Machine Learning 1.12: Belief Propagation

Tom S. F. Haines
T.S.F.Haines@bath.ac.uk

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Markov Chain

It's a chain!
Markov network (usual representation):

$$x_0 - x_1 - x_2 - x_3 \cdots \cdots$$

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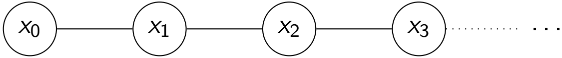
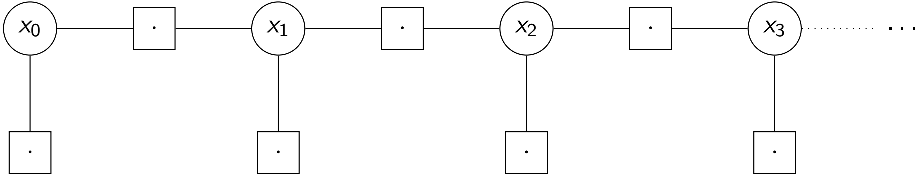


Factor graph:



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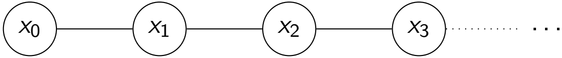

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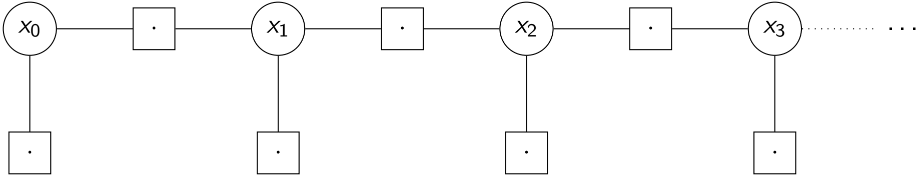
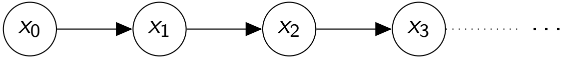


Bayesian network:



Markov?

- "*Next state depends only on the previous state*"
- Alternatively, "*state has no memory of the past*".
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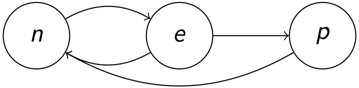
- Examples:
    - Much of physics, chemistry etc. Most simulations are Markov.
    - Stock market and other financial things.
    - Speech recognition.
    - Data transfer over noisy channels.

$$\vdots$$

Finite State Machines

- Crossroad traffic lights as a finite state machine:



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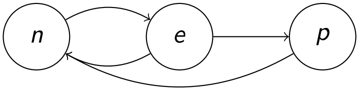

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- Crossroad traffic lights unrolled as a Markov random chain:

$$n \rightarrow e \rightarrow n \rightarrow e \rightarrow p \rightarrow n \cdots\cdots\cdots \;\ldots$$

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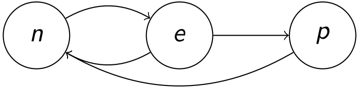

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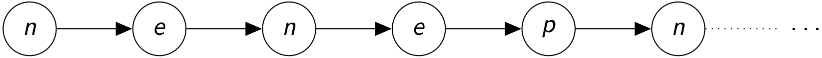

- Finite state machines (FSM) can run on Markov chains.
- Markov chains are probabilistic; FSMs are usually not.
- Markov chains can have infinite states.

Transition Matrix

A Markov random chain **can** be represented by:

- A set of n states, $\{s_0, s_1, \ldots, s_{n-1}\}$.
- A transition matrix $T \in \mathbb{R}^{n \times n}$, such that $P(s_a \rightarrow s_b) = T_{ab}$.
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- Set a prior, e.g. a uniform Dirichlet distribution (conjugate).
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Explicitly:

- Fill T with 1.
- For each $s_a \rightarrow s_b$ in data increment T_{ab}.
- Normalise each row of T.

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Simulation

To draw from the model:

- Not part of Markov chain:
  - Select a start state (fit categorical to data).
  - Select a length (fit distribution to data length, e.g. Poisson).

  (can introduce start and stop states instead)

- $x_{i+1} \sim P(x_i \rightarrow)$ for $i \in [1 \ldots \texttt{length} - 1]$.

Fake Place Names

Steps:

- Download UK place names from `https://www.paulstenning.com/uk-towns-and-counties-list/`
- Learn transition matrix.
- Learn categorical distribution over starting letter.
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- Draw “plausible” place names:

ackilin	ckleryhtz	harkbrdy borthr	genongigropo
burie	ftonsest	ad onff	mveyldid
beston	kilelerg	crynhal	coeerigent
kqomironam	wey ok mbu	licalst	borayeamol

- Extension: n-gram model, e.g. $n = 2 \implies$ States are two previous letters.

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Observations

Real problems:

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Possible model queries:

- Maximum Likelihood (ML)/Maximum a Posteriori (MAP) – the most probable state sequence.
- Marginals – per node distribution over state.

