# PRACTICE PROBLEMS ON MATHEMATICS BASED ON CURRENT TIFR GS MATHEMATICS PAPER

## TRUE-FALSE TYPE QUESTIONS

**Topics :**

Theory Of Equations

Number Theory

Linear Algebra

Abstract Algebra

Real Analysis

Topology

Set , Combinatorics & Probability

Differential Equations

## TRUE/FALSE TYPE: —

1. T/F :- The equation $\cos(e^x) = 2^x + 2^{-x}$ has no real roots.

Sol. False

For all $x$, $-1 \le \cos(e^x) \le 1$.

But $\forall x$, $2^x + 2^{-x} > 1$.

So, for every $x$, $\cos(e^x) \ne 2^x + 2^{-x}$.

2. T/F :- The polynomial $x^6 - 27x^4 - 13x^3 + 11x^2 + 29x$ has exactly one real root.

Sol. False

The given polynomial is of degree 6. So, it may have 6 roots at most. Since complex roots come in pairs, so it can't have exactly one real root.

3. T/F :- Equation $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$ has 4 roots in $\mathbb{Z}_7$.

Sol. False $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$

The given equation is not satisfied by any values of above other than 1, since $7 \equiv 0 \pmod 7$.

4. T/F :- Each root of the equation $y^4 - 22y^3 + 13y^2 - 230y + 61 = 0$ is greater than each root of the equation $x^4 - 6x^3 - 38x^2 - 3x + 17 = 0$ by 4.

Sol. True Here for this equation $x^4 - 6x^3 - 38x^2 - 3x + 17 = 0$, let $\alpha, \beta, \gamma, \delta$ be four roots.

$$\text{So,} \quad \alpha + \beta + \gamma + \delta = \frac{-(-6)}{1} = 6.$$

Since it is given that in equation $y^4 - 22y^3 + 13y^2 - 230y + 61 = 0$ that each root is greater by p, so roots will be

$$\alpha + p, \beta + p, \gamma + p, \delta + p.$$

$$\therefore \quad (\alpha + \beta + \gamma + \delta) + 4p = \frac{-(-22)}{1} = 22.$$

$$\Rightarrow \boxed{p = 4}$$

5. T/F :- The sum of square of roots of $x^2+(2-\lambda)x+1=\lambda$ is at least 1.

Sol. TRUE Given equation is $x^2+(2-\lambda)x+(1-\lambda)=0$

Let $\alpha,\beta$ be the roots of the above equation then

$$\alpha+\beta = -\frac{(2-\lambda)}{1}$$

$$\alpha\beta = \frac{1-\lambda}{1}$$

$$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \lambda^2-2\lambda+2$$

For least value, $f(\lambda) = \lambda^2-2\lambda+2$

$$= (\lambda-1)^2+1 \geq 1.$$

6. T/F :- $\alpha,\beta$ be the roots of $x^2+x+1=0$ then $\alpha^{2012}+\beta^{2012}=0$.

Sol. FALSE

$\alpha,\beta$ be the two roots.

Here, $\alpha = \frac{-1+\sqrt{3}i}{2}$, $\beta = \frac{-1-\sqrt{3}i}{2}$.

Changing it into polar form, we get,

$$r\cos\theta = -\tfrac{1}{2}$$

$$r\sin\theta = \sqrt{3}/2$$

$$\therefore r^2(\sin^2\theta+\cos^2\theta) = 1$$

$$\Rightarrow r = 1.$$

$$\therefore \alpha = 1, \beta = 1.$$

$$\therefore \alpha^{2012}+\beta^{2012} = 2.$$

7. T/F :- The roots of $x^2-2x\cos n\theta+1=0$ is the $n^{th}$ power of the roots of $x^2-2x\cos\theta+1=0$.

Sol. TRUE $x^2-2x\cos\theta+1=0$

$$\text{Then } x = \frac{-(-2\cos\theta)\pm\sqrt{(-2\cos\theta)^2-4}}{2\times1}$$

$$= \cos\theta \pm i\sin\theta$$

So, $\alpha = \cos\theta + i\sin\theta$, $\beta = \cos\theta - i\sin\theta$

$$\therefore \alpha^n+\beta^n = (\cos\theta+i\sin\theta)^n + (\cos\theta-i\sin\theta)^n$$

$$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$= 2\cos n\theta.$$

Similarly, $\alpha^n\beta^n = \cos^2 n\theta + \sin^2 n\theta = 1$.

$\therefore$ Equation is :- $x^2-(\alpha^n+\beta^n)x+\alpha^n\beta^n=0$

$$\Rightarrow x^2 - 2\cos n\theta . x + 1 = 0.$$

8. **T/F:-** If an unknown polynomial is divided by $(x-1)$ and $(x-2)$ we obtain the remainder 2 and 1, respectively. Then the resulting remainder from the division of that unknown polynomial by $(x-1)(x-2)$ is 2.

**Sol.** False

Suppose $P(x)$ leaves remainder 2 when divided by $(x-1)$.

i.e. $P(1)=2$. Similarly $P(2)=1$.

Then when $P(x)$ is divided by $(x-1)(x-2)$, then

$P(x) = \{q(x)\}(x-1)(x-2) + r(x)$; where $q(x)$ is quotient, $r(x)$ is remainder.

i.e. $r(x) = ax+b$.

$P(x) = (x-1)(x-2)q(x) + (ax+b)$

$\Rightarrow P(1) = 2 \qquad P(2) = 1$

$\Rightarrow a+b=2 \qquad \Rightarrow 2a+b=1$

$\therefore a=-1,\ b=3.$

So, the remainder term is $(3-x)$.

9. **T/F:-** The equation $5x^3-11x^2+12x-2=0$ has no integral root.

**Sol.** TRUE

Constant term $=-2$

Divisors of constant term are $\pm 1, \pm 2$.

So, possible values of integral roots are $\pm 1, \pm 2$.

Now, $f(1) = 5-11+12-2 \neq 0$

$f(-1) \neq 0$

$f(2) \neq 0$

$f(-2) \neq 0$

Hence, there is no integral root of the equation.

10. **T/F:-** If $P(x) = ax^2+bx+c$ and $Q(x) = -ax^2+bx+c$ where $ac \neq 0$ then $P(x)Q(x)$ has at least two real roots.

**Sol.** TRUE

$P(x)Q(x) = (ax^2+bx+c)(-ax^2+bx+c)$

Now, $D_1 = b^2-4ac, \quad D_2 = b^2+4ac$

So, $D_1+D_2 = 2b^2 \geq 0.$

$\therefore$ At least one of $D_1$ and $D_2$ is positive, then it has at least two real roots.

11. T/F :- Equation $x^4+ax^3+bx^2+cx+d=0$ have two pairs of equal root if $c^2=a^2d$.

Sol. TRUE Let the equation has two pairs of equal root.

$\alpha, \alpha, \beta, \beta$.

$\therefore \alpha+\alpha+\beta+\beta=-a$ and $\alpha^2\beta^2=(-1)^4 \cdot d$

$\therefore \alpha+\beta=-\frac{a}{2}$. $\therefore \alpha^2\beta^2=d$.

and $\alpha^2(\beta+\beta)+\beta^2(\alpha+\alpha)=-c$

$\Rightarrow 2\alpha\beta(\alpha+\beta)=-c$.

$\Rightarrow 4\alpha^2\beta^2(\alpha+\beta)^2=c^2$

$\Rightarrow 4d\left(-\frac{a}{2}\right)^2=c^2$

$\Rightarrow a^2d=c^2$.

12. T/F :- If $a<b<c<d$ then the roots of the equation $(x-a)(x-c)+2(x-b)(x-d)=0$ are real and distinct.

Sol. TRUE

Let $f(x)=(x-a)(x-c)+2(x-b)(x-d)$

$f(a)>0$

$f(b)<0$

$f(c)<0$

$f(d)>0$

$\therefore \exists$ two real and distinct roots one in the interval $(a,b)$ and other in $(c,d)$.

13. T/F :- If $a,b,c$ are the roots of $x^3+px^2+qx+r$ then $\sum\frac{b^2+c^2}{bc}=0$.

Sol. FALSE Here $a+b+c=-p$

$ab+bc+ca=q$

$abc=-r$

$$\sum\frac{b^2+c^2}{bc}=\frac{(a+b+c)(ab+bc+ca)-3abc}{abc}$$

$$=\frac{-pq+3r}{-r}$$

$$=\frac{pq}{r}-3.$$

In general, if the polynomial is of order $n$, then

$$\sum\frac{b^2+c^2}{bc}=\frac{pq}{r}-n.$$

14. T/F:- If $P(x)$ be a non-constant polynomial $\ni P(n) = P(-n)\ \forall n \in \mathbb{N}$.
Then $P'(0) = 0$.

Sol. TRUE $P(n) = P(-n)$

$\Rightarrow P(x)$ is an even function of $x$.

Then $P(x)$ have all the even powers of $x$.

$\Rightarrow P'(x)$ does not have constant term.

$\Rightarrow P'(0) = 0$.

15. T/F:- If the sum of roots of $2x^3 - 3x^2 + Kx - 1$ is 1, then $\frac{1}{2}, \frac{1}{2}(1+\sqrt{3}i), \frac{1}{2}(1-\sqrt{3}i)$ are the roots of the equation.

Sol. FALSE. sum of the roots $= 1\frac{1}{2} \neq 1$.

16. T/F:- $x2^x = 1$ has at least one positive root not exceeding 1.

Sol. TRUE

$f(x) = x2^x - 1$.

$f(-1) = -3/2$

$f(0) = -1 < 0$

$f(1) = 1 > 0$

So, $\exists$ at least one positive root in $(0,1)$.

17. T/F:- If $c, d$ are the roots of $(x-a)(x-b) - K = 0$ then $a, b$ are the roots of $(x+c)(x+d) + K = 0$.

Sol. FALSE $x^2 - (a+b)x + ab - K = 0$

$\Rightarrow c + d = a + b$.

Now, considering the equation $x^2 + (c+d)x + K = 0$

Let $\alpha, \beta$ be two roots of the equation.

$\therefore \alpha + \beta = -(c+d)$

$= -(a+b)$

So, roots of the second equation will be $-a$ and $-b$.

18. T/F:- For all positive integer m, the roots of $m^2x^2+2mx+1=0$ are positive integers.

Sol. FALSE

$$(mx+1)^2=0$$
$$\Rightarrow x=-\frac{1}{m}.$$

As the roots are positive, so, m is negative.

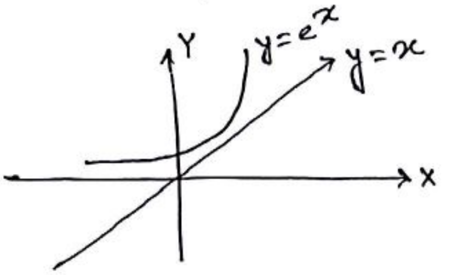
19. If a, b, c are the roots of the equation $x^3-px^2+qx-r=0$ then $(a+b)(b+c)(c+a)=pqr$. → T/F.

Sol. FALSE

$$a+b+c=\frac{-(-p)}{1}=p.$$
$$ab+bc+ca=q$$
$$abc=r$$
$$\therefore (a+b)(b+c)(c+a)=(p-a)(p-b)(p-c)$$
$$=p^3-p^2(a+b+c)+p(ab+bc+ca)-abc$$
$$=pq-r.$$

20. T/F:- Equation $x=e^x$ has at least one real solution.

Sol. FALSE




$x=\log x$ has no solution since there is no intersecting points.

21. T/F:- There do not exist any non-negative integer solution of the equation $y^4+6xy^2-8x=0$.

Sol. FALSE

$$x=\frac{y^4}{8-6y^2}\geq 0$$
$$\therefore 6y^2<8$$
$$\Rightarrow y^2<\frac{4}{3}$$

As y is non-negative, so $y=0$ or 1.

For $y=0$, $x=0$

for $y=1$, $x=\frac{1}{2}\notin I$, set of all integers.

So, only one non-negative integer solution exists.

22. T/F :- Equation $\frac{1}{x-\alpha^2} + \frac{1}{x-\beta^2} + \frac{1}{x-\gamma^2} = x$ can't have complex root.

Sol. TRUE

Let $a+ib$, $a-ib$ are two roots of the equation, then

$$\frac{1}{(a-\alpha^2)+ib} + \frac{1}{(a-\beta)^2+ib} + \frac{1}{(a-\gamma^2)+ib} = a+ib \quad \cdots\cdots (1)$$

$$\frac{1}{(a-\alpha^2)-ib} + \frac{1}{(a-\beta^2)-ib} + \frac{1}{(a-\gamma^2)-ib} = a-ib \quad \cdots\cdots (2)$$

Add (1) from (2), we get $2a$.

23. T/F :- $x=-1$ be the repeated root of $x^3+3x^2+3x+1=0$

Sol. TRUE

$$f(x) = (x+1)^3$$
$$f'(x) = 3(x+1)^2 = 0$$
$$\therefore x = -1, -1.$$

24. T/F :- The sum of the roots of the equation $\begin{vmatrix} 1-x & 2 & 3 & 4 \\ 1 & 2-x & 3 & 4 \\ 1 & 2 & 3-x & 4 \\ 1 & 2 & 3 & 4-x \end{vmatrix} = 0$ is zero.

Sol. FALSE Sum of the roots $= (-1)^4 \cdot \frac{1+2+3+4}{(-1)(-1)(-1)(-1)}$

$$= 10.$$

25. T/F :- Let $P(X) = a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n}$ be a polynomial in a real variable $x$ with $0 < a_0 < a_1 < a_2 < \cdots < a_n$. Then $P(X)$ is bounded.

Sol. FALSE

$$P(X) = a_0 + a_1X^2 + a_2X^4 + \cdots + a_nX^{2n}$$
$$P'(X) = 2a_1X + 4a_2X^3 + \cdots + 2a_n \cdot nX^{2n-1}$$
$$P''(X) = 2a_1 + 12a_2X^2 + \cdots + 2n(2n-1)a_n \cdot X^{2n-2}$$

For maximum or minimum of $P(X)$, we put

$$P'(X) = 0 \Rightarrow X = 0.$$

$\therefore P''(0) = 2a_1 > 0$

$\therefore$ minimum point of $P(X)$.

$\therefore x=0$ is the minimum point of $P(X)$.

So, $P(X)$ is not bounded above.

TRUE/FALSE TYPE:-

26. T/F The equation $77x+65y+20z=1001$ has an integral solution.

Sol$^n$: $(x,y,z)\equiv(13,0,0)$ is an integer solution of the equation. there may be more integral solution.

The answer is True.

27. T/F $(n^5-n)$ is divisible by 30.

Sol$^n$: True 5 is a prime number

$$n^5-n=n(n^{5-1}-1)$$
$$a^{p-1}\equiv 0 \pmod p$$
$$\therefore n^{5-1}-1\equiv 0 \pmod 5$$

$\therefore n^5-n$ is divisible by 5.

$$n^5-n = n(n-1)(n+1)(n^2+1)$$

Here, $(n^5-n)$ involve product of 3 consecutive integers

$$(n-1),\ n,\ (n+1)$$

Hence, it is divisible by $3!$, i.e. 6

Hence, $(n^5-5)$ is divisible by 30.

28. T/F The product of 2n consecutive negative integers is divided by $(2n)!$

Sol$^n$: True Product of 2n consecutive negative integers

$$=(-r)(-r-1)(-r-2)\cdots\cdots(-r-2n+1)$$
$$=(-1)^{2n}r(r+1)(r+2)\cdots\cdots(r+2n-1)$$
$$=(-1)^{2n}\frac{(r+2n-1)!}{(r-1)!}$$
$$=(-1)^{2n}\frac{(r+2n-1)!\,(2n)!}{(r-1)!\,(2n)!}$$
$$= {}^{r+2n-1}C_{2n}\cdot(2n)!$$

This is divisible by $(2n)!$

29. T/F The product of $r$ consecutive integers is divisible by $(r+1)!$

Sol$^n$: False The product of $r$ consecutive integers are

$$= n(n-1)(n-2)\cdots(n-r+1)$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \times r!$$

$$= {}^nC_r \times r!$$

$$= {}^nP_r \text{ is necessarily an integer.}$$

Therefore, the product is divisible by $r!$, not by $(r+1)!$

30. T/F If $p$ is a real number such that $0<p<1$ and $x, y$ are real numbers such that $x<y$ then $p^x < p^y$.

Sol$^n$: False $0<p<1$

$$x<y \Rightarrow p^x > p^y.$$

31. T/F Equation $2x \equiv 3 \pmod{20}$ has infinitely many solutions.

Sol$^n$: False $(2x-3)$ is divisible by 20.

which is not possible for any value of $x$. As 3 does not divide 20. $\therefore$ it has no solutions

32. T/F $n^{21} \equiv n \pmod{30}$ for every integer $n$.

Sol$^n$: True Let $p$ be a prime, then

$$n^{k(p-1)} \equiv n \pmod p \text{ for any integer } k \text{ \& } n.$$

By Fermat's little theorem, $n^{p-1} \equiv 1 \pmod p$

$$\text{So, } n^{k(p-1)+1} = \left(n^{p-1}\right)^k \cdot n \equiv n \pmod p$$

Let us take $p = 2, 3, 5$

$$n^{20(2-1)+1} \equiv n \pmod 2$$

$$n^{10(3-1)+1} \equiv n \pmod 3$$

$$n^{5(5-1)+1} \equiv n \pmod 5$$

In other words, $(n^{21}-n)$ is divisible by 2, 3, 5 and since these numbers are pairwise relatively prime. $(n^{21}-n)$ is divisible by their product $2\times3\times5 = 30$,

i.e., $n^{21} \equiv n \pmod{30}$.

