Course Review - End of the Course

MATH350 – Statistical Inference

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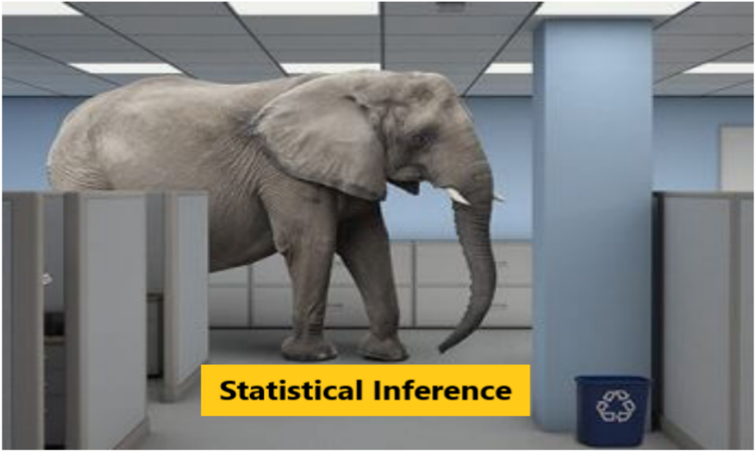
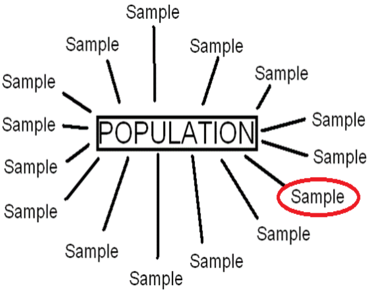
Course page: https://github.com/ctanujit/MATH350
Course for BSc Mathematics and Data Science Students.

> *"When there's an elephant in the room introduce him."*
>
> *- Randy Pausch, The Last Lecture (2008).*



Overview of Statistical Inference



Given a sample from a population, how do we make inferences about the population?

- Point & Interval Estimation (both Frequentist & Bayesian approaches)
- Hypothesis testing (both Frequentist & Bayesian approaches)

Syllabus covered

Statistical Inference:

- Point Estimation
- Interval Estimation
- Hypothesis Testing
- Linear Models and ANOVA
- Nonparametric Inference
- Bayesian Inference

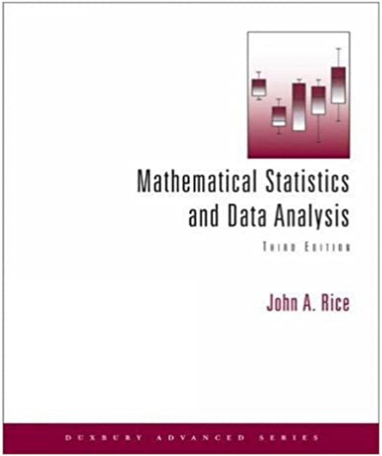
Lab Session & Projects:

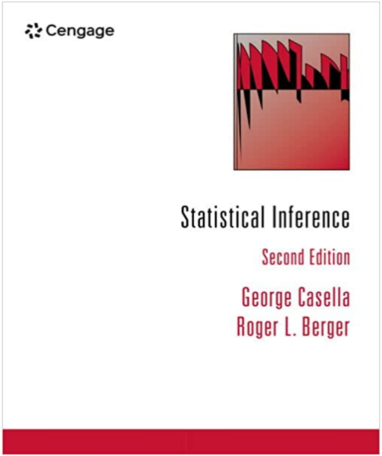
- Implementations in R
- Simulations in R
- MCMC in R
- Course Project using Real-world Data

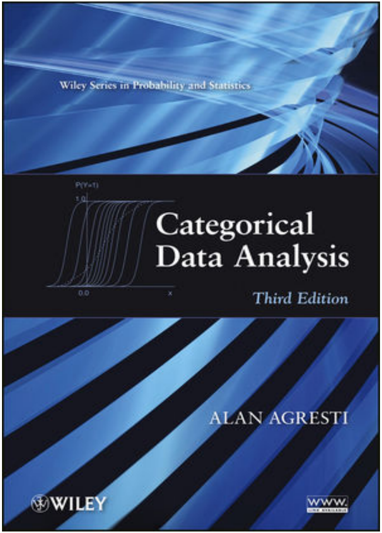
Evaluation:

