1. **Problem**
   Let $X_1, \ldots, X_n \sim f(x|\theta)$ with
   $$f(x|\theta) = \exp\{-(x-\theta)\}, \ x \geq \theta.$$
   The method of moments estimator for θ is
   (a) $\bar{x}$
   (b) $\bar{x} - 1$
   (c) $\min\{X_1, \ldots, X_n\}$
   (d) $\max\{X_1, \ldots, X_n\}$

   Solution
   First we find the moment generating function of X.
   $$\begin{aligned} M(t) &= \int_\theta^\infty e^{tx} e^{-(x-\theta)} dx \\ &= e^\theta \int_\theta^\infty e^{-x(1-t)} dx \\ &= \frac{e^\theta}{1-t} [-e^{-x(1-t)}]_\theta^\infty \\ &= \frac{e^\theta}{1-t} e^{-\theta(1-t)} \\ &= \frac{e^{t\theta}}{1-t}, \ t < 1. \\ M'(t) &= \frac{\theta e^{t\theta}}{1-t} + \frac{e^{t\theta}}{(1-t)^2} \\ M'(0) &= \theta + 1 \end{aligned}$$
   For the MoM estimator we set $\theta + 1 = \bar{x} \Rightarrow \theta = \bar{x} - 1$.

2. **Problem**
   Let $X_1, \ldots, X_n \sim f(x|\theta)$ with
   $$f(x|\theta) = \exp\{-(x-\theta)\}, \ x \geq \theta.$$
   The maximum likelihood estimator for θ is
   (a) $\bar{x}$
   (b) $\bar{x} - 1$
   (c) $\min\{X_1, \ldots, X_n\}$
   (d) $\max\{X_1, \ldots, X_n\}$

   Solution
   Because the support of the pdf (the values x for which it is positive) depends on θ, we write the pdf using the indicator function as
   $$f(x|\theta) = \exp\{-(x-\theta)\} 1(x \geq \theta),$$
   where $1(x \geq \theta) = 1$ if $x \geq \theta$ and 0 otherwise. Then, the likelihood function is
   $$\begin{aligned} L(\theta) &= \prod_{i=1}^n f(x_i|\theta) \\ &= \exp\{-\sum x_i + n\theta\} 1(x_1 \geq \theta) \cdots 1(x_n \geq \theta) \\ &= \exp\{-\sum x_i + n\theta\} 1(x_{(1)} \geq \theta), \end{aligned}$$

where $x_{(1)} = \min\{x_1, \ldots, x_n\}$. We observe that the exponential term is an increasing function of θ, so as θ increases, this term increases as well. However, when θ exceeds $x_{(1)}$ the indicator function drops to 0, so the likelihood is maximised when $\theta = x_{(1)}$.

3. **Problem**
Let $X_1, \ldots, X_n \sim \mathrm{N}(0, \theta)$ where θ denotes the variance parameter. We wish to test

$$\mathrm{H}_0\colon\ \theta = \theta_0 \text{ v.s. } \mathrm{H}_1\colon\ \theta < \theta_0.$$

A hypothesis test would reject H_0 if

(a) $\bar{x} < c^*$
(b) $\bar{x} > c^*$
(c) $\sum x_i^2 < c^*$
(d) $\sum x_i^2 > c^*$

Solution
We consider the equivalent hypothesis test

$$\mathrm{H}_0\colon\ \theta = \theta_0 \text{ v.s. } \mathrm{H}_1\colon\ \theta = \theta_1 \text{ for some } \theta_1 < \theta_0.$$

The pdf is

$$f(x|\theta) = (2\pi\theta)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\theta}x^2\right\}.$$

The likelihood is

$$\begin{aligned} L(\theta|\mathbf{x}) &= \prod_{i=1}^{n} f(x_i|\theta) \\ &= (2\pi\theta)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\theta}\sum x_i^2\right\}. \end{aligned}$$

Next, we calculate the likelihood ratio

$$\begin{aligned} T &= \frac{L(\theta_1|\mathbf{x})}{L(\theta_0|\mathbf{x})} \\ &= \left(\frac{\theta_1}{\theta_0}\right)^{-\frac{n}{2}} \exp\left\{\left(\frac{1}{2\theta_0} - \frac{1}{2\theta_1}\right)\sum x_i^2\right\}. \end{aligned}$$

Because $\theta_0 > \theta_1 \Rightarrow \frac{1}{2\theta_0} - \frac{1}{2\theta_1} < 0$ so the likelihood ratio is a **decreasing** function of $\sum x_i^2$. Therefore, for a rule that rejects the null hypothesis if $T > c$, there is an equivalent rule of the form $\sum x_i^2 < c^*$.