33. T/F $\frac{n^2!}{(n!)^n}$ is an integer, $n \in N$.

Sol$^n$: True Here, $n^2$ objects are distributed in $n$ groups, each group containing $n$ identical objects.

$\therefore$ Number of arrangements

$$= {}^{n^2}C_n \cdot {}^{n^2-n}C_n \cdot {}^{n^2-2n}C_n \cdot {}^{n^2-3n}C_n \cdot \ldots\ldots {}^{n}C_n$$

$$= \frac{(n^2)!}{n!\,(n^2-n)!} \cdot \frac{(n^2-n)!}{n!\,(n^2-2n)!} \cdot \ldots\ldots \frac{n!}{n!\,1}$$

$$= \frac{(n^2)!}{(n!)^n} = \text{integer (as number of arrangement has to be integer)}$$

34. T/F If $a, b > 0$, $0 < p < 1$ then $(a+b)^p < a^p + b^p$

Sol$^n$: True Let $f(x) = (1+x)^p - 1 - x^p$, $x > 0$.

$$f'(x) = p\left\{(1+x)^{p-1} - x^{p-1}\right\}$$

Now, $1 + x > x$

$\Rightarrow (1+x)^{p-1} < x^{p-1}$ as $0 < p < 1 \Rightarrow p-1 < 0$

$\Rightarrow (1+x)^{p-1} - x^{p-1} < 0$.

$\therefore f'(x) < 0$

$\therefore f(x)$ is a decreasing function.

Now, $(1+x)^p - (1+x^p) < 0 \Rightarrow (1+x)^p < 1 + x^p$.

Putting, $x = a/b$.

$\Rightarrow a^p + b^p > (a+b)^p$.

35. T/F When $8^{103}$ is diveded by 103, the remainder is 8.

Sol$^n$: True Fermat's theorem states that if $p$ is a prime and $a$ is any integer, then $a^p \equiv a \pmod p$

So, here $8^{103} \equiv 8 \pmod{103}$

as 103 is a prime number.

36. T/F $9^7 + 7^9$ is divisible by 64.

Sol$^n$: True $9^7 + 7^9 = (1+8)^7 - (1-8)^9$

$= (1 + {}^7C_1 \cdot 8 + \ldots + 8^7) - (1 - {}^9C_1 \cdot 8 + \ldots - 8^9)$

$= 8({}^7C_1 + {}^9C_1) + 8^2(\ldots\ldots)$

$= 8 \times 16 + \ldots\ldots \mid 64$

37. T/F $2^{60}$ does not leave remainder when divided by 7.

Sol$^n$: False $2^{60} = (2^3)^{20} = (1+7)^{20} = 1 + {}^{20}C_1 \cdot 7 + {}^{20}C_2 \cdot 7^2 + \dots + 7^{20}$

$\therefore$ the remainder is 1.

38. T/F $\exists$ more than 250 integral solution of $x_1 x_2 x_3 x_4 = 770$.

Sol$^n$: True $770 = 2 \cdot 5 \cdot 7 \cdot 11$

2 can be assigned in 4 ways among $x_1, x_2, x_3, x_4$.
Sameas 5 " " " " 4 " " "

& so on.

$\therefore$ The no. of positive integral solution of $x_1 x_2 x_3 x_4 = 770$ is $= 4^4 = 256$.

39. $(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{4})(1+\frac{1}{5}) \dots$ tends to $\frac{1}{2}$. T/F

Sol$^n$: True $S = \frac{1}{2} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{5}{6} \dots$
$= \frac{1}{2}$

40. T/F If $n$ is relatively prime to 72, then $n^{12} \equiv 1 \pmod{72}$

Sol$^n$:- True [Euler's Theorem:

If $n$ be a positive integer and $a$ is prime to $n$, then
$$a^{\phi(m)} \equiv 1 \pmod{m}]$$

$72 = 8 \times 9$ ; 8, 9 are relatively prime to $n$.

$m = 8$ $\qquad$ $m = 9$

$\phi(8) = 8(1 - \frac{1}{2}) = 4$ $\qquad$ $\phi(9) = 9(1 - \frac{1}{3}) = 6$.

$\therefore n^4 \equiv 1 \pmod{8}$ $\qquad$ $n^6 \equiv 1 \pmod{9}$

$\Rightarrow n^{12} \equiv 1 \pmod{8}$ $\qquad$ $\Rightarrow n^{12} \equiv 1 \pmod{9}$

Since 8 & 9 are relatively prime,
we have $n^{12} \equiv 1 \pmod{72}$

[Note: If $x \equiv y \pmod{m_1}$ and $x \equiv y \pmod{m_2}$ and $m_1$ and $m_2$ are relatively prime then $x \equiv y \pmod{m_1 m_2}$]

41. T/F If $x = \sqrt[3]{9}$ , $y = \sqrt[4]{11}$ , $z = \sqrt[6]{17}$ then $x > y > z$.

Solⁿ: True $x = 9^{1/3} = 9^{4/12} = (9^4)^{1/12} = (6489)^{1/12}$

$y = (11)^{1/4} = 11^{3/12} = (11^3)^{1/12} = (1331)^{1/12}$

$z = (17)^{1/6} = (17)^{2/12} = (289)^{1/12}$.

$\therefore \quad x > y > z$

42. T/F $n^4 + 4$ is composite for $n > 1$.

Solⁿ: True $n^4 + 4 = (n^2+2)^2 - 4n^2 = (n^2+2-2n)(n^2+2+2n)$

So, $n^4+4$ is a composite number.

43. T/F If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$ $\forall x \in (1, \infty)$

Sol. False $\forall x > 1$, $(x-1) > 0$.

Now, $\log_{0.3}(x-1) < \log_{0.09}(x-1) = \log_{(0.3)^2}(x-1)$

$\Rightarrow \log_{0.3}(x-1) < \frac{1}{2}\log_{0.3}(x-1)$

$\Rightarrow (x-1)^2 > (x-1)$

$\Rightarrow (x-1)(x-2) > 0$

$\Rightarrow x > 1, x > 2$.

$\therefore x \in (2, \infty)$.

44. T/F:- $x^7 + x + 2 \equiv 0 \pmod 7$ does not have integer solutions.

Sol. By Fermat's little theorem, we have

FALSE

$6^7 \equiv 6 \pmod 7$

$6^7 - 6 \equiv 0 \pmod 7$

$6^7 - 6 + 14 \equiv 0 \pmod 7$

$6^7 + 8 \equiv 0 \pmod 7$

$6^7 + 6 + 2 \equiv 0 \pmod 7$

$\Rightarrow$ 6 is a root of the given congruence.

45. T/F:- If $p$ be a prime and $a$ is prime to $p$ then
$a^{p^2-p} \equiv 1 \pmod{p^2}$.

Sol. TRUE $a^{p-1} \equiv 1 \pmod p$

$\therefore a^{p-1} = 1 + tp$ for some integer $t$.

$a^{p^2-p} = (1+tp)^2 = 1 + kp^2$, where $k$ is an integer.

Consequently, $a^{p^2-p} \equiv 1 \pmod{p^2}$.

# LINEAR ALGEBRA



## TRUE / FALSE TYPE:-

46. T/F:- A is a 3x3 matrix of rank 2 then the system of equation $A\underset{\sim}{x} = \underset{\sim}{b}$ has unique solution.

Sol. FALSE

Note:- 1. A be a matrix of order $m \times n$, then $A_{m\times n} \underset{\sim}{x}_{n\times 1} = \underset{\sim}{0}_{m\times 1}$ will have a non-trivial solution iff Rank(A) < n.

2. Rank(A) = r (< n) then $A\underset{\sim}{x} = \underset{\sim}{0}$ has exactly (n−r) independent non-trivial solution.

Now, A system of linear equations have unique solution iff
$$R(A|b) = R(A) = 3.$$
But here R(A) = 2, so the system is inconsistent.
Hence, it does not have any solution.

47. T/F:- A is 3x3 matrix of rank 2 then the system of equation $A\underset{\sim}{x} = \underset{\sim}{b}$ has unique solution.

Sol. FALSE

A system of linear equations have unique solution iff
$$\text{rank}(A|b) = \text{rank}(A) = 3$$
But here rank(A) = 2, so the system is inconsistent.
Hence, it does not have any solution.

48. T/F:- A matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with real entries if $\det(A) \geq 0$ then it has repeated eigen values.

Solⁿ: TRUE

$$|A - \lambda I| = 0$$
$$\Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$|A| \geq 0$$
$$\Rightarrow (ad - bc) \geq 0$$

$$\Rightarrow (a-\lambda)(d-\lambda) - bc = 0$$
$$\Rightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$
This equation gives equal roots if $(a+d)^2 = 4.1.(ad-bc)$

49. T/F:- A subset V of $\mathbb{R}^3$ consisting of vectors $(x_1, x_2, x_3)$ satisfying $x_1^2+x_2^2+x_3^2=1$ is a subspace of $\mathbb{R}^3$.

Sol. FALSE

Let $(x_1, x_2, x_3)$, $(y_1, y_2, y_3) \in \mathbb{R}^3$ and

$x_1^2+x_2^2+x_3^2=1$ and $y_1^2+y_2^2+y_3^2=1$

$\alpha, \beta \in \mathbb{R}$

So, $\alpha(x_1, x_2, x_3)+\beta(y_1, y_2, y_3) = (\alpha x_1+\beta y_1, \alpha x_2+\beta y_2, \alpha x_3+\beta y_3)$

Now, $(\alpha x_1+\beta y_1)^2+(\alpha x_2+\beta y_2)^2+(\alpha x_3+\beta y_3)^2$

$= \alpha^2+\beta^2+2\alpha\beta(x_1y_1+x_2y_2+x_3y_3)$

$\neq 1$

So, V is not a vector subspace.

50. T/F:- Let A be a 2×2 matrix with complex entries. The number of 2×2 matrices A with complex entries satisfies the equation $A^3=-I$ is infinite.

Sol. TRUE $A^3-I=0$

$\Rightarrow (A-I)(A^2+A+I)=0$

There may be infinite choices.

51. T/F:- If A be an n×n matrices such that $P^{-1}AP>0$, where P is an invertible matrix of order n×n. Then eigen values of A are positive.

Sol. True $P^{-1}AP>0$

$\Rightarrow$ $P^{-1}AP$ is a diagonal matrix whose diagonal entries are eign values of A and are positive.

52. T/F:- Let A be an n×n mtx. over real numbers such that $AB=BA$ $\forall$ n×n mtx. B. Then A only can be an identify mtx.

Sol$^n$: False $AB=BA$

In general mtx. multiplication is not commutative except for identify and zero matrix.

So, A, B can be either identify or zero mtx.

53. T/F $S = \{(1, i, 0), (2i, 1, 1), (0, 1+i, 1-i)\}$ forms a basis for complex vector space.

Sol. True

$$\begin{bmatrix} |A| \neq 0 \Rightarrow \text{LIN} \\ |A| = 0 \Rightarrow \text{LD} \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & i & 0 \\ 2i & 1 & 1 \\ 0 & 1+i & 1-i \end{vmatrix} = -2i - 2i(i - i^2) = -2i - 2i^2 + 2i^3 = 2 \neq 0$$

$\therefore$ These vectors are Linearly independent and generate the complex vector space.

So, $S$ forms a basis.

54. T/F If $\omega$ be a non-real cube root of unity. Then eigenvalues of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1/\omega & 1/\omega^2 \\ 1 & 1/\omega^2 & 1/\omega^4 \end{bmatrix}$$ are real.

Sol. True $\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix}$ is the inverse of $\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1/\omega & 1/\omega^2 \\ 1 & 1/\omega^2 & 1/\omega^4 \end{bmatrix}$

$\therefore$ $3B(M)B^{-1}$ = given expression $= 3M = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ The diagonal mtx. $M$ have the eigen values $(3, -3, 0$

55. T/F If $U$ and $W$ are subspace of vector space $V$ if $\dim(V) = 12$, $\dim(U) = 6$, $\dim(W) = 8$. Then $\dim(U \cup W) \geq 8$ and $\dim(V \cap W) \leq 6$.

Sol. False Given, $\dim(V) = 12$, $\dim(U) = 6$ and $\dim(W) = 8$

Now $\dim(U \cup W) \geq 8$ as $\dim(W) \geq \dim(U)$

$\dim(V \cap W) \leq 8$ as $\dim(W) \leq \dim(V)$

56. T/F For each value of $K$ the set $\{(1-K, K, 1-K), (0, 2-3K, 2), (1-K, -1, 0)\}$ form a basis of $\mathbb{R}^3$.

Sol. False $\begin{vmatrix} 1-K & K & 1-K \\ 0 & 2-3K & 2 \\ 1-K & -1 & 0 \end{vmatrix} = 3K^3 - 10K^2 + 11K - 4 = (K-1)(3K^2 - 7K + 4) \neq 0$, as it forme a basis of $\mathbb{R}^3$

$\Rightarrow$ For $K = 1$, it does not form a basis.

57. T/F If $A$ is an $n\times n$ matrix and if $\text{rank}(A) < n$ then $A\underset{\sim}{x} = B$ has exactly $r$ solutions.

Sol. False $A\underset{\sim}{x} = \underset{\sim}{b}$

Reduce the argumented matrix $[A : \underset{\sim}{b}]$ and $A$ to echelon form by elementary matrix operations to find rank $[A]$ & rank$[A|b]$

i) If we have $\text{rank}(A) \neq \text{rank}(A|b)$, then the system of equations is inconsistent, no solution.

ii) If we have $\text{rank}(A) = \text{rank}(A|b)$ = the no. of variables The system is consistent and may have an unique solution.

iii) $\text{rank}(A) = \text{rank}(A|b) <$ no. of variables then the system has infinite no. of solutions.

58. T/F $A^{-1} = A^2 - 2A - I$ for $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Sol. True

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 2 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -\lambda^3 + 2\lambda^2 + \lambda + 1 = 0$$

C-H theorems says that every square matrix satisfies its own characteristic equation.

$$\therefore -A^3 + 2A^2 + A + I = 0$$
$$\Rightarrow A^3 - 2A^2 - A = I$$
$$\Rightarrow A[A^2 - 2A - I] = I$$
$$\Rightarrow A^{-1} = A^2 - 2A - I$$

59. T/F The dimension of vector space $V = \{(a, b, c, d) : b - 2c + d = 0\}$ is two.

Sol. False $b - 2c + d = 0 \Rightarrow c = \dfrac{b+d}{2}$

$a, b, d$ are free variables.

$$\underset{\sim}{x} = \left(a, b, \frac{b+d}{2}, d\right)$$
$$= a(1, 0, 0, 0) + b(0, 1, 1/2, 0) + d(0, 0, 1/2, 1)$$

$\boxed{\dim(V) = 3}$ there are $\{(1,0,0,0), (0,0,1/2,0), (0,0,1/2,1)\}$ total 3 LIN vectors which generate the vector space.

60. T/F Let a, b, c, d and e be non-zero, then the Matrix $M$ is non-singular for all distinct a, b, c, d and e, where $M = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}$

Sol False

$$|M| = ace - bcd$$

To be non-singular, $|M| \neq 0 \Rightarrow ace \neq bcd$

$$\Rightarrow ae \neq bd, \ c \neq 0$$

61. T/F If $U = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then rank of $U^2$ is 2.

Sol True

$$U^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$U^2$ is also an echelon matrix. rank of $U^2 = 2$.

62. T/F If $A = [a_{ij}]$ be any square matrix and $a_{ij} = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } i > j \\ -1, & \text{if } i < j \end{cases}$

then $\det(A) = 1$

Sol False $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = 0$

Alt. A is a skew symmetric matrix and we know the value of the determinant of any odd order skew-symmetric matrix is zero.

