MATHEMATICS FOR ISI ENTRANCE

• 1. How many zeros are at the end of 1000! ?

ANS:- The number of two's is enough to match each 5 to get a 10. So,

$$\begin{aligned} 5^1 &\rightarrow 200 \\ 5^2 &\rightarrow 40 \\ 5^3 &\rightarrow 8 \\ 5^4 &\rightarrow 1 \\ \hline & 249 \end{aligned}$$

∴ Thus, 1000! ends with 249 zeros.

• 2. The product of the first 100 positive integers ends with (a) 21 zeros, (b) 22 zeros, (c) 23 zeros, (d) 24 zeros.

ANS:-

$$\begin{aligned} 5^1 &\rightarrow 20 \\ 5^2 &\rightarrow 4 \\ \hline & 24 \end{aligned}$$

(d) 24 zeros.

• 3. The four digit number aabb is a square. Find it.

ANS:- $aabb = n^2$

then $n^2 = 1100a + 11b$
$= 11(100a + b)$
$= 11(99a + a + b)$

Since, n^2 is divisible by 11^2, we see that $11 \mid (a+b)$, i.e. $a+b = 11$. Since n^2 is a square, b can't be 0, 1, 2, 3, 5, 7 or 8. Checking the remaining we see that

$7744 = 88^2$.

• 4. 1000.....001 with 1961 zeros is composite (not prime).

ANS:- The number $= 10^{1962} + 1$
$= (10^{654})^3 + 1$

So, it is divisible by $(10^{654} + 1)$.

• 5. Find the smallest positive integer n, so that $999999 \cdot n = 111\ldots11$

ANS:-

$$(10^6 - 1)n = (10^k - 1)/9$$

$$\Rightarrow n = \frac{10^k - 1}{9(10^6 - 1)} \text{ with } k = 6m$$

then $n = \left(1 + 10^6 + \cdots + 10^{6(m-1)}\right)/9$

The numerator becomes a multiple of 9 with $m = 9$. thus the smallest n is $n = \dfrac{10^{54} - 1}{9(10^6 - 1)}$.

6. $a_1 = a_2 = 1$, $a_3 = -1$, $a_n = a_{n-1} \cdot a_{n-3}$. Find a_{1964}.

ANS:-
$$a_1 = a_2 = 1,\ a_3 = a_4 = a_5 = -1,\ a_6 = 1, a_7 = -1$$
$$\underbrace{+1, +1, -1, -1, -1, 1, -1}_{\text{period}}, \underbrace{1, 1, -1, -1, -1, 1, 1}, \ldots$$

since $1964 = (7 \times 280) + 4 = 7 \times 280 + 4$, thus we have
$$a_{1964} = -1.$$

7. Show that $1982 \mid 222\ldots\ldots22$ (1980 two).

ANS:- To show: $991 \mid 111\ldots\ldots11$ (1980 one)

Now, $111\ldots11$
$$= \frac{1}{9}\left[10^{1980} - 1\right]$$
$$= \frac{1}{9}\left[(10^{990}+1)(10^{990}-1)\right]$$

Now, $(10^{991-1} - 1) \mid 991$, since $10^{n-1} - 1 \mid n$, (n = odd) by induction.

This proves the assertion.

8. Prove that the number 1280000401 is composite.

ANS:- $1280000401 = a^7 + a^2 + 1$ when $a = 20$

the polynomial $a^7 + a^2 + 1$ has the factor $a^2 + a + 1$, since $\omega^7 + \omega^2 + 1 = \omega^2 + \omega + 1$, where ω is the 3rd root of unity.

Hence, the number is divisible by 421.

9. If r and s are any two rational numbers, prove that $r+s$ and $r.s$ are rational numbers.

ANS:- r and s are rational numbers.

$r = \frac{p_1}{q_1}$, $s = \frac{p_2}{q_2}$ when p_1, p_2, q_1, q_2 are non-zero integers.

$r + s = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$, it is of the form $\frac{p}{q}$, p and q are integers

so $(r+s)$ is rational number.

$r.s = \frac{p_1 p_2}{q_1 q_2}$, for the same reason, provided before,

rs is rational number.

