

STATISTICAL PROCESS CONTROL

BY

TANUJIT CHAKRABORTY

Indian Statistical Institute

Mail : tanujitisi@gmail.com

STATISTICAL QUALITY CONTROL

Introduction: — The most important word in the term 'Statistical Quality Control' is quality.

A Quality and Quality Control: — The quality of a product is the most important property that one desires while purchasing it. A product is of good quality if it meets the required specifications, otherwise not. By quality, we mean an attribute of the product that determines its suitability or fitness for use.

Quality control is a powerful productivity technique for effective diagnosis of lack of quality in any of the materials, processes, machines, etc. Quality control covers all the factors and processes of production which may be broadly classified as follows :

(i) Quality of materials. Material of good quality will result in smooth processing thereby reducing the waste and increasing the output. It will also give better finish to end products.

(ii) Quality of manpower. Trained and qualified personnel will give increased efficiency due to the better quality production through the application of skill and also to reduce production cost and waste.

(iii) Quality of machines. Better quality equipment will result in efficient work due to lack of scarcity of breakdowns and thus reduce the cost of defectives.

(iv) Quality of management. A good management is imperative for increase in efficiency, harmony in relations, growth of business and markets.

Chance and Assignable causes of Variation

Variation in the quality of manufactured product in the repetitive process in industry is inherent and inevitable. These variations are broadly classified as being due to two causes viz.,

- (i) Chance causes, and
- (ii) assignable causes,

(i) Chance Causes: Some "stable pattern of variation" or "a constant cause system" is inherent in a manufacturing process. This pattern results from some minor causes on this variation to which no reason can be assigned, and is of random nature. Therefore, these causes of variation are known as chance causes. The variation due to these causes is beyond the control of human hand and cannot be prevented or eliminated under any circumstances. One has got to allow for variation within this suitable pattern, usually termed as allowable variation. This type of variation is tolerable and does not affect the quality and the utility of the process. The range of such variation is known as natural tolerance of the process.

(ii) Assignable Causes: Sometimes, the products show marked deviation from the given specifications of a product. This affects the utility of the product and causes worry to the manufacturer. Such a major variation from the specifications may be due to various reasons, such as, defective raw materials, faulty equipment, negligence of the operators, wrong or improper handling of the machines, etc. These causes are non-random and known as so called 'assignable causes'. These causes can be identified and eliminated and are to be discovered in a defective production process. This type of variation due to assignable causes is termed as preventable variation.

Q What do you mean by SQC?

By Statistical Quality Control we mean the various statistical methods used for the maintenance of quality in a continuous flow of manufactured products. The main purpose of SQC is to derive statistical methods for separating allowable variation from preventable variation, so that we may take appropriate steps as quickly as possible whenever assignable causes are operating in the process. The elimination of assignable causes of erratic fluctuations is described as bringing a process under control. A production process is said to be in a state of statistical control, if it is governed by chance causes alone, in the absence of assignable causes of variation.

"SQC is simply a statistical method for determining the extent of which quality goals are being met without necessity. Checking every item produced and for indicating whether or not the variations which occur are exceeding normal expectations. SQC also enables us to decide whether to reject or accept a particular product." — Grant.

Uses of S.Q.C. :- We briefly outline some of the advantages that might result when a process is brought in good statistical control.

1. The act of getting a process in statistical control involves the identification and elimination of assignable causes of variation and possibly the inclusion of good ones viz., new material or methods.
2. It tells us when to leave a process alone and when to take action to correct troubles, thus preventing frequent and unnecessary adjustments.
3. If a process in control is not good enough, we shall have to make more or less a radical change in the process - just meddling with it won't help.
4. It provides better quality assurance at lower inspection cost.
5. The very presence of a quality control scheme in a plant improves and alerts the personnel. Such a scheme is likely to breed 'quality consciousness' throughout the organisation which is of immense long run value.
6. S.Q.C. reduce waste of time and material to the absolute minimum by giving an early warning about the occurrence of defects.

Remark:-

1. An S.Q.C. department is, thus, an essential part of a modern plant, and its important functions are as follows;
- (i) Evaluation of quality standards of incoming materials, products in process and of finished goods.
 - (ii) Judging the conformity of the process to established standards and taking suitable action when deviations are noted.
 - (iii) Evaluation of optimum quality obtainable under given conditions.
 - (iv) Improvement of quality and productivity through process control and experimentation.
2. Advantages of Quality control in industry:

Planned collection of data, analysis and interpretation

↓
Improvement in product quality and design

Reduction in scrap

Saving in excess use of materials

Reduction in inspection

Quality consciousness

Greater consumer satisfaction

↓

Enhanced Productivity

The meaning of Control: — Variability is of two types

- systematic, which is attributable to assignable causes
- random, which is due to a number of small independent causes within a system of causes, i.e. due to chance causes.

When we have eliminated all assignable causes of variation which is economical to eliminate, there still remains a type of variability which may behave statistically in a way that we call random and this due to chance causes.

Thus, if all non-random types of variation have been eliminated, then we have a manufacturing process operating in a random manner and consequently the probability distn. of the random variation can be obtained. A process that is operating with only chance causes of variation is said to be in statistical control.

A process that is operating in the presence of assignable causes is said to out of Control.

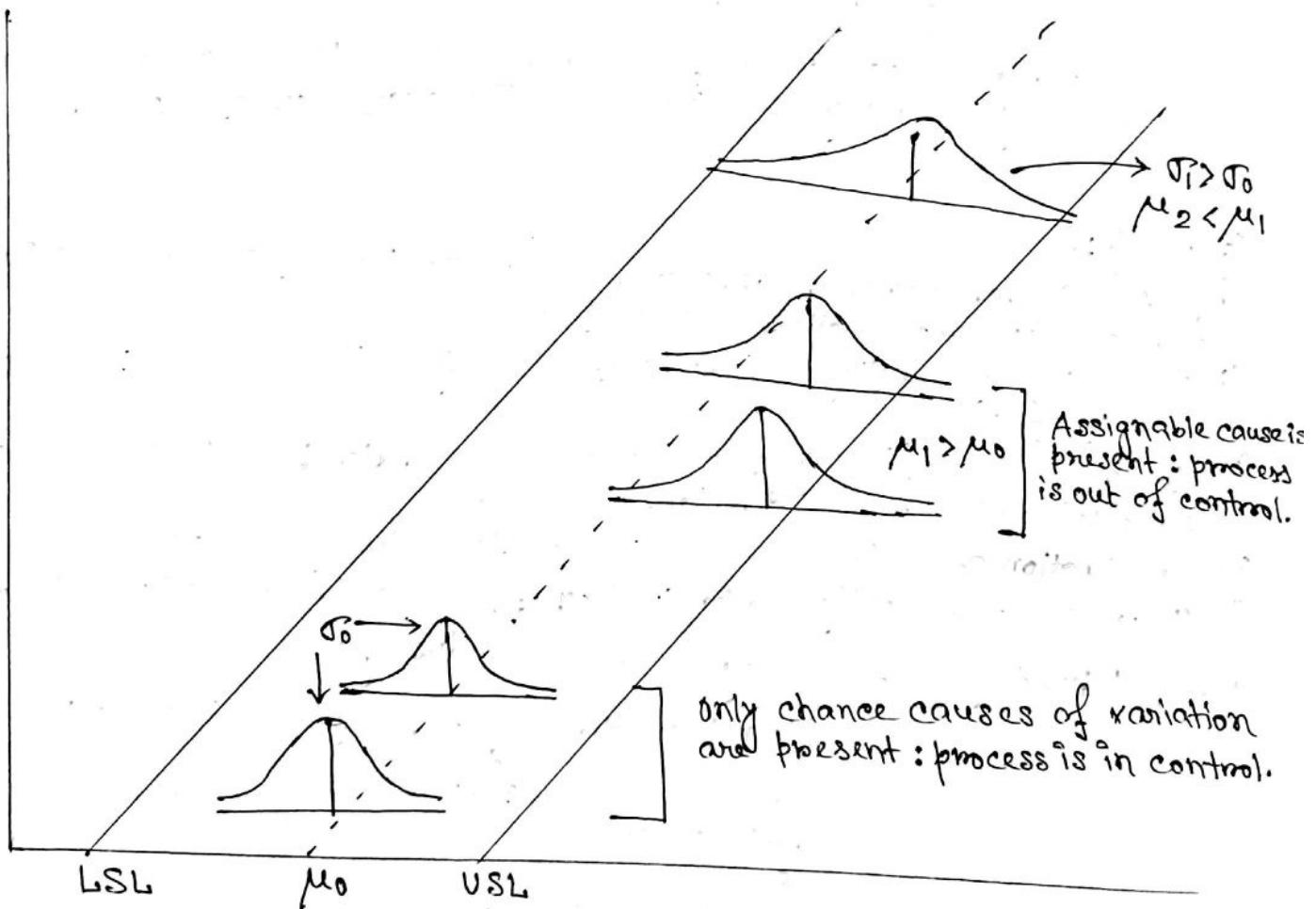


Fig: Chance and assignable causes of variation

Process control and Product control:

It is apparent that a manufacturing is faced with two quality control problems:

- (i) His manufacturing process should be so controlled that the proportion of defective units is not excessive.
- (ii) He should not ship out lots that contain an excessive proportion of defective pieces. We should refer to these two aspects of quality control as (a) Process Control (b) Product Control or lot control.

It is important to realise that the process may be in satisfactory control, so that the number of defective items will not be excessive for the entire output over a long period of time, individual lots, occasionally may not be satisfactory and the objectives of process control and product control are distinct. The primary object of process control is to keep the process in control. The main statistical tool is the control chart. The primary object of product control is to decide whether to accept or reject a lot on the basis of evidence afforded by one or more samples drawn at random from the lot in question and it is achieved through sampling inspection.

If the process is kept in control, product control is made more economical. If the process is in control one can make a valid estimate of the quality being manufactured. Knowledge of the process quality, in turn, may enable one to select the most economical sampling inspection plan.

(a) Process Control:

A process that is operating with only chance causes is said to be 'in control' and a process that is operating in the presence of assignable causes is said to be out of control. A major objective of process control is to quickly detect the occurrence of assignable causes of process shifts so that the investigation of the process and the corrective action may be undertaken before many nonconforming units are manufactured.

The question to be answered by the "process control" is :

"Do the samples show statistical control?" \Leftrightarrow "Do the samples indicate a stable pattern of variation?" \Leftrightarrow "Is there one popn. from which the samples appear to come?"

In quality control in manufacturing, the answer "No, this is not a constant-cause system", leads to a hunt for an assignable cause of variation, and an attempt to remove it, if possible. The answer, "Yes, this is a constant-cause system", leads to leaving the process alone, making no effort to hunt for causes of variation.

Control charts:

Shewhart's control chart provides a powerful tool of discovering and correcting the assignable causes of variation outside the "stable pattern" of chance causes, thus enabling us to stabilize and control our processes at desired performance; and thus being the process under statistical control. A typical control chart is shown in the figure, which is a graphical display of a quality characteristic that has been measured from a sample Versus the sample numbers.

A typical control chart consists the following three horizontal lines:

- (1) A central line (CL), indicating the desired standard or the level of the process.
- (2) Upper control limit (UCL), indicating the upper limit of tolerance.
- (3) Lower control limit (LCL), indicating the lower limit of tolerance.

In the control chart, UCL and LCL are usually plotted as dotted lines and CL is plotted as a bold line.

We may give a general model for a control chart.

Let T be a (sample) statistic that measures some quality characteristic of interest.

Suppose that $E(T) = \mu_T$, and, $\text{Var}(T) = \sigma_T^2$, when the process is in control.

Then CL, UCL, LCL become

$$UCL = \mu_T + L\sigma_T$$

$$CL = \mu_T$$

$$LCL = \mu_T - L\sigma_T;$$

where, L is the 'distance' of the control limits from the central line, expressed in standard deviation units.

This general theory of control charts was first proposed by Dr. Walter Shewhart, and control charts developed according to the principles are often called Shewhart control charts.

The appropriateness of 3- σ limits:

3- σ limits :
$$\left. \begin{array}{l} UCL = \mu_T + 3\sigma_T \\ CL = \mu_T \\ LCL = \mu_T - 3\sigma_T \end{array} \right\}$$
, were proposed by Dr. Shewhart

for his control charts for various considerations, the main being probabilistic considerations.

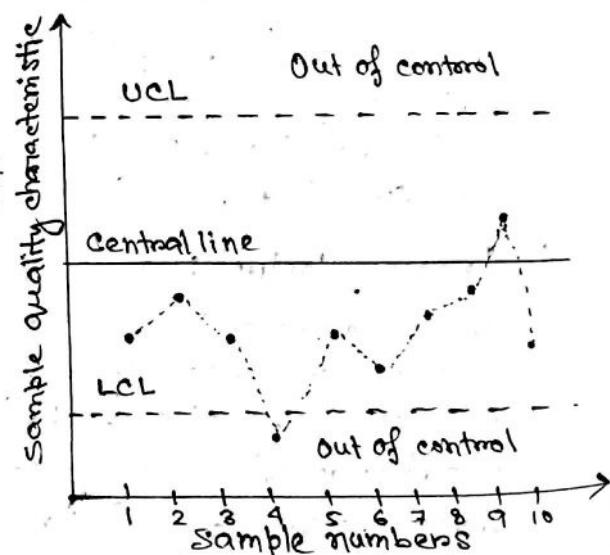


Fig:- Outline of a control chart

Here T is a statistic that measures some quality characteristic of the popn. If the process is in control, then, let $E(T) = \mu_T$ and $V(T) = \sigma_T^2$ and the fluctuations in the value of T from the sample to sample should be due to random variation alone.

Note that, by Chebyshov's inequality, $P[|T - \mu_T| < 3\sigma_T] > 1 - \frac{1}{9}$

$$\Leftrightarrow P[\mu_T - 3\sigma_T < T < \mu_T + 3\sigma_T] > \frac{8}{9} \approx 0.9$$
, in the case

where the process is in control, whatever the distn. of T may be.

In particular, in case T is normally distributed and the process is in control, $P[\mu_T - 3\sigma_T < T < \mu_T + 3\sigma_T] = 2\Phi(3) - 1 = 0.9973$

$\Rightarrow P[|T - \mu_T| > 3\sigma_T] = 0.0027$; that is, the probability that a random value of T goes outside the $3-\sigma$ limits is 0.0027, which is very small.

The rule for establishing the control limits depends on the control over the two types of errors — (i) the errors of hunting for trouble or assignable causes when it is absent (ii) the errors of not hunting for trouble or assignable causes when it is really present.

It has been pointed out that as long as the samples are really random samples from one population (or, from a process which is in control), the observed value of T will nearly always fall within $3-\sigma$ limits. Also, the probability of type I error, i.e. the probability of indication of out of control when the process is in control, i.e.

$P[|T - \mu_T| > 3\sigma_T \text{ [the process is in control]}] = 0.0027$, if T is normally distributed. The $3-\sigma$ limits seldom make the errors of indicating an assignable cause of variation when there is no assignable causes present. Therefore, if the observed T for the i^{th} sample lies between $\mu_T - 3\sigma_T$ and $\mu_T + 3\sigma_T$, it is taken to be a fairly good indication of non-existence of assignable causes of variation

at the time when the i^{th} sample was taken and if the observed T for the i^{th} samples lies outside the $3-\sigma$ limits, it is considered to be a danger signal indicating that some assignable cause has present and it must be identified and eliminated.