So, $|A| = 0$

63. T/F Let $P$ be a non-singular matrix and $1+P+P^2+\dots+P^n=0$ then $P^{-1}=P^n$

Sol. True $1+P+P^2+\dots+P^n=0$

$$\Rightarrow 1\cdot\frac{(P^{n+1}-I)}{(P-I)}=0 \Rightarrow P^{n+1}=I \quad \text{i.e. } P^n=P^{-1}.$$

64. T/F For any real $k$ the system of linear equations

$$2x-5ky+6z=0$$
$$kx+2y-2z=0$$
$$2x+2y-kz=0$$

is not consistent.

Sol. False $A\underset{\sim}{x}=\underset{\sim}{0}$ is said to be a homogeneous equation

$\Rightarrow \underset{\sim}{x}=0$ is a trivial solution.

$\Rightarrow$ A homogeneous system of linear equations is necessarily consistent.

65. T/F If $A=\underset{\sim}{x}\underset{\sim}{y}^T$ and $\underset{\sim}{x}$ and $\underset{\sim}{y}$ be non-zero column of order $5\times1$ and $7\times1$. Then $\operatorname{rank}(A)\le 7$.

Sol. False $\therefore$ A is of order $5\times7$.

$\therefore \operatorname{rank}(A)\le 5$

66. $V=\{p(x)\,|\,p(-1)=0\}$ be the vector space on real number field.

Sol. True Let $f,g\in V$

then $f(-1)=0$, $g(-1)=0$

We have, $(af+bg)(-1)=af(-1)+bg(-1)=0$

$\Rightarrow af+bg\in V$

$\Rightarrow V$ be a vector space.

67. T/F $\begin{bmatrix}2&1\\4&2\end{bmatrix}$ is diagonalisable.

Sol True Since ch. equation of the matrix is given by

$$\begin{vmatrix}2-\lambda&1\\4&2-\lambda\end{vmatrix}=0 \Rightarrow \lambda^2-4\lambda=0 \Rightarrow \lambda=0,4.$$

Since both eigen values are different so we get linearly independent eigen vectors of it so it is diagonalisable.

68. T/F If A is skew-symmetric as well as symmetric then A could not be null mtx.

Sol False $a_{ij} = a_{ji}$ and $a_{ij} = -a_{ji}$

and $a_{ii} = 0$ $\therefore a_{ij} = 0 \ \forall \ i \neq j$

$\therefore$ A must be a null mtx.

69. T/F Eigen values of $(A^4 + 3A - 2I)$ can't be determined if Eigen values of A are given.

Sol False If eigen values of A are 1, 2, 3.

then eigen values of $A^4 + 3A - 2I$ will be 2, 20, 88.

By C-H Theorem, " Every mtx. satisfies its own characteristic equation".

70. T/F the sum of eigen values of A and B are the sum of eigenvalues of A+B.

Sol True $A = \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix}$, $\lambda_1 = 1$, $\lambda_2 = 3$

$B = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$, $\lambda_3 = -2$, $\lambda_4 = 1$

$\therefore \sum_1^4 \lambda_i = 3$

$A+B = \begin{bmatrix} 2 & 8 \\ 0 & 1 \end{bmatrix}$ $\therefore \lambda_5 = 2$, $\lambda_6 = 1$

$\therefore \sum_{i=5}^{6} \lambda_i = 3$, i.e. they are equal.

71. T/F There is no vector space of dimension 1.

Soln False Since $\left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} : a \in \mathbb{R} \right\}$ is a vector space of dimension 1.

72. T/F Let $T_1$ and $T_2$ be linear transformations on $\mathbb{R}^3$ defined as

$T_1(x_1, x_2) = (x_2, x_1)$

and $T_2(x_1, x_2) = (x_1, 0)$

then $T_1T_2 = T_2T_1$.

Sol False $T_1T_2(x_1, x_2) = T_1(x_1, 0) = (0, x_1)$

$T_2T_1(x_1, x_2) = T_2(x_2, x_1) = (x_2, 0)$

$\therefore T_1T_2 \neq T_2T_1$

73. T/F A linear mapping on $R^2$ defined by

$T(x, y) = (a_1x + b_1y, a_2x + b_2y)$ s.t. $a_1b_2 - a_2b_1 = 0$

then T is invertible.

Sol False. T is invertible iff

$$T = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$ is non-singular.

i.e. $a_1b_2 - b_1a_2 \neq 0$

but we have $a_1b_2 - a_2b_1 = 0$, So T is not invertible.

74. T/F $\lambda\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ be the eigen vector corresponding to the smallest eigen-value of the matrix $\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$

Sol True $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & 0 \\ 5 & 4-\lambda & 0 \\ 3 & 6 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(4-\lambda)(1-\lambda) = 0$$

i.e. $\lambda = 1, 3, 4$.

Taking $\lambda = 1$ $\therefore [A - I]$

$$= \begin{bmatrix} 3-1 & 0 & 0 \\ 5 & 4-1 & 0 \\ 3 & 6 & 1-1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 3 & 0 \\ 3 & 6 & 0 \end{bmatrix} = B$$

$\therefore$ rank$(B) = 2 < 3$

$\therefore 3 - 2 = 1$ non-zero solution.

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 5 & 3 & 0 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2x_1 = 0 \\ 5x_1 + 3x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{cases}$$

$\Rightarrow x_1 = 0, x_2 = 0$

$\therefore$ The eigen vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ;

where $x_3 = \lambda$ (let)

75. T/F If $A$ is nilpotent then $I+A$ is non singular.

Sol True E.g. $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ , $I+A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$\therefore |I+A| = 2 \neq 0$

i.e. $I+A$ is non- singular.

76. T/F A nilpotent mtx is singular.

Sol True $A^m = 0$ , where $A$ is a nilpotent mtx of order $m$.

$\therefore \det(A^m) = [\det A]^m = 0$

$\Rightarrow \det(A) = 0 \Rightarrow A$ is singular.

77. T/F If $A = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ then the roots of the equation $\det(A - xI) = 0$ are $-\lambda_i \ \forall 1 \leq i \leq n$

Sol False If $A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

Since $A$ is diagonal mtx.

$$\therefore |A - xI| = \begin{vmatrix} \lambda_1 - x & 0 & \cdots & 0 \\ 0 & \lambda_2 - x & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n - x \end{vmatrix} = 0$$

$\Rightarrow (\lambda_1 - x)(\lambda_2 - x)\cdots(\lambda_n - x) = 0 \Rightarrow x = \lambda_1, \lambda_2, \dots, \lambda_n$.

78. T/F $V = \{(a,b,c,d) : a = d, b = 2c\}$ is a vector space of dimension 3.

Sol False Let $c = 0, d = 1$ and $c = 1, d = 0$

so, $(0,2,1,0), (1,0,0,1)$ form a basis of $V_2$.

$\therefore \dim(V) = 2$

79. <u>T/F</u> If T be linear transformation defined as

$$T(v_1, v_2, v_3) = (v_2, v_3, v_1) \text{ then } T^{100}(v) = T(v)$$

<u>Sol</u> <u>True</u> $T(v_1, v_2, v_3) = (v_2, v_3, v_1)$

$$T^2(v_1, v_2, v_3) = T(v_2, v_3, v_1) = (v_3, v_1, v_2)$$

$$T^3(v_1, v_2, v_3) = T(v_3, v_1, v_2) = (v_1, v_2, v_3)$$

$$T^3 = T$$

$$\therefore T^{100} = (T^{33})^3 \cdot T = T$$

$$\Rightarrow T^{100}(v) = T(v)$$

80. <u>T/F</u> $\{(0,1,\alpha), (\alpha,1,0), (1,\alpha,1)\}$ be basis of $\mathbb{R}^3$ for every real $\alpha$.

<u>Sol</u> <u>False</u> $\begin{vmatrix} 1 & \alpha & 1 \\ 1 & 1 & \alpha \\ \alpha & 0 & 1 \end{vmatrix} \neq 0 \Rightarrow \alpha(\alpha-2) \neq 0 \Rightarrow \alpha \neq 0, \alpha \neq 2.$

81. <u>T/F</u> Matrix $M = \begin{bmatrix} -1 & 1+i & 2i & 9 \\ 1-i & 3 & 4 & 7-i \\ -2i & 4 & 5 & i \\ 9 & 7+i & -i & 7 \end{bmatrix}$ have no real eigen value.

<u>Sol</u> <u>False</u> Since $(\bar{M}^T) = M$

$\Rightarrow$ M is Hermition matrix.

and eigen value(s) of Hermiton matrix are real.

82. <u>T/F</u> Let $T: R^2 \to R^3$ be a linear transformation s.t,

$$T\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

then $T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

<u>Sol</u> <u>False</u> $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = a\begin{bmatrix} -1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

i.e. $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -a+2b \\ 2a-b \end{bmatrix}$

i.e. $-a+2b = 2$

$2a-b = 1$

$a = 4/3, \quad b = 5/3$

$$\therefore T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{4}{3}T\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{5}{3}T\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \frac{4}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \frac{5}{3}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

83. T/F $W = \{P(x) \in V : P(a) = P(1-a) : a \in \mathbb{R}\}$ is a vector sub-space of vector space of polynomial with real coefficients.

Sol True $W = \{P(x) \in V : P(a) = P(1-a) ; a \in \mathbb{R}\}$

Let, $P_1(x), P_2(x) \in W$ $\quad P_1(a) = P_1(1-a); P_2(-a) = P_2(1-a)$

for some constant $\alpha, \beta \in \mathbb{R}$

$\alpha P_1(x) + \beta P_2(x) \in W$

$\Rightarrow \alpha P_1(a) + \beta P_2(a) = \alpha P_1(1-a) + \beta P_2(1-a)$

$\Rightarrow W$ be a vector subspace of $V$.

84. T/F $\{(i, i, 0), (2i, 1, 1), (0, 1+i, 1-i)\}$ forms a basis for $V_3(C)$

Sol True

$$\begin{vmatrix} i & 2i & 0 \\ i & 1 & 1+i \\ 0 & 1 & 1-i \end{vmatrix} = 4-2i \neq 0.$$

$\therefore$ The vectors are LIN

So, they form a basis for $V_3(C)$.

85. T/F Dimension of vector space $\{(x_1, x_2, x_3, x_4) : 3x_1 - x_2 \cdot x_3 = 0\}$ is 3.

Sol True $\quad 3x_1 = x_2 + x_3 \Rightarrow x_1 = \frac{x_2 + x_3}{3}$

$\therefore \underset{\sim}{x} \in S$

$\Rightarrow \underset{\sim}{x} = \left(\frac{x_2 + x_3}{3}, x_2, x_3, x_4\right)$

$= x_1 (0,0,0,0) + x_2 (1/3, 1, 0, 0) + x_3 (1/3, 0, 1, 0) + x_4 (0, 0, 0, 1)$

So, $\{(1/3, 1, 0, 0), (1/3, 0, 1, 0), (0, 0, 0, 1)\}$ forms a basis for $S$.

$\therefore \dim(V) = 3$.

86. T/F Let $T : V \to V$ be a linear map such that the rank$(T) = 1$ then $T^2 = aT$.

Sol True Since, rank$(T) = 1$, $T(V)$ is one dimensional, $T(V) = \{w\}$, say

Let $T(w) = aw$. Then for any $v \in V$, let

Then $T(v) = bw$

$T^2(v) = bT(w) = b\,aw = aT(v)$.

$\therefore T^2 = aT$.

87. T/F If $B$ is idempotent matrix and $A = I - B$, then

$AB = 0 = BA$

Sol True $B^2 = B$

$A^2 = (I-B)^2 = (I-B)(I-B) = I - B - B + B^2 = I - 2B + B^2$

$= I - B$

$= A$

$\therefore$ $A$ is idempotent

$\therefore AB = (I-B)B = IB - B^2 = B - B^2 = B^2 - B^2 = 0$

Similarly, $BA = 0$

88. T/F All the eigen values of the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ lies in the disc $|\lambda - 1| \leq 1$.

Sol$^n$ False $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1+\lambda) \left[ -(1-\lambda)^2 + 4 \right] = 0$$

$\therefore \lambda = -1$, $-\lambda^2 + 2\lambda + 3 = 0$

$$\therefore \lambda = \frac{-2 \pm \sqrt{4+12}}{-2}$$

$$= -3, 1$$

$\therefore \lambda = -1, 1, -3$

But here, $|\lambda - 1| \leq 1$

$\Rightarrow -1 \leq \lambda - 1 \leq 1$

$\Rightarrow 0 \leq \lambda < 2$

But $\lambda = 1, -1, -3$ do not lie in the disc.

89. <u>T/F</u> Rank of linear transformation $T : R^3 \to R^3$ s.t,

$T(x, y, z) = (y, 0, z)$ is two

<u>Sol</u>. True $T(1,0,0) = (0,0,0)$ [ y, z are free variables, so Rank (T) = 2 ]

$T(0,1,0) = (1,0,0)$

$T(0,0,1) = (0,0,1)$

i.e. $\{(1,0,0), (0,0,1)\}$ is the basis of range of T.

i.e. dim (Range of T) = 2

i.e. Rank(T) = 2.

90. <u>T/F</u> Let V be vector space of all functions from R into R & $W = \{ f \mid f(3) = 1 + f(-4) \}$ be the subspace of V.

<u>Sol</u> False $f, g \in W$. The $f(3) = 1 + f(-4)$, $g(3) = 1 + g(-4)$

$$(af+bg) f(3) = a[1+f(-4)] + b[1+g(-4)]$$
$$= a + b + (af+bg)(-4)$$
$$\neq 1 + (af+bg)(-4)$$

Hence, W is not a subspace of V.

91. <u>T/F</u> Let $T : R^2 \to R^2$ be a map defined by $T(x, y) = (x+y, x-y)$, then rank$(T) = 2$

<u>Sol</u>$^n$: True The corresponding matrix to T is

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \therefore \text{rank}(T) = \text{rank}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = 2$$

92. <u>T/F</u> If $V = \{ (x, y, z, w) : x+y+z-3w = 0, x-y+z-w = 0, x-7y+z+5w = 0 \}$. Then V is a subspace of $R^4$ with dimension 2.

<u>Sol</u>$^n$: True Given that, $\{(1,1,1,-3), (1,-1,1,-1), (1,-7,1,5)\}$ is a subspace of the vector space $R^4$, then we will find the basis of $R^4$, let $a, b, c \in \mathbb{R}$

$\Rightarrow a(1,1,1,-3) + b(1,-1,1,-1) + c(1,-7,1,5) = (0,0,0,0)$

$a+b+c = 0$ — identical (i)
$\Rightarrow a-b-7c = 0$ — (ii)
$a+b+c = 0$ — identical (i)
$-3a-b+5c = 0$ — (iii)

Solving (ii) & (iii) $\Rightarrow a = 3c$

i.e. $a = 0$, $c = 0$

is the only sol$^n$.

i.e. $\{(1,1,1,-3), (1,-7,1,5)\}$ form an LI. set and span the vector space $\mathbb{R}^4$.

i.e. These two vectors form a basis of $V$.

$\therefore \dim(V) = 2$.

(T/F) 93. $\{\alpha, \beta, \gamma\}$ is basis of $V(F)$ then $\{\alpha+\beta, \alpha-\beta, \alpha-2\beta+\gamma\}$ is also a basis for $V(R)$

Solⁿ: True Let $a, b, c$ be scalars such that

$$a(\alpha+\beta) + b(\alpha-\beta) + c(\alpha-2\beta+\gamma) = 0$$

i.e. $(a+b+c)\alpha + (a-b-2c)\beta + c\gamma = 0$

But $\alpha, \beta, \gamma$ are LIN then $\begin{aligned} a+b+c &= 0 \\ a-b-2c &= 0 \\ c &= 0 \end{aligned}$

The only solution of these equations are

$a = b = c = 0$

So, hence $\{\alpha+\beta, \alpha-\beta, \alpha-2\beta+\gamma\}$ are LIN.

So, they forms a basis for $V(R)$.

94. T/F $W = \{(x,y,z) : x,y,z \in \mathbb{Q}\}$ is a subspace of $V_3(R)$.

Solⁿ: False Let $\alpha = (1/2, 3/5, 7)$

& $a = \sqrt{3}$, a real no.

then $a\alpha = \sqrt{3}(1/2, 3/5, 7) \notin W$

$\sqrt{3}/2, 3\sqrt{3}/5, 7\sqrt{3}$ are not rational nos.

Hence, $W$ is not closed under scalar multiplication

Consequently, it does not form a vector space of $V_3(\mathbb{R})$

95. T/F:- If A is an $n\times n$ matrix, with each entries 1, then $\lambda = 1$ be its one of the eigen value.

Sol. FALSE Taking $n=3$, so $|A - \lambda I| = 0$ gives

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow 3\lambda^2 - \lambda^3 = 0 \Rightarrow \lambda = 0, 0, 3.$$

So, 0 and $n$ be the eigenvalues of $A_{n\times n}$.