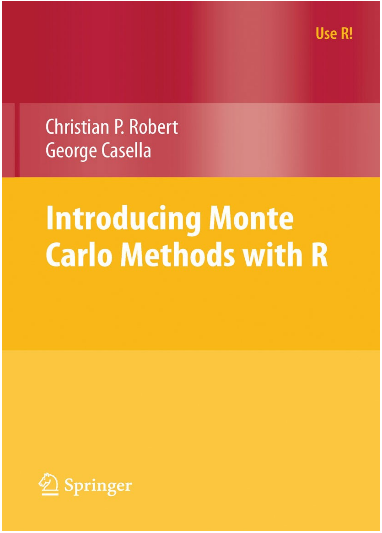
- Course Project (50%)
- Final Exam (50%)

"Teaching two separate courses, one on theory and one on data analysis, seems to me artificial" - John A. Rice.






Wiley Series in Probability and Statistics
Categorical Data Analysis
Third Edition
ALAN AGRESTI
WILEY


Use R!
Christian P. Robert
George Casella
Introducing Monte Carlo Methods with R
Springer

Course Review

We started with a fundamental assumption:

Data is a realization of a random process.

The goal throughout this course has been to use the observed data to draw inferences about the underlying process or probability distribution.
We discussed:

- Hypothesis testing-deciding whether a particular “null hypothesis" about the underlying distribution is true or false
- Estimation-fitting a parametric model to this distribution and/or estimating a quantity related to the parameters of this model
- Standard errors and confidence intervals-quantifying the uncertainty of these estimates

Hypothesis testing

Goal: Accept or reject a null hypothesis H_0 based on the value of an observed test statistic T.

Question #1: How to choose a test statistic T ?

Question #2: How to decide whether H_0 is true/false based on T ?

In a simple-vs-simple testing problem, there is a "best" answer to *Question # 1*, which is the likelihood ratio statistic

$$L(X_1, \ldots, X_n) = \frac{f_0(X_1, \ldots, X_n)}{f_1(X_1, \ldots, X_n)}$$

We can equivalently use any monotonic 1-to-1 transformation of this statistic, which is simpler to understand in many examples (e.g. the total count for Bernoulli coin flips).

Hypothesis testing

In problems with composite alternatives, there is oftentimes not a single most powerful test against all of the alternative distributions. We instead discussed popular choices of T for several examples:

- Testing goodness of fit: Histogram methods (Pearson's chi-squared statistic), QQ-plot methods (one-sample Kolmogorov-Smirnov statistic)
- Testing if a distribution is centered at 0: One-sample t-statistic, signed-rank statistic, sign statistic
- Testing if two samples have the same distribution or mean: Two-sample t-statistic, rank-sum statistic
- Testing if the parameters of a model satisfy additional constraints (i.e. belong to a sub-model): Generalized likelihood ratio statistic

Hypothesis testing

Question #2: How to decide whether H_0 is true/false based on T?

We adopted the frequentist significance testing framework: Control the probability of type I error (falsely rejecting H_0) at a target level α, by considering the distribution of T if H_0 were true.

To do this for each test, we either derived the null distribution of T exactly (e.g. t-test), appealed to asymptotic approximations (e.g. the χ^2 distribution for the GLRT), or used computer simulation (e.g. permutation two-sample tests).

Estimation

Goal: Fit a parametric probability model to the observed data. For IID data $X_1, \ldots, X_n \sim f(x \mid \theta)$, we discussed three methods:

- Method of moments: Equate the sample mean of X_i, X_i^2, etc. to the theoretical mean of X, X^2, etc., and solve for θ
- Maximum likelihood: Pick θ to maximize $\prod_{i=1}^n f(X_i \mid \theta)$, or equivalently $\sum_{i=1}^n \log f(X_i \mid \theta)$
- Bayesian inference: Postulate a prior distribution $f_\Theta(\theta)$ for θ, compute the posterior

$$f_{\Theta|X}(\theta \mid X_1, \ldots, X_n) \propto f_\Theta(\theta) \prod_{i=1}^n f(X_i \mid \theta)$$

  and estimate θ by e.g. the posterior mean.