10. If a, b are positive real variables whose sum is a constant λ, then the minimum value of $\sqrt{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)}$ is

(a) $\lambda - 1/\lambda$ (b) $\lambda + \frac{2}{\lambda}$ (c) $1 + \frac{2}{\lambda}$ (d) none.

ANS:- (c) $E^2 = 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{ab} = \frac{a+b+1}{ab}+1 = 1+\frac{\lambda+1}{ab}$, it will be min. when ab is maximum. Now we know that if sum of two quantities is constant, then their product is maximum when the quantities are equal.

$$\therefore a+b=\lambda \Rightarrow a=b=\frac{\lambda}{2}.$$

$$\therefore E^2 = 1+\frac{\lambda+1}{\frac{\lambda^2}{4}} = \left(\frac{\lambda+2}{\lambda}\right)^2 \Rightarrow E = 1+\frac{2}{\lambda},$$ which is the required result.

Shortcut:- (c) $\sqrt{(1+1/a)(1+1/b)}$ will be minimum when a and b will take the maximum value.

$a+b=\lambda$, then the max. value of a and b is $a=b=\lambda/2$, putting these, we get,

$$\left(\sqrt{(1+1/a)(1+1/b)}\right)_{min} = \sqrt{\left(1+\frac{2}{\lambda}\right)\left(1+\frac{2}{\lambda}\right)} = 1+\frac{2}{\lambda}.$$

11. If a, b, c are positive integers $\ni$ $abc+ab+bc+ca+a+b+c=1000$. Find the value of a+b+c.

ANS:- $abc+ab+ac+bc+a+b+c=1000$

$\Rightarrow a(bc+b+c) + a+b+c+bc+1 = 1000+1$

$\Rightarrow a(bc+b+c+1) + (b+c+bc+1) = 1001$

$\Rightarrow (a+1)(b+1)(c+1) = 1001 = 13 \times 7 \times 11$

$\Rightarrow a=12$

$\Rightarrow b=6$ $\therefore a+b+c = 28.$

$\Rightarrow c=10$

12. Find minimum value of $2^{\cos x}+2^{\sin x}$, $0 \leq x \leq 2\pi$.

ANS:- By AM $\geq$ GM inequality, we have

$$2^{\cos x}+2^{\sin x} \geq 2\sqrt{2^{\cos x+\sin x}} = 2 \cdot 2^{\frac{\sin x+\cos x}{2}}$$

N.T. the RHS will be minimum if $\sin x+\cos x$ is minimum, i.e. when both $\sin x$ and $\cos x$ are minimum, i.e. at $x = 225°$,

$\sin x = -\frac{1}{\sqrt{2}} = \cos x$

$$\therefore 2^{\cos x}+2^{\sin x} \geq 2 \cdot 2^{\frac{-\frac{2}{\sqrt{2}}}{2}} = 2 \cdot 2^{-\frac{1}{\sqrt{2}}}$$

$\therefore$ the minimum value is $2^{1-\frac{1}{\sqrt{2}}}$.

13. $f(x) = \dfrac{e^{2x-1}}{1+e^{2x-1}}$. Then $f\left(\dfrac{1}{1234}\right) + f\left(\dfrac{3}{1234}\right) + \cdots\cdots + f\left(\dfrac{1231}{1234}\right) + f\left(\dfrac{1233}{1234}\right) = ?$

ANS:- $f\left(\frac{1}{2}\right) = \dfrac{e^{1-1}}{1+e^{1-1}} = \dfrac{1}{2}$.

Now, $f(x) + f(1-x) = 1$.

So, $f\left(\dfrac{1}{1234}\right) + f\left(1 - \dfrac{1}{1234}\right) = 1$.

$\vdots$

Now there are 308 terms upto $f\left(\dfrac{615}{1234}\right) + f\left(1 - \dfrac{615}{1234}\right) = 1$.

Now, $f\left(\dfrac{617}{1234}\right) + f\left(1 - \dfrac{617}{1234}\right) = 1$

$\Rightarrow f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) = 1$

i.e. $f\left(\dfrac{617}{1234}\right) = 1/2$

$\therefore f\left(\dfrac{1}{1234}\right) + f\left(\dfrac{3}{1234}\right) + \cdots\cdots + f\left(\dfrac{615}{1234}\right) + f\left(\dfrac{617}{1234}\right) + f\left(\dfrac{619}{1234}\right) + \cdots\cdots + f\left(\dfrac{1233}{1234}\right)$

$= 308 + 0.5 = 308.5$

14. Let $P(x)$ be a polynomial of degree 11 such that $P(x) = \dfrac{1}{x+1}$ for $x = 0(1)11$.

Then $P(12) = ?$

(a) 0, (b) 1, (c) $\dfrac{1}{13}$, (d) None of these.