# ABSTRACT ALGEBRA



## TRUE/FALSE TYPE :-

96. T/F:- Set $S = \{-2, -1, 1, 2\}$ forms a group w.r.t. multiplication.

Sol. FALSE

$(-2)\cdot(2) = -4 \notin S$.
So, closure property is not satisfied.
Hence, $S$ is not a group.

97. T/F:- A polynomial in $\mathbb{Z}_2[X]$ has a factor $(x+1)$ when it has an even number of non-zero coefficients.

Sol. TRUE

Let $P(X) = a_0 + \cdots + a_n x^n \in \mathbb{Z}_2[X]$.
By the factor theorem, $(x+1)$ is a factor of $P(X)$ iff $P(1) = 0$.
(Remember that $x-1 = x+1$ in $\mathbb{Z}_2[X]$).
Now, $P(1) = a_0 + a_1 + \cdots + a_n$, which is zero in $\mathbb{Z}_2$ iff $P(X)$ has an even number of non-zero coefficients.

98. T/F:- Any group of prime order can have proper subgroups.

Sol. FALSE

Theorem:- Any group of prime order can have no proper subgroups.

Proof:- Let $O(G) = p$, where $p$ be a prime.
Let $H$ be a subgroup of $G$ and let $O(H) = m$.
$\therefore$ By Lagrange's theorem, we know that $O(H) \mid O(G)$.
$\therefore$ $m$ is a divisor of $p$.

So, either $m = 1$, or $m = p$.
Now, $m = 1 \Rightarrow O(H) = 1 \Rightarrow H = \{e\}$
and $m = p \Rightarrow O(H) = O(G)$.
$\Rightarrow H = G$.

Either $H = \{e\}$ or $H = G$,
i.e. $H$ is not a proper subgroup of $G$.

99. <u>T/F</u>:- $P(X) = X^4 - 3X^2 + 2X + 1$ is irreducible in $Q[X]$.

<u>Sol.</u> TRUE

$P(X)$ can't be factored into quadratics.
The only possible rational roots of $P(X)$ are $\pm 1$. However, there are no roots, so $P(X)$ has no linear factor.
Therefore, if it has factor, it must be factored into two quadratics.
By Gauss's lemma, these factors can be chosen to have integral coefficients.
But, here it is not possible.
So, $P(X)$ is irreducible in $Q[X]$.

100. <u>T/F</u>:- Let $G$ be a finite group of order $n$ and let $a \in H$ then $a^n = e$ where, $H$ is a subgroup of $G$.

<u>Sol.</u> TRUE

By Lagrange's theorem, the order of each subgroup of a finite group is a divisor of the order of the group.

$\Rightarrow$ if $o(H) = m$ then $m | n$.
for each, $a \in H$, $a^m = e$.

$\Rightarrow a^n = e$.

101. <u>T/F</u>:- Let $a$ be the generator of group $G$, $o(G) = 35$, then $o(a^{15})$ is 3.

<u>Sol.</u> FALSE

$$o(G) = 35$$
$$\Rightarrow a^{35} = e$$
$$\Rightarrow a^{15} = a^{5 \times 3}$$
$$\Rightarrow (a^{15})^7 = (a^{5\times 3})^7$$
$$= (a^{5 \times 7})^3$$
$$= e^3$$
$$= e.$$
$$\Rightarrow o(a^{15}) = 7.$$

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101. T/F:- Suppose that $a$ is an element of the group $G$ of order $n$, Suppose that $a^K = e$ and that $K < n$. Then $K$ divides $n$.

Solution:- FALSE

Suppose that $a$ is an element of order 5 in a group of order 20 (for, example, the number 4 in $\mathbb{Z}_{20}$). Then $a^{15} = e$, and $15 < 20$ but 15 does not divide 20.

102. T/F:- If $H$ and $K$ are subgroups of a group $G$, then $H \cap K$ is a subgroup as well.

Solution:- True

We need to show that $H \cap K$ contains the identity, is closed under group operation and that each element in $H \cap K$ has its inverse also in $H \cap K$.

Since $e$ belongs to each $H$ and $K$, then $e$ belongs to $H \cap K$.

Suppose that $x, y \in H \cap K$, Then both $x$ and $y$ belong to $H$ and both belong to $K$ as well. Since they are both in $H$ and $H$ is closed, $xy$ is in $H$, and similarly, since they are both in $K$, $xy$ is in $K$. Thus, $xy \in H \cap K$ and so $H \cap K$ is closed.

Now, suppose that $x$ is any element of $H \cap K$. Then since $x$ is in $H$ and $H$ is a subgroup, $x^{-1} \in H$. Similarly, $x^{-1} \in K$ and so $x^{-1} \in H \cap K$.

103. T/F:- Every cyclic group is abelian.

Solution:- True

Suppose that $G$ is a cyclic group that is generated by the element $g$. Let $x$ and $y$ be arbitrary elements of $G$, We must show that

$$xy = yx.$$

Since $G$ is generated by $g$, there must exist integers $r$ and $s$ such that

$$x = g^r, \quad y = g^s, \text{ But then } xy = g^{r+s} = g^s \cdot g^r = yx.$$

104. **T/F:-** Note that $\phi(5) = \phi(8) = \phi(10) = \phi(12) = 4$. Then the groups $U(5)$, $U(8)$, $U(10)$, $U(12)$ are isomorphic to one another.

**Solution:-** FALSE

Since, $U(8)$ and $U(12)$ are both isomorphic to the Klein 4-group (and hence to each other), while $U(5)$ and $U(10)$ are both cyclic.

105. **T/F:-** Let $G$ be a group and $\mathbb{Z} = \{a \in G : ag = ga \ \forall \ g \in G\}$. Then $\mathbb{Z}$ is a subgroup of $G$.

**Solution:-** TRUE We need to show that $\mathbb{Z}$ contains the identity, is closed under the group operation and that each element in $\mathbb{Z}$ has its inverse also in $\mathbb{Z}$.

Since, $ex = xe = x$ for any $x$ in $G$, $e$ satisfies the condition to belong to $\mathbb{Z}$.

If $a$ and $b$ belong to $\mathbb{Z}$, then $(ab)x = a(bx) = a(xb) = (ax)b$
$= (xa)b = x(ab)$ for any $x$ in $G$

and so $ab$ satisfies the condition to belong to $\mathbb{Z}$. Finally if $a$ is in $\mathbb{Z}$, then $a^{-1}x = (x^{-1}a)^{-1} = (ax^{-1})^{-1} = xa^{-1}$ and so $a^{-1}$ satisfies the condition to belong to $\mathbb{Z}$.

Note that we have used the property $(xy)^{-1} = y^{-1}x^{-1}$ here as well as the fact that $(x^{-1})^{-1} = x$.

**Note:-** the group $\mathbb{Z}$ is called the Center of $G$.

106. **T/F:-** If every proper subgroup of a group $G$ is cyclic, then $G$ must be itself cyclic.

**Solution:-** FALSE

No, the Kline 4-Group is a counter example.

107. T/F:- The groups $\mathbb{Z}_3 \times \mathbb{Z}_2$ and $\mathbb{Z}_6$ are isomorphic.

Solution:- It is enough to show that $\mathbb{Z}_3 \times \mathbb{Z}_2$ is of order 6.
We can use either $(1,1)$ or $(2,1)$ as generators:

TRUE

$$(1,1)$$
$$(1,1)+(1,1)=(2,0)$$
$$(1,1)+(2,0)=(0,1)$$
$$(1,1)+(0,1)=(1,0)$$
$$(1,1)+(1,0)=(2,1)$$
$$(1,1)+(2,1)=(0,0)$$

The second version of the test had $\mathbb{Z}_2 \times \mathbb{Z}_3$.

108. T/F:- In the ring $\mathbb{Z}/7\mathbb{Z}$ the equation $x^4+4 \equiv 0$ has exactly two roots.

Solution:- FALSE $\quad x = \{0,1,2,3,4,5,6\}$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $x^4$ | 0 | 1 | 2 | 4 | 4 | 2 | 1 |
| $x^4+4$ | 4 | 5 | 6 | 1 | 1 | 6 | 6 |

$\Rightarrow$ $x^4+4 \equiv 0$ has no roots in $\mathbb{Z}/7\mathbb{Z}$.
i.e. $x^4+4$ is irreducible in $\mathbb{Z}_7[x]$.

109. T/F:- Let two functions be $f(x)=x^3+x^2+x+1$ and $g(x)=x^3+x+1$ are irreducible over $\mathbb{Z}_2$.

Solution:- $x=\{0,1\}$. $\quad f(x)=x^3+x^2+x+1 \qquad g(x)=x^3+x+1$

TRUE

$f(0)=1 \qquad g(0)=1$

$f(1)=4 \qquad g(1)=3.$

110. T/F:- $S_7$ has a subgroup of order 11.

Solution:- FALSE $\quad |S_7| = 7! = 7 \times 5 \times 3^2 \times 2^4$

If H were a subgroup of $S_7$ of order 11, then 11 divides $|S_7|$.
But in the prime decomposition of $|S_7|$ there is no 11. Hence, there is no subgroup of order 11.

111. T/F :- A homomorphism from a simple group is one-to-one.

Sol$^n$ :- True

Let $G$ be a simple group and $f : G \longrightarrow G'$ is homomorphism so the Kernel $f$ is a normal subgroup of $G$ are $G$ and $\{e\}$.

Either $\ker f = G$ or $\ker f = \{e\}$

$\Rightarrow$ Either $f$ is trivial or $f$ is one-one.

112. T/F :- $S_4$ is the permutation group with four symbols, it has exactly one subgroup of order 12.

Sol$^n$ :- True

Let $H$ be any subgroup of order 12 in $S_4$. Let $H \neq A_4$.

Then $H$ contains an odd permutation.

Thus, $H$ has 6 odd and 6 even permutations.

$\Rightarrow H \cap A_4$ is a subgroup of $A_4$ of order 6.

$\Rightarrow A_4$ has subgroup of group 6.

But, that is not possible by above problem.

Hence, the result.

113. T/F :- If $g \in A_5$ and $g^7 = e$ then $g = e$.

If $g^7 = e$ then either $g = e$ or $g$ is of order 7.

Sol$^n$ :- True If $g^7 = e$ then either $g = e$ or $g$ is of order 7.

Since, $g^m = e$ implies that the order of $g$ divides $m$.

As, $m = 7$ is a prime the only possibility.

if $g = e$ is that the order of $g$ is 7, it would follows that 7 divides $o(A_5)$ which is not true.

Hence, $g = e$.

114. T/F :- $f(x) = x^{p-1} + x^{p-2} + \dots + x + 1$ is reducible in $\mathbb{Q}[x]$ when $p$ is prime.

Sol$^n$ :- True $(x-1)f(x) = x^p - 1$.

let, $(x-1) = y$.

$y f(y+1) = (y+1)^p - 1 = y^p + \binom{p}{1} y^{p-1} + \binom{p}{2} y^{p-2} + \dots + \binom{p}{p-1} y$.

now, $\binom{p}{i}$ is divisible by $p$.

dividing the expression of $y f(y+1)$ by $y$ shows that $f(y+1)$ satisfied the conditions of the Eienstein Criterion, hence that is an irreducible polynomial.
It follows that $f(x)$ is irreducible polynomial.

115. T/F :- $S_3$ is an abelian group of order 6.

Sol$^n$ :- False.

$S_3$ be a group of permutation of order 6.
But, $S_3$ does not allow commutative rule.
$\Rightarrow S_3$ is not abelian.

116. T/F :- A cyclic of order $n$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Sol$^n$ :- True If $G$ & $H$ have finite order $n$.

$G = \{e, g, g^2, \ldots, g^{n-1}\}$ and $H = \{e, h, h^2, \ldots, h^{n-1}\}$

Define, the bijection $f : G \to H$ by $f(g^r) = h^r$ for $r = 0(1)\overline{n-1}$

if $r$ and $s$ lie between $0$ and $\overline{n-1}$.
let $r+s = kn+l$; $0 < l < \overline{n-1}$. Then

$$f(g^r . g^s) = f(g^{r+s}) = f(g^{kn+l}) = f((g^n)^k . g^l)$$
$$= f(e^k . g^l)$$
$$= f(g^l)$$
$$= h^l$$

and $f(g^r) = h^r h^s = h^{r+s} = h^{kn+l} = e^k h^l = h^l$

Hence, $f$ is an isomorphism.
and $\mathbb{Z}/n\mathbb{Z}$ is a cyclic group of order $n$ so it is isomorphic to any cyclic group of order $n$.

117. T/F :- $[12]$ is a inverse of $[49]$ in $\mathbb{Z}$.

Sol$^n$ : False Let $[x] = [49]^{-1}$ in $\mathbb{Z}_{53}$. Then

$[49] . [x] = [1]$ ; that is

$49x \equiv 1 \pmod{53}$.

we can solve this convergence by solving the equation $49x - 1 = 53y$, where $y \in \mathbb{Z}$. By using the euclidean algorithm we have

$$53 = 1 \cdot 49 + 4 \quad \text{and} \quad 49 = 12 \cdot 4 + 1$$

Hence, $GCD(49, 53) = 1 = 49 - 12 \cdot 4 = 49 - 12(53-49)$
$= 13 \cdot 49 - 12 \cdot 53$

Therefore, $13 \cdot 49 \equiv 1 \pmod{53}$

and $[49]^{-1} = [13]$ in $\mathbb{Z}_{53}$.

118. T/F :- If the ring $R = \mathbb{Z} = [\sqrt{n}] = \{a + b\sqrt{n} : a, b \in \mathbb{Z}\}$; a map $f : R \to R$ defined by $f(a+b\sqrt{n}) = a - b\sqrt{n}$, then $f$ is isomorphism.

Solⁿ: True To see that $f$ is a homomorphism,

Note that, $f\big((a+b\sqrt{n}) + (c+d\sqrt{n})\big)$

$$= f\big((a+c) + \sqrt{n}\,(b+d)\big) = (a-b\sqrt{n}) + (c-d\sqrt{n})$$
$$= f(a+b\sqrt{n}) + f(c+d\sqrt{n})$$

and $f\big((a+b\sqrt{n})(c+d\sqrt{n})\big) = f\big((ac+nbd) + \sqrt{n}(bc+ad)\big)$

$$= (ac+nbd) - (b+ad)\sqrt{n}$$
$$= (a-b\sqrt{n})(c-d\sqrt{n})$$
$$= f(a+b\sqrt{n}).\, f(c+d\sqrt{n})$$

Now, Note that, $f\big(f(a+b\sqrt{n})\, f(c+d\sqrt{n})\big) = a + b\sqrt{n}$.

i.e., $f \circ f$ is the identity.

Thus, $f$ is a bijection and therefore is isomorphism.

119. T/F :- Let $G$ be a group of order 30 and $A, B$ be the subgroup of $G$ of order 2 and 5 respectively, then $o\left(\frac{G}{AB}\right) = 3$.

Sol$^n$: True $A$ and $B$ are normal subgroup of $G$ and $o(A) = 2$, $o(B) = 5$.

Since, 2 and 5 are relatively prime so

$o(A \cap B) = 1$.

and $o(AB) = \frac{o(A).o(B)}{o(A \cap B)} = 10$.

Now, $A$ and $B$ are normal subgroups,

So, $AB$ is normal subgroup in $G$.

So, $o\left(\frac{G}{AB}\right) = \frac{o(G)}{o(AB)} = \frac{30}{10} = 3$

120. T/F :- The set of $2\times 2$ symmetric matrices under matrix multiplication and addition is a ring.

Sol$^n$: False Since the product of two symmetric matrices needs not to be symmetic.

e.g. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}$

Thus, symmetric matrix do not form a ring.

121. T/F :- If $H, K$ be subgroups of group $G$, s.t. $O(H) = 10$, $o(K) = 12$, then if $o(H \cap K) \neq 0$ then $O(HK) = 12$.

Sol$^n$ :- False $o(H \cap K) = 2$ as $\gcd(10, 12) = 2$

$\therefore o(HK) = \frac{o(H).o(K)}{o(H \cap K)} = \frac{10\times 12}{2} = 60$

if $H \cap K = \{e\}$ then $o(HK) = o(H).o(K)$.

122. T/F :- There are finite no. of proper subgroups of $(\mathbb{Z}, +)$.

Sol$^n$: False Since. $(\mathbb{Z}, +)$ is itself a infinite order group so it has infinite no. of subgroups. Consequently, it has infinite no. of proper subgroups.

123. T/F :- Every subgroup of an abelian group is normal.

Sol$^n$: True Let $G$ be abelian, $H$ any subgroup of $G$. Then, if $g \in G$ and $h \in H$, $g^{-1}hg = h$; for since $G$ is abelian

so $gh = hg$.