In stead of using $3-\sigma$ limits, we may use other limits, as for e.g., if

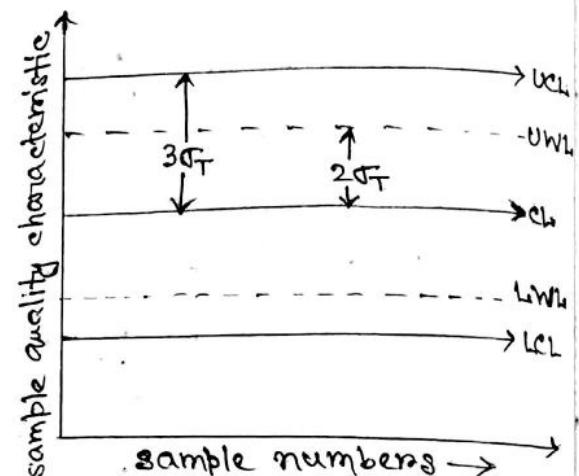
$$L = 3.09 \text{ then: } UCL = \mu_T + 3.09\sigma_T$$

$$CL = \mu_T$$

$$LCL = \mu_T - 3.09\sigma_T$$

with probability of type I errors 0.002.

Warning limits: ~ The outer limits - say, 3σ limits — are usual action limits. The inner limits, usually at two-sigma, are called warning limits. If one or more points fall between the warning limits and the control limits, or very close to the warning limit, we should be suspicious that the process may not be operating properly, one possible action to take when this occurs is to increase the sample size so that more information about the process can be obtained quickly. Process schemes that change the sample size depending on the position of the current sample value is called variable sample size.



Rational Subgroups : ~ [CU]

To explain this term, suppose that we are using some control chart to detect changes in process quality. A fundamental idea in the use of control charts is the collection of sample data according to what Shewhart called the rational subgroups. The rational subgroup concept means that subgroups or samples be selected so that if assignable causes are present, the chance for differences between subgroups will be maximized, while the chance for differences due to these assignable causes within a subgroup will be minimized. The use of such subgroups would tend to reveal assignable causes of variation.

When control charts are applied to production processes, the time order of production is logical basis for rational subgrouping. Each sample consists of units that were produced at the same time (or, as closely together as possible). It minimizes the chance of variability due to assignable causes within a sample, and it maximizes the chance of variability between the samples if assignable causes are present. Time order is frequently a good basis for forming subgroups because it allows us to detect assignable causes that occur over-time.

There are other bases for forming rational subgroups. For example, a group of machines in a factory may have different variation, and it may be necessary to have different subgroups for different machines, or for different operators or different shifts.

The rational subgroup concept is very important. The proper selection of subgroups requires careful consideration of the process, with the objective of obtaining as much useful information as possible from the control chart analysis.

Control charts for Variables : — When dealing with a quality characteristic that is a variable, it is usually necessary to monitor both the mean value of the quality characteristic and its variability. Control of the process mean quality level is done with the control chart for mean or \bar{x} chart. Process variability can be monitored with either a control chart for the standard deviation, called the S chart, or a control chart for the range, called a R chart. The R chart is more widely used. Usually, separate \bar{x} and R charts are maintained for each quality characteristic of interest.

The four types of situation that may be encountered here are : (i) the process is in control, (ii) the mean is out of control but not the variability, (iii) the variability is out of control but not the mean, (iv) both mean and variability are out of control.

We have assumed that the distribution of the quality characteristic is normal. However, the above assumption is still approximately correct if the underlying distribution is non-normal, because of the Central-limit theorem.

A. Control charts for \bar{x} and R : — Suppose that a quality characteristic is normally distributed with mean μ and standard deviation σ , where both μ and σ are usually unknown. If x_1, x_2, \dots, x_n is a sample of size n , then the sample mean is

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} \text{ and } \bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

Control charts for mean on \bar{x} chart : — [C.V.]

Case I : Standards given

Suppose that the values for μ and σ are specified as μ_0 and σ_0 . Then the control chart for \bar{x} is given by

$$LCL = \mu \bar{x} - 3\sigma \bar{x} = \mu_0 - 3\frac{\sigma_0}{\sqrt{n}} = \mu_0 - A\sigma_0$$

$$CL = \mu \bar{x} = \mu_0 = \mu_0$$

$$UCL = \mu \bar{x} + 3\sigma \bar{x} = \mu_0 + 3\frac{\sigma_0}{\sqrt{n}} = \mu_0 + A\sigma_0, \text{ where, } A = \frac{3}{\sqrt{n}}.$$

Case II : Standards not given

In practice, we usually will not know μ and σ . Therefore, they must be estimated from preliminary samples taken when the process is thought to be in control. Suppose that 'm' samples are available, each containing n observations on the quality characteristic. Let $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m$ be the means of the samples. Then an unbiased estimator of μ is the grand mean

$$\bar{\bar{x}} = \frac{\bar{x}_1 + \dots + \bar{x}_m}{m}.$$

Let R_1, R_2, \dots, R_m be the ranges of the m samples. The average range is

$$\bar{R} = \frac{R_1 + R_2 + \dots + R_m}{m}.$$

[The RV $W = \frac{R}{\sigma}$ is called the relative range and $E(W) = d_2$, a function of the sample size n . Consequently, an estimator of σ is $\frac{R}{d_2}$.]

Then, $\hat{\sigma} = \frac{\bar{R}}{d_2}$ is an unbiased estimator of σ .

If we use $\bar{x} = \frac{\sum x_i}{n}$ as an estimator of μ and $\frac{\bar{R}}{d_2}$ as an estimator of σ , then the control chart for \bar{x} is given by

$$LCL = \bar{x} - \frac{3}{d_2 \sqrt{n}} \bar{R} = \bar{x} - A_2 \bar{R}$$

$$CL = \bar{x} = \bar{x}$$

$$UCL = \bar{x} + \frac{3}{d_2 \sqrt{n}} \bar{R} = \bar{x} + A_2 \bar{R}, \text{ where } A_2 = \frac{3}{d_2 \sqrt{n}}$$

tabulated for various sample sizes.

[C.U.]

Control chart for range on R-chart: — Assuming that the quality characteristic is normally distributed, then the relative range $W = \frac{R}{\sigma}$ has mean $E(W) = d_2$ and $\text{Var}(W) = d_3$.

Then $\mu_R = E(R) = d_2 \sigma$ and $\sigma_R = d_3 \sigma$.

Case-I : Standard given

To construct the R-chart with a standard value σ_0 of σ . Then the control chart for the range is given by

$$LCL = \mu_R - 3\sigma_R = d_2 \sigma_0 - 3d_3 \sigma_0 = D_1 \sigma_0$$

$$CL = \mu_R = d_2 \sigma_0$$

$$UCL = \mu_R + 3\sigma_R = d_2 \sigma_0 + 3d_3 \sigma_0 = D_2 \sigma_0, \text{ where } D_1 = d_2 - 3d_3$$

are tabulated for different values of 'n'.

Case-II : Standard not given

Hence the process s.d. σ is unknown. To determine the control limits we need an estimator of μ_R as well as σ_R .

Since $\mu_R = d_2 \sigma$ and $\sigma_R = d_3 \sigma$, hence $\hat{\sigma} = \frac{\bar{R}}{d_2}$ is an unbiased estimator of σ and consequently $\hat{\sigma}_R = d_3 \frac{\bar{R}}{d_2}$ is an unbiased estimator of σ_R .

Hence, the control chart for the range is given by

$$LCL = \hat{\mu}_R - 3\hat{\sigma}_R = \bar{R} - \frac{3d_3}{d_2} \bar{R} = D_3 \bar{R}$$

$$CL = \hat{\mu}_R = \bar{R}$$

$$UCL = \hat{\mu}_R + 3\hat{\sigma}_R = \bar{R} + \frac{3d_3}{d_2} \bar{R} = D_4 \bar{R}, \text{ where } D_3 = \left(1 - \frac{3d_3}{d_2}\right), D_4 = \left(1 + \frac{3d_3}{d_2}\right)$$

are tabulated for different values of 'n'.

B. Control charts for \bar{x} and S : ~*

The R chart is relatively insensitive to shifts in the process S.d. for small samples. Larger samples would seem to be more effective but we also know that the range method for estimating the standard deviation drops dramatically in efficiency as n increases. Consequently, for large n , say, $n > 10$, it is probably best to use control charts based on S instead of R.

Construction :

If the quality characteristic x is normally distributed with mean μ and standard deviation σ . If x_1, x_2, \dots, x_n be a sample of size n , then $\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$ and $\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$, where $S^2 = \frac{1}{(n-1)} \sum_i (x_i - \bar{x})^2$ is the sample variance.

We also have, $E(S) = C_4 \sigma$ and $\text{Var}(S) = \sigma^2(1 - C_4^2)$

$\Leftrightarrow \mu_S = C_4 \sigma$ and $\sigma_S = \sigma \sqrt{1 - C_4^2}$, where C_4 is a constant that depends on ' n '.

Control chart for \bar{x} :

Case-I : Standards given

Suppose that standard values of μ and σ are given, say, μ_0 and σ_0 , then

$$LCL = \mu_0 - 3 \frac{\sigma_0}{\sqrt{n}} = \mu_0 - A \sigma_0$$

$$CL = \mu_0 = \mu_0$$

$$UCL = \mu_0 + 3 \frac{\sigma_0}{\sqrt{n}} = \mu_0 + A \sigma_0, \text{ where } A = \frac{3}{\sqrt{n}}.$$

Case-II : Standards not given

If no standards are given for μ and σ , then it must be estimated by analyzing the past data. Suppose that m preliminary samples are available, each of size n , and let \bar{x}_i, s_i be the mean and the s.d. of the i^{th} sample.

* When subgroup size n is moderately large (say $n > 10$ or 12). Range may not be a good measure of variation. It is desirable to estimate the variation using standard deviation.

Define, $\bar{\bar{x}} = \frac{1}{m} \sum_{i=1}^m \bar{x}_i$, $\bar{s} = \frac{1}{m} \sum_{i=1}^m s_i$.

Note that $E(\bar{\bar{x}}) = \mu$ and $E\left(\frac{\bar{s}}{C_4}\right) = \sigma$. Hence $\hat{\mu} = \bar{\bar{x}}$ and $\hat{\sigma} = \frac{\bar{s}}{C_4}$

are unbiased estimators of μ and σ .

Hence the control chart for \bar{x} (based on \bar{s}) is given by :

$$LCL = \hat{\mu} - 3 \cdot \frac{\hat{\sigma}}{\sqrt{n}} = \bar{\bar{x}} - \frac{3}{C_4 \sqrt{n}} \bar{s} = \bar{\bar{x}} - A_3 \bar{s}$$

$$CL = \hat{\mu} = \bar{\bar{x}} = \bar{\bar{x}}$$

$$UCL = \hat{\mu} + 3 \cdot \frac{\hat{\sigma}}{\sqrt{n}} = \bar{\bar{x}} + \frac{3}{C_4 \sqrt{n}} \bar{s} = \bar{\bar{x}} + A_3 \bar{s}, \text{ where } A = \frac{3}{C_4 \sqrt{n}}$$

Control chart for s :

Case-I : Standard given

Suppose that a standard value σ_0 is given, say, σ_0 .

The control chart for s is :

$$LCL = \mu_s - 3\sigma_s = C_4 \sigma_0 - 3\sqrt{1-C_4^2} \sigma_0 = B_3 \sigma_0$$

$$CL = \mu_s = C_4 \sigma_0 = C_4 \sigma_0$$

$$UCL = \mu_s + 3\sigma_s = C_4 \sigma_0 + 3\sqrt{1-C_4^2} \sigma_0 = B_4 \sigma_0$$

Case-II : Standard not given

Hence σ is unknown and $\hat{\sigma} = \frac{\bar{s}}{C_4}$ is an unbiased estimator of σ .

Therefore, the control chart for s is given by :-

$$LCL = \hat{\mu}_s - 3 \cdot \hat{\sigma}_s = C_4 \hat{\sigma} - 3 \sqrt{1-C_4^2} \cdot \hat{\sigma} = \bar{s} - \frac{3 \sqrt{1-C_4^2}}{C_4} \cdot \bar{s} = B_3 \bar{s}$$

$$CL = \hat{\mu}_s = C_4 \hat{\sigma} = \bar{s}$$

$$UCL = \hat{\mu}_s + 3 \cdot \hat{\sigma}_s = C_4 \hat{\sigma} + 3 \sqrt{1-C_4^2} \cdot \hat{\sigma} = \bar{s} + \frac{3 \sqrt{1-C_4^2}}{C_4} \cdot \bar{s} = B_4 \bar{s}$$

$$\text{where, } B_3 = 1 - \frac{3}{C_4} \sqrt{1-C_4^2}, B_4 = 1 + \frac{3}{C_4} \sqrt{1-C_4^2}$$

X-R Chart: Methodology :-

- Decide on Total Number of samples N . ($N > 19$)
- Decide on Subgroup size n . ($n > 3$)
- Decide on Frequency of Sampling (eg: once in a hour, once in 2 hours, etc.)

Interpretation of \bar{x} and R charts | Analysis of patterns of control charts

charts :— In interpreting patterns on \bar{x} chart, we must first determine whether or not the R chart is in control. Some assignable causes show up on both the \bar{x} and R charts. If both the \bar{x} and R charts exhibit a non-random pattern, the best strategy is to eliminate the R chart assignable causes first. In many cases, this automatically will eliminate the non-random pattern on the \bar{x} -chart. Never attempt to interpret to \bar{x} -chart when the R chart indicates an out-of-control condition. Situations exist where R-chart is in state of control but \bar{x} -chart is not.

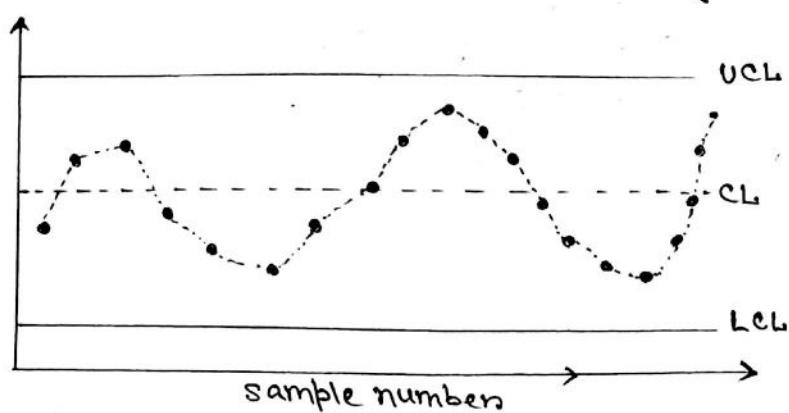
A control chart may indicate an out-of-control condition either when one or more points fall beyond the control limits or when the plotted points exhibit some non-random pattern of behavior. If the points are truly random, we should expect a more even distn. of them above and below the central line. In general, we define a run as a sequence of observations of same type. In addition, to runs up & runs down, we could define the types of the observations as those above and below the central line, respectively.

A run of length 8 or more points has a very low probability of occurrence in a random sample of points. Consequently, any type of run of length 8 or more is often taken as a signal of an out-of-control state.