ANS:-(a) $P(x) = \dfrac{1}{x+1}$

$\Rightarrow (x+1)[P(x)] - 1 = c(x-0)(x-1)\cdots(x-11)$

Putting $x = -1$,

$0 - 1 = c(-1)(-2)\cdots(-12)$

$\Rightarrow c = -\dfrac{1}{12!}$

$\therefore [P(x)](x+1) - 1 = -\dfrac{1}{12!}(x-0)(x-1)\cdots(x-11)$

$\Rightarrow P(12)\,13 - 1 = -\dfrac{1}{12!}\,12.11.\ldots.2.1$

$\Rightarrow P(12)\,13 - 1 = -1$

$\Rightarrow P(12) = 0$.

15) Let $R = \dfrac{48^{52} - 46^{52}}{96^{26} + 92^{26}}$. Then R satisfies

(a) $R < 1$, (b) $23^{26} < R < 24^{26}$, (c) $1 < R < 23^{26}$,
(d) $R > 24^{26}$

ANS:-
(b) $R = \dfrac{(2.24)^{52} - (2.23)^{52}}{(4.24)^{26} + (4.23)^{26}}$

$$= \frac{2^{52}\left(24^{52} - 23^{52}\right)}{4^{26}\left(24^{26} + 23^{26}\right)}$$

$$= \frac{2^{52}}{2^{52}} \cdot \frac{\cancel{\left(24^{26} + 23^{26}\right)}\left(24^{26} - 23^{26}\right)}{\cancel{\left(24^{26} + 23^{26}\right)}}$$

$$= 24^{26} - 23^{26} < 24^{26}$$

Also, $R = 24^{26} - 23^{26}$

$$= (1+23)^{26} - 23^{26}$$

$$= \cancel{23^{26}} + {}^{26}C_1 . 23^{25} + {}^{26}C_2 . 23^{24} + \cdots + 1 - \cancel{23^{26}}$$

$$= 26 . 23^{25} + {}^{26}C_2 . 23^{24} + \cdots + 1$$

$$> 26.23^{25} > 23.23^{25} = 23^{26}$$

$\therefore\ 23^{26} < R < 24^{26}$.

16) The number of pairs of integers (m,n) satisfying $m^2 + mn + n^2 = 1$ is

(a) 8, (b) 6, (c) 4, (d) 2.

ANS:- (b) $m^2 + mn + n^2 - 1$

The equation is symmetric in m and n, we make the substitution $u = m+n$ and $v = m-n$

so that $u^2 + v^2 = 2(m^2 + n^2)$, $u^2 - v^2 = 4mn$

Multiplying the given equation by 4, we have

$$4m^2 + 4mn + 4n^2 = 4$$

$$\Rightarrow 4(m^2 + n^2) + 4mn = 4$$

$$\Rightarrow 2(u^2 + v^2) + u^2 - v^2 = 4$$

$$\Rightarrow 3u^2 + v^2 = 4$$

Set $u^2 = x$, $v^2 = y$ with $x, y \geq 0$, then we get

$$3x + y = 4$$

The ordered pairs (x, y) satisfying the above equation in integers are $(0,4)$ and $(1,1)$. We have,

$u^2 = 0$ and $u^2 = 1$
$v^2 = 4$, and $v^2 = 1$

i.e. $u=0, v=2$; $u=0, v=-2$;
$u=1, v=1$; $u=1, v=-1$;
$u=-1, v=1$; $u=-1, v=-1$;

giving 6 ordered pair solutions (m,n) viz.

$(1,-1), (-1,1), (1,0), (0,1), (0,-1), (-1,0)$.

17. From a group of 20 persons, belonging to an association, a president, a secretary and three members are to be elected for the executive committee. Find the number of ways this can be done.