Then as $h \in H$ so $g^{-1}hg \in H$; Thus $H$ is a normal subgroup of $G$.

124. T/F :- The group of roots (real or complex) of the equation $x^n - 1 = 0$, where $n$ is a natural no. forms a cyclic group.

Sol$^n$:- True Root of the equation $x^n - 1 = 0$ given by $n^{th}$ roots of unity for any natural no. $n$ and it forms a cyclic group.

for $n=3$, it is $(\{1, \omega, \omega^2\}, \times)$

for $n=4$, it is $(\{1, -1, i, -i\}, \times)$

and so on for every natural $x$.

125. T/F :- If $a(x) = x^2 + 2x + 3$, $b(x) = 3x^2 + 2x$, $c(x) = 2x + 2$ be three members of the ring $I_4[x]$ over the ring $I_4$ of integer modulo 4, then $\deg[a(x) + b(x)] = 0$, $\deg(c[x]) = 1$.

Sol$^n$:- True Since, $a(x) + b(x) = 4x^2 + 4x + 3$

$= 0.x^2 + 0.x + 3$ over the ring $I_4[x]$ of modulo 4.

$= 3$

$\therefore \deg\left[a(x)+b(x)\right] = 0$

and $c(x) = 2x+2$

$\therefore \deg\left[c(x)\right] = 1$ .

126. T/F :- The entire classes modulo 7 forms an abelian group under multiplication.

Solⁿ :- True Let $A = \{[1], [2], \ldots, [6], [7]\}$

be the set of reside class modulo 7.

Here, A is closed under matrix multiplication.

i.e., $[a][b] = [b]\cdot[a]$

and every element has inverse.

127. T/F :- If p is an odd prime then a group of order 2p can't be cyclic.

Solⁿ : False The group of order 6 is cyclic, where $p=3$

By Cauchy Theorem : If G contains an element 'a' of order p and an element 'b' of order 2, then G consists of the 2p elements. and G is cyclic.

128. T/F :- ∃ only two irreducible polynomials of degree less than or equal to 4 over $\mathbb{Z}_2$.

Solⁿ :- False The irreducible polynomials of degree ≤ 4 over $\mathbb{Z}_2$ are $x$, $x+1$, $x^2+x+1$, $x^3+x+1$, $x^3+x^2+1$, $x^4+x+1$, $x^4+x^3+1$, $x^3+x^4+x^2+x+1$.

129. T/F :- $(\mathbb{Z}_{2p}, +)$, where p is prime is a field.

Solⁿ :- False Since p is prime but 2p is not prime so $\mathbb{Z}_{2p}$ does not make a field with addition and multiplication.

130. T/F :- $f(x) = x^5 - 5$, $g(x) = x^3+1$, $h(x) = x^4+x^2-1$ are irreducible over $\mathbb{Z}_3$.

Solⁿ :- False $\quad \mathbb{Z}_3 = \{0, 1, 2\}$

$$f(x) = x^5 - 5$$
$$f(0) = -5, \quad f(1) = -4, \quad f(2) = 27$$

$$g(x) = x^3+1$$
$$g(0) = 1, \quad g(1) = 2, \quad g(2) = 9$$
$$h(x) = x^4+x^2-1$$
$$h(0) = -1, \quad h(1) = 1, \quad h(2) = 19.$$

131. T/F :- If F be a field containing 11 elements then $\alpha^{10} = 1 \ \forall \ \alpha \in F$.

Solⁿ : True Since, F be a field containing 11 elements and it is a field of order 11.

$\therefore (F, \{0\}, \cdot)$ is abelian group of order 10.

$\Rightarrow \forall \ \alpha \in F, \ \alpha^{10} = 1$.

132. T/F :- If G is a finite group of order n then for any $a \in G$ $\exists$ some positive integers $r$, $1 \leq r \leq n$, s.t. $a^r = e$

Solⁿ : True

133. T/F :- In a ring zero and unit elements are idempotent.

Solⁿ :- True Since if $(R, +, \cdot)$ be a ring usual addition and multiplication, its zero element is 0 and unit element is 1.

$$0+0 = 0^2 = 0$$
$$1 \cdot 1 = 1^2 = 1$$

$\Rightarrow$ 0 and 1 are always idempotent.

134. T/F:- Let $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$ be the group under matrix addition and H be the subgroup of G consisting of matrices with even entries. Then the order of the quotient group G/H is 12.

Solution:- FALSE

$\mathbb{Z}$ : The set of all integers.

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

$$H = \left\{ \begin{pmatrix} e & f \\ g & h \end{pmatrix} : e, f, g, h \in \text{even } \mathbb{Z} \right\}$$

As, $Z/Z_2$ is of order 2, so as 2X2 matrix is of dimension 4.

$$O(G/H) = O\left( \frac{Z \times Z \times Z \times Z}{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2} \right)$$

$$= 2 \times 2 \times 2 \times 2$$

$$= 16.$$

135. T/F:- Let G be a group of order 7 and $\phi(x) = x^4, x \in G$. Then $\phi$ is one-to-one.

Solution:- TRUE $\phi(x) = x^4$ for modulo 7 with addition is

$\Rightarrow$ $\phi(0) = 0$ $\phi(1) = 4$ $\phi(2) = 1$ $\phi(3) = 5$

$\phi(4) = 2$ $\phi(5) = 6$ $\phi(6) = 3$.

So, $\phi$ is one-to-one.

136. T/F:- A ring R has maximal ideals if R is infinite.

Solution:- FALSE

A ring R has maximal ideals if R is finite with at least two elements.

137. T/F:- In a group $\{1, 2, \ldots, 16\}$ under the operation of multiplication modulo 17, the order of the element 3 is 12.

Solution:- FALSE $[3]^4_{\times_{17}} = [13]_{\times_{17}}$

$$[13]^2_{\times_{17}} = [16]_{\times_{17}}$$

$$[16]^2_{\times_{17}} = [1]_{\times_{17}}$$

$$\Rightarrow [3]^{16}_{\times_{17}} = [1]$$

$\Rightarrow$ order of 3 is 16.

138. T/F:- For $n \in \mathbb{N}$, let $n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$. Then the number of units of $\mathbb{Z}/11\mathbb{Z}$ and $\mathbb{Z}/12\mathbb{Z}$ are same.

Solution:- FALSE As $\mathbb{Z}/11\mathbb{Z}$ is a field and 11 is irreducible in $\mathbb{Z}$.
Hence, all non-zero elements of $\mathbb{Z}/11\mathbb{Z}$ are invertible.
i.e. number of units of $\mathbb{Z}/11\mathbb{Z}$ are 10.
On the other hand, $\mathbb{Z}/12\mathbb{Z}$ is not a field, but a commutative ring whose units are given by the number of elements which are relatively prime to 12.
By Euler's formula, $\phi(12) = 12\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) = 4$.

i.e. number of units of $\mathbb{Z}/12\mathbb{Z}$ is 4.

139. The number of elements of $S_5$ (the symmetric group on 5 letters)
T/F:- which are their own inverses equals 11.

Solution:- Total no. of such elements $^5C_2 + 1 = 10 + 1 = 11$.

TRUE

140. T/F:- The number of $2\times2$ matrices over $\mathbb{Z}_3$ (the field with three elements) with determinant 1 is 24.

Solution:- TRUE $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $a, b, c, d \in \{0, 1, 2\}$.

$$|ad - bc| = 1.$$

let $ad = 1$, $bc = 0$ or, $ad = 0$, $bc = 1$

$ad = 1$ either through $a = 1, b = 1$
$a = 2, b = 2$

| | |
|---|---|
| $b=0$, | $c=1$ |
| $b=0$, | $c=0$ |
| $b=1$, | $c=0$ |
| $b=2$, | $c=0$ |
| $b=0$, | $c=1$ |

Total no. of such matrices $= 2\times2\times5 = 20$.

Now, also $ad = 2$, $bc = 1$ or $ad = 1$, $bc = 2$ the $|ad - bc| = 1$

No. of such cases $= 2+2 = 4$. $\therefore$ Total no. of cases $= 24$.

141. T/F:- Let $G$ be a finite group and $H$ be a normal subgroup of $G$ of order 2. Then the order of the centre $G$ is a multiple of 2.

Sol. TRUE As centre of $Z$ is a sub-group of $G \Rightarrow O(Z) \mid O(G)$.

Also element of $H$ belong to $Z \Rightarrow H \subseteq Z$.

i.e. $O(Z) \geq 2$.

Also, $O(H)$ is even, i.e. $O(G)$ is an even integer $\geq 2$.

142. T/F:- Let $G$ be an abelian group of order 10. Let $S = \{g \in G : g^{-1} = g\}$. Then the number of non-identity element in $S$ is 0.

Sol. FALSE $\{[0],[1],[2],[3],[4],[5],[6],[7],[8],[9]\}$ is an abelian group of order 10.

$[0]^{-1} = [0]$ & $[5]^{-1} = [5]$

$[0]$ is identity and $[5]$ is non-identity element.

143. T/F:- Let $R$ be the ring of polynomials over $\mathbb{Z}_2$ and let $I$ be the ideal of $R$ generated by the polynomial $x^3+x+1$. Then the number of elements in the quotient ring $R/I$ is 2.

Sol. FALSE $x^3+x+1$ is irreducible in $\mathbb{Z}_2$.

So, $\frac{R}{I} = 2 \times 2 \times 2 = 8$.

144. T/F:- Consider the ring $R = \{a+ib : a,b \in \mathbb{Z}\}$ with usual addition and multiplication. Then the invertible elements of $R$ is $\pm 1, \pm i$.

Solution. TRUE $R = \{a+ib : a,b \in \mathbb{Z}\}$

For $a+ib$, $c+id$ will be the inverse if $(a+ib)(c+id) = 1+i0$

$\Rightarrow (ac-bd) + i(ad+bc) = 1+i.0$

$\Rightarrow ac-bd = 1, \quad ad+bc = 0$.

So, inverse of 4 elements are $\pm 1, \pm i$ exist only.

TRUE / FALSE TYPE:-

145. T/F If $f: \mathbb{R} \to \mathbb{R}$ be differentiable function and $f(1) = 4$

then $\lim_{x \to 1} \int_4^{f(x)} \frac{2t}{x-1} dt = 8f'(1)$.

Sol True

$$L = \lim_{x \to 1} \left[\frac{t^2}{x-1}\right]_4^y$$

$$= \lim_{x \to 1} \frac{y^2 - 16}{x-1}$$

$$= \lim_{x \to 1} \frac{2y \frac{\partial y}{\partial x}}{1} \quad \text{[By L'Hospital Rule]}$$

$$= \lim_{x \to 1} 2y f'(x) = 2 \times 4 f'(1) = 8f'(1)$$

146. T/F For the function $f(x) = x \cos 1/x$, $x \geq 1$

$f(x+2) - f(x) > 2 \ \forall \ x \in [1, \infty)$

Sol. True

$$f'(x) = \frac{1}{2} \sin 1/x + \cos 1/x \quad , x > 1$$
$$> 1$$

$$\therefore f'(x) = \frac{f(x+2) - f(x)}{2} > 1 \quad , \text{by LMVT}$$

$$\Rightarrow f(x+2) - f(x) > 2 \quad \forall x \in [1, \infty)$$

147. T/F Series $\frac{2}{1^2} - \frac{3}{2^2} + \frac{4}{3^2} - \frac{5}{4^2} + \cdots$ is conditionaley convergent

Soln True $\sum_1 u_n = \frac{2}{1^2} - \frac{3}{2^2} + \cdots$

$$\sum_1 |u_n| = \frac{2}{1^2} + \frac{3}{2^2} + \cdots = \sum_1 \frac{n+1}{n^2}$$
$$= \sum_1 v_n \text{ , say}$$

$$\sum_1 w_n = \sum_1 \frac{1}{n}$$

$$\therefore \lim_{n \to \infty} \frac{v_n}{w_n} = \lim_{n \to \infty} \left[\frac{n+1}{n^2} \times \frac{n}{1}\right] = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1.$$

$\sum_1 w_n$ diverges $\therefore \sum_1 v_n$ diverges

$\because$ Now the series $\sum_1 u_n$ has alternatively positive and negative terms.
Hence, $\sum_1 u_n$ is conditionally convergent.

148. T/F $x^{1/x}$ is decreasing function where $x>e$.

Sol True $f'(x) = x^{1/x}\dfrac{1-\log_e x}{x^2}$

$f(x)$ is decreasing if $f'(x) < 0$

$\Rightarrow \log_e x > 1 \Rightarrow x > e$.

149. T/F $x > \ln(1+x)$, $x>0$

Sol True $f(x) = x - \ln(1+x)$

$\therefore f'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0 \ \forall\ x>0$

$\therefore x > \ln(1+x)$.

150. T/F $\lim\limits_{n\to\infty}\left(\dfrac{1-2+3-4+\cdots 2n}{\sqrt{n^2+1}}\right)$ does not exist.

Sol False $\lim\limits_{n\to\infty}\left(\dfrac{1+3+5+\cdots+(2n-1)}{\sqrt{n^2+1}}\right) - \lim\limits_{n\to\infty}\left(\dfrac{2+4+\cdots+2n}{\sqrt{n^2+1}}\right)$

$= \lim\limits_{n\to\infty}\dfrac{\frac{n}{2}\left[2+(n-1)2\right]}{n(1+1/n)^{1/2}} - \lim\limits_{n\to\infty}\dfrac{\frac{n}{2}\left[4+(n-1)\cdot 2\right]}{n(1+1/n)^{1/2}}$

$= -1$.

151. T/F $f(x)$ be a constant function then it is Riemann integrable.

Sol True $f(x) = c \ \forall\ x\in[a,b]$

then $\int_a^b f(x)dx = c(b-a)$;

Then $f \in R[a,b]$

152. T/F Series $\sum_1^\infty \dfrac{1}{n^{1+1/n}}$ divergent.

Soln: True $\lim\limits_{n\to\infty}\dfrac{u_n}{v_n} = \lim\limits_{n\to\infty}\dfrac{1}{n^{1/n}} = 1$; $u_n = \dfrac{1}{n^{1+1/n}}$

$v_n = \dfrac{1}{n}$

As $\sum_1 v_n$ diverges so does $\sum_1 u_n$

153. T/F $1 - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \cdots$ is convergent.

Solⁿ: True $u_n = \frac{1}{(n+1)^3}(1+2+\cdots+n) = \frac{n}{2(n+1)^2}$

$\therefore u_n > u_{n+1} > 0 \ \forall n$, by Leibnitz test $\sum_1^\infty (-1)^{n-1} u_n$ is convergent

$\therefore \lim u_n = 0$

$\therefore$ The series is convergent.

154. T/F $\sin x + \tan x > 2x \quad (0 < x < \pi/2)$

Solⁿ: True $f(x) = \sin x + \tan x - 2x$

$f'(x) = \cos x + \sec^2 x - 2 = \cos x + \frac{1}{\cos^2 x} - 2$

$f'(x) \geq 0 \quad \forall x \in (0, \pi/2)$

Hence, $\sin x + \tan x > 2x$

155. T/F $f(x) = \begin{cases} \frac{1}{2^n}, & x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \quad n = 0, 1, 2, \ldots \\ 0, & x = 0 \end{cases}$

Then $f(x)$ is Riemann integrable on $[0, 1]$.

Solⁿ: True Here, $f(0) = 0$, $f(1) = 1$

Then the lower bound & upper bound for $f(x)$ in $[0,1]$ are 0 and 1, respectively.