Although runs are an important measure of non-random behaviour of a control chart, other types of patterns may also indicate an out-of-control condition:

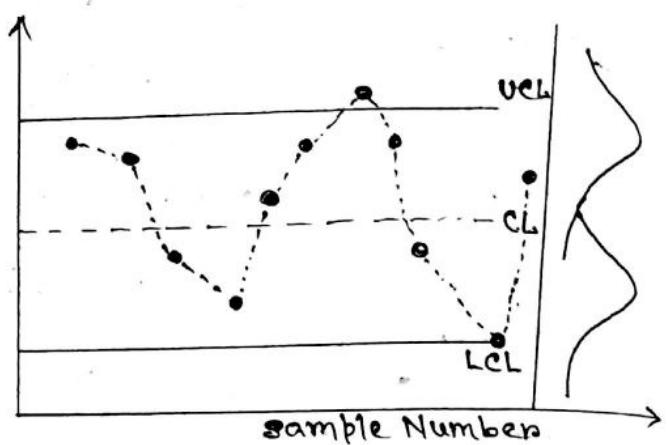
- (i) Cyclic Patterns occasionally appear on the control chart. Such a pattern may indicate a problem with the process such as operator fatigue, raw material deliveries, heat or stress build up, etc. Although the process is not really out of control, the yield may be improved by elimination or reduction of the source of variability.

R charts will sometimes reveal cycles because of maintenance schedules, operators fatigue, or tool wear.



cyclic pattern

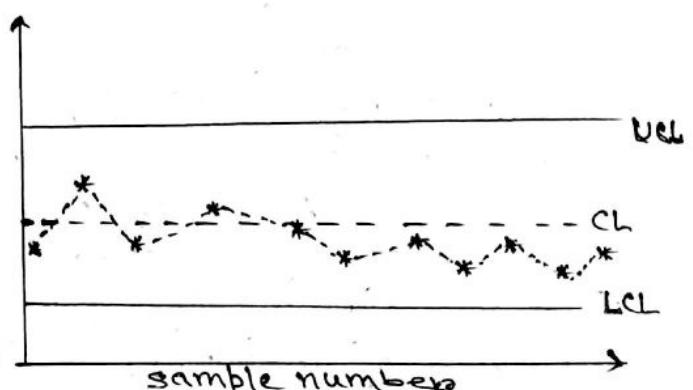
(ii) A mixture pattern is indicated when the plotted points tend to fall near or slightly outside the control limits, with relatively few points near the central line. A mixture pattern is generated by two or more overlapping distributions, generating the process output.



A mixture pattern

(iii) A shift in process level is illustrated in the following figure:

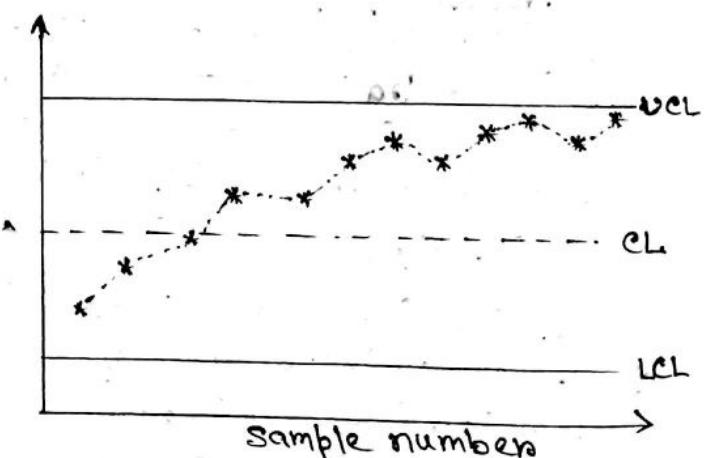
These shifts may result from the introduction of new workers, methods, raw materials, or machines.



A shift in process

(iv) A trend or continuous movement in one direction, is shown on the control chart:

Trends are usually due to a gradual wearing out or deterioration of a tool or some other critical process component.



A trend in process level

The effect of non-normality on \bar{x} and R (or S) charts:

A fundamental assumption in the development of \bar{x} and R (or S) charts is that the underlying distn. of the quality characteristic is normal. In many situations we may have reason to doubt the validity of this assumption. Now if we know the form of the underlying distn., it is possible to derive the sampling distn. of \bar{x} and R (or S), and to obtain exact probability limits for the control charts. This approach could be difficult in some cases, and most analysts would probably prefer to use the standard approach based on the normality assumption if they felt that the effect of departure from this assumption was not serious. However, we may know nothing about the form of the underlying distribution, and then our only choice may be to use the normal theory result. In either case, we would be interested in knowing the effect of departures from normality on the control chart for \bar{x} and R (or S).

Operating characteristic and Average run length of Control-Chart

For a control chart, define $\beta(\theta) = P[\text{a sample is taken from 'in-control' process as decided by the control chart, when the true process parameter is } \theta]$. The $\beta(\theta)$, considered as a function of true process parameter θ , is called the OC function of the control chart.

The ability of the \bar{x} and R charts to detect shift in process quality is described by their OC functions or curves.

Consider the OC curve for an \bar{x} -chart with s.d. σ (known). If the mean shifts from the 'in-control' value - say, μ_0 - to another value $\mu_1 = \mu_0 + k\sigma$, the probability of not detecting this shift on the first subsequent sample (or β -risk) is

$$\beta = P[LCL \leq \bar{x} \leq UCL \mid \mu = \mu_1 = \mu_0 + k\sigma]$$

since $\bar{x} \sim N(\mu_1, \frac{\sigma^2}{n})$, and the upper and lower control limits are

$$LCL = \mu_0 - \frac{3\sigma}{\sqrt{n}}, \quad UCL = \mu_0 + \frac{3\sigma}{\sqrt{n}}, \quad \text{we have}$$

$$\beta = P[\mu_0 - \frac{3\sigma}{\sqrt{n}} < \bar{x} < \mu_0 + \frac{3\sigma}{\sqrt{n}} \mid \mu = \mu_0 + k\sigma]$$

$$= P\left[\frac{\sqrt{n}\{\bar{x} - (\mu_0 + k\sigma)\}}{\sigma} < \frac{(3 - k\sqrt{n})}{\sigma}\right]$$

$$= \Phi(3 - k\sqrt{n}) - \Phi(-3 - k\sqrt{n})$$

The probability that such a shift will be detected on the first subsequent sample is $(1-\beta)$. To construct OC curve for the \bar{x} -chart, plot β -risk against the magnitude of shift, for a given sample size(n).

Average Run Length (ARL) : ~ The OC curve does not give an entirely fair comparison between two control charts.

Note that the probability that the shift will be detected on the first sample is $(1-\beta)$. The probability that the shift will be detected on the r th subsequent sample is $\beta^{r-1}(1-\beta)$.

The expected number of samples taken to detect the shift is simply the average run length (ARL) on

$$ARL = \sum_{r=1}^{\infty} r \beta^{r-1} (1-\beta) = \frac{1}{1-\beta} \quad (*)$$

It is also convenient to express the performance of the control chart in terms of its average time to signal (ATS). If samples are taken at fixed intervals of time that are 'h' hours apart, then

$$ATS = (ARL) \times h \quad (**)$$

The equations. (*) and (**) can be used to evaluate the performance of the control charts.

If may also be useful to express the ARL in terms of the expected numbers of individual units sampled - say I - rather than the numbers of samples taken to detect a shift. If the sample size is n , the relationship between I and ARL is

$$I = nARL$$

Examples :-

(1) i) Show that p_n , the probability of the mean of a random sample of size n exceeding $UCL = \mu_0 + 3\sigma/\sqrt{n}$, when the population mean has shifted to $\mu_0 + k\sigma$ is $\Phi(k\sqrt{n}-3)$.

ii) If the r th sample mean is the first to exceed UCL , show that $E(r) = 1/p_n$.

Solution:-

$$\begin{aligned} i) p_n &= P[\bar{x} > UCL] = P[\bar{x} > \mu_0 + 3\frac{\sigma}{\sqrt{n}}] \\ &= P\left[\frac{\bar{x} - (\mu_0 + k\sigma)}{\sigma/\sqrt{n}} > 3 - k\sqrt{n}\right], \text{ since } \\ &= \Phi(k\sqrt{n}-3), \quad \bar{x} \sim N(\mu_0 + k\sigma, \frac{\sigma^2}{n}) \end{aligned}$$

ii) If the r th sample mean is the first to exceed the UCL , the preceding $(r-1)$ sample means must be $\leq UCL$. Thus if Y is the random variable such that $Y = r(1, 2, \dots)$ implies that the r th sample mean is the first to exceed UCL then $Y \sim Geo(p_n)$, i.e. $P[Y=r] = (1-p_n)^{r-1} p_n$

$$\text{Then, } E(Y) = \frac{1}{p_n}.$$

(2) Show that the probability that at least one of the two points \bar{x} and R goes outside the control limits is :

$$1 - [\Phi(\sqrt{n}T + 3\beta) - \Phi(\sqrt{n}T - 3\beta)] [P\left(\frac{R}{\sigma} \leq D_2\beta\right) - P\left(\frac{R}{\sigma} \leq D_1\beta\right)]$$

where $\beta = \sigma'/\sigma$, $T = \frac{\mu' - \mu}{\sigma'}$, assuming that the control charts are based on μ' and σ' as standards, where the actual values of these parameters are μ and σ respectively.

Solution:- The probability that at least one of the two points \bar{x} and R goes outside the control limits

$$= 1 - P[\text{none of the points } \bar{x} \text{ and } R \text{ goes outside the control limits}]$$

$$= 1 - P[LCL_{\bar{x}} \leq \bar{x} \leq UCL_{\bar{x}}, LCL_R \leq R \leq UCL_R]$$

$$= 1 - P[LCL_{\bar{x}} \leq \bar{x} \leq UCL_{\bar{x}}] P[LCL_R \leq R \leq UCL_R]$$

$$= 1 - P\left[\frac{\mu' - \frac{3\sigma'}{\sqrt{n}} - \mu}{\sigma/\sqrt{n}} \leq \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leq \frac{\mu' + 3\sigma'/\sqrt{n} - \mu}{\sigma/\sqrt{n}}\right] \times$$

$$P[D_1\sigma' \leq R \leq D_2\sigma']$$

[Since in normal population \bar{x} and R are independently distributed]

$$= 1 - \left\{ \Phi(\sqrt{n}T + 3P) - \Phi(\sqrt{n}T - 3P) \right\}^x$$

$$\times \left\{ P\left(\frac{R}{\sigma} \leq D_2 P\right) - P\left(\frac{R}{\sigma} \leq D_1 P\right) \right\}$$

(3) Let p_n is the probability of the mean of a sample of size n falling outside the control limits. Show that

(a) the probability that at most x samples are to be taken for r points to go out of control is

$$1 - \sum_{s=0}^{r-1} \binom{x}{s} p_n^s (1-p_n)^{x-s}$$

(b) The probability that exactly x samples are to be taken for r points to go out of control is

$$\left(\frac{p_n}{1-p_n}\right)^r \cdot \binom{x-1}{r-1} (1-p_n)^x, x \geq r.$$

Solution:-

(a) Let Y be the RV which represents the number of points (sample means) falling outside the control limits in x samples. Then $Y \sim \text{Bin}(x, p_n)$.

Hence the probability 'p' that in x samples the number of points going out of the control limits is greater than or equal to r is the required probability.

$$\therefore p = P[Y \geq r] = 1 - P[Y < r]$$

$$= 1 - \sum_{s=0}^{r-1} \binom{x}{s} p_n^s (1-p_n)^{x-s}$$

- (b) The extent 'E' that exactly x samples are required for r points to go out of control limits, happens if
 (i) the r th point goes out of control limits at the x th sample and

(b) X : the no. of samples required for r points to go out of control limits.

$X \sim \text{Negative Binomial}(r, p_n)$.

$$\therefore \text{Required probability} = \binom{x-1}{r-1} p_n^r (1-p_n)^{x-r}, \text{ if } x \geq r.$$

$$= \binom{x-1}{r-1} \left(\frac{p_n}{1-p_n}\right)^r (1-p_n)^x, x \geq r$$

CONTROL CHART FOR ATTRIBUTES :

A defective or non conforming item is a unit of product that does not satisfy one or more of the specifications for that product. Each specific point at which a specification is not satisfied results in a defect or non conformity.

Usually a unit is considered defective when it is qualitatively unsatisfactory. It may be usable, but have a major defect or too many minor defects.

[A]. Procedures with constant sample size]: [C.V]

The control chart for fraction defective or non-conforming :

The fraction defective is defined as the ratio of the numbers of defective items in a population to the total numbers of items in that popn..

The statistical principles underlying the control chart for fraction non-conforming or defectives are based on the binomial distribution. Suppose the production process is operating in a stable manner, such that the probability that any unit will not conform to specifications is p , and successive units produced are independent. Then each unit produced is a realization of a Bernoulli random variable with parameter p . If a random sample of n units of product is selected, and if D is the number of units of product that are non conforming, then D has a binomial distn. with parameters n and p , i.e.

$$P[D=d] = \binom{n}{d} p^d (1-p)^{n-d}, d=0,1,2,\dots,n.$$

Note that $E(D) = np$, $V(D) = np(1-p)$.

The sample fraction conforming or defective is defined as the ratio of the numbers of non-conforming units in the sample D to the sample size n ; that is, $\hat{p} = \frac{D}{n}$.

Again, $\mu_{\hat{p}} = E(\hat{p}) = p$ and $\sigma_{\hat{p}}^2 = V(\hat{p}) = V\left(\frac{D}{n}\right) = \frac{1}{n^2} V(D) = \frac{p(1-p)}{n}$.

Because the chart monitors the process function non-conforming ' p ', it is also called the p -chart.

Development of the Control chart : — If T is a statistic that measures a quality characteristic, and if the mean of T is μ_T and the variance of T is σ_T^2 , then the general structure of Shewhart control chart is as follows :

$$\begin{aligned} UCL &= \mu_T + 3\sigma_T \\ CL &= \mu_T \\ LCL &= \mu_T - 3\sigma_T \end{aligned} \quad \left. \begin{array}{l} \text{ } \\ \text{ } \\ \text{ } \end{array} \right\} \rightarrow 3\text{-sigma limits.}$$

(1) Standard value is given :

Suppose that the true fraction defective p in the production process is known or is a standard value specified by management.

Then the central line and control limits of the fraction defective control chart are:

$$UCL = \mu_p^+ + 3\sigma_p^+ = p + 3\sqrt{\frac{p(1-p)}{n}}$$

$$CL = \mu_p^+ = p$$

$$LCL = \mu_p^+ - 3\sigma_p^+ = p - 3\sqrt{\frac{p(1-p)}{n}}.$$

(2) No standard given :

When the process fraction defective p is not known, then it must be estimated from observed data. The usual procedure is to select m preliminary samples, each of size n . Then if there are D_i defective units in the i^{th} sample, we compute the fraction defective in the i^{th} sample as

$$\hat{p}_i = \frac{D_i}{n}, i=1, 2, \dots, m.$$

and the average of these individual sample fractions defective is

$$\bar{p} = \frac{1}{m} \sum_{i=1}^m \hat{p}_i = \frac{1}{mn} \sum_{i=1}^m D_i. \text{ The statistic } \bar{p} \text{ estimates the unknown fraction non-conforming (defective) } p.$$

The central line and control limits of the control chart for fraction non-conforming (defective) are:

$$UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

$$CL = \bar{p}$$

$$LCL = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

The control chart for the number of defectives on the np control chart :

It is also possible to base a control chart on the numbers of defectives rather than the fraction defective. This is often called np-control chart.

(1) Standard given : p is given.

$$UCL = np + 3\sqrt{np(1-p)}$$

$$CL = np$$

$$LCL = np - 3\sqrt{np(1-p)}$$

(2) Standard not given :

If the process fraction defective p is not known then it must be estimated from observed data.