ANS:- $\quad {}^{20}C_1 \times {}^{19}C_1 \times {}^{18}C_3$ OR $\dfrac{20!}{1!\,1!\,3!\,15!}$

18. The sum of the digits of the number $100^{13}-26$, written in decimal notation is

(a) 227, (b) 218, (c) 228, (d) 219

ANS:- (a) $\quad 10^{26}-26$

$$= \underbrace{100\cdots\cdots0}_{26\text{ zeros}} - 26$$

$$= \underbrace{999\cdots\cdots}_{24\ 9\text{'s}}974$$

$\therefore$ The sum of the digits $= 24\times9+7+4$
$= 227$.

19) Let $S = \{(x_1,x_2,x_3) \mid 0 \le x_i \le 9$ and $x_1+x_2+x_3$ is divisible by $3\}$. Then the number of elements in S is

(a) 334, (b) 3133, (c) 327, (d) 336

ANS:- (a) With each (x_1,x_2,x_3) identify a three digit code, where leading zeros are allowed. We have a bijection between S and the set of all non-negative integers less than or equal to 999 divisible by 3.

The no. of numbers between 1 and 999, inclusive, divisible by 3 is $\left(\dfrac{999}{3}\right)=333$.

Also, '0' is divisible by 3.

Hence, the number of elements in S is $= 333+1 = 334$

Scanned by CamScanner

20) Let x and y be positive reals with $x<y$. Also $0<b<a<1$. Define $E=\log_a\left(\frac{y}{x}\right)+\log_b\left(\frac{x}{y}\right)$

Then E can't take the value

(a) -2 (b) -1 (c) $-\sqrt{2}$ (d) 2

ANS:- (d) $E=\log_a\left(\frac{y}{x}\right)+\log_b\left(\frac{x}{y}\right)$

$$\cancel{\frac{\log\frac{y}{x}-\log\frac{x}{y}}{\log a.\log b}}$$

$$=\frac{\log\frac{y}{x}}{\log a}-\frac{\log\frac{y}{x}}{\log b}$$

$$=\log\left(\frac{y}{x}\right)\left\{\frac{1}{\log a}-\frac{1}{\log b}\right\}$$

$$=\log\left(\frac{y}{x}\right)\left\{\frac{\log b-\log a}{(\log a)(\log b)}\right\}$$

$$=\log\left(\frac{y}{x}\right).\frac{\log\left(\frac{b}{a}\right)}{(\log a)(\log b)}$$

$$=-\log\left(\frac{y}{x}\right).\frac{\log\left(\frac{a}{b}\right)}{(\log a)(\log b)}$$

Now $0<a<1$, $0<b<1$

$\therefore \log a$ and $\log b$ are both negative.

Also, $\frac{y}{x}>1$ and $\frac{a}{b}>1$. Thus $\log\left(\frac{y}{x}\right)$ and $\log\left(\frac{a}{b}\right)$ are both positive.

Finally E turns out to be a negative value, so, E can't take the value '2'.

21) The great common divisor (gcd) of $2^{2^{22}}+1$ and $2^{2^{222}}+1$ is

(a) 1 (b) $2^{2^{22}}+1$ (c) $2^{2^{11}}-1$ (d) $2^{2^{21}}-1$

ANS:- (a) Let $F_n=2^{2^n}+1$, with $m<n$

$$F_n-2=2^{2^n}+1-2$$
$$=2^{2^n}-1$$
$$=(2^{2^{n-1}})^2-1$$
$$=(2^{2^{n-1}}+1)(2^{2^{n-1}}-1)$$
$$=(2^{2^{n-1}}+1)(2^{2^{n-2}}-1)(2^{2^{n-2}}+1)$$
$$=(2^{2^m}+1)(2^{2^m}-1)(2^{2^{n-1}}+1)=\lambda F_m$$

Now, $F_n-\lambda F_m=2$

let $d\,|\,F_n$ and $d\,|\,F_m$ then $d\,|\,2$. Then $d=1$ or 2. But F_m & F_n are both odd. Hence gcd $=1$.

22) A function f is said to be odd if $f(-x) = -f(x)\ \forall x$. Which of the following is not odd ?