- Joint distribution – full distribution over unknown states (rarely practical).
- Conditional draw – draws from above.

(covering first two, ignoring second two)

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Notes:

- Usually given as a *hidden Markov model*
- Improvements: (Above was state of the art in $\sim$2012)
  - Other features, e.g. Perceptual Linear Prediction.
  - Deep learning instead of a GMM for phoneme probability.
  - Recurrent neural networks for language model, and for phoneme probability.

  Note: None of these replace the Markov chain!

Marginals

- **Forward-backwards algorithm**.
- A kind of dynamic programming.

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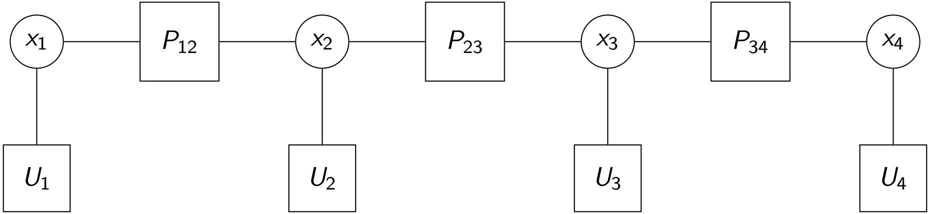
- Marginal is defined as:

$$b_i(x_i) = \sum_{\cdot / x_i} P(x_1, \ldots, x_N)$$

  where:
  - $x_i, i \in [1, \ldots, N]$ are the random variables.
  - $b_i(x_i)$ is the belief.
  - $\cdot / x_i$ means all variables except for x_i.
  - $P(x_1, \ldots, x_N)$ is the joint distribution over all variables.

Example



- x_i – Random variable.
- U_i – Unary, a factor on one RV.
- P_i – Pairwise, a factor on two RVs.

Message Passing

- Solve by sending messages along edges.
- Two messages per edge, one in each direction.
- Problem solved when all messages sent.

- Two kinds of message:
  - Random variable $\rightarrow$ Factor
  - Factor $\rightarrow$ Random variable

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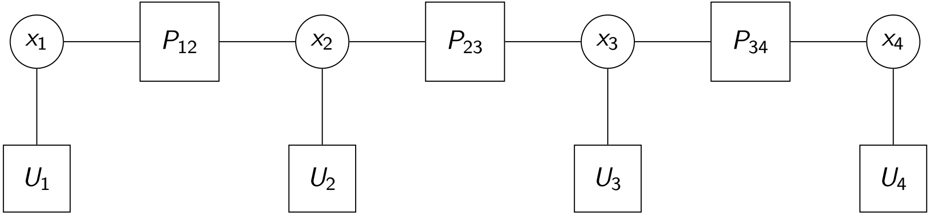
- $\therefore$ Two passes: Forward (left to right) and backwards (right to left).

Example Forwards



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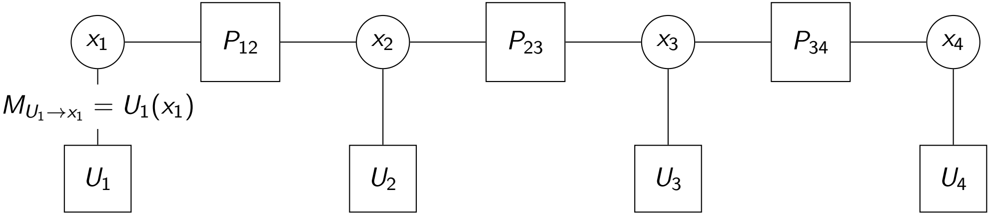

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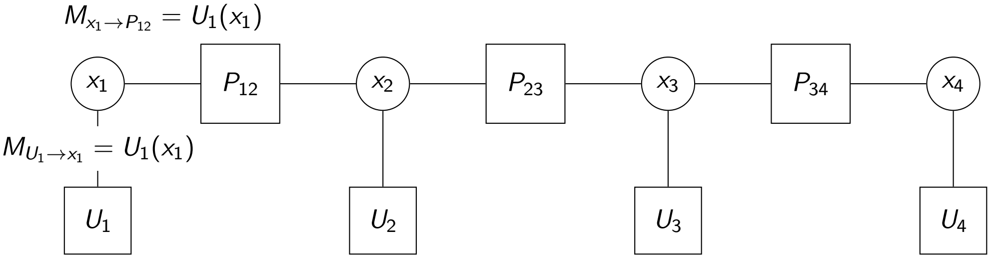

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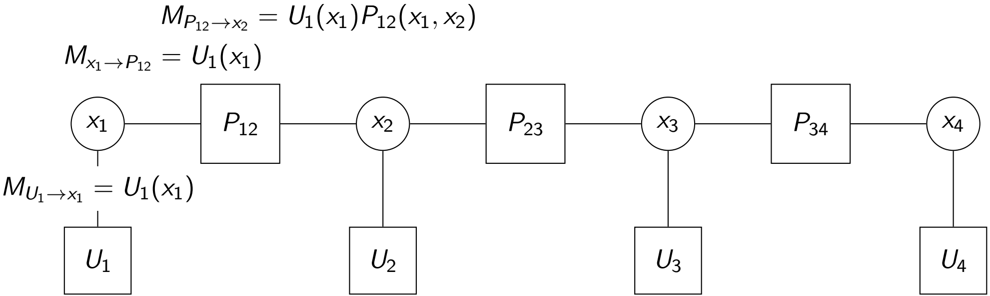