Then $\bar{p} = \frac{1}{mn} \sum_{i=1}^m D_i = \frac{1}{m} \sum_{i=1}^m \hat{p}_i$ can be used to estimate p . Then

$$UCL = n\bar{p} + 3\sqrt{n\bar{p}(1-\bar{p})}$$

$$CL = n\bar{p}$$

$$LCL = n\bar{p} - 3\sqrt{n\bar{p}(1-\bar{p})}$$

• Remark:

(1) Note that p or np can never be negative. Hence, if LCL , in p chart or np -chart, comes out negative, then it is to be taken as zero.

(2) When a control chart for defectives instead of means and range is used, much of the information is thrown away, because we utilize only the information that the measurement is on or is not within a specified range of values, rather than its actual value. The sample must thus be larger to provide a test of the same power. Using a large sample may, however, be more economical. It is generally cheaper to use some sort of a gage that tells whether the object conforms to standard, and then count the number of defectives, than to weigh or measure the object, record the observations, and compute their mean and range.

(3) Care must be exercised in interpreting points that plot below the lower control limit. These points often do not represent a real improvement in process quality. Frequently, they are caused by errors in the inspection process from inadequately trained inspectors, or from improper inspection equipment.

B. Variable Sample size : In some applications of the control chart for fraction defective, the sample is a 100% inspection of process output over some period of time. Since different numbers of units could be produced in each period, the control chart would then have a variable sample size. There are three approaches to constructing and operating a control chart with a variable sample size.

p-chart :

(a) Variable-width Control limits :

The first approach is to determine control limits for each individual sample that are based on the specific sample size. That is if the i th sample is of size n_i , then the upper and lower control limits for p-chart are $\bar{p} \pm 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n_i}}$. Note that the width of the control limit is inversely proportional to the square root of the sample size.

(b) Control limits based on an Average sample size :

The second approach is to base the control chart on an average sample size, resulting in an approximate set of control limits. This assumes that future sample sizes will not differ greatly from those previously observed. If this approach is used, the control limits will be constant.

Therefore, the approximate control limits for p-chart are:

$$UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{\bar{n}}}, \quad LCL = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{\bar{n}}},$$

where \bar{n} is average sample size and \bar{p} is the average fraction defective based on all the samples.

(c) The standard control chart :

The third approach to dealing with variable sample size is to use a "standardized" control chart, where the points are plotted in standard deviation units. Such a control chart has the central line at zero, the $UCL = 3$, $LCL = -3$.

The variable plotted on the chart is

$$Z_i = \frac{\hat{p}_i - \bar{p}}{\sqrt{\frac{\bar{p}(1-\bar{p})}{n_i}}} \text{ or } \frac{\bar{p}_i - \bar{p}}{\sqrt{\frac{\bar{p}(1-\bar{p})}{n_i}}} \text{ where } \bar{p} \text{ (given)} \text{ or } \hat{p} \text{ is}$$

the estimate of the process fraction defective in the in-control state.

np-chart:

(a) variable-width control limits : ~

$$UCL = n_i \bar{p} + 3 \sqrt{n_i \bar{p}(1-\bar{p})}$$

$$CL = n_i \bar{p}$$

$$LCL = n_i \bar{p} - 3 \sqrt{n_i \bar{p}(1-\bar{p})}$$

(b) Control limits based on average sample size : ~

$$UCL = \bar{n} \bar{p} + 3 \sqrt{\bar{n} \bar{p}(1-\bar{p})}$$

$$CL = \bar{n} \bar{p}$$

$$LCL = \bar{n} \bar{p} - 3 \sqrt{\bar{n} \bar{p}(1-\bar{p})}$$

(c) The standardised control chart : ~

$$UCL = 3, CL = 0, LCL = -3$$

The variable plotted on the chart is

$$Z_i = \frac{d - n_i \bar{p}}{\sqrt{n_i \bar{p}(1-\bar{p})}} \text{ or } \frac{d - \bar{n} \bar{p}}{\sqrt{\bar{n} \bar{p}(1-\bar{p})}}$$

Choice between chart for p and chart for np : ~ Whenever subgroup (sample) size is variable, the control chart must show the fraction defective rather than the numbers of defectives. If actual numbers of defectives were plotted the central line on the np -chart (as well as limits) would need to be changed with every change in sample size. When the sample size is constant, both the charts are equivalent.

Control charts for defects or non-conformities : ~ It is possible to develop control charts for either the total number of defects in a unit or the average number of defects per unit.

The Poisson distribution is used with two types of data in quality control work
 (i) for defectives when n is large and p is small,
 (ii) for defects per unit of output.

Essentially, this requires that the number of opportunities for defects be indefinitely large and that the probability of occurrence of a non-conformity at any location be small and constant.

A. Control charts for Constant sample size : ~

In most cases, the inspection unit will be a single unit of product, although this is not necessarily always so. The inspection unit is simply an entity for which it is convenient to keep records. It could be a group of 5 or 10 units.

Suppose that defects or non-conformities occur in this inspection unit according to the Poisson distribution; i.e.

$$P[X = x] = e^{-c} \cdot \frac{c^x}{x!}, x = 0, 1, 2, \dots$$

where X is the no. of defects and $c > 0$ is the parameter.

Note that $E(X) = c = V(X)$.

Therefore, a control chart for defects with 3σ limits could be defined as follows :

$$UCL = E(X) + 3\sqrt{V(X)} = c + 3\sqrt{c}$$

$$CL = E(X) = c$$

$$LCL = E(X) - 3\sqrt{V(X)} = c - 3\sqrt{c}$$

The c-chart or control chart for defects : —

Standard given : Assuming that the standard value of c is available.

$$UCL = c + 3\sqrt{c}$$

$$CL = c$$

$$LCL = c - 3\sqrt{c}$$

Should these calculation yield a negative value for the LCL, set $LCL = 0$ as LCL can't be negative.

Standard not given : If no standard is given, then c may be estimated as the observed average numbers of defects in a preliminary sample of inspection units — say, $\bar{c} = \frac{1}{m} \sum_{i=1}^m c_i$; where c_i is the no. of defects in the i^{th} inspection unit. In this case, the control chart is given by

$$UCL = \bar{c} + 3\sqrt{\bar{c}}$$

$$CL = \bar{c}$$

$$LCL = \bar{c} - 3\sqrt{\bar{c}}$$

The u-chart or the control chart for the average number of defects per unit : There is no reason why the sample size must be restricted to one inspection unit. In fact, we could often prefer to use several inspection units in the sample, thereby increasing the area of opportunity for the occurrence of defects.

The sample size should be chosen according to statistical considerations such as - cost, probability of detecting a process shift.

If we find \bar{x} , total defects in the sample of n inspection units, then the average number of defects per inspection unit is

$$u = \frac{\bar{x}}{n}.$$

$$\text{Here, } x \sim P(c). \quad E(u) = \frac{c}{n}, \quad V(u) = \frac{c}{n^2}.$$

$$\Rightarrow \mu_u = \frac{c}{n}, \quad \sigma_u = \frac{\sqrt{c}}{n}.$$

(i) If we have taken m samples of size ' n ', we make estimates of the parameters $\hat{\mu}_u = \bar{u} \Leftrightarrow \hat{\mu}_u = \frac{\sum c_i}{\sum n} = \frac{1}{n} \bar{c}$; where $\bar{c} = \frac{1}{m} \sum_{i=1}^m c_i$

$$\text{and } \hat{\sigma}_u = \frac{\sqrt{\bar{c}}}{n} = \frac{\sqrt{\bar{u}n}}{n} = \sqrt{\frac{\bar{u}}{n}}.$$

(ii) If c is given \Leftrightarrow if $u' = \frac{c'}{n}$ is given, then $\mu_u = u'$, $\sigma_u = \sqrt{\frac{u'}{n}}$.

Standard given : If c is given, say $u' = \frac{c'}{n}$, then the control limits are:

$$UCL = \mu_u + 3\sigma_u = u' + 3\sqrt{\frac{u'}{n}}.$$

$$CL = \mu_u$$

$$= u'$$

$$LCL = \mu_u - 3\sigma_u = u' - 3\sqrt{\frac{u'}{n}}, \text{ from (ii)}$$

Standard not given :

If no standard ^{is} given, then from (i), we get

$$UCL = \hat{\mu}_u + 3\hat{\sigma}_u = \bar{u} + 3\sqrt{\frac{\bar{u}}{n}}$$

$$CL = \hat{\mu}_u$$

$$= \bar{u}$$

$$LCL = \hat{\mu}_u - 3\hat{\sigma}_u = \bar{u} - 3\sqrt{\frac{\bar{u}}{n}}.$$

B. Control charts for variable sample size: —

When a 100% inspection of the product is observed, the number of inspection units in a sample will usually not be constant. For example the inspection of rolls of cloth or papers often leads to a situation in which the size of the sample varies, because not all rolls are exactly the same length or width. If a control chart for defects (c chart) is used in this situation, both the central line and control limits will vary with the sample size — such a control chart for non-conformities per unit (u -chart). This control chart will have a constant central line; however, the control limits will vary inversely with the square root of the sample size n .

(i) U -chart:

$$UCL = \bar{u} + 3 \sqrt{\frac{\bar{u}}{n_i}}$$

$$CL = \bar{u}$$

$$LCL = \bar{u} - 3 \sqrt{\frac{\bar{u}}{n_i}} ; \text{ where, } \bar{u} = \sum_{i=1}^m c_i / \sum_{i=1}^m n_i$$

(ii) Use of control limits based on the average sample size:

$$\bar{n} = \frac{1}{m} \sum_{i=1}^m n_i$$

$$UCL = \bar{u} + 3 \sqrt{\frac{\bar{u}}{\bar{n}}}$$

$$CL = \bar{u}$$

$$LCL = \bar{u} - 3 \sqrt{\frac{\bar{u}}{\bar{n}}}$$

(iii) Use of a standardized Control chart (This is the preferred option):

Here, we plot the standardized statistic :

$$Z_i = \frac{u_i - \bar{u}}{\sqrt{\frac{\bar{u}}{n_i}}} \text{ on a control chart with}$$

$$LCL = -3, UCL = 3 \text{ and } CL = 0.$$

Process Capability and Modified Control charts :

Statistical techniques can be useful throughout the product cycle, including development activities prior to manufacturing, in quantifying process variability, in analyzing this variability relative to product requirements or specifications. This general activity is called process capability analysis.

Natural Tolerance Limits : ↪ [C.U.]

Process capability refers to the uniformity of the process. It is customary to take the six-sigma spread in the distn. of the product quality characteristic as a measure of the process capability. If μ and σ are the process average and process standard deviation respectively, then the limits $\mu \pm 3\sigma$ (three sigma above and below the mean) are called the 'Natural Tolerance Limits'. The upper and lower "natural tolerance limits" of the process fall at $\mu + 3\sigma$ and $\mu - 3\sigma$, respectively, that is,

$$UNTL = \mu + 3\sigma$$

$$LNTL = \mu - 3\sigma$$

The width ' 6σ ' which is the inherent variability of the process is given a special name Natural tolerance.

For normal distribution, only 0.27% of the process output will fall outside natural tolerance limits. If the distribution of process output is non-normal then the percentage of output falling outside $\mu \pm 3\sigma$ may differ considerably from 0.27%.

If μ and σ are unknown the $\hat{\mu} \pm 3\hat{\sigma}$ are the estimates of the natural tolerance limits, where $\hat{\mu} = \bar{X}$, $\hat{\sigma} = \bar{R}/d_2$ on $\frac{s}{c}$.

Specification Limits :

It might happen that even though the process is in statistical control as exhibited by control limits / chart, the consumer may not be satisfied with the products. The specification limits, are determined (externally) by the management, the manufacturing engineers, the customers such that a product having quality outside the specification limits is considered as unsatisfactory, one should have knowledge of 'inherent variability' of the process while setting specifications, but remember that there is no mathematical or statistical relationship between the control limits and specification limits.

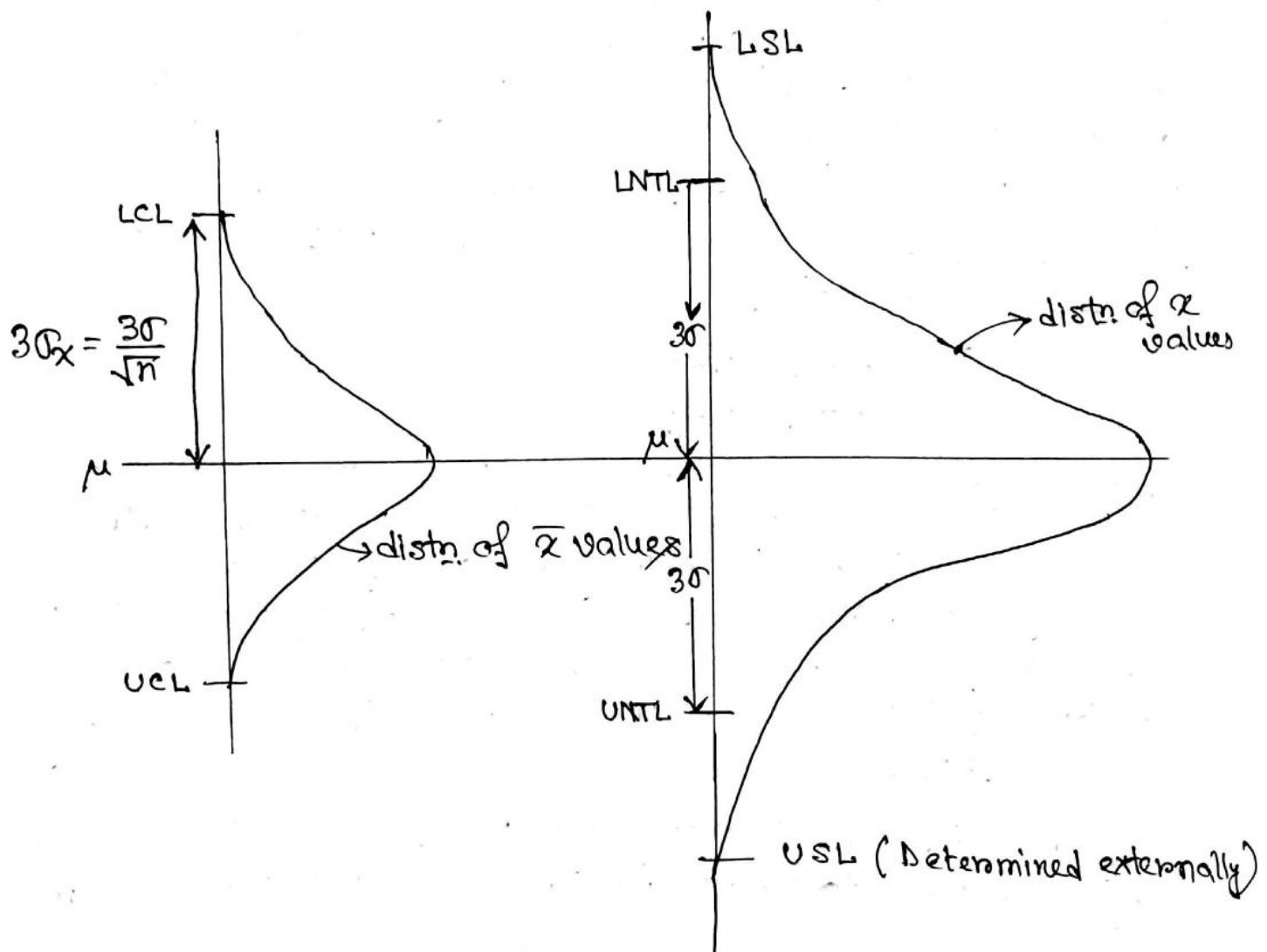


Fig:- Relationship of natural tolerance limits, control limits and specification limits

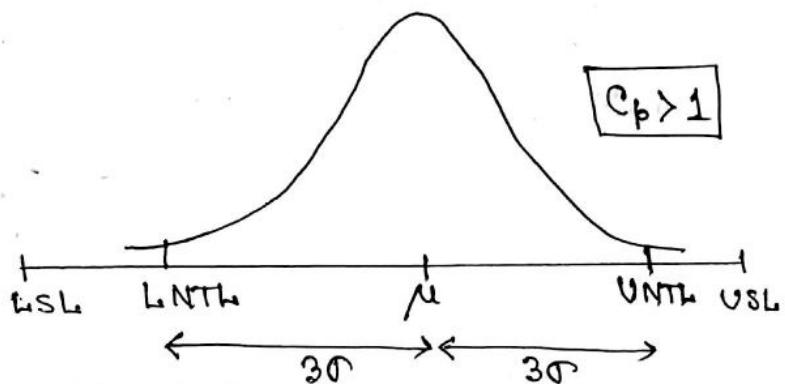
Process Capability Ratio: — Another way to express process capability is in terms of the process capability ratio (PCR) C_p , which for a quality characteristic with both USL and LSL is

$$C_p = \frac{USL - LSL}{6\sigma}$$

Case-(I): $C_p > 1 \Leftrightarrow USL - LSL > 6\sigma$.