$\therefore$ The function is bounded & monotonically increasing in $[0, 1]$. So, $f(x)$ is Riemann integrable on $[0, 1]$

156. T/F $u_n = \frac{1}{2} + \frac{1}{2.4} + \cdots + \frac{1}{2.4\cdots(2n)}$, then $\lim_{n\to\infty} u_n < 2$

Solⁿ: True $1/2 < 1$

$\frac{1}{2.4} < \frac{1}{2^2}$

$\frac{1}{2.4.6} < \frac{1}{2^3}$

$\vdots \qquad \vdots$

$$u_n < 1 + \frac{1}{2^2} + \frac{1}{2^3} + \cdots < \frac{1-\left(\frac{1}{2}\right)^n}{1-1/2}$$

$$\therefore \lim_{n\to\infty} u_n < \lim_{n\to\infty} 2\left(1-\left(\frac{1}{2}\right)^n\right)$$

$$\therefore \lim_{n\to\infty} u_n < 2$$

157. T/F The series $\sum_{1}^{\infty} \frac{1}{\sqrt{n+1}}$ divergent.

Soln: True $\quad x_n = \frac{1}{\sqrt{n+1}}$, $\quad y_n = \frac{1}{\sqrt{n}}$

$$\therefore \frac{x_n}{y_n} = \frac{1}{\sqrt{1+1/n}}$$

$\therefore \lim \frac{x_n}{y_n} = 1$, by comparison test $\sum x_n$ diverges

158. T/F $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \cdots + (n-1)^2 - n^2 = 1+2+\cdots+n$

Soln: False Case I: When $n$ is even,

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (n-1)^2 - n^2$$

$$= (1-2)(1+2) + (3-4)(3+4) + \cdots + (n-1-n)(n-1+n)$$

$$= -\left(1+2+3+\cdots+(n-1)+n\right) = -\frac{n(n+1)}{2}$$

Case II: When $n$ is odd

$$(1^2-2^2) + (3^2-4^2) + \cdots + \left\{(n-2)^2 - (n-1)^2\right\} + n^2$$

$$= (1-2)(1+2) + (3-4)(3+4) + \cdots + \left\{(n-2)-(n-1)\right\}\left\{(n-2)+(n-1)\right\} + n^2$$

$$= -\frac{(n-1)n}{2} + n^2$$

$$= \frac{n(n+1)}{2}.$$

159. T/F $\lim_{n\to\infty} n \log \frac{\log(n+1)}{\log n} = 0$

Sol$^n$: True

$$L_t = \lim_{n\to\infty} \frac{\log(1+1/n)}{1/n} \quad \lim_{n\to\infty} \frac{\log n}{\log(1+1/n)} \log\left(1+\frac{\log(1+\frac{1}{n})}{\log n}\right)$$

$$\times \lim_{n\to\infty} \frac{1}{\log n}$$

$= 1.1.0$ $\quad$ as $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$

$= 0$

160. T/F If $f$ is bounded and integrable on $[a,b]$ and $k$ is a number such that $|f(x)| \le k \ \forall\, x \in [a,b]$ then $\left|\int_a^b f\, \partial x\right| \le k|b-a|$

Sol$^n$: True $\quad -k \le f(x) \le k$

$$\Rightarrow -k \le m \le f(x) \le M \le k.$$

For $b \ge a$, $\quad -k(b-a) \le m(b-a) \le \int_a^b f\, \partial x$

$$\le M(b-a) \le k(b-a)$$

$$\Rightarrow \left|\int_a^b f\, \partial x\right| \le k(b-a)$$

If $b < a$, then we finally have

$$\left|\int_a^b f\, \partial x\right| \le k|b-a|$$

161. T/F $\frac{1^m + 2^m + \dots + n^m}{n} \le 1 \ \forall\, m > 1, \ n \in \mathbb{N}$

Sol$^n$: False For $m>1$, AM of the $m^{th}$ power $>$ $m^{th}$ power of AM

$$\Rightarrow \frac{1^m + 2^m + \dots + n^m}{n} > \left(\frac{1+2+3+\dots+n}{n}\right)^m$$

$$\Rightarrow \frac{1^m + 2^m + \dots + n^m}{n} > \left(\frac{n+1}{2}\right)^m.$$

162. T/F $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{6n}\right)=\log 6.$

Solⁿ: True $\lim_{n\to\infty}\sum_{r=1}^{5n}\frac{1}{n+r} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{5n}\frac{1}{1+r/n} = \int_0^5\frac{dx}{1+x}$

$= \log 6.$

163. T/F If $f_n(x)=\frac{x^2}{x^2+(1-nx)^2}$, $0\le x\le 1$ ; $n=1,2,3,\ldots$

$\{f_n\}$ converges uniformly on $[0,1]$.

Solⁿ: False $|f_n(x)|\le 1$, So, $\{f_n\}$ is uniformly bounded on $[0,1]$

$\therefore \lim_{n\to\infty} f_n(x)=0.$ $(0\le x\le 1)$

but $f_n(1/n)=1$ $(n=1,2,3,\ldots)$

So, no subsequence can converge uniformly on $[0,1]$

164. T/F $\lim_{x\to 0}\log|x|=\infty$

Solⁿ: False Given, $G>0$, choose $\delta=e^{-G}$

Now, if $0<|x-0|<\delta$

we have $|x|<e^{-G}$, $\log|x|<-G$ consequently

$\lim_{x\to 0}\log|x|=-\infty$

165. T/F Series, $1+2-3+1+2-3+1+2-3+\cdots$ is convergent

Solⁿ: False The given series is

$\sum u_n = 1+2-3+1+2-3+\cdots$

$S_{3n}=(1+2-3)+(1+2-3)+(1+2-3)+\cdots=0$

$\therefore \lim_{n\to\infty}S_{3n}=0$

$S_{3n+1}=(1+2-3)+(1+2-3)+(1+2-3)+\cdots+(1+2-3)+1=1$

$\therefore \lim_{n\to\infty}S_{3n+1}=1.$

$$S_{3n+2} = (1+2-3)+(1+2-3)+\dots+(1+2-3)+(1+2) = 3$$

$$\therefore \lim_{n\to\infty} S_{3n+2} = 3.$$

The limit does not exist because the sum of infinite terms are 0, 1 & 3. Hence the given series is oscillatory.

166. F/T If $I_n = \int_1^e (\ln x)^n \, dx$, then $I_8 + 8I_7 = I_{10} + 10 I_9$

Soln: True Let $\ln x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$.

$$\therefore I_n = \int_0^1 t^n e^t \, dt = t^n e^t \Big|_0^1 - n \int_0^1 t^{n-1} e^t \, dt$$

$$= e - n I_{n-1}$$

$$\Rightarrow I_n + n . I_{n-1} = e$$

$$\therefore I_8 + 8 I_7 = e = I_{10} + 10 I_9.$$

167. F/T $f(x) = |x|$ is continuous but not differentiable at $x = 0$

Soln: True $|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$

$$Rf'(0) = \lim_{h\to 0^+} \frac{(x+h) - x}{h} = 1.$$

$$Lf'(0) = \lim_{h\to 0^-} \frac{-(x+h) - x}{h} = -1.$$

$\therefore |x|$ is continuous but not differentiable at $x=0$

168. T/F If the sequence $U_n$ be such that $|U_{n+1} - U_n| < \epsilon$ $\forall n$ where $0 < \alpha < 1$ then $U_n$ convergent sequence.

Soln: True We know that, $|U_{n+p} - U_n| < \epsilon \; \forall n \geq m$ is a Cauchy sequence for all $p = 1, 2, 3, \dots$

We know Cauchy sequence is always convergent.

Here, $p=1$, so the statement is true.

169. T/F The factor of derivative of the function $\int_{x^2}^{x^2+1} e^{-t^2} dt$ at $x=1$ is $(1-e^2)$.

(False) Sol$^n$: $\frac{\partial}{\partial x} \int_{x^2}^{x^2+1} e^{-t^2} \partial t$

$= e^{-(x^2+1)^2} \frac{\partial}{\partial x}(x^2+1) - e^{-x^4} \frac{\partial}{\partial x}(x^2)$

$= 2xe^{-x^4}\left(e^{-2x^2-1} - 1\right)$

at $x=1$, $\frac{\partial}{\partial x} \int_{x^2}^{x^2+1} e^{-t^2} \partial t = 2e^{-1}\left(e^{-2-1} - 1\right) = 2e^{-4}(1-e^3)$

So, $(1-e^2)$ is not a factor of derivative at $x=1$

170. T/F If $Q(x) = \int_{1/x}^{\sqrt{x}} \sin t^2 \, \partial t$ then $Q'(x)$ at $x=1$ is $\frac{3}{2}\sin 1$.

Sol$^n$: True $Q'(x) = \sin x \frac{\partial}{\partial x}(\sqrt{x}) - \sin \frac{1}{x^2} \frac{\partial}{\partial x}\left(\frac{1}{x}\right)$

$= \sin x \frac{1}{2\sqrt{x}} + \frac{1}{x^2} \sin \frac{1}{x^2}$

$Q'(1) = \sin 1 \cdot \frac{1}{2} + \sin 1. = \frac{3}{2} \sin 1.$

171. T/F A non-decreasing sequence which is not bounded above diverges to infinity.

Sol$^n$: True Suppose $\{s_n\}_{n=1}^{\infty}$ is non-decreasing but not boundend above.

Given, $M > 0$, we must find $m_0 \in I \ni$

$s_n > M \quad (n \geq m_0)$

Now, since $M$ is not an upperbound for $\{s_1, s_2, \ldots\}$ there must exist $m_0 \in I \ni s_{m_0} > M$. Then for this $m_0$, the sequence diverges to infinity.

172. T/F Every sequence has a monotonic subsequence.

Sol$^n$: True

Example: Consider the sequence $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, \ldots\}$

See that $\{u_1, u_2, u_4, u_6, u_8, \ldots\}$ is a monotonic increasing sequence and $\{u_3, u_5, u_7, \ldots\}$ is a monotonic decreasing sequence.

So, Every sequence has a monotonic subsequence.

173. T/F Sequence $\{U_n\}$ converges to 3 if $U_0 = 0$ and $U_n = \sqrt{6+U_{n-1}}$

Soln: True $\quad l^2 = 6 + l \Rightarrow l = 3.$

174. T/F If $\underset{x\to a}{Lt}\{f(x)\,g(x)\}$ exists then both $\underset{x\to a}{Lt} f(x)$ and $\underset{x\to a}{Lt}\, g(x)$ exist.

Soln: False If $f(x) = \dfrac{x-a}{|x-a|}$ and $g(x) = \dfrac{|x-a|}{x-a}$

Then $f(x)\,g(x) = 1$.

$\therefore \underset{x\to a}{Lt}\{f(x)\,g(x)\} = 1.$

But $\underset{x\to a}{Lt} f(x)$ & $\underset{x\to a}{Lt}\, g(x)$ do not exist.

175. T/F The series $\sum_{n=1}^{\infty}\left(\sqrt{n^3+1} - \sqrt{n^3}\right)$ converges.

Soln: True

$$U_n = \frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}, \quad v_n = \frac{1}{n^{3/2}}$$

$$\therefore \underset{n\to\infty}{Lt}\frac{U_n}{v_n} = \underset{n\to\infty}{Lt}\left(\frac{\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}}{1/n^{3/2}}\right)$$

$$= \underset{n\to\infty}{Lt}\frac{1}{\sqrt{1+\frac{1}{n^3}}+\sqrt{1}} = \frac{1}{\sqrt{1}+\sqrt{1}} = \frac{1}{2} > 0$$

By comparison test, as $\sum_1 v_n$ is power series with $p = 3/2 > 1$, there $\sum_1 v_n$ converges, so does $\sum_1 U_n$.

$\therefore \sum_1 U_n$ is also convergent.

176. T/F The series $\dfrac{1}{1^{3r+2}} + \dfrac{1}{2^{3r+2}} + \cdots$ is divergent for $r > 0$.

Soln: False $\sum_{n=1}^{\infty}\dfrac{1}{n^p}$ converges when $p > 1$

$\therefore$ if $(3r+2) > 1$, then the given series is convergent

$\Rightarrow r > -1/3$

$\therefore r > 0 \Rightarrow \sum_{n=1}^{\infty}\dfrac{1}{n^{3r+2}}$ is convergent.

177. T/F $f(x)=\begin{cases} x, & x \text{ is rational} \\ 1-x, & x \text{ is irrational} \end{cases}$

is discontinuous everywhere.

Solⁿ: False For any $x=a$

$$\underset{x\to a}{\mathrm{Lt}}\, f(x) = \underset{x\to a}{\mathrm{Lt}}\, x = a \text{ (when } x\to a \text{ through rational values)}$$

and $$\underset{x\to a}{\mathrm{Lt}}\, f(x) = \underset{x\to a}{\mathrm{Lt}}\, (1-x) = 1-a \text{ (when } x\to a \text{ through irrational values)}$$

Now, $\underset{x\to a}{\mathrm{Lt}}\, f(x)$ exists when $a = 1-a \Rightarrow a = 1/2$

$\therefore$ Hence, $f(x)$ is continuous at $x = 1/2$.

178. T/F If $\sum_{n=1}^{\infty} a_n$ converges to zero, then $\underset{n\to\infty}{\mathrm{Lt}}\, \dfrac{a_n - 1}{n\sqrt{n}} = 0$.

Solⁿ: False $\underset{n\to\infty}{\mathrm{Lt}}\, a_n = 0$.

$$\therefore \underset{n\to\infty}{\mathrm{Lt}}\, \frac{a_n-1}{n\sqrt{n}} = ?$$

$$\underset{n\to\infty}{\mathrm{Lt}}\, a_n = 0 \Rightarrow \underset{n\to\infty}{\mathrm{Lt}}\, (a_n - 1) = -1$$

$$\Rightarrow \underset{n\to\infty}{\mathrm{Lt}}\, \frac{a_n-1}{n^{1/n}} = \underset{n\to\infty}{\mathrm{Lt}}\, \frac{-1}{n^{1/n}} = -1.$$

179. T/F If $A = \{1, 1/2, 1/3, \cdots\}$ then it is itself the set of boundary points.

Solⁿ: False l.u.b $=1$, g.l.b $=0$

$\{0, 1, 1/2, 1/3, \cdots\}$ is the set of boundary points of A.

180. T/F Series $\sum \dfrac{n^{n^2}}{(n+1)^{n^2}}$ is convergent

Solⁿ: True $u_n = \dfrac{n^{n^2}}{(n+1)^{n^2}} > 0 \quad \forall n$

$$\underset{n\to\infty}{\mathrm{Lt}}\, \sqrt[n]{u_n} = \mathrm{Lt}\, \frac{n^n}{(n+1)^n} = \mathrm{Lt}\, \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.$$

∴ By root test the series

$$\sum \frac{n^{n^2}}{(n+1)^{n^2}} \text{ is convergent.}$$

[ Root test: $\sum u_n$ is convergent if $\lim u_n^{1/n} < 1$

$\sum u_n$ " divergent if $\lim u_n^{1/n} > 1$ ]

181.

T/F $y = |x|^3$ is continuous and differentible at $x=0$.

Solⁿ: True $y = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases}$

is continuous.

$$\frac{\partial}{\partial x}(y) = \begin{cases} 3x^2, & x \geq 0 \\ -3x^2, & x < 0 \end{cases}$$

$$\therefore \left.\frac{\partial y}{\partial x}\right|_{x=0} = 0$$

∴ It is differentiable at $x=0$.

182.

T/F If $x = \sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$ then $1 < x < 2$

Solⁿ: True $x = \sqrt{1+x}$

$$x^2 - x - 1 = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore x = \frac{1+\sqrt{5}}{2} \not< 1$$

$\sqrt{5} = 2.4$ ∴ $\frac{1+2.4}{2} = 1.7$.

183.

∴ $1 < x < 2$

T/F The series $\sum \frac{1}{n\sqrt{n^2-1}}$ is convergent.

Solⁿ: True $u_n = \frac{1}{n\sqrt{n^2-1}}$, $v_n = \frac{1}{n^2}$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{1}{n^2}}} = 1.$$

Since $\sum v_n$ converges, so does $\sum u_n$. [By Limit Comparison Test].

184. T/F $\sin x^2$ is unimormly continnous on $\mathbb{R}$.

Solⁿ : False $|x-y| < \delta$

$\Rightarrow |\sin x^2 - \sin y^2| < \epsilon \ \forall \ x, y \in \mathbb{R}$.

$x^2 = 2n\pi$, $y^2 = (4n+1)\pi/2 = 2n\pi + \pi/2$ ; $n \in \mathbb{N}$

$\therefore |x-y| = \dfrac{1}{\sqrt{2n\pi} + \sqrt{2n\pi + \pi/2}} \to 0$ as $n \to \infty$

$\therefore |\sin x^2 - \sin y^2| = 1 \nless \epsilon$.

$\therefore$ $\sin x^2$ though continnous and bounded on $\mathbb{R}$ but is not uniformly continnous on $\mathbb{R}$.

185. T/F If $K$ be a continnous real valued function on $[0,1]^2$ then $f(x) = \int_0^1 K(x,t)\,\partial t$ is uniformly continnous.