We illustrated how the method of maximum likelihood generalizes to regression models with covariates, where the data $Y_1, \ldots, Y_n$ are independent but not identically distributed.

Estimation

We discussed the accuracy of estimators in terms of several finite n properties:

- The bias $\mathbb{E}_\theta[\hat{\theta}] - \theta$
- The variance $\mathrm{Var}_\theta[\hat{\theta}]$
- The MSE $\mathbb{E}_\theta\left[(\hat{\theta} - \theta)^2\right] = \text{bias}^2 + \text{variance}$

We also discussed asymptotic properties, as $n \to \infty$ and when the true parameter is θ:

- Consistency: $\hat{\theta} \to \theta$ in probability
- Asymptotic normality: $\sqrt{n}(\hat{\theta} - \theta) \to \mathcal{N}(0, v(\theta))$ in distribution for some variance $v(\theta)$
- Asymptotic efficiency: $\hat{\theta}$ is asymptotically normal, and the limiting variance is $v(\theta) = I(\theta)^{-1}$, where $I(\theta)$ is the Fisher information (of a single observation)

The definition of asymptotic efficiency was motivated by the Cramer-Rao lower bound: Under mild smoothness conditions, for any unbiased estimator $\hat{\theta}$ of θ, its variance is at least $\frac{1}{n}I(\theta)^{-1}$.

A major result was that the MLE is asymptotically efficient:

$$\sqrt{n}(\hat{\theta} - \theta) \to \mathcal{N}\left(0, I(\theta)^{-1}\right)$$

We showed informally that Bayes estimators, asymptotically for large n, approach the MLE — an implication is that Bayes estimators are usually also asymptotically efficient.

On the other hand, method-of-moments estimators are oftentimes not asymptotically efficient and have a larger mean-squared error than the other two procedures for large n.

We discussed standard error estimates and confidence intervals both when the model is correctly specified and when it is not.

In a correctly specified model, we can derive the variance $v(\theta)$ of the estimate $\hat{\theta}$ in terms of the true parameter θ, and estimate the standard error by the plugin estimate $\sqrt{v(\hat{\theta})}$.

For the MLE, asymptotic efficiency implies that $v(\theta) \approx \frac{1}{n}I(\theta)^{-1}$ for large n. In the setting of non-IID data $Y_1, \ldots, Y_n$ in regression models, we used the Fisher information of all n samples, $I_Y(\theta)^{-1}$, in place of $\frac{1}{n}I(\theta)^{-1}$.

For other estimators, we can sometimes derive $v(\theta)$ directly from the form of $\hat{\theta}$, perhaps using asymptotic approximations like the CLT and delta method.

In an incorrectly specified model, we studied the behavior of the MLE $\hat{\theta}$ and interpreted the parameter θ that it tries to estimate as the probability distribution in the model "closest in KL-divergence" to the true distribution of data.

We derived a more general formula for the variance of $\hat{\theta}$, and showed how this variance may be estimated by a "sandwich" estimator.

Alternatively, we discussed the nonparametric bootstrap as a simulation-based approach for estimating the standard error that is also robust to model misspecification.

Standard errors and confidence intervals

To estimate a function $g(\theta)$, we may use the plugin estimate $g(\hat{\theta})$.

- If $\hat{\theta}$ is asymptotically normal, then $g(\hat{\theta})$ is usually also asymptotically normal, and its asymptotic variance may be derived using the delta method.
- If $\hat{\theta}$ is asymptotically efficient (e.g. the MLE), then $g(\hat{\theta})$ is also asymptotically efficient for $g(\theta)$.

For any quantity θ, an approximate level $100(1-\alpha)\%$ confidence interval for θ may be obtained from any asymptotically normal estimator $\hat{\theta}$ and an estimate $\hat{\mathrm{se}}$ of its standard error:

$$\hat{\theta} \pm z(\alpha/2)\hat{\mathrm{se}}$$

Most confidence intervals that we constructed in this class were of this form. (The accuracy of such intervals should be checked by simulation if n is small.)

Further Comments

We were informal in our discussion of regularity conditions required for asymptotic efficiency of the MLE, asymptotic χ^2 distribution of the GLRT statistic $-2 \log \Lambda$, etc. For parametric models having differentiable likelihood function and common support, you don't need to check regularity conditions when applying these results.