This implies that natural tolerance limits in the process are well inside the USL and LSL. In such a case almost all the products will conform to specifications as long as the process is in statistical control. The larger the C_p , the greater is the likelihood of getting good product without assistance from any control chart. This will imply that the process is too good for the product, less costly processing or material could be allowed or it may also be worthwhile to 'squeeze' the specification limits, to produce a product superior to the one originally intended.

Here, the process mean can sometimes be allowed to vary over an interval without appreciably affecting the overall performance of the process.



Modified Control Chart:

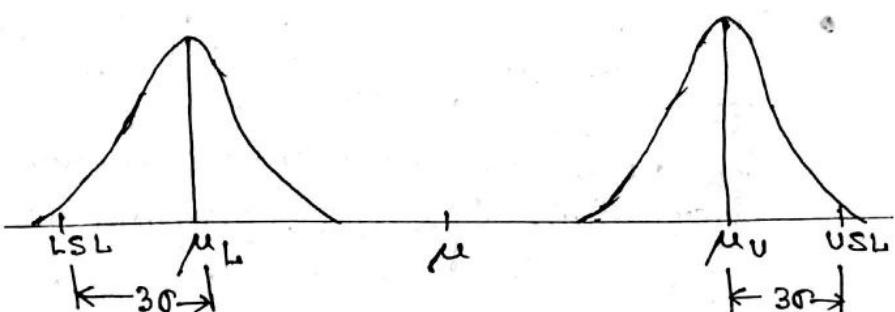
When this situation occurs, we can use a modified control charts or reject charts.

In effect, μ is allowed to vary over an interval, say, $\mu_L \leq \mu \leq \mu_U$ — where, μ_L and μ_U are chosen as the smallest and largest permissible values of μ , respectively, until they reach at a danger point. To specify the control limits (reject limits) for a modified \bar{x} -chart, we will assume that the process output is normally distributed.

From the figure, we have

$$\mu_L = LSL + 3\sigma$$

$$\mu_U = USL - 3\sigma$$



Hence the control limits for the modified chart or the reject limits are:

$$URL_{\bar{x}} = \mu_U + 3\sigma/\sqrt{n} = USL - 3\sigma + \frac{3\sigma}{\sqrt{n}} = USL - 3\sigma(1 - \frac{1}{\sqrt{n}})$$

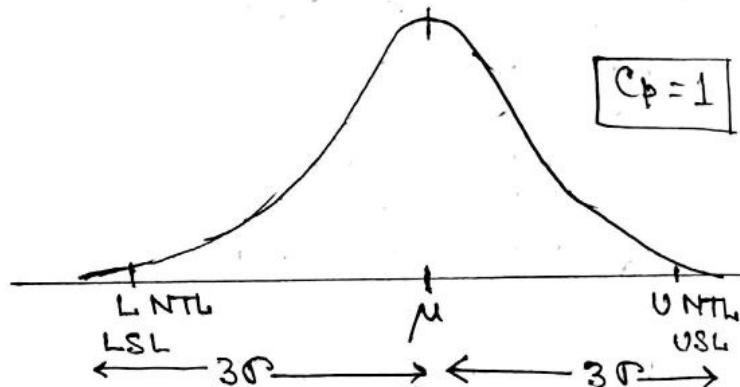
$$RL_{\bar{x}} = \mu_L - 3\sigma/\sqrt{n} = LSL + 3\sigma - \frac{3\sigma}{\sqrt{n}} = LSL + 3\sigma(1 - \frac{1}{\sqrt{n}})$$

These reject limits, when used in place of control limits, are called modified control limits.

To design a modified control chart, we must have a good estimate of σ available. If the process variability shifts, then the modified control limits are not appropriate. Consequently, an R or an S chart should always be used in conjunction with the modified control chart.

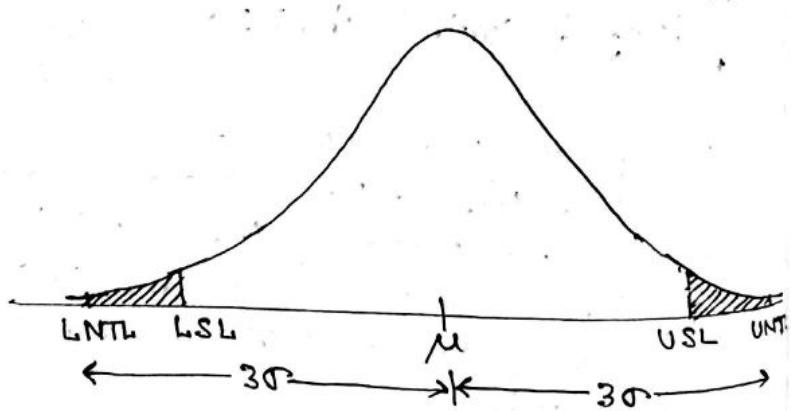
Case-II: $C_p = 1 \Leftrightarrow USL - LSL = 6\sigma$

For a normal distr., this would imply about 0.27% non-conforming units.



Case III: $C_p < 1 \Leftrightarrow USL - LSL < 6\sigma$

In this case, the process is very yield-sensitive, and a large number of non-conforming limits will be produced.



- Process Performance Index :- (C_{pk}) $C_{pk} = \min [C_{pl}, C_{pu}]$, where $C_{pl} = \frac{\mu - LSL}{3\sigma}$, $C_{pu} = \frac{USL - \mu}{3\sigma}$.

C_{pk} checks whether the process is centered at the middle of the specification. $C_{pk} < 1$, performance is not OK.

$C_{pk} = 1 \Leftrightarrow C_{pu} = C_{pl} = C_{pk} = C_p$, otherwise $C_{pk} < C_p$, then performance is not optimum.

(b) Product control : — The object of product control is to decide whether to accept or reject a lot on the basis of evidence afforded by one or more samples drawn at random from the lot in question.

Lot acceptance sampling plans refers to the use of sampling inspection by a purchaser to decide whether to accept or reject a lot of given product. Acceptance sampling plans are often designed so as to accomplish at least two of the following objectives :

- (1) the probability of rejecting a good lot is some specified value (Producer's risk).
- (2) the probability of accepting a bad lot is some specified value (Consumer's risk).
- (3) the average quality of goods shipped out shall not be worse than some specified standard.
- (4) the amount of inspection (consistent with the conditions imposed) shall be minimized.

Advantages & disadvantages of Sampling : —

When acceptance sampling is constructed with 100% inspection, it has the following advantages : —

- (1) It is usually less expensive because there is less inspection.
- (2) There is less handling of the product, hence reduced damage.
- (3) It is applicable to destructive testing.
- (4) Fewer personnel are involved in inspection activities.
- (5) It often greatly reduces the amount of inspection errors.

Acceptance sampling also has several disadvantages : —

- (1) There are risks of accepting 'bad' lots and rejecting 'good' lots.
- (2) Less information is usually generated about the product or about the process.
- (3) Acceptance sampling requires planning and documentation of the acceptance-sampling procedure whereas 100% inspection does not.

Lot Acceptance Sampling for Attributes :

Acceptance sampling is concerned with inspection and decision making regarding products, one of the oldest aspects of quality assurance. A typical example on application of acceptance sampling is as follows:

A company receives a shipment of product from a vendor. A sample is taken from the lot, and some quality characteristic of the units in the sample is inspected. On the basis of the information in this sample, a decision is made regarding lot disposition. Usually, this decision is either to accept or to reject the lot. Accepted lots are put for sale (production); rejected lots may be returned to the vendor or may be subjected to some other lot-disposition action.

A sampling plan may be either the acceptance-rejection or the acceptance-rectification type.

Acceptance - rejection Inspection Plan :

In this plan, the lot is accepted or rejected on the basis of the sample(s) drawn from the lot and rejected lot is returned to the vendor. The accepted lot, after replacing the defective items in the drawn sample, is put for sale/ production.

Acceptance-Rectifying Inspection Plan :

In this plan, the lot is accepted or rejected on the basis of the sample(s) drawn from the lot and the rejected lot is subjected to corrective action. This generally takes the form of 100% inspection or screening of rejected lots, with all discovered defective items either removed for subsequent rework or replaced from a set of known good items. Such sampling programs are called rectifying inspection programs, because the inspection activity affects the final quality of the outgoing product. The rejected lots will be screened, and their final fraction defective will be zero.

Notions:-

Producers: Any person, company or department that sells or prepares goods to be received by another person or company or another department of the same business.

Consumer: The recipient of product. It may be a buyer, or another department of the producer.

\bar{p} : Process average or fraction defective turned out by a process over a long period of time.

p : The fraction defective in a lot.

Acceptable quality level (AQL), p_1 : A relatively small fraction defective. The AQL represents the poorest level of quality for the vendor's process that the consumer could consider to be acceptable as a process average. More specially a lot with this fraction defective (p_1) is a lot of sufficiently good quality that we do not wish to reject if more often than a specified small proportion (usually 1%, 5%) of the time. Usually,

$$P[\text{Rejecting a lot of quality } p_1] = 0.05$$

$$\Rightarrow P[\text{accepting a lot of quality } p_1] = 0.95$$

' p_1 ' is known as the Acceptance Quality Level and a lot of this quality is considered as satisfactory by the consumer.

Lot Tolerance Proportion or Percent Defective (LTPD), p_t : A relatively large fraction defective. The LTPD is the lot quality which is considered to be bad by the consumer, the consumer is not willing to accept lots having proportion defective p_t or greater. 100 p_t is called Lot Tolerance Percentage Defective. In other words, this is the quality level which the consumer regards as rejectable and is usually abbreviated as R.Q.L. (Rejecting Quality Level). A lot of quality p_t stands to be accepted some arbitrary and small fraction of time (usually 5%, 10%).

Operating - Characteristic (OC) Function : for an acceptance-sampling plan, define $L(p) = P[\text{accepting a lot when the fraction defective of the lot is } p]$. The $L(p)$, considered as a function of the fraction defective of the lot (p); is called the OC function of the sampling plan. The curve obtained by plotting $L(p)$ against p is called the OC curve and it is an important measure of the acceptance-sampling

Process Average Fraction Defective (\bar{p}): \bar{p} represents the quality turned out by the manufacturing process over a long period of time. The process average of any manufactured product is obtained by finding the percentage of defectives in the product over a fairly long time.

Producer's risk: Any acceptance sampling plan for acceptance-rejection has certain risk on the part of the producer—the producer has to face the situation that some good lots will be rejected. The probability of rejecting a lot, with a fraction defective p_1 (AQL), under the acceptance-rejection sampling plan, is called the producer's risk.

Clearly, in terms of OC function we have $\{1 - L(p_1)\}$ as the producer's risk and it is denoted by ' α ' or P_p .

Consumer's risk: The consumer has also to face the situation sometimes that a bad lot will be accepted, on the basis of an acceptance-rejection sampling plan. The probability of accepting a lot with fraction defective p_t (LTPD), under the acceptance-rejection sampling plan, is called the consumer's risk.

Clearly, in terms of OC function, we have $L(p_t)$ as the consumer's risk and it is denoted by ' β ' or P_c .

Rectifying Inspection Plans: Acceptance-sampling programs requires corrective action when lots are rejected. This takes the form of 100% inspection of rejected lots, with all discovered defective items replaced from a stock of known good items. Such sampling programs are called rectifying inspection programs, because the inspection activity affects the final quality of the outgoing products. The two important points related to rectifying inspection plans are:

Average Outgoing Quality (AOQ): AOQ is the expected fraction defective, after replacing good items for defective ones in rejected lots and in samples taken from accepted lots, in a lot. It is the average value of lot quality that could be obtained over a long sequence of lots from a process with fraction defective p , that results from the application of the rectifying inspection. Average outgoing quality will vary as the function defective of the incoming lots varies. The curve that plots AOQ against incoming lot quality, is called AOQ Curve.

Remark:- The fraction defective (p) of a incoming lot on the quality of a lot before inspection, is termed as 'incoming quality' of the lot. The fraction defective of the lot after inspection is termed as 'outgoing quality' of the lot.

Average Outgoing Quality Limit (AOQL) :-

The maximum value of the average outgoing quality (AOQ), the maximum being taken const. the incoming quality (p), is called the average outgoing quality limit (AOQL). Symbolically,

$$AOQL = \text{Max}_{0 \leq p \leq 1} \{ AOQ(p) \} .$$

Average Sample Numbers (ASN): The ASN is the expected value of the sample size required for coming to a decision about the acceptance or rejection in an acceptance-rejection sampling plan.

Obviously, it is a function of the incoming lot quality ' p '. The curve that plots ASN against incoming lot quality ' p ', is called an ASN curve.

Average Amount of Total Inspection [ATI]: Another important measure relative to acceptance - rectifying inspection is the total amount of inspection required by the sampling program; the expected numbers of items inspected in a lot to arrive at a decision in an acceptance - rectification sampling inspection plan calling for 100% inspection of the rejected lots is called Average Total inspection (ATI). Obviously, ATI is a function of the incoming lot quality (p).

□ We observe that —

$$ATI = ASN + (\text{Average size of inspection of the remainder in the rejected lots})$$

Thus, if the lot is accepted on the basis of the sampling inspection plan, then $ATI = ASN$, otherwise $ATI > ASN$. In other words, ASN gives the average numbers of units inspected per accepted lot.

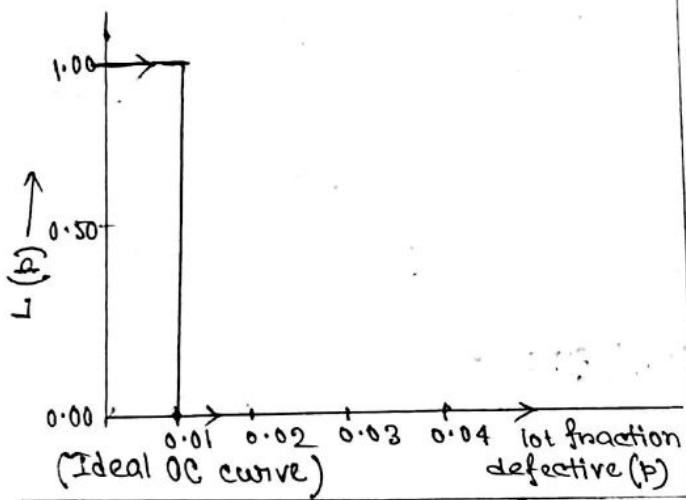
For example, if a single sampling acceptance-rejection plan is used, the numbers of items inspected from each lot will be the the corresponding sample size n

$$\text{i.e. } ASN = n,$$

and this will be true independently of the quality of the submitted lots.