4. **Problem**
Let $X_1, \ldots, X_n \sim \mathrm{N}(0, \theta)$ where θ denotes the variance parameter. We wish to test

$$\mathrm{H}_0\colon\ \theta = 1 \text{ v.s. } \mathrm{H}_1\colon\ \theta = 2.$$

A rule that rejects the null hypothesis at level α is of the form $\sum X_i^2 > c$ where the critical value c is such that

(a) $P(Y > c) = \alpha$ where $Y \sim \mathcal{X}_{n-1}^2$.
(b) $P(Y < c) = \alpha$ where $Y \sim \mathcal{X}_{n-1}^2$.
(c) $P(Y > c) = \alpha$ where $Y \sim \mathcal{X}_n^2$.

(d) $P(Y < c) = \alpha$ where $Y \sim \mathcal{X}_n^2$.

Solution
Assuming H_0 is true, let $Y = \sum X_i^2$. Then, $Y \sim \mathcal{X}_n^2$ and we reject H_0 if $Y > c$, so c must be chosen such that $P(Y > c) = \alpha$ where $Y \sim \mathcal{X}_n^2$.

5. **Problem**
Let $X_1, \ldots, X_n \sim \mathrm{N}(0, \theta)$ where θ denotes the variance parameter. We wish to test

$$\mathrm{H}_0\text{: } \theta = 1 \text{ v.s. } \mathrm{H}_1\text{: } \theta = 2.$$

Our decision is to reject H_0 if $\sum X_i^2 > 1.2$. Let $Y \sim \mathcal{X}_n^2$. The probabilities of Type I and Type II errors are

(a) $P(Y > 1.2)$. / $P(Y < 1.2)$. / $P(Y > 0.6)$. / $P(Y < 0.6)$.
(b) $P(Y > 1.2)$. / $P(Y < 1.2)$. / $P(Y > 0.6)$. / $P(Y < 0.6)$.

Solution
Assuming H_0 is true, let $W = \sum X_i^2$. Then, $W \sim \mathcal{X}_n^2$ and we reject H_0 if $W > 1.2$, so $P(\text{Type I}) = P(Y > 1.2)$ where $Y \sim \mathcal{X}_n^2$.
Assuming H_1 is true, let $W = \sum X_i^2$. Then, $W/2 \sim \mathcal{X}_n^2$ and we accept H_0 if $W < 1.2$, so $P(\text{Type II}) = P(W < 1.2) = P(W/2 < 0.6) = P(Y < 0.6)$ where $Y \sim \mathcal{X}_n^2$.

6. **Problem**
Let $X_1, \ldots, X_n \sim \text{Exponential}(\theta)$ where θ denotes the rate parameter with

$$f(x|\theta) = \theta \exp\{-\theta x\},\ x > 0,\ \theta > 0$$

and assume a gamma prior $\theta \sim \text{Gamma}(\alpha, \beta)$, i.e.

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} \exp\{-\beta\theta\}.$$

The posterior distribution for θ is

(a) $\text{Gamma}(\alpha, \beta)$
(b) $\text{Gamma}(\alpha + n, \beta + n\bar{x})$
(c) $\text{Gamma}(\alpha + n - 1, \beta + n\bar{x})$
(d) $\text{Gamma}(n, n\bar{x})$

Solution
The likelihood is

$$\begin{aligned} f(\mathbf{x}|\theta) &= \prod_{i=1}^n f(x_i|\theta) \\ &= \theta^n \exp\{-\theta \sum x_i\} \\ &= \theta^n \exp\{-\theta n\bar{x}\}. \end{aligned}$$

Multiplying this with the prior, we have,

$$\begin{aligned} f(\mathbf{x}|\theta)\pi(\theta) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{n+\alpha-1} \exp\{-(\beta + n\bar{x})\theta\} \\ &= C\theta^{A-1} \exp\{-B\theta\} \end{aligned}$$

with $A = \alpha + n$ and $B = \beta + n\bar{x}$ and C does not depend on θ. This can be recognised as a form of the gamma distribution with shape A and rate B. Therefore, the posterior distribution of θ is

$$\theta|\mathbf{x} \sim \text{Gamma}(\alpha + n, \beta + n\bar{x}).$$

7. **Problem**
The following 3 observations

$$5.7,\ 3.34,\ 0.35$$

are a random sample from the Exponential(θ) distribution where θ denotes the rate parameter with

$$f(x|\theta) = \theta \exp\{-\theta x\},\ x > 0,\ \theta > 0$$

and assume a gamma prior $\theta \sim \text{Gamma}(\alpha, \beta)$, i.e.

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} \exp\{-\beta\theta\},$$

with $\alpha = 1$ and $\beta = 1$.

Use the data to calculate the Bayesian estimator for θ. (Give 3 decimal points of accuracy.)

Solution
It can be seen that the posterior distribution for θ is

$$\theta|\mathbf{x} \sim \text{Gamma}(A, B),$$

where $A = \alpha + n = 1 + 3 = 4$, and
$B = \beta + \sum x_i = 1 + 9.39 = 10.39$.

The mean of the gamma distribution is $A/B = 4/10.39 = 0.385$.