(a) $f(x+y) = f(x)+f(y)\ \forall\ x, y$

(b) $f(x) = \dfrac{xe^{x/2}}{1+e^{x}}$

(c) $f(x) = x - [x]$

(d) $f(x) = x^2\sin x + x^3\cos x$

ANS:- (c) $f(x+y) = f(x)+f(y)\ \forall\ x, y$.

Let $x = y = 0$

$\Rightarrow f(0) = f(0) + f(0)$

$\therefore f(0) = 0$

Replacing y with $-x$, we have

$f(x-x) = f(x) + f(-x)$

$\Rightarrow f(0) = f(x) + f(-x)$

$\Rightarrow f(x) + f(-x) = 0$

$\Rightarrow f(-x) = -f(x)$

Thus f is odd.

Again for $f(x) = \dfrac{xe^{x/2}}{1+e^{x}}$

$$f(-x) = \frac{(-x)(e^{-x/2})}{1+e^{-x}}$$

$$= \frac{(-x)(e^{-x/2})\cdot e^{x}}{1+e^{x}}$$

$$= -\frac{xe^{x/2}}{1+e^{x}} = -f(x)$$

$\therefore f$ is odd.

$f(x) = x - [x]$ is not odd.

Counter example:-

$f(-2.3) = -2.3 - [-2.3]$

$= -2.3 - (-3)$

$= 3 - 2.3$

$= 0.7$

$f(2.3) = 2.3 - [2.3]$

$= 2.3 - 2$

$= 0.3$

$\therefore f(2.3) \neq f(-2.3)$

thus f is not odd.

$f(x) = x^2\sin x + x^3\cos x$

$f(-x) = -x^2\sin x - x^3\cos x$

$= -f(x)$

$\therefore f$ is odd here.

23) The $\lim_{x\to 0} \frac{\cos x - \sec x}{x^2(1+x)}$ is

(a) -1, (b) 1, (c) 0, (d) does not exist.

Ans:- (a) $\lim_{x\to 0} \frac{\cos^2 x - 1}{\cos x (x^2(x+1))}$

$= \lim_{x\to 0} \frac{-\sin^2 x}{\cos x (x^2)(x+1)}$

$= -\lim_{x\to 0} \frac{1}{\cos x} \left(\frac{\sin x}{x}\right)^2 \cdot \frac{1}{(x+1)}$

$= -1 \cdot 1 \cdot 1.$

$= -1.$

24) Let S be the set of all 3-digit numbers such that
(i) the digits in each number are all from the set $\{1, 2, 3, \ldots, 9\}$
(ii) exactly one digit in each numbers is even
The sum of all numbers in S is
(a) 96100 (b) 133200 (c) 66600 (d) 99800

Ans:-(b) The sum of the digits in unit place of all the numbers in S will be same as the sum in tens or hundreds place.
The only even digit can have any of the three positions, i.e. $^3C_1$, ways.
And the digit itself has 4 choices (2, 4, 6 or 8). The other two digits can be filled in $5\times4 = 20$ ways.
Then the number of numbers in $S = 240$
Number of numbers containing the even digit in units place
$= 4\times5\times4 = 80$
So, the other 160 numbers have digits 1, 3, 5, 7 or 9 in units place, with each digit appearing $\frac{160}{5} = 32$ times.

Sum in units place $= 32(1+3+5+7+9) + 20(2+4+6+8)$

$= 32 \cdot 5^2 + 20\times2\times\frac{4\times5}{2}$

$= 32\times25 + 20\times20$

$= 1200$

$\therefore$ The sum of all numbers $= 1200(1+10+10^2)$

$= 1200\times111$

$= 133200$

Scanned by CamScanner

25) Let $y = \frac{x}{x^2+1}$. Then $y^4(1)$ is equals
(a) 4, (b) −3, (c) 3, (d) −4

Ans: (b) Simply differentiating would be tedious.
We take advantage of 'i', the square root of '−1'.

$$y = \frac{x}{x^2+1} = \frac{1}{2}\left\{\frac{1}{(x-i)} + \frac{1}{(x+i)}\right\}$$

$$\frac{d^4y}{dx^4} = \frac{1}{2}\left\{\frac{4!}{(x-i)^5} + \frac{4!}{(x+i)^5}\right\}$$

Note that, $\frac{d^n}{dx^n}\left\{\frac{1}{x+a}\right\} = \frac{(-1)^n n!}{(x+a)^{n+1}}$