OC curve: This curve plots the probability of accepting the lot when the fraction defective (or the incoming quality) of the lot is p , $L(p)$, for different values of ' p '. The curve shows that the probability that a lot submitted with a certain fraction defective will be accepted; that is, the OC curve displays the discriminatory powers of the sampling plan.

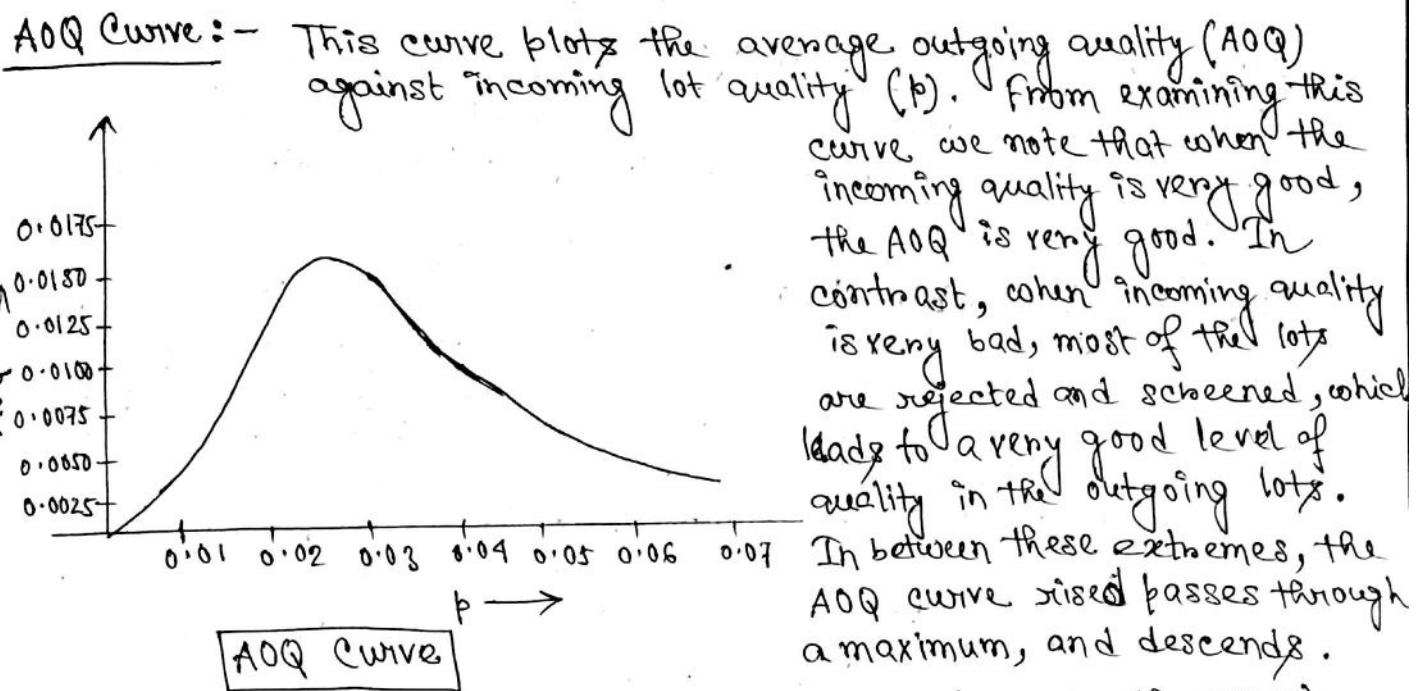
A sampling plan that discriminated perfectly between good and bad lots would have an OC curve that looks like:



The OC curve runs horizontally at a probability of acceptance $L(p)=1.00$ until a level of quality that is considered 'bad' is reached; at which point the curve drops vertically to a probability of acceptance $L(p)=0.00$ and then the curve runs horizontally again for all lot fraction defective greater than the undesirable level. In such a sampling plan, if

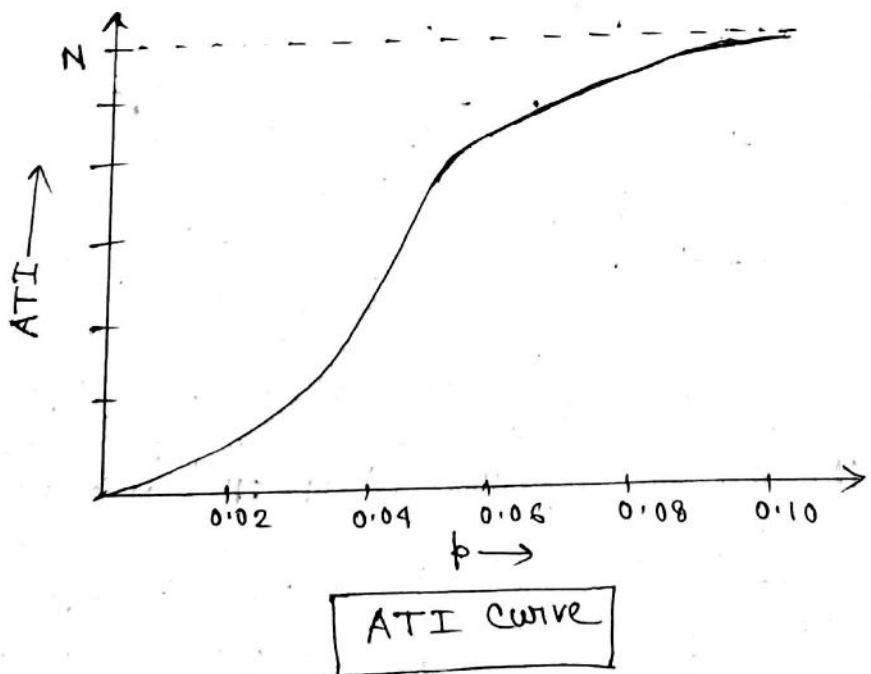
exists, all lots of 'bad' quality would be rejected and all lots of 'good' quality would be accepted.

Unfortunately, the ideal OC curve can almost never be obtained in practice. In theory, it could be realized by 100% inspection, if the inspection were error free. The ideal OC curve shape can be approached, however, by increasing the sample size. Thus, the precision with which a sampling plan differentiates between good and bad lots increases with the size of the sample. The greater is the slope of the OC curve, the greater is the discriminatory power.



The maximum ordinate on the AOQ curve represents the worst possible average quality that would result from the rectifying inspection program and this point is called the average outgoing quality limit (AOQL).

ATI Curve:- The average total inspection in a lot is an acceptance-rectification plan is a function of incoming lot quality. If p increases, ATI increases. A typical ATI curve looks like:



for Attributes

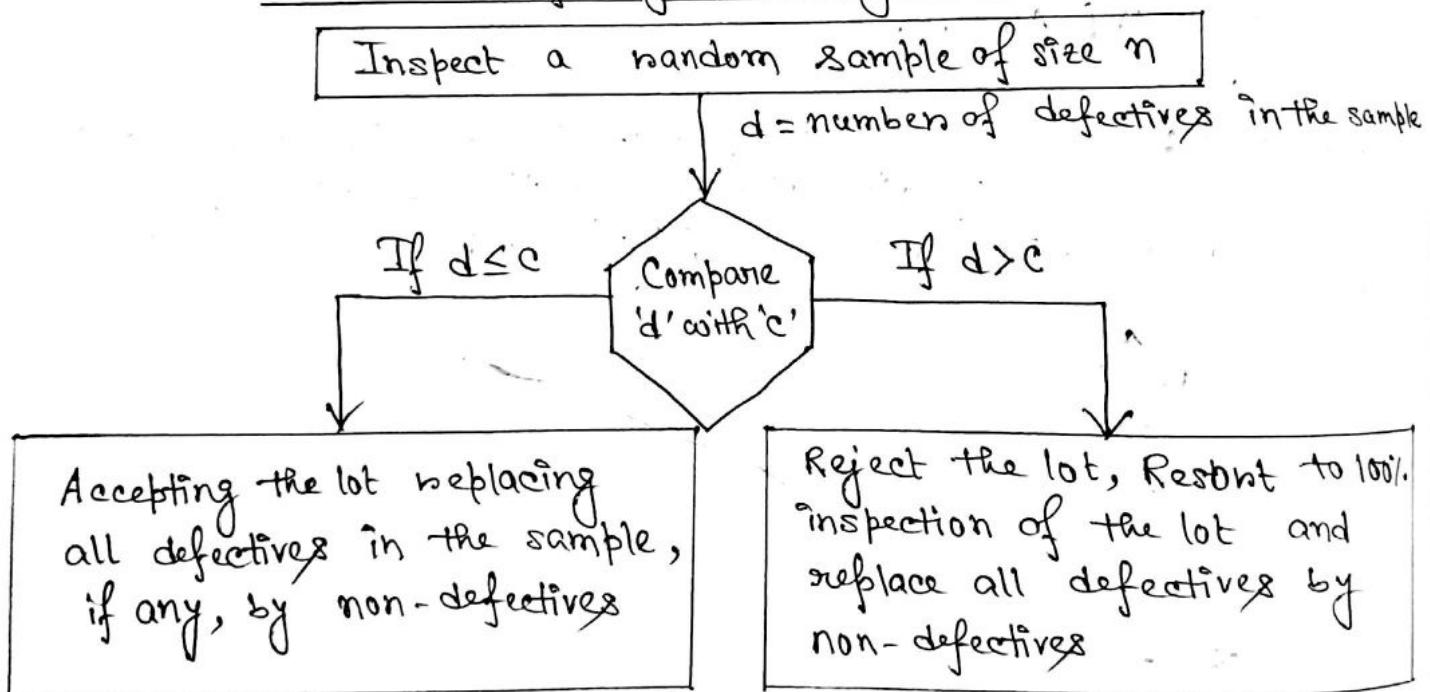
Types of Sampling Plans: Acceptance sampling plans are classified according to the sampling methods in three chief methods: Single sampling, Double sampling and sequential sampling.

A. Single-Sampling Plans: — suppose that a lot of size N has been submitted for inspection.

A single sampling plan is defined by the sample size n and the acceptance number c . The procedure would operate as follows: Select n items at random from the lot. If there are ' c ' or fewer defectives in the sample, accept the lot, and if there are more than c defective items in the sample, reject the lot. The statistical problem is to determine n and c so as to provide the desired protection.

A single sample acceptance rectification sampling plan for attributes is described as follows:

Flow chart of single sampling Rectification Plan



OC - Function: In a lot of incoming quality ' p ', that is, in a lot with fraction defective p , the number of defectives is Np and non-defectives is $N - Np = N(1-p)$. Then, the no. of defectives d in a random sample of size n follows a Hypergeometric distribution with parameters (n, N, p) .

Then the probability of accepting a lot of incoming quality p is

$$L(p) = P_a(p) = P[d \leq c] = \sum_{d=0}^c \frac{\binom{Np}{d} \binom{N-Np}{n-d}}{\binom{N}{n}}$$

AOQ and AOQL: ~ If p is the incoming lot quality, there will be no defectives remain in a lot of size N if $d > c$ and if $d \leq c$, the number of defectives in a lot of size N is $(Np - d)$. Thus, the mean of the numbers of defectives remain after sampling inspection is given by :

$$m = \sum_{d=0}^c (Np - d) \binom{Np}{d} \binom{N-Np}{n-d} / \binom{N}{n} + 0$$

The expected fraction defective remains after inspection, i.e. AOQ is given by

$$AOQ = \bar{p} = \frac{m}{N} = \sum_{d=0}^c \left(p - \frac{d}{N} \right) \cdot \frac{\binom{Np}{d} \binom{N-Np}{n-d}}{\binom{N}{n}}$$

Subject to variation in p , $AOQ(\bar{p})$ has a maximum value, \bar{p}_L which is termed as AOQL.

ATI : ~ The total amount of inspection consists of two parts:

- (1) a sample from each lot, whether it is accepted or rejected.
- (2) the rest of the items in the rejected lots. therefore the average (expected) total inspection in a lot, when the process average is F is the sum of (i) the sample size 'n' and (ii) the remainder of rejected lots, $N-n$ multiplied by the probability of obtaining a sample with more defectives than the acceptance number.

If the process average fraction defective in a lot is \bar{p} as claimed by the producer, then the average total inspection(ATI) per lot is :

$$ATI = n + (N-n) P[d > c | p = \bar{p}]$$

$$\text{Hence, } ATI = n + (N-n)(1 - L(F))$$

$$= n + (N-n) \{ 1 - Pa(\bar{F}) \}; \text{ where } Pa(\bar{F}) \text{ is the}$$

lot acceptance probability when the lot incoming quality is \bar{p} .

Plans classified according to type of protection : — [cu]

1. Lot quality protection on LTPD plan:

Consumer's requirement fixes the values of P_c , the consumer's risk and P_t , the lot tolerance fraction defective, where N is always fixed. If P_t be the lot tolerance fraction defective, the expression for P_c is

$$P_c = P[\text{Accepting a lot of quality } P_t]$$

$$P_c = L(P_t) \text{ or } Pa(P_t)$$

$$\therefore \text{Consumer's risk} = \sum_{d=0}^c \binom{Np_t}{d} \binom{N-Np_t}{n-d} / \binom{N}{n} \quad (*)$$

For given values of P_c and p_t , the equation (*) which involves two unknowns n and c is satisfied by various pairs of values of n and c .

If \bar{p} is the producer's process average, the producer's risk is given by

$$P_p = P[\text{rejecting a lot of quality } \bar{p}]$$

$$P_p = 1 - Pa(\bar{p})$$

$$\therefore \text{Producer's risk} = 1 - \sum_{d=0}^c \binom{N\bar{p}}{d} \binom{N-N\bar{p}}{n-d} / \binom{N}{n}.$$

Then ATI is given by,

$$ATI = n + (N-n)(1 - Pa(\bar{p})) \quad (**)$$

To safeguard producer's interest also, out of these possible pairs of (n, c) satisfying (*), one involving the minimum ATI as given by (**) is chosen. The solution, however, is theoretically very difficult to obtain. Dodge and Romig, by applying numerical methods, have prepared extensive tables for minimising values of n, c for $P_c = 0.10$ and different values of \bar{p} .

Hence, it is possible to design rectifying inspection plan (that is, to find the values of n and c) that gives a specified level of protection (P_c) at the LTPD (p_t) point and that minimizes the ATI for a specified process average (\bar{p}).

2. AOQL Plan: → [CV]

Hence, the consumer's interests are taken care of by specifying the AOQL, so that no matter how bad the fraction defective is in the coming lots, he will never have a worse quality level on the average than $AOQL \times 100\% \text{ defective}$.

