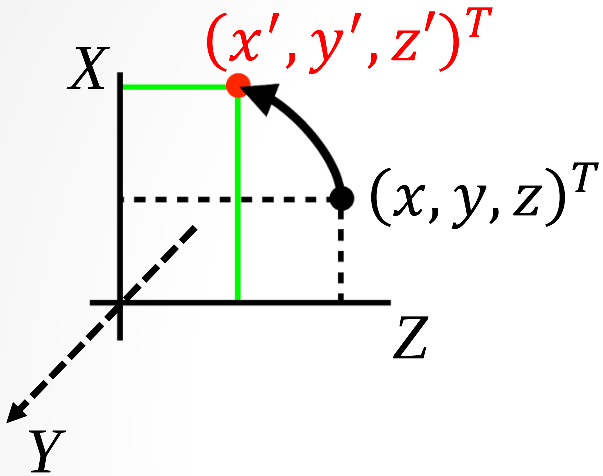
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Downarrow \quad \begin{matrix} x \rightarrow z \\ y \rightarrow x \\ z \rightarrow y \end{matrix}$$



$$\begin{bmatrix} z' \\ x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ x \\ y \end{bmatrix}$$

3D Rotations



$$\begin{bmatrix} z' \\ x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ x \\ y \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

3D Rotation Matrices

$$\mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

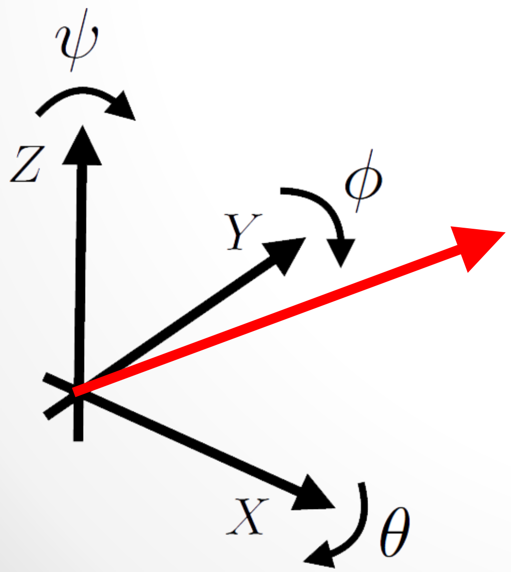
Overview

- 3D Rotation About x/y/z Axis
- **3D Arbitrary Rotation Representations**
  - **Euler angles**
  - **Axis-angle**
  - **Quaternions**

Euler Angles

- How do we generate an arbitrary rotation matrix? (rotation about arbitrary axis passing through origin)

- Simple idea: use an ordered combination of rotations about the X, Y and Z axes

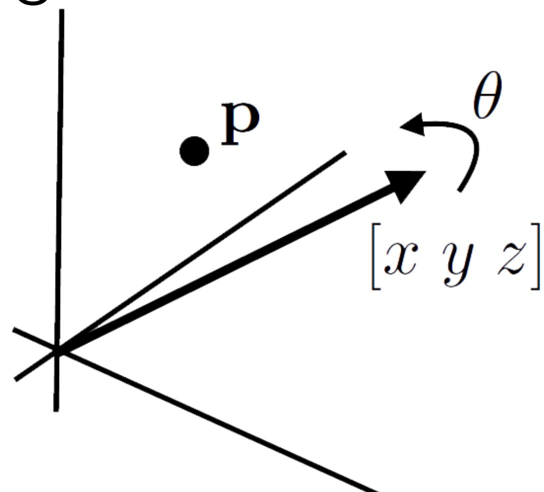


$$\mathbf{R} = \mathbf{R}_z(\psi)\mathbf{R}_y(\phi)\mathbf{R}_x(\theta)$$

e.g. first X, then Y then Z
(all variations used)

Axis-Angle Rotations

- We can represent any rotation directly by a rotation axis and angle



Consider rotating a point **P** by an angle θ about an axis direction $[x,y,z]$

- It can be shown that

$$\mathbf{R} = \begin{bmatrix} x^2(1-c)+c & xy(1-c)-zs & xz(1-c)+ys \\ yx(1-c)+zs & y^2(1-c)+c & yz(1-c)-xs \\ zx(1-c)-ys & zy(1-c)+xs & z^2(1-c)+c \end{bmatrix}$$

where $c = \cos\theta$ and $s = \sin\theta$

Quaternions

- Quaternions are 4-dimensional entities, represented as

$$\mathbf{q} = q_0 + iq_1 + jq_2 + kq_3$$

with *i*, *j* and *k* (imaginary units) having the properties

$$i^2 = j^2 = k^2 = ijk = -1$$

- Quaternions are actually an extension to complex numbers
- This allows us to add, subtract and multiply quaternions

Quaternions

- **Unit Quaternions**

$$|\mathbf{q}| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1$$

- This represent rotations in a manner similar to axis-angle representation:

$$\text{angle: } \theta = 2\cos^{-1} q_0 \qquad \text{axis: } [\; q_1 \;\; q_2 \;\; q_3 \;]$$

- Quaternions have the advantage that rotation interpolation is made simple (next lecture)

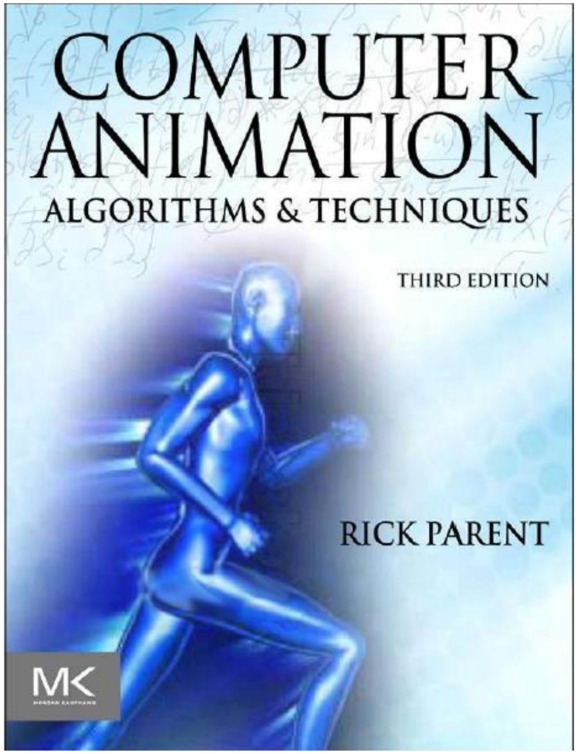
More about Rotation Matrix

- If R is a rotation matrix, then it is also an orthogonal matrix, i.e.

$$R^T R = I$$

- **Q:** How many degrees of freedom does a rotation matrix have?

More about 3D Rotations



Chapter 2 and Appendix B.3.4