So, $y^4(x) = \frac{4!}{2}\left\{\frac{1}{(x-i)^5} + \frac{1}{(x+i)^5}\right\}$.

$$y^4(1) = 12\left\{\frac{1}{(1-i)^5} + \frac{1}{(1+i)^5}\right\}$$

$$= 12\left\{\frac{1-i}{(-2i)^3} + \frac{1+i}{(2i)^3}\right\}$$

$$= 12\left\{\frac{1-i}{8i} + \frac{1+i}{-8i}\right\}$$

$$= 12\left(-\frac{1}{8} - \frac{1}{8}\right) = -3.$$

26) The number of real roots of the equation
$1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\cdots + \frac{x^7}{7} = 0$ (without factorial)
is (a) 7, (b) 5, (c) 3, (d) 1.

ANS: (d) Let f has a minimum at $x = x_0$, where
$f(x) = 1 + \frac{x}{1} + \frac{x^2}{2} + \cdots + \frac{x^6}{6}$;
then $f'(x_0) = 0$

$$\Rightarrow 1 + x_0 + x_0^2 + x_0^3 + x_0^4 + x_0^5 = 0$$

$$\Rightarrow \frac{x_0^6 - 1}{x_0 - 1} = 0$$

$$\Rightarrow \frac{(x_0^3-1)(x_0^3+1)}{(x_0-1)} = 0$$

$$\Rightarrow (x_0^2 + x_0 + 1)(x_0^2 - x_0 + 1)\cdot(x_0 + 1) = 0$$

which has a real root $x_0 = -1$

But, $f(-1) = 1 - 1 + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \frac{1}{6} > 0$

The $f(x) > 0$ and hence f has no real zeros.

Now let, $g(x) = 1 + \frac{x}{1} + \frac{x^2}{2} + \cdots + \frac{x^7}{7}$

An odd degree polynomial has at least one real root.
If our polynomial g has more than one zero, say x_1, x_2,
then by Rolle's theorem in (x_1, x_2) we have 'x_3' such that

$$g'(x_3) = 0$$

$$\Rightarrow 1 + x_3 + x_3^2 + \cdots + x_3^6 = 0$$

But this has no real zeros.
Hence the given polynomial has exactly one real zero.

27) Let $f(m,n) = 36^m - 5^n$, with $m>0, n>0$. The smallest value of $|f(m,n)|$ is

(a) 1 , (b) 11 , (c) 5 , (d) 9 .

ANS:- (b) The unit digit of 5^n is 5 for all positive integers n and the unit digit of 36^m is 6.
The unit digit of $|f(m,n)|$ is 1 or 9 depending upon which of 5^n and 36^m is greater.
The least value of $|f(m,n)|$ can be 1.

Note that $|f(m,n)| = 1 \Rightarrow 36^m - 5^n = \pm 1$

Modulo '5' the above congruence reads

$$1 \equiv \pm 1 \pmod 5, \text{ i.e. } 2 \equiv 0 \pmod 5 \text{ [impossible]}$$

The next possible value can be 9.

$$36^m - 5^n = \pm 9$$
$$\Rightarrow 36^m \pm 9 = 5^n$$

which reads modulo '3' [impossible]

$$0 \equiv (-1)^m \pmod 3$$

Proceeding this way the next possible value of $|36^m - 5^n|$ will be 11.

$$|f(1,2)| = 36 - 25 = 11.$$

28) Let x, y, z be different from 1 satisfying $x+y+z = 2007$, $xy+yz+zx = 4011$.
Then the value of $\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z}$ is

(a) 0 , (b) 1 , (c) 2008 , (d) $\frac{1}{2008}$.

Soln. :→ (a)

$$\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z}$$

$$= \frac{3 - 2(x+y+z) + (xy+yz+zx)}{(1-x)(1-y)(1-z)}$$

$$= \frac{3 - 2\times 2007 + 4011}{(1-x)(1-y)(1-z)} = 0$$

29. The number a7389b, a, b are digits, is divisible by 72, then a+b equals
(a) 10, (b) 9, (c) 11, (d) 12

ANS:- (b) $72 = 8 \times 9$, and 8 and 9 are coprime. As the number a7389b is divisible by 72, it is divisible by 9 and 8 both. For divisibility by 8, the last three digits must be divisible by 8, i.e. $800 + 90 + b \mid 8$., so $(b+2) \mid 8$; $\therefore b = 6$.