If p be the incoming lot quality of a lot of size N , the AOQ is given by

$$AOQ = \tilde{p} = \sum_{d=0}^c \left(p - \frac{d}{N} \right) \binom{Np}{d} \binom{N-Np}{n-d} / \binom{N}{n} \quad (***)$$

Given N and \tilde{p}_L (AOQL), it is possible to select several pairs of values n and c that will give \tilde{p} as defined in (***) having approximately the same value of \tilde{p}_L ; as a safeguard to producer's interests we select the pair (n, c) which minimizes ATI as defined in (**), i.e. $ATI = n + (N-n)(1 - Pa(\tilde{p}))$, for a specified value of \tilde{p} :

Hence, it is possible to choose the rectifying sampling plan that has a specified AOQL (\tilde{p}_L) and, in addition, yields a minimum ATI at a particular level of process level (\bar{p}).

B. Double-Sampling Plans: — A double sampling plan is a procedure in which, under certain circumstances, a second sample is required before the lot can be sentenced. A double sampling plan is defined by four parameters:

n_1 = size of the 1st sample.

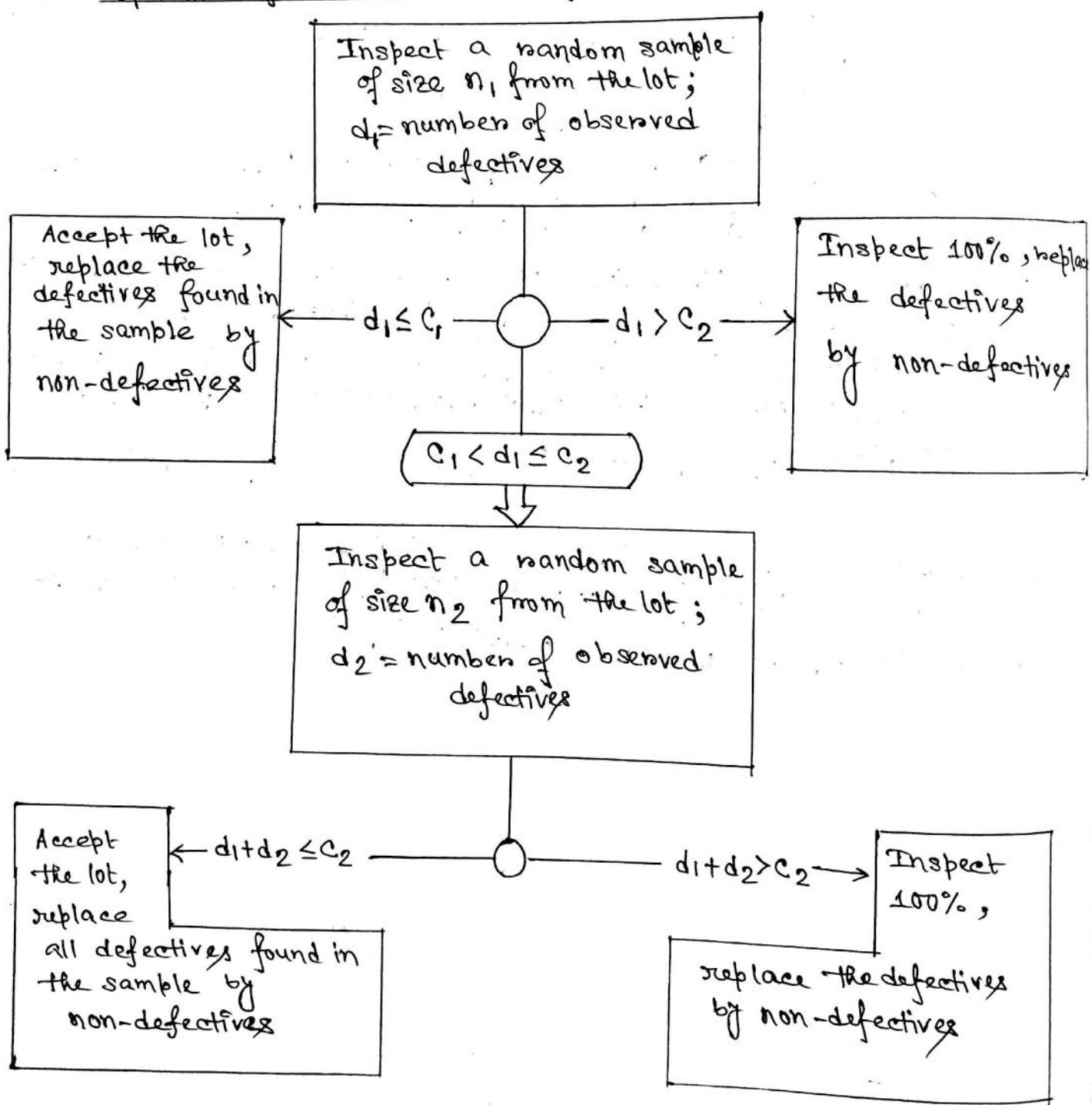
c_1 = acceptance numbers of the first sample.

n_2 = size of the 2nd sample.

c_2 = acceptance numbers for both samples.

Operation of the double-sampling plan:

Acceptance - Rectification



The OC curve: If $P_a(p)$ denotes the probability of acceptance on the combined samples, and $P_a^I(p)$, $P_a^{II}(p)$ denote the probability of acceptance on the 1st and 2nd samples, respectively, of a lot of incoming quality p , then

$$\begin{aligned} P_a(p) &= P_a^I(p) + P_a^{II}(p) \\ &= P[d_1 \leq c_1] + P[c_1 < d_1 \leq c_2, d_1 + d_2 \leq c_2] \\ &= \sum_{d_1=0}^{c_1} f(d_1, p) + \sum_{d_2=0}^{c_2-d_1} \sum_{d_1=c_1+1}^{c_2} f(d_1, p) g(d_2, p | d_1) \end{aligned}$$

where, $f(d_1, p)$ is the probability of getting ' d_1 ' defectives in the 1st sample and $g(d_2, p | d_1)$ is the conditional probability of finding d_2 defectives in the second sample under the condition that d_1 defectives have already appeared in the 1st sample.

thus $f(d_1, p) = \binom{Np}{d_1} \binom{N-Np}{n_1-d_1} / \binom{N}{n_1}$ and

$$g(d_2, p | d_1) = \binom{Np-d_1}{d_2} \binom{N-n_1-(Np-d_1)}{n_2-d_2} / \binom{N-n_1}{n_2}$$

Hence,

$$P_a(p) = \sum_{d_1=0}^{c_1} \frac{\binom{Np}{d_1} \binom{N-Np}{n_1-d_1}}{\binom{N}{n_1}} + \sum_{d_2=0}^{c_2-d_1} \sum_{d_1=c_1+1}^{c_2} \frac{\binom{Np}{d_1} \binom{N-Np}{n_1-d_1} \binom{Np-d_1}{d_2} \binom{N-n_1-Np+d_1}{n_2-d_2}}{\binom{N}{n_1} \binom{N-n_1}{n_2}}$$

Consumer's risk & Producer's risk:

The consumer's risk is $P_c = P[\text{accepting a lot of quality } p_t]$
 $= P_a(p_t)$;

the producer's risk is $P_p = 1 - P_a(p_t)$.

ATI: Since (i) only n_1 items will be inspected if $d_1 \leq c_1$ and the probability is $P_a^I(p)$.

(ii) (n_1+n_2) items will be inspected if the lot is accepted on the basis of the second sample and its probability is $P_a^{II}(p)$, and

(iii) the entire lot of N items will be inspected if the lot is rejected and the probability of this is $\{1 - P_a(p)\}$

Then, the average total inspection (ATI) is given by

$$ATI = n_1 P_a^I(p) + (n_1+n_2) P_a^{II}(p) + N \{1 - P_a(p)\}$$

$$= n_1 + n_2 \{1 - P_a^I(p)\} + (N - n_1 - n_2) \{1 - P_a(p)\},$$

using $P_a(p) = P_a^I(p) + P_a^{II}(p)$

[In an acceptance-rejection double sampling plan, the numbers of items inspected for a lot is either n_1 , when the lot is accepted or rejected on the basis of the 1st sample, or (n_1+n_2) when a 2nd sample of size n_2 is drawn. Thus the expected sample size for a decision is given by]

$$\text{ASN} = n_1 P_1 + (n_1+n_2)(1-P_1) = n_1 + n_2(1-P_1),$$

where, P_1 is the probability of a decision (acceptance or rejection of the lot) on the basis of the 1st sample

$$= P(d_1 \leq c_1 \text{ or } d_1 > c_2) = 1 - P[c_1 < d_1 \leq c_2]$$

$$= 1 - \sum_{d_1=c_1+1}^{c_2} \frac{\binom{Np}{d_1} \binom{N-Np}{n_1-d_1}}{\binom{N}{n_1}},$$

AOQ :-

$$\text{AOQ} = \frac{[P_a^I(p) \cdot \{N-n_1\} + P_a^{II}(p) \{N-n_1-n_2\}]}{N}$$

The maximum value of this AOQ with respect to p is the AOQL in the double sampling plan.

Designing Double-Sampling Plans :-

It is often necessary to be able to design a double sampling plan that has a specified OC-curve — the values to be determined here are n_1, n_2, c_1 and c_2 . There are two approaches for determining these values — LTPD plan, or, AOQL plan.

The Dodge-Romig tables give double sampling plans that have either a specified p_0 or a specified AOQL and yield minimum ATI at the given values for the process average.

Comparison of Double Sampling and Single Sampling plans:

- (1). The principal advantage of a double-sampling plan over single sampling is that it may reduce the total amount of required inspection. Suppose that the 1st sample taken under a double-sampling plan that offers the consumers the same protection. In all cases, then, in which a lot is accepted or rejected on the first sample, the cost of inspection will be lower for double sampling than it would be for single sampling. It is also possible to reject a lot without complete inspection of the second sample (this is called curtailment on the 2nd sample). Consequently, the use of double sampling can often result in lower total inspection cost.
- (2). Furthermore, in some situations, double-sampling plan has the psychological advantage of giving a lot a second chance. This may have some appeal to the vendor but there is no real advantage to double sampling plans can be chosen so that they have the same OC curve.
- (3). Unless curtailment is used on the 2nd sample, under some circumstances double sampling may require more total inspection than would be required in a single sampling plan that offers the same protection.
- (4). The double-sampling is administratively more complex than a single-sampling, which may increase the opportunity for the occurrence of inspection errors.

Sampling inspection by Variables :

Consider a variables sampling plan to control the lot or process fraction non-conforming. Since the quality characteristic is a variable, there will exist either a LSL, an USL or both, than define the acceptable values of this parameter.

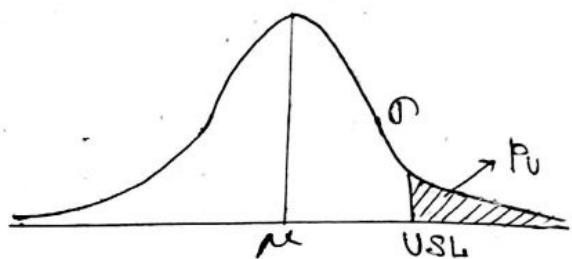
Let the quality characteristic x is $N(\mu, \sigma^2)$.

- (i) If only the USL is given, then an item is considered non-conforming iff $x > \text{USL}$.

Then the fraction defective in the

$$p_U = P[x > \text{USL}]$$

$$= 1 - \Phi\left(\frac{\text{USL} - \mu}{\sigma}\right)$$



- (ii) If only LSL is given, then an item is considered defective iff $x < \text{LSL}$, and the lot fraction defective is

$$p_L = P[x < \text{LSL}] = \Phi\left(\frac{\text{LSL} - \mu}{\sigma}\right).$$

- (iii) When there are double specification limits, then an item is considered defective if $x < \text{LSL}$ or $x > \text{USL}$ and the lot fraction defective is

$$p_U + p_L = 1 - \Phi\left(\frac{\text{USL} - \mu}{\sigma}\right) + \Phi\left(\frac{\text{LSL} - \mu}{\sigma}\right)$$

sampling inspection provides us with estimates of p_L and p_U or equivalently, of μ and σ , if, estimates \hat{p}_L and \hat{p}_U exceed a specified maximum value M , reject the lot otherwise accept it.

Case I : Variable inspection with known s.d. (σ) :-

When σ is known, there exists MVUEs. of p_U and p_L , viz.

$$\hat{p}_U = 1 - \Phi\left(\sqrt{\frac{n}{n-1}} \left(\frac{\text{USL} - \bar{x}}{\sigma}\right)\right) \text{ and } \hat{p}_L = \Phi\left(-\sqrt{\frac{n}{n-1}} \left(\frac{\bar{x} - \text{LSL}}{\sigma}\right)\right)$$

- (i) If only USL is given, the lot is accepted if the estimate \hat{p}_U is small, i.e., if $\hat{p}_U \leq M$ (say) $\Leftrightarrow \frac{\text{USL} - \bar{x}}{\sigma} \geq K$ $\Leftrightarrow \bar{x} + K\sigma \leq \text{USL}$. Note that, M is a quantity determined in accordance with the specified prob. of type I errors and $K = \sqrt{\frac{n-1}{n}} T_M$.

- (ii) If only LSL is given, the lot is accepted iff

$$\hat{p}_L \leq M \Leftrightarrow \text{iff } \frac{\bar{x} - \text{LSL}}{\sigma} \geq K \Leftrightarrow \text{iff } \bar{x} - K\sigma \geq L.$$

(iii) If both specification limits are given, then the lot will be accepted iff $\hat{p}_U + \hat{p}_L \leq M$; otherwise, it will ^{not} be accepted.

The values of k , corresponding to the lot size, the sample size and specified acceptance quality level (with prob. of wrong rejection $\alpha = 0.05$), are given in tables A and K of Bowker and Groode's book (1).

Case II: Variable inspection with unknown s.d. (σ) :-

Let $s^2 = \frac{1}{n-1} \sum_i (x_i - \bar{x})^2$ is the sample variance.

(i) For the upper specification limit, the lot ^{is} accepted iff $\hat{p}_U \leq M \Leftrightarrow \frac{USL - \bar{x}}{s} > k^*$ $\Leftrightarrow \bar{x} + k^*s \leq USL$,

here k^* being a more complicated than the k in the previous case.

(ii) For a given LSL ; the lot is accepted iff $\hat{p}_L \leq M \Leftrightarrow \frac{\bar{x} - L}{s} \geq k^*$
 $\Leftrightarrow \bar{x} - k^*s \geq L$.

(iii) For two-sided specification, the lot will be accepted iff $\hat{p}_U + \hat{p}_L \leq M$.

The value of k for given lot size, sample size and acceptable quality level (with $\alpha = 0.05$), is obtainable from Table A and B of Bowker and Groode's book (1).

Advantages of Variable Sampling : —

1. The variable acceptance-sampling plan that has the same protection as an attribute acceptance-sampling plan would require less sampling. The measurements data required by a variables sampling plan would probably cost more per observation than the collection of attributes data. However, the reduction in sample size obtained may more than offset this increased cost. When destructive testing is employed, variables sampling is particularly useful in reducing the costs of inspection.
2. A second advantage is that measurement data usually provide more information about the manufacturing process than do attributes data.
3. A final point to be emphasized is that when acceptable quality levels are very small, the sample sizes required by attributes sampling plan are very large. Under these circumstances there may be significant advantages in switching to variables measurement.

Disadvantages of Variable Sampling : —

1. Primary disadvantage is that the distn. of the quality characteristic must be known.
2. Most standard variables acceptance-sampling plans assume that the distn. of the quality characteristic is normal, but the quality characteristic may not have normal distn..
3. For each quality characteristic, a separate sampling plan must be employed.