Solⁿ : True We have, $|f(x) - f(y)| \le \int_0^1 |K(x,t) - K(y,t)|\,\partial t$

uniform continuity of $K$ on $[0,1]^2$ means that given $\epsilon > 0$ $\exists$ $\delta > 0$ $\ni$ $|K(x,s) - K(x,t)| < \delta$ provided

$\partial\left((x,s)(y,t)\right) < \delta$ so if $|x-y| < \delta$

then $|f(x) - f(y)| < \epsilon$

186. T/F

$$\lim_{n\to\infty} \left\{ \frac{n}{n + \frac{\sqrt[3]{n}}{n + \frac{\sqrt[3]{n}}{n + \frac{\sqrt[3]{n}}{n + \cdots}}}} \right\} = 3\sqrt{2}$$

Solⁿ: False $y = \dfrac{n}{n + \frac{1}{n^{2/3}} + \frac{n}{n + \sqrt[3]{n} + \cdots \infty}}$

$= \dfrac{n}{n + \frac{y}{n^{2/3}}}$

$\Rightarrow y = \dfrac{n^{5/3}}{n^{5/3} + y} \Rightarrow y^2 + y\,n^{5/3} - n^{5/3} = 0$

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$$\Rightarrow y = \frac{-n^{5/3} + \sqrt{n^{10/3} + 4n^{5/3}}}{2} \quad (\because y > 0)$$

$$= \frac{2}{\sqrt{1 + \frac{4}{n^{5/3}}} + 1}$$

$$\therefore \lim_{n \to \infty} y = \frac{2}{1+1} = 1.$$

187. T/F $S_n = \frac{n^n}{(n+1)(n+2)\cdots\cdots(n+n)}$ then $\{S_n\}^{1/n}$ diverges.

Solⁿ: False $\lim_{n \to \infty} \{S_n\}^{1/n} = \lim_{n \to \infty} \frac{S_{n+1}}{S_n}$

$$= \lim_{n \to \infty} \frac{(n+1)^{n+1} \cdot (n+1)(n+2)\cdots(2n)}{(n+1)(n+2)\cdots\cdots(2n+1)(2n+2)n^n}$$

$$= \lim_{n \to \infty} \frac{(1+1/n)^n (1+1/n)^n}{(2+1/n)(2+2/n)} = \frac{e^2}{4} > 1.$$

$\therefore S_n$ diverges.

188. T/F The series $\sum_1 u_n$ whose $n^{th}$ term $u_n = \{(n^3+1)^{1/3} - n\}$ converges.

Solⁿ: True $u_n = \{(n^3+1)^{1/3} - n\}$

$$= \left\{n\left(1+\frac{1}{n^3}\right)^{1/3} - n\right\}$$

$$= n\left\{(1+1/n^3)^{1/3} - 1\right\}$$

$$= n\left\{\frac{1}{3} \cdot \frac{1}{n^3} + \cdots\right\} = \frac{1}{3n^2} + \cdots\cdots$$

$$v_n = \frac{1}{n^2}$$

$$\therefore \lim_{n \to \infty} \frac{u_n}{v_n} = \frac{1}{3} \neq 0$$

Therefore as $\sum v_n$ converges, so does $\sum_1 u_n$

189. T/F $\{f_n\}$ be a sequence of continuous function on $\mathbb{R}$ and $\{f_n\}$ converges uniformly to the function $f$ on $\mathbb{R}$ then $f$ is continuous and bounded.

Sol$^n$: True Real valued function on the set of real numbers $\mathbb{R}$ which converges uniformly to the function $f$ on $\mathbb{R}$ if $f_n$ $(n=1,2,\ldots)$ is continuous on $\mathbb{R}$ then $f$ is also continuous on $\mathbb{R}$, and as it converges, so it must be bounded.

190. T/F $\int_0^1 e^{-x^2} \partial x \leq 1$

Sol$^n$: True $\int_0^1 e^{-x^2} \partial x$

$$e^{-x} \leq e^{-x^2} = \frac{1}{e^{x^2}} \leq 1 .$$

$$\therefore \int_0^1 e^{-x} \partial x \leq \int_0^1 e^{-x^2} \partial x \leq \int_0^1 1 . \partial x$$

$$\therefore 1 - \frac{1}{e} \leq \int_0^1 e^{-x^2} \partial x \leq 1 .$$

191. T/F $f(x) = x^n |x|$ is differentiable at origin for $n$ be any real no.

Sol$^n$: True $f(x) = \begin{cases} x^{n+1} , & 0 \leq x \\ -x^{n+1} , & x < 0 \end{cases}$

$$\therefore \frac{\partial}{\partial x} f(x) = \begin{cases} (n+1)x^n , & x \geq 0 \\ -(n+1)x^n , & x < 0 \end{cases}$$

$\therefore$ at $x=0$, the function is differentiable

192. T/F Function $x^3 - 6x^2 - 36x + 7$ increases for $-2 < x < 6$.

Sol$^n$: False for increasing function, $f'(x) > 0$

$$\Rightarrow 3x^2 - 12x - 36 > 0$$

$$\Rightarrow (x-6)(x+2) > 0$$

$$\therefore x < -2 \text{ or } x > 6.$$

193. T/F The series $\frac{(\log 2)^2}{2^2} + \frac{(\log 3)^2}{3^2} + \cdots$ is convergent.

Solⁿ: True $\sum u_n = \sum_{i=2}^{\infty} \frac{(\log i)^2}{i^2} \geqslant \sum \frac{1}{i^2} = \sum v_n$

$\sum v_n$ converges, so does $\sum u_n$

Note: Power series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p \in \mathbb{R}$ converges iff $p > 1$.

194. T/F If $\sum u_n$ is a series of positive terms then the converges of $\sum u_n$ implies the convergence of $\sum (-1)^n u_n$

Solⁿ: True Here $\sum u_n$ is a series of positive term. If $\sum u_n$ is convergent then the alternating series $\sum (-1)^n u_n$ must be convergent.

195. T/F $\lim_{n\to\infty} \frac{n^c}{(1+p)^n} = c$ $\forall$ real $c$ and $p > 0$

Solⁿ: False $\lim_{n\to\infty} \left(\frac{x_{n+1}}{x_n}\right) = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p \frac{1}{1+p}$

$= \frac{1}{1+p} < 1.$

$\therefore \lim_{n\to\infty} \frac{n^p}{(1+p)^n} = 0$

196. T/F $\sum 2^{-n+(-1)^n}$ is convergent.

Solⁿ: True Here, $U_n = 2^{-n+(-1)^n}$

$\lim U_n^{1/n} = \lim \left(2^{-n+(-1)^n}\right)^{1/n}$

$= \frac{1}{2} < 1 \Rightarrow$ Convergence

Therefore, by root test the series is convergent.

197. T/F $f(x) = \frac{1}{x}$ $(x>0)$ is uniformly continuous in $(0,1]$.

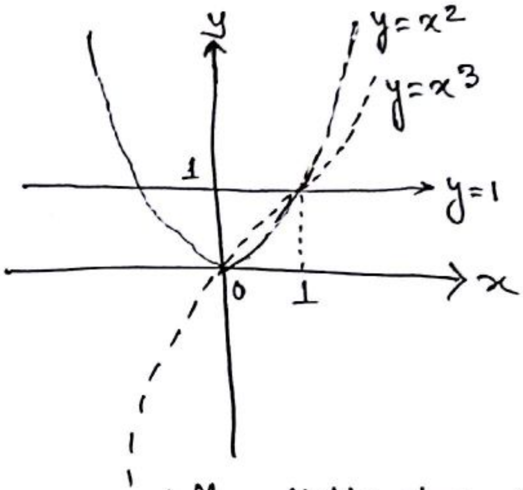
Sol$^n$. False $f(x)$ is uniformly continuous on $[1,\infty)$

[Theorem : Let $I = [a,b]$ be a closed and bounded interval and a function $f : I \to \mathbb{R}$ be continuous on $I$ then $f$ is uniformly continuous on $I$.]

198. T/F If $f(x) = \min\{1, x^2, x^3\}$ then it is not differentiable exactly at one point.

Sol$^n$. True at $x=1$, $f(x)=1$

So, the graph will be




So, $f(x)$ is continuous for $x \in \mathbb{R}$ and not differentiable at $x=1$ due to sharp edge.

199. T/F $f(x) = \begin{cases} e^{-1/x^2} \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is differentiable at $x=0$

Sol$^n$: True

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x}$$

$$= \lim_{x\to 0} \frac{e^{-1/x^2} \sin 1/x}{x}$$

$x \to 0$
$x^2 \to 0$
$z = \frac{1}{x^2} \to \infty$

$$= \lim_{z\to\infty} e^{-z} \lim_{z\to\infty} z \sin z$$

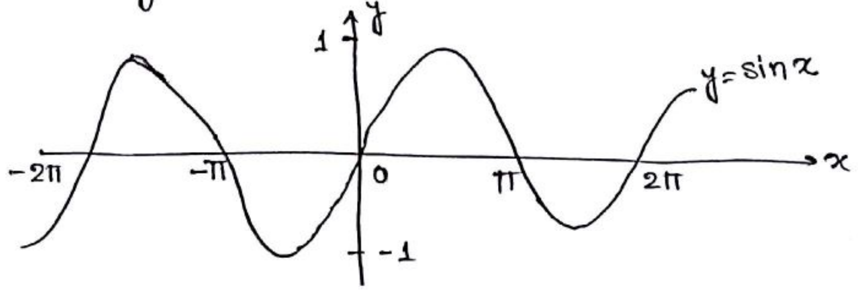
[ow, use expansion]

$$= 0$$

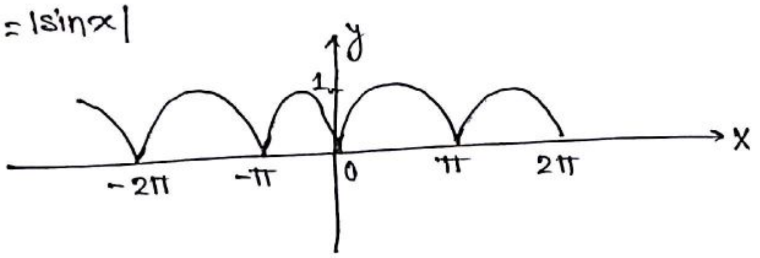
$f$ is differentiable at $x=0$

200. **T/F:-** $f(x) = 1 + |\sin x|$ is continuous everywhere.

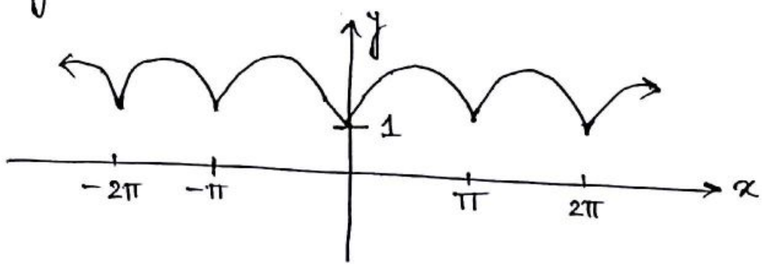
**Sol.** **TRUE** $y = \sin x$




$y = |\sin x|$




$y = 1 + |\sin x|$




Clearly, $y = 1 + |\sin x|$ is continuous $\forall x$ but differentiable at infinite number of points.

201. **T/F:-** If $\{S_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are bounded sequence of real numbers such that $S_n \le t_n$ $(n \in I)$. Then $\lim\limits_{n\to\infty} \inf S_n \le \lim\limits_{n\to\infty} \sup t_n$.

**Sol.** **TRUE** let, $\lim \inf S_n = h$, $\lim \inf t_n = i$
$\lim \sup S_n = K$, $\lim \sup t_n = K$.

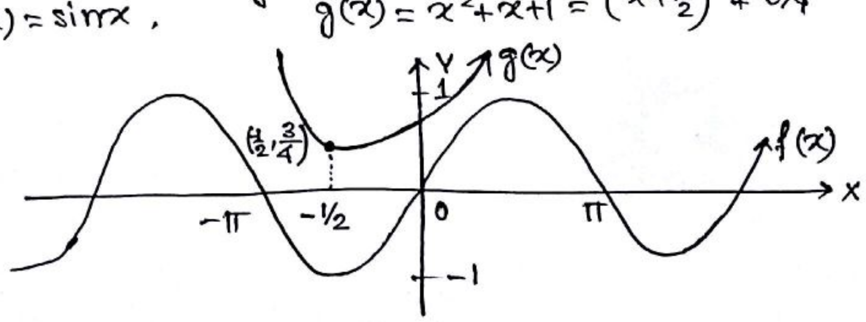
$$h \le S_n \le K \le i \le t_n \le K$$

$$\therefore \lim_{n\to\infty} \inf S_n \le \lim_{n\to\infty} \sup t_n.$$

202. **T/F:-** There is always one solution of the equation $\sin x = x^2 + x + 1$.

**Sol.** $f(x) = \sin x$, $g(x) = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + 3/4$

**FALSE**



203. T/F:- If $4a+2b+c=0$ then the equation $3ax^2+2bx+c=0$ has at least one real root lying in $(0,2)$.

Sol. TRUE  Since, $f(x)$ is continuous & differentiable in $(0,2)$.
By Rolle's theorem, $\exists$ a value of $x$ in $(0,2)$ $\ni$ $f'(c)=0$ $\forall$ $c \in (0,2)$.
So, there is a root in the interval.

204. T/F:- Let $a$ be a non-zero real number, then $\lim\limits_{x\to a} \frac{1}{x^2-a^2}\int_a^x \sin(t^2)\,dt = 0$.

Sol. FALSE

The given expression $\lim\limits_{x\to a}\frac{1}{x^2-a^2}\int_a^x \sin(t^2)\,dt$ is of the form $\left(\frac{0}{0}\right)$.

By L. Hospital Rule, $\lim\limits_{x\to a}\frac{\sin x^2 \cdot 1 - 0}{2x} = \frac{1}{2a}\cdot \sin a^2$.



205. T/F:- The function $f(x,y) = \begin{cases} \exp\{(-|x-y|)/(x^2-2xy+y^2)\} & \text{if } x\neq y \\ 0 & \text{if } x=y \end{cases}$

is continuous at $(0,0)$.

Sol. TRUE  Let $0<\epsilon<1$ and let $\epsilon_1 = -\frac{1}{\log\epsilon}$.
For $x\neq y$, $|x-y|/(x^2-2xy+y^2) = \frac{1}{|x-y|}$, we have
$|x-y| \le |x|+|y| \le 2\sqrt{x^2+y^2} < \epsilon_1$ if $x^2+y^2<\delta^2$.
Whenever $\delta^2 = \frac{\epsilon_1^2}{4}$. Therefore,
for $x\neq y$, $x^2+y^2<\delta^2$, we have $|x-y|/(x^2-2xy+y^2) > \frac{1}{\epsilon_1}$,
i.e., $-|x-y|/(x^2-2xy+y^2) < -\frac{1}{\epsilon_1} = \log\epsilon$.
That gives, $\exp\{-|x-y|/(x^2-2xy+y^2)\} < \epsilon$ whenever $x\neq y$ and $x^2+y^2<\delta$.
So, $|f(x,y)-f(0,0)|<\epsilon$ whenever $x^2+y^2<\delta^2$.

TRUE/FALSE TYPE:-

206. T/F:- If $A = [0,1)$ then $\text{int}\, A = (0,1)$ and $\text{ext}\, A = (1,\infty)$.

Sol. FALSE $\quad \text{int}(A) = \{x \in A : S_r(X) \subseteq A,$ for some $r > 0\}$

Here $\text{int}\, A = (0,1)$, i.e. points 0 and 1 are not interior points.

Again $\text{ext}\, A = (-\infty, 0) \cup (1, \infty)$

i.e. points 0 and 1 are not exterior points.

207. T/F:- A set in discrete metric space is open.

Sol. TRUE $\quad$ Let $G$ be any non-empty subset of the discrete space $(X, d)$ and $x$ be any point of $G$. Then open sphere $S_r(X)$ with $r \leq 1$ is the singleton set $\{X\}$ which is contained in $G$, i.e. each point of $G$ is the centre of some open sphere contained in $G$.

In particular, each singleton set is open.

208. T/F:- The closure of any set is closed set.

Sol. TRUE $\quad$ Let $X$ be a point of closure of $\bar{E}$.

Then for any $\delta > 0$, $\exists\, y \in \bar{E} \ni |x-y| < \delta/2$.

Same as $|y-z| < \delta/2$; where $z \in \bar{E}$.

$$\therefore |x-z| \leq |x-y| + |y-z| < \delta.$$

i.e. for $\delta > 0$; $z \in E \ni |x-z| < \delta$.

$$\Rightarrow x \in \bar{E}.$$

209. T/F:- On the real line with the usual metric the singleton set $\{x\}$ is open.

Sol. FALSE $\quad$ For the metric space $(R, d)$ each open sphere $S_r(X)$ is the bounded open interval $(X-r, X+r)$ and for a very small value of $r$ this sphere contained in $\{x\}$, but $R\{x\}$ is open then the singleton set is closed.

210. T/F :- $\{F_n : n \in \mathbb{N}\}$ where $F_n = [\frac{1}{n}, 1]$ be a family of sets then $\bigcup_{n=1}^{\infty} F_n$ is closed.

Sol. FALSE    Given family $= \{F_n : n \in \mathbb{N}\}$ ; $F_n = [1/n, 1]$

$\bigcup_{n=1}^{\infty} F_n = (0, 1]$ which is neither closed nor open.