For divisibility by 9, the sum of the digits

$a+7+3+8+9+b$ should be divisible by 9.

i.e. $a+7+3+8+9+6 \equiv 0 \pmod 9$

$\Rightarrow a+6 \equiv 0 \pmod 9$

$\Rightarrow a \equiv -6 \pmod 9$

$\Rightarrow a \equiv 3 \pmod 9$

$\therefore a = 3$ only. Hence $a+b=9$.

30. In a special version of chess, a rook moves either horizontally or vertically on the chessboard. The number of ways to place 8 rooks of different colours on a 8×8 chessboard such that no rook lies on the path of the other rook at the start of the game is

(a) $8 \times \underline{|8}$, (b) $\underline{|8} \times \underline{|8}$, (c) $2^8 \times \underline{|8}$, (d) $2^8 \times {}^{64}C_8$.

ANS:-

(b) The first rook can be placed in any row in 8 ways and in any column in 8 ways. So, it has 8^2 ways to be disposed off. Since no other rook can be placed in the path of the first rook, a second rook can be placed in 7^2 ways for there now remains only 7 rows and 7 columns. Counting in this manner, the number of ways $= 8^2 . 7^2 . 6^2 . \dots 1^2 = (8!)^2$

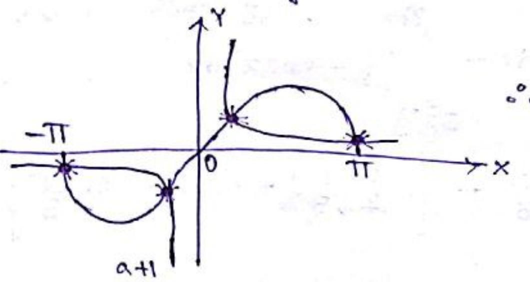
31) How many roots are there between $-\pi$ and π of the equation

$$\frac{2}{3}\sin x . x = 1$$

Soln. → $\frac{2}{3}x.\sin x = 1$

$$\Rightarrow \sin x = \frac{3}{2x}$$

Now, draw the curve of $y = \sin x$ and $y = \frac{3}{2x}$ on $xy = 3/2$



∴ There are 4 real roots.

32) $\underset{a}{\text{Max}}\ I(a) = \int_{a-1}^{a+1} e^{-|x|}dx$

Soln. → $I(a) = \int_{a-1}^{a} e^{-|x|}dx + \int_{a}^{a+1} e^{-|x|}dx$

Let $0<a<1$, Then, $\int_{a-1}^{a} e^{-|x|}dx = \int_{a-1}^{0} e^{x}dx + \int_{0}^{a} e^{-x}dx$

$$= \left[e^{x}\right]_{a-1}^{0} + \left[-e^{-x}\right]_{0}^{a}$$

$$= 1 - e^{a-1} - (e^{-a} - 1)$$

And, $\int_{a}^{a+1} e^{-|x|}dx = -e^{-x}\Big|_{a}^{a+1} = e^{-a} - e^{-a-1}$

$$\therefore I(a) = 2 - (e^{a-1} + e^{-a-1})$$

$$\therefore \frac{d}{dx}I(a) = 0 \Rightarrow e^{a-1} = e^{-a+1}$$

$$\Rightarrow a = 0$$

Also, $-1<a<0$; $\int_{a-1}^{a} e^{x}dx = e^{a} - e^{a-1}$

and $\int_{a}^{a+1} e^{-|x|}dx = \int_{a}^{0} e^{x}dx + \int_{0}^{a+1} e^{-x}dx$

$$= (1 - e^{a}) - (e^{a+1} - 1)$$

$$= 2 - e^{a} - e^{a+1}$$

$$\therefore I(a) = 2 - e^{a-1} - e^{a+1}$$

$$\therefore I'(a) = 0 \Rightarrow a = 0$$

∴ $I(a)$ is maximum at $a = 0$.