QUESTION & ANSWERS (C.U. PAPER)

4.(a) Derive in details double sampling plan for attributes and derive the quantity by which you protect consumer from inferior product under plan. (10)

(b) S.T. in a single sampling inspection plan by attribute, under suitable assumption, OC curve is given by

$$e^{-np} \sum_{x=0}^c (np)^x / x!$$

where, n denotes the sample size, c acceptance numbers & p lot fraction defective.

Ans:- (b)

$$P_a(p) = \sum_{x=0}^c \frac{\binom{Np}{x} \binom{N-Np}{n-x}}{\binom{N}{n}} = \sum_{x=0}^c P[X=x]$$

$$P[X=x] = \binom{n}{x} p^x q^{n-x}, \quad \text{as } N \rightarrow \infty;$$

$q = (1-p)$ [Binomial approximation to Hypergeometric distn.]

now, if $n \rightarrow \infty, p \rightarrow 0$

$$P[X=x] = \frac{e^{-np} \cdot np^x}{x!} \quad [\text{Poisson approximation to Binomial distn.}]$$

Hence, the OC curve is given by,

$$e^{-np} \sum_{x=0}^c \frac{(np)^x}{x!}, \text{ where } N \rightarrow \infty, n \rightarrow \infty, p \rightarrow 0,$$

(c) (i) Determine the probability limit 0.1 (i.e. the prob. is 0.1) that without the change in the universe a point will fall above the UCL or fall below the LCL for \bar{x} and R charts assuming the parent population to be $N(1, 2)$ and sample size to be 2.

(ii) Suppose the samples are actually being taken from the $N(1, 2, 2.4)$ population. In that case find the expected number of samples to be drawn to reach the conclusion that the process is not in control w.r.t either of the quality characteristic (as soon as a sample point goes outside the control limits, you conclude the process is not in control).

ANS: (i) Let (x_1, x_2) be a r.s. from $N(1, 2)$.

Let 0.1 probability limits for \bar{x} chart are $L_{\bar{x}}$ and $U_{\bar{x}}$.

$$\begin{aligned} \text{Then } 0.1 &= 1 - P[L_{\bar{x}} < \bar{x} < U_{\bar{x}}] \\ &= 1 - P\left[\frac{L_{\bar{x}} - 1}{\sqrt{2}} < \frac{\bar{x} - 1}{\sqrt{2}} < \frac{U_{\bar{x}} - 1}{\sqrt{2}}\right] \\ &= 1 - \left[\Phi\left(\frac{U_{\bar{x}} - 1}{\sqrt{2}}\right) - \Phi\left(\frac{L_{\bar{x}} - 1}{\sqrt{2}}\right) \right] \end{aligned}$$

$$\therefore \Phi\left(\frac{U_{\bar{x}} - 1}{\sqrt{2}}\right) - \Phi\left(\frac{L_{\bar{x}} - 1}{\sqrt{2}}\right) = 0.9 \quad \left[\begin{array}{l} \text{as } x \sim N(1, 2) \\ \therefore \bar{x} \sim N\left(1, \frac{2}{2}\right) \end{array} \right]$$

Let $U_{\bar{x}}$ and $L_{\bar{x}}$ are symmetric about $E(\bar{x}) = 1$.

$$\text{Then } U = 1 + k$$

$$L = 1 - k$$

$$\therefore \Phi(k) - \Phi(-k) = 0.9$$

$$\Rightarrow 2\Phi(k) = 1.9$$

$$\Rightarrow \Phi(k) = 0.95$$

$$\Rightarrow k = \gamma_{0.05} = 1.65 \quad (\text{from table})$$

Hence the 0.1 prob. limits for \bar{x} are $1 \pm k = -0.65$ and $+2.65$.

$$\text{Range} = |X_1 - X_2| = R$$

$$X_1 - X_2 \sim N(0, 2)$$

$$\Rightarrow \frac{X_1 - X_2}{2} \sim N(0, 1)$$

$$\text{Now, } P[0 < R < U_R] = 0.9$$

$$\Rightarrow P[-U_R < X_1 - X_2 < U_R] = 0.9$$

$$\Rightarrow P\left[-\frac{U_R}{2} < \frac{X_1 - X_2}{2} < \frac{U_R}{2}\right] = 0.9$$

$$\Rightarrow \Phi\left(\frac{U_R}{2}\right) - \Phi\left(-\frac{U_R}{2}\right) = 0.9$$

$$\Rightarrow \Phi\left(\frac{U_R}{2}\right) = 0.95$$

$$\Rightarrow \frac{U_R}{2} = 1.65 \quad \therefore U_R = 3.3$$

\therefore 0.1 prob. limits for R are 0 and 3.3.

(ii) Consider a sample point lies outside either of the control chart as a success.

Let Z be the no. of success require to get the 1st success.

p = Probability of success,

$$= 1 - P[-0.65 < \bar{X} < 2.65, 0 < R < 3.3],$$

$$= 1 - P[-0.65 < \bar{X} < 2.65] P[0 < R < 3.3],$$

$$= 1 - P[-0.65 < \bar{X} < 2.65].P[|X_1 - X_2| < 3.3],$$

[for normal sample \bar{X} and $S = \frac{1}{2}|X_1 - X_2|$ are independently distributed as $\bar{X} \sim N(1.2, \frac{2.4}{12})$ and $X_1 - X_2 \sim N(0, 4.8)$.]

$$\therefore p = 1 - \left\{ \Phi\left(\frac{2.65 - 1.2}{\sqrt{1.2}}\right) - \Phi\left(\frac{-0.65 - 1.2}{\sqrt{1.2}}\right) \right\} \left\{ 2 \Phi\left(\frac{3.3 - 0}{\sqrt{4.8}}\right) - 1 \right\}$$

$$= \underline{\hspace{2cm}}, \quad [\text{Use table}]$$

Here $Z \sim \text{Geometric}(p)$

\therefore Required expected no. of samples $= E(Z) = \frac{1}{p} = \underline{\hspace{2cm}}$.

2.(a) In connection with deriving optimum sampling inspection plan define the following terms (Illustrate your answers with an example)

(i) OC, (ii) AOQL.

(b) Derive a double sampling inspection plan by variable.

Describe the usefulness of the plan.

(c) Describe the uses of Indian Standard Sampling Inspection Plans.

Ans:- (c) Use of IS sampling plans:—

These sampling plans have been prepared by the Bureau of Standards, New Delhi and are being widely used.

i) These plans are intended primarily for a continuing series of lots sufficient to allow the switching rules to be applied which provide for (a) an automatic protection to the consumer should a deterioration occur by tightened inspection or discontinuance of inspection, (b) an incentive to reduce inspection costs should consistently good quality be achieved.

ii) These plans may also be used for lots in isolation but in this case the OC curves should be consulted to find a plan to yield the desired protection. Sample sizes are designated by code letters for particular lot size and the prescribed inspection levels. Three types of plans—single, double and multiple are available.

3. Large batches of screws are subject to a single sampling plan with $n=60, c=2$. If the process average $\bar{p} = 0.01$, does this lot accept batches of high quality with $(p \leq \bar{p})$ high probability? If the proportion of defectives in a batch is 0.05, what is the chance of accepting the batch?

Ans:-

Since the batch size is large, n is large, then provided we restrict values of p , we can express the OC curve in terms of

$$L(p) = \sum_{d=0}^{2} e^{-60p} \frac{(60p)^d}{d!}$$

$$= e^{-60p} \left[1 + 60p + \frac{(60p)^2}{2!} \right] \quad (*)$$

Substituting $p = \bar{p} = 0.01, L(\bar{p}) = 0.977$

thus, the plan accepts batches of high quality ($p \leq \bar{p}$) with high probability (≥ 0.977)

Now, substituting $p = 0.05$ in (*) gives

$$L(0.05) = 0.423.$$

Thus if a batch contains 5% defective screws, the chance of it being accepted is only 0.423.

5. The lifetime of electric bulbs in a large batch is $N(600, 250)$ (in hours) when the manufacturing process is operating under the specified norms (under control). The retailer considers the bulbs defective if the mean life-time is less than 500 hours.
- Calculate the percentage of defective bulbs in a large batch produced when the manufacturing process is under control.
 - A random sample of size 20 is taken. Find the value of acceptability constant which ensures that such a batch of bulbs (with mean life-time 600 hours) could be accepted with probability 0.95.
 - For the value of n and k in (b), calculate the chance of accepting a bulb containing 5% defective bulbs.

Ans:- (a) $P = \Phi\left(\frac{L-\mu}{\sigma}\right) = \Phi\left(\frac{500-600}{50}\right) = \Phi(-2)$

Thus, 2.28% of the bulbs are defective when the process is under control.

(b) When $L = 500$, $P = 0.0228$, $Z_p = -2$, then

$$\begin{aligned} L(p) &= \Phi(-\sqrt{n}(k + Z_p)) = P(\text{batch is accepted} | p) \\ &= \Phi(-\sqrt{20}(k-2)) \\ &= 0.95 \\ \therefore &= \Phi(1.645) \end{aligned}$$

Hence $k = 1.632$

(c) When $p = 0.05$,

$$\begin{aligned} 0.05 &= \Phi(Z_p) = \Phi(-1.645) \\ \therefore Z_p &= -1.645. \end{aligned}$$

$$\begin{aligned} \text{So, } L(p) &= \Phi(-\sqrt{20}(1.632 - 1.645)) \\ &= \Phi(-0.0581) \\ &= 0.4769 \end{aligned}$$

$\therefore p = 0.477$

Hence, the chance of accepting a batch with 5% defective bulbs is 0.477.

SPC (Statistical Process Control)

Calculation for Control Limits

- Notations:

UCL — Upper Control Limit
 LCL — Lower Control Limit
 CL — Central Line
 n — sample size

PCR — Process Capability Ratio = $\frac{USL-LSL}{6\sigma}$,
 σ — Process standard deviation

\bar{x} — Average of Measurements
 $\bar{\bar{x}}$ — Average of Averages
 R — Range
 \bar{R} — Average of Ranges

USL — Upper Specification Limit
 LSL — Lower Specification Limit

- Variables Data (\bar{x} and R and S charts):

\bar{x} control chart:

$$\begin{aligned}
 \text{(standard) } UCL &= \bar{\bar{x}} + A_2 \bar{R} & \text{given) } UCL = \mu_0 + A\sigma_0 \\
 \text{(not given) } CL &= \bar{\bar{x}} & CL = \mu_0 \\
 \text{(not given) } LCL &= \bar{\bar{x}} - A_2 \bar{R} & LCL = \mu_0 - A\sigma_0
 \end{aligned}
 , \text{ where } A_2 = \frac{3}{d_2 \sqrt{n}}, \text{ where } A = \frac{3}{\sqrt{n}}$$

R control chart:

$$\begin{aligned}
 \text{(standard) } UCL &= \bar{R} D_4 & UCL = D_2 \sigma_0 \\
 \text{(not given) } CL &= \bar{R} & CL = \sigma_0 \\
 \text{(not given) } LCL &= \bar{R} D_3 & LCL = D_1 \sigma_0 \\
 \text{where, } D_3 &= \left(1 - \frac{3d_3}{d_2}\right), D_4 = \left(1 + \frac{3d_3}{d_2}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{(standard given) } UCL &= D_2 \sigma_0 \\
 \text{(standard given) } CL &= \sigma_0 \\
 \text{(standard given) } LCL &= D_1 \sigma_0 \\
 \text{where, } D_1 &= d_2 - 3d_3, D_2 = d_2 + 3d_3
 \end{aligned}$$

S control chart:

$$\begin{aligned}
 \text{(standard not given) } UCL &= B_4 \bar{s} & \text{(standard given) } UCL = B_6 \sigma_0 \\
 \text{(standard not given) } CL &= \bar{s} & CL = C_4 \sigma_0 \\
 \text{(standard not given) } LCL &= B_3 \bar{s} & LCL = B_5 \sigma_0 \\
 \text{where, } B_3 &= 1 - \frac{3}{c_4} \sqrt{1 - c_4^2} & \text{where, } B_5 = c_4 - 3 \sqrt{1 - c_4^2} \\
 B_4 &= 1 + \frac{3}{c_4} \sqrt{1 - c_4^2} & B_6 = c_4 + 3 \sqrt{1 + c_4^2}
 \end{aligned}$$

- Attribute Data (p, np, c and u control chart):

	p (fraction)	np (no. of defectives)	c (count of defectives)	u (average no. of defects per unit)
CL	\bar{p}	$n\bar{p}$	\bar{c}	\bar{u}
UCL	$\bar{p} + 3 \sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$	$n\bar{p} + 3 \sqrt{n\bar{p}(1-\bar{p})}$	$\bar{c} + 3\sqrt{\bar{c}}$	$\bar{u} + 3\sqrt{\frac{\bar{u}}{n}}$
LCL	$\bar{p} - 3 \sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$	$n\bar{p} - 3 \sqrt{n\bar{p}(1-\bar{p})}$	$\bar{c} - 3\sqrt{\bar{c}}$	$\bar{u} - 3\sqrt{\frac{\bar{u}}{n}}$
Notes	If n varies, use \bar{n} or individual n_i	n must be a constant	n must be a constant	If n varies, use \bar{n} or individual n_i

Appendix G

Table of Control Chart Constants

n	d_2	d_3	C_4	\bar{X} and R Charts			\bar{X} and S Charts		
				A_2	D_3	D_4	A_3	B_3	B_4
2	1.128	0.8525	0.7979	1.880	—	3.267	2.659	—	3.267
3	1.693	0.8884	0.8862	1.023	—	2.574	1.954	—	2.568
4	2.059	0.8798	0.9213	0.729	—	2.282	1.628	—	2.266
5	2.326	0.8798	0.9400	0.577	—	2.114	1.427	—	2.089
6	2.534	0.8480	0.9515	0.483	—	2.004	1.287	0.030	1.970
7	2.704	0.8332	0.9594	0.419	0.076	1.924	1.182	0.118	1.882
8	2.847	0.8198	0.9650	0.373	0.136	1.864	1.099	0.185	1.815
9	2.970	0.8078	0.9693	0.337	0.184	1.816	1.032	0.239	1.761
10	3.078	0.7971	0.9727	0.308	0.223	1.777	0.975	0.284	1.716
11	3.173	0.7873	0.9754	0.285	0.256	1.744	0.927	0.321	1.679
12	3.258	0.7785	0.9776	0.266	0.283	1.717	0.886	0.354	1.646
13	3.336	0.7704	0.9794	0.249	0.307	1.693	0.850	0.382	1.618
14	3.407	0.7630	0.9810	0.235	0.328	1.672	0.817	0.406	1.594
15	3.472	0.7562	0.9823	0.223	0.347	1.653	0.789	0.428	1.572
16	3.532	0.7499	0.9835	0.212	0.363	1.637	0.763	0.448	1.552
17	3.588	0.7441	0.9845	0.203	0.378	1.662	0.739	0.466	1.534
18	3.640	0.7386	0.9854	0.194	0.391	1.607	0.718	0.482	1.518
19	3.689	0.7335	0.9862	0.187	0.403	1.597	0.698	0.497	1.503
20	3.735	0.7287	0.9869	0.180	0.415	1.585	0.680	0.510	1.490
21	3.778	0.7272	0.9876	0.173	0.425	1.575	0.663	0.523	1.477
22	3.819	0.7199	0.9882	0.167	0.434	1.566	0.647	0.534	1.466
23	3.858	0.1759	0.9887	0.162	0.443	1.557	0.633	0.545	1.455
24	3.895	0.7121	0.9892	0.157	0.451	1.548	0.619	0.555	1.445
25	3.931	0.7084	0.9896	0.153	0.459	1.541	0.606	0.565	1.435

Copyright ASTM International, used by permission.