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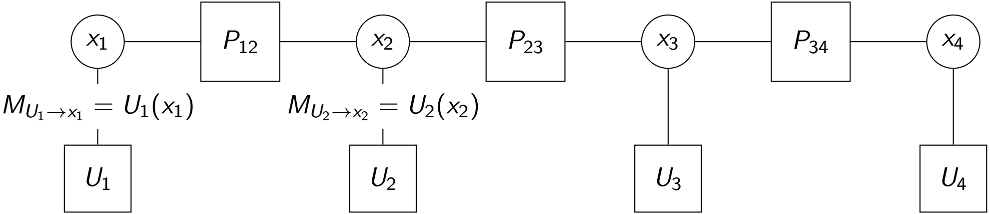

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$$M_{x_2 \to P_{23}} = U_1(x_1)P_{12}(x_1, x_2)U_2(x_2)$$

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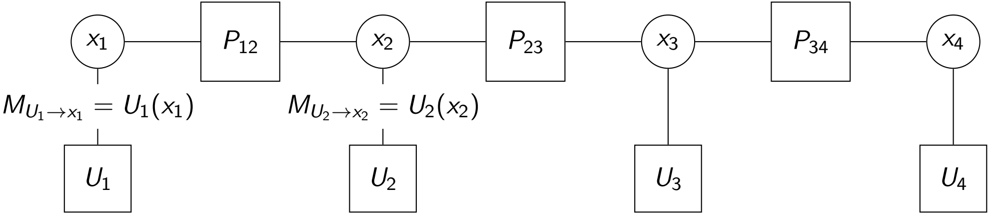

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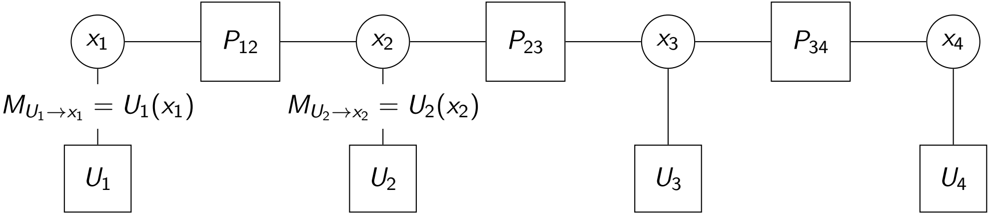

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- Messages only passed when all dependent messages received.
- Can be simplified: Messages passed between RVs directly.

Equation

- Messages sent by RVs:

$$M_{v\to F} = \sum_{x\notin F}\left[\prod_{G\in N_v/F} M_{G\to v}\right]$$

where $i \in N_v/F$ is all neighbours except the message destination.

- Messages sent by factors:

$$M_{F\to v} = \sum_{x\notin v}\left[F(\cdot)\prod_{u\in N_F/v} M_{u\to F}\right]$$

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- Best to record which RV won each max. Draws are common!
- $\therefore$ Only forward pass required.
- Can be implemented as a recursive function.

Dynamic Programming Notes

- Dynamic programming:
  "Any algorithm where the solution can be found recursively by solving slightly smaller problems first."
- Forwards-backwards and Viterbi are both instances of this; many others, e.g. Levenshtein distance.
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- **Kalman smoothing** = Gaussian distributions
  (Forwards-backwards or Viterbi – give same answer!)
- **Kalman filtering** = Online version where only the past is factored in.

Dynamic Programming Tricks

- Alignment: You know the data labels but not where they are.
- Solution: Labels represent where in the sequence, 0 probability of transitioning to any but the next.

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- n-gram: States are dependent on not just neighbours but the neighbours of their neighbours.
- Explode states to encode history.

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- sum-product = generalisation of forward-backwards.
- max-product = generalisation of Viterbi.

- min-sum = identical to max-product, but performed in negative log space (costs).

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- Loopy belief propagation approximately solves graphs with loops.
  Ignores dependencies and just keeps sending messages!
  (better choices, particularly graph cuts. Some situations where it can be solved precisely (Ising/Potts models and GBP))

Summary

- Markov chain.
- Dynamic programming.
- Belief propagation.

Further Reading

- Very pragmatic tutorial on BP on grids of RVs:
  "Efficient Belief Propagation for Early Vision", by Felzenszwalb & Huttenlocher:
  `https://cs.brown.edu/~pff/papers/bp-cvpr.pdf`

- Voice recognition as described above:
  Hidden Markov Model Toolkit (HTK) `http://htk.eng.cam.ac.uk/`