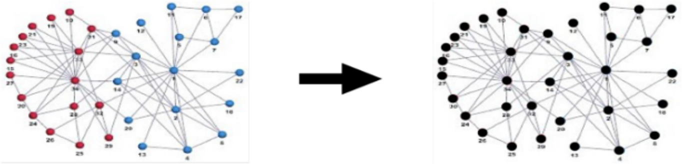
One last example: Beyond the MLE

We live in a time when there is a convergence of ideas and interchange of tools across quantitative disciplines. There is a rich interplay between statistical inference and optimization, algorithms, machine learning, and information theory.

Here is one last example which illustrates how the idea of the MLE, in a seemingly simple problem, connects to interesting questions in a variety of other fields of study.

One last example: Beyond the MLE

- n people, belonging to two distinct communities of equal size $n/2$, are connected in a social network.
- Every pair of people is connected (independently) with probability p if they are in the same community, and with probability q if they are in different communities, where $q < p$.
- We can see the network of connections, but we cannot see which person belongs to which community.



Question: Suppose (for simplicity) that we know the values p and q. How can we infer who belongs to which community?

One last example: Beyond the MLE

Let $S \subset \{1, \ldots, n\}$ denote community 1 , and S^c denote community 2.

Let Same be the set of pairs $\{i, j\}$ such that $i, j \in S$ or $i, j \in S^c$, and let Different be the set of pairs $\{i, j\}$ such that one of i, j belongs to S and the other to S^c.

Our observed data are the connections in this network:

$$A_{ij} = \begin{cases} 1 \text{ if } \{i, j\} \text{ are connected in the network} \\ 0 \text{ otherwise} \end{cases}$$

Under our model, A_{ij} are independent Bernoulli random variables with $A_{ij} \sim \text{Bernoulli}(p)$ if $\{i, j\} \in$ Same and $A_{ij} \sim \text{Bernoulli}(q)$ if $\{i, j\} \in$ Different.

One last example: Beyond the MLE

The likelihood function is

$$\begin{aligned} \text{lik}(S) &= \prod_{\{i,j\} \in \text{Same}} p^{A_{ij}} (1-p)^{1-A_{ij}} \prod_{\{i,j\} \in \text{Different}} q^{A_{ij}} (1-q)^{1-A_{ij}} \\ &= \prod_{\{i,j\} \in \text{Same}} (1-p) \left(\frac{p}{1-p} \right)^{A_{ij}} \prod_{\{i,j\} \in \text{Different}} (1-q) \left(\frac{q}{1-q} \right)^{A_{ij}} \end{aligned}$$

Each observed connection (where $A_{ij} = 1$) contributes a factor of $p/(1-p)$ to the likelihood if $\{i,j\} \in$ Same, and a factor of $q/(1-q)$ if $\{i,j\} \in$ Different.

Since $p > q$, the MLE $\hat{S}$ is given by the partition of the people into two communities that minimizes the number of observed connections between communities (or, equivalently, maximizes the number of connections within communities).

One last example: Beyond the MLE

More formally, the MLE $\hat{S}$ solves the optimization problem

$$\begin{aligned} &\text{minimize } \sum_{i \in S} \sum_{j \in S^c} A_{ij} \\ &\text{subject to } |S| = \left|S^c\right| = n/2 \end{aligned}$$

This is a well-known problem in computer science, called the **minimum graph bisection** problem.

Unfortunately, this problem is known to be NP-complete —- it is widely believed that no computationally efficient computer algorithm can compute this MLE (for all possible realizations of the network).

One last example: Beyond the MLE

This leads to a number of interesting questions:

- Can we approximately solve this optimization problem, and prove that our answer is not too far off?
- Are there other algorithms (not directly based on this optimization) that can yield a good estimate of S?
- What is a lower bound for the error (expected fraction of people assigned to the incorrect community) achievable by any estimator?
- How robust are our algorithms to the modeling assumptions, and how well do they generalize to settings with more than two communities?

These questions have attracted the attention of people working in statistics, mathematics, computer science, statistical physics, and optimization, and they remain an active area of research today.

End of Course

> “If you’re brave enough to say goodbye, life will reward you with a new hello.”
>
> – *Paulo Coelho*