211. T/F :- Every compact subset A of a metric space is bounded.

Sol. TRUE    $A \subseteq \bigcup_{x \in A} S_1(x)$

Let $M = \max d(x_i, x_j)$ ; $1 \le i \le j \le n$.

Then for any two element $x_i$ and $x_j$, we have

$$x \in S_1(x_i) \quad , \quad y \in S_1(x_j)$$

$\therefore$ By Triangle inequality, $d(x,y) \le d(x, x_j) + d(x_i, x_j) + d(x_j, y)$

$$\le 1 + M + 1 = M + 2.$$

Hence A is bounded.

212. T/F :- If S be a compact subset of $\mathbb{R}$ and let $f : S \to \mathbb{R}$ be continuous mapping then $f$ is continuous uniformly.

Sol. TRUE    Suppose that $f$ is not uniformly continuous.

Then $\exists$ an $r > 0$ $\ni$ for every $n$, there are $x_n \in S$ and $y_n \in S$ $\ni$

$$|x_n - y_n| < 1/n$$

$$\text{but } |f(x_n) - f(y_n)| > r.$$

Since, S is compact, every sequence in S must have a convergent subsequence.

Hence, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converge to $x \in S$.

But, it is evident that $\{y_n\}$ also converges to $x$.

However, it is not true that,

$$\lim_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} f(y_{n_k}) \text{ since } |f(x_{n_k}) - f(y_{n_k})| > r,$$

for every $k$. This contradicts the hypothesis.

Hence, $f$ must be uniformly continuous.

213. T/F:- Every open interval need not to be open set.

Sol. FALSE

The definition of open set is given as follows:
Let $G$ be a subset of $R$ is said to be open if for every point $p \in G$ $\exists$ an open interval $I$ such that $p \in I \in G$.
This is equivalent of saying that $G$ is open iff for every $p \in G$ $\exists$ an $\epsilon$ neighbourhood $N(p, \epsilon) = (p+\epsilon, p-\epsilon)$ such that $N(p, \epsilon) \in G$.
$\Rightarrow$ Every interval is an open set in $G$.

214. Suppose $f: (0,1) \to (0,1)$ is a continuous map. Then $f$ has a fixed point.
T/F:-

Sol. FALSE

Counter Example:- $f(x) = x^2$. For $c \in (0,1)$,
Then, $c - f(c) = c - c^2 = c(1-c) < c$.
Since $0 < 1-c < 1$. Hence, $f(c) \neq c$ for all $c \in (0,1)$, so $f$ has no fixed points.

215. T/F:- For any $g \in G$ $\exists$ a unique element $x \in G$ such that $x \cdot g = g \cdot x = e$; this element is called the inverse of $g$

Sol.:- TRUE

Let $g \in G$. By definition of a group,
we know $\exists$ at least one element $h \in G$ such that $hg = gh = e$.
Suppose there exists two such elements $h_1$ and $h_2$. Then
$h_1 = h_1 e = h_1(gh_2) = (h_1 g) h_2 = e h_2 = h_2$.
Hence, the element $x \in G$ such that $gx = xg = e$ is unique.

TRUE/FALSE TYPE:-

216. T/F:- A pair of fair dice is rolled together till a sum of either 5 or 7 is obtained. the probability that 5 comes before 7 is $1/3$.

Sol. FALSE

A: The event that a sum of 5 occurs
B: The event that a sum of 7 occurs
C: Neither a sum of 5 nor a sum of 7 occurs.

$$P(A)=\frac{4}{36}=\frac{1}{9}\ ;\ P(B)=\frac{6}{36}=\frac{1}{6}\ ;\ P(C)=\frac{26}{36}=\frac{13}{18}.$$

Thus, P(A occurs before B)

$$=P(A)+P(C)P(A)+P(C)P(C)P(A)+\cdots$$

$$=P(A)\left[1+P(C)+[P(C)]^2+\cdots\right]$$

$$=P(A).\frac{1}{1-P(C)}$$

$$=\frac{1/9}{1-13/18}=\frac{2}{5}.$$

217. T/F:- There are four machines and it is known that exactly two are faulty. they are tested one by one in a random order till both the faulty machines are identified then the probability that only two tests are needed is $1/6$.

Sol. TRUE

The probability that only two tests are needed
= (Probability that the first machine tested is faulty) × (Prob. that the second machine tested is faulty)

$$=\frac{2}{4}\cdot\frac{1}{3}$$

$$=\frac{1}{6}.$$

218. T/F:- $\exists$ infinitely many non-negative integer solution of $x+y+z=18$.

Sol. FALSE

We can view each solution, say $x=3,\ y=7,\ z=8$ as a combination of $r=18$ objects consisting of 3 a's, 7 b's, and 8 c's, where are $M=3$ kinds of objects a's, b's, c's.

Then no. of combinations $= {}^{r+M-1}C_{M-1} = {}^{20}C_2 = 190$.

219. T/F:- $\exists$ a set $A \subset \{1, 2, \ldots, 200\}$ with 130 elements such that 130 can't be expressed as a sum of two elements in A.

Sol. TRUE Since in A we can take all the elements greater than 130 so, 71 elements from 130 to 200 and now we have 1 to 129 elements in hand of which we can choose 59 elements for A.

Take elements in A in such a way that if X is in A then 130 − X will not be in A, we can take 1 to 65 easily.

We will get more than 130 elements.

220. T/F:- Three numbers are chosen at random without replacements from the set $A = \{x \mid 1 \le x \le 10 : x \in \mathbb{N}\}$. The probability that the minimum of the chosen number is 3 and maximum is 7 is $1/3$.

Sol. FALSE $n(S) = {}^{10}C_3$.

Minimum = 3, Maximum = 7.

Then we have to choose another number from 4, 5, 6.

So, $n(E) = {}^{3}C_1$.

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{{}^{3}C_1}{{}^{10}C_3} = \frac{1}{40}.$$

221. T/F:- The set of all complex numbers is closed but not open.

Sol. FALSE

Since every neighbourhood of each element contains infinitely many points of C. Hence it is open and closed both.

222. T/F:- If A and B are two events $\ni P(A) = P(B) = \lambda$ then $P(A \cap B) \le \lambda^2$.

Sol. FALSE $S = \{HH, HT, TH, TT\}$

$A = \{HT, TH\} = B$

$P(B) = P(B) = \frac{1}{2} = \lambda.$

$A \cap B = A$ ; $P(A \cap B) = P(A) = \frac{1}{2} > \lambda^2$

$\therefore P(A \cap B) \nleq \lambda^2$.

223. T/F:- $A = \{1, 2, 3, 4, \ldots, 20\}$ be a set. A number is chosen at random from the set A and it is found that it is a prime then the probability that it is more than 10 is $1/5$.

Sol. TRUE No. of primes between 10 to 20 is 4.

So, required prob. $= \frac{4}{20} = \frac{1}{5}$.

224. T/F:- If $I = \{1, 2, 3, 4, 5\}$, The total no. of possible functions on I, where $f(f(i)) \neq i$ is 44.

Sol. TRUE Total no. of required functions = No. of dearrangement of 5 objects

$= 5!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$

$= 44$.

225. T/F:- Let $x = 33^n$, $n \in \mathbb{N}$. Then the probability that the value of $x$ will have 3 in units place is $1/4$.

Sol. TRUE

$n(S)$ = the no. of ways on which $33^n$ can have a digit in the units place

$33^1 = 3$ (last digit)

$33^2 = 9$ (last digit)

$33^3 = 7$ (last digit)

$33^4 = 1$ (last digit)

$n(S) = 4$.

$n(E) = 1$.

$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}$.

226. T/F:- Four numbers from $\{0, 1, 2, \ldots, 9\}$ are multiplied together then the probability that the product is divisible by 5 or 10 is less than $\frac{1}{2}$.

Sol. FALSE The divisibility of the product of four numbers depend upon the value of the last digit of each number. The last digit of a number can be any one of the ten digits $0, 1, 2, \ldots, 9$. Since the ~~total~~ no. of ways of selecting last digit of four number is $10 \times 10 \times 10 \times 10 = 10^4$ ways.

If the product of the four number is not divisible by 5, 10; then the number of choices for the last digit for each number is 8 (excluding 0 and 5). So, the favourable number of ways $= 8^4$.

227. T/F:– 17 students give the guarantee that 5 of them belong to the same class, where there are 4 classes.

Sol. TRUE Here $n=4$ classes are the pigeon holes and $k+1=5$, $k=4$. Thus among $kn+1=17$ students (pigeons), 5 of them belong to the same class.

228. T/F:– Two integers $x$ and $y$ are chosen from set $\{0,1,2,\dots,10\}$ Then order of the set $S=\{(x,y): |x-y|>5\}$ is 30.

Sol. TRUE Since $x$ and $y$ can take values from 0 to 10. So, the total number of ways of selecting $x$ and $y$ is $11^2=121$.

Now, $|x-y|>5$

so, $x-y<-5$
$>5$

There are 30 pairs of values of $x$ and $y$ satisfying these two inequalities.

229. 20 cards are numbered 1 through 20 are placed face down on a

T/F:– table, cards are selected one at a time and turned over until 10 cards are chosen. If two of the cards add up to 21, the player wins. There are 20 ways (possible ways) to win the game?

Sol. FALSE $(1,20)$, $(2,19)$, $(3,18)$, ...., $(10,11)$

There are 10 possible ways to win this game.

230. T/F:– The probability that the roots of the equation $x^2+px+q=0$ are real is greater than $1/2$, where $p,q \in \{1,2,3,\dots,10\}$.

Sol. TRUE $x^2+px+q=0$

If the roots are real, then $p^2-4q \geq 0 \Rightarrow p^2 \geq 4q$.

$p,q \in \{1,2,3,\dots,10\}=S$

When $p=1$, no value of $q$ from S satisfy the condition.

$p=2$, $q=1$ will satisfy

$p=3$, $q=1,2$ " "

$p=4$, $q=1,2,3,4$ " "

$p=5$, $q=1,2,3,4,5,6$ " "

$p=6$, $q=1,2,3,\dots,9$ " "

$p=7$, all values of $q$ will satisfy.

$\therefore$ Sum of these selections is $= 1+2+4+6+9+10+10+10+10$
$= 62$.

Total no. of selections $= 10\times10 = 100$ ways.

$\therefore$ Required Probability $= \frac{62}{100} > \frac{1}{2}$.

# DIFFERENTIAL EQUATIONS



## TRUE / FALSE TYPE:-

231. T/F :- The ODE $2y(y'+2) - xy'^2 = 0$ has the singular solution $y=0$.

Solution:- False  Let us take $y' = c$

Then the equation reduces to $2y(c+2) - xc^2 = 0$

$$\Rightarrow xc^2 - 2yc - 4y = 0$$

which is quadratic in nature.

Now, $B^2 - 4AC = 0$ gives $(-2y)^2 - 4x.(-4y) = 0$

$$\Rightarrow 4y(y+4x) = 0$$

$$\Rightarrow y = 0, -4x.$$

232. T/F :- $\frac{dy}{dx} = \sqrt{|y|}$, $0 < y < 10$, $y(0) = 0$ has infinite number of solutions.

Solution:- True

$$\int \frac{dy}{\sqrt{|y|}} = \int dx$$

$$\Rightarrow -\frac{1}{2}|y|^{1/2} + c = x$$

Now, $y(0) = 0$ gives c remaining undetermined.

So, the ODE has infinite number of solutions.

233. T/F :- The solution of the ODE $\frac{dy}{dx} = \frac{x}{y}$, $y(0) = 0$ is unbounded.

Solution:- False

$$\frac{dy}{dx} = \frac{x}{y}$$

$$\Rightarrow y\,dy - x\,dx = 0$$

$$\Rightarrow y^2 - x^2 = c \text{ , on integration}$$

$y(0) = 0$ gives $c = 0$.

$\therefore$ $y = x$ is the solution of the ODE.

234. T/F :- If $f$ is a differentiable function, define $Df(x) = \frac{1}{2x} \cdot \frac{df}{dx}$. Given $n \in \mathbb{N}$ and an $n$-times differentiable function $f(D^n f)(x)$ by successively applying $D$ for $n$ times to the function $f$ then $\left[D^8\left(e^{-x^2}\right)\right]_{x=0} = -e^{-1}$.

Solution:- False

$$f(x) = e^{-x^2}$$

$$Df(x) = \frac{1}{2x} \cdot \frac{d}{dx}(f)$$

$$De^{-x^2} = \frac{1}{2x}\left(-2x \cdot e^{-x^2}\right)$$

$$= -e^{-x^2}.$$

$$\therefore D^2 e^{-x^2} = \frac{1}{2x}\left(2x e^{-x^2}\right)$$

$$= e^{-x^2}$$

$$\vdots$$

$$D^8 e^{-x^2} = e^{-x^2}$$

At $x = 0$, $D^8\left(e^{-x^2}\right) = 1 \neq -e^{-1}$.

235. T/F :- The first order differential equation of the family of circles of fixed radius $r$ with centres on the $x$-axis is

$$y^2\left(\frac{dy}{dx}\right)^2 + y^2 = r^2.$$

Solution:- True

Let the equation of the given circle with radius $r$ and whose centre is on $X$-axis, at $(a,0)$ is

$$(x-a)^2 + y^2 = r^2$$

Differentiating w.r.t. $x$, we get

$$2(x-a) + 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow y\frac{dy}{dx} = -(x-a)$$

$$\Rightarrow \left(y\frac{dy}{dx}\right)^2 = (x-a)^2$$

$$\Rightarrow y^2\left(\frac{dy}{dx}\right)^2 = r^2 - y^2$$

$$\Rightarrow y^2\left(\frac{dy}{dx}\right)^2 + y^2 = r^2.$$

236. T/F :- The differential equation $\frac{dy}{dx} = \frac{y-1}{x}$ ; $y(0)=1$ have infinite number of solutions.

Solution:— True

$$\int \frac{dy}{y-1} = \int \frac{dx}{x}$$

$$\Rightarrow \quad y-1 = xc$$

$y(0)=1 \Rightarrow x=0, y=1.$

$\therefore$ We see that 'c' can't be determined.

So, the equation has infinite solutions.

237. T/F :- $y : \mathbb{R} \to \mathbb{R}$ be an infinitely many times differentiable function which satisfies $y'' + y' - y \geq 0$, $y(0) = y(1) = 0$ if $y(x) \geq 0$ for all $x \in [0,1]$. Then the differential equation have unique solution.

Solution:— True

Since $y'' + y' - y \geq 0$

$$\Rightarrow (D^2 + D - 1)y \geq 0$$

$$\therefore y \leq c_1 e^{\frac{-1+\sqrt{5}}{2}x} + c_2 e^{\frac{-1-\sqrt{5}}{2}x}$$

For $y(0)=0$ implying $c_1 = -c_2$

$y(1)=0$ giving $c_1 = 0$.

$\Rightarrow y \leq 0.$

But given $y \geq 0 \ \forall \ x \in [0,1]$

So, $y(x) = 0 \ \forall \ x \in [0,1]$

It is the unique solution.

238. T/F :- Solution of the ODE $xy\left(\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} = 0$ have exactly one arbitrary constant.

solution:- False

Since the given differential equation is of order 2 so its solution contain two arbitrary constants.

239. T/F:- $y = x^2/4$ is a singular solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right) + y = 0$.

Solution:- TRUE    Let us take $\frac{dy}{dx} = p$

So, $p^2 - px + y = 0$

Differentiating w.r.t. $x$, we have

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\Rightarrow p = p + x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\Rightarrow (x - 2p)\frac{dp}{dx} = 0$$

$$\Rightarrow x = 2p$$

$$\Rightarrow 2dy = x\,dx$$

$$\Rightarrow y = \frac{x^2}{4}, \text{ on integration.}$$

240. T/F:- $y = \phi(x) - 1 + e^{-\phi(x)}$ be the solution of the differential equation $\frac{dy}{dx} + y\frac{d\phi}{dx} = \phi(x)\frac{d\phi}{dx}$.

Solution:- TRUE

The given differential equation is of the form : $\frac{dy}{dx} + Py = Q$

where, $P = \frac{d\phi}{dx}$, $Q = \frac{d\phi}{dx} \cdot \phi(x)$

I.F. is given by, $\text{I.F.} = e^{\int \frac{d\phi}{dx} dx} = e^{\int d\phi}$

So, the required solution is :

$$ye^{\phi(x)} = \int \phi(x)\frac{d\phi}{dx} e^{\phi(x)} dx + c$$

$$= \phi(x)\left(e^{\phi(x)}\right) - e^{\phi(x)} + c$$

$$\therefore y = \phi(x) - 1 + e^{-\phi(x)}.$$



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