33. $\int_0^{\pi} \left|\frac{1}{2}+\cos x\right| dx = ?$

Soln.: $\Rightarrow$ Note that, $\left|\frac{1}{2}+\cos x\right| = \begin{cases} \frac{1}{2}+\cos x & \text{if } \cos x > -\frac{1}{2} \\ -\frac{1}{2}-\cos x & \text{if } \cos x < -\frac{1}{2} \end{cases}$

$$= \begin{cases} \frac{1}{2}+\cos x & \text{if } x < \frac{2\pi}{3} \\ -\left(\frac{1}{2}+\cos x\right) & \text{if } x > \frac{2\pi}{3} \end{cases}$$

$$I = \int_0^{2\pi/3} \left(\frac{1}{2}+\cos x\right) dx - \int_{2\pi/3}^{\pi} \left(\frac{1}{2}+\cos x\right) dx$$

$$= \frac{1}{2}\cdot\frac{2\pi}{3} + \sin\frac{2\pi}{3} - \left[\frac{1}{2}\left(\frac{\pi}{3}\right) + \sin\pi - \sin\frac{2\pi}{3}\right]$$

$$= \frac{\pi}{6} + \sqrt{3}.$$

34. The value of $\int_0^1\int_0^1 \{\text{Min}(x,y) - xy\}\, dx\, dy = ?$

Ans:- $\int_0^1\int_0^1 \text{Min}(x,y)\,dx\,dy = \underset{x<y}{\int_0^1\int_0^1} \text{Min}(x,y)\,dx\,dx + \underset{x>y}{\int_0^1\int_0^1} \text{Min}(x,y)\,dx\,dy$

$$= \int_0^1\int_0^y x\,dx\,dy + \int_0^1\int_0^x y\,dx\,dy$$

$$= \frac{1}{3}$$

and $\int_0^1\int_0^1 xy\,dx\,dy = \frac{1}{4}$

$\therefore I = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

35. Given that $\sum a_n$ converges $(a_n > 0)$; show that $\sum a_n^3 \sin n$ converges.

ANS:- since $\sum a_n$ converges, we have $\lim_{n\to\infty} n.a_n$ converges.

i.e. $|n.a_n| \le 1$ for $n \ge M$ (say)

$\Rightarrow n.a_n < 1 \; [\because a_n > 0]$

$\Rightarrow a_n < \frac{1}{n}$

$\therefore a_n^3 < \frac{1}{n^3} \Rightarrow a_n^3 \sin n \le \frac{1}{n^3}\sin n \le \frac{1}{n^3}$

$\Rightarrow \sum a_n^3 \sin n \le \sum \frac{1}{n^3}$

$\because$ RHS converges so LHS will also converge.

36. 3 balls are distributed to 3 boxes at random. Find the no. of way in which we get at most 1 box empty.

Ans:-

Zero box empty + 1 box empty

$= 3$ balls in 3 boxes $+ \left\{ \binom{3}{1} \times 3 \text{ balls in 2 boxes} \right\}$

$= 3! + 3 \times (3)_2 = 24$

37. If a circle intersects a hyperbola $y = \frac{1}{x}$ at 4 distinct points $\{(x_i, y_i) : i = 1(1)4\}$, then Prove that $x_1 x_2 = y_3 y_4$.

ANS:- Let the circle be $x^2 + y^2 = a^2$

hyperbola $y = \frac{1}{x}$ is given.

For points of contact, $x^2 + \frac{1}{x^2} = a^2$

$\Rightarrow x^4 - x^2 a^2 + 1 = 0$

It has 4 roots, so, $x_1 x_2 x_3 x_4 = 1$

Now, we know $x = \frac{1}{y}$

i.e. $x_3 = \frac{1}{y_3}$ & $x_4 = \frac{1}{y_4}$

$\therefore x_1 x_2 = y_3 y_4$.

38. How many real roots do $x^4 + 12x - 5 = 0$ have?

ANS:- $f(x) = 0 \Rightarrow x^4 + 12x - 5 = 0$

$f(-x) = x^4 - 12x - 5$

38. If f is a continuous function on $\mathbb{R} \ni f(x+y) = f(x) + f(y) \ \forall x, y \in \mathbb{R}$ then $f(0) = 0 \ \forall x \in \mathbb{R} \longrightarrow$ T/F.

Sol. False $f(x+y) = f(x) \cdot f(y)$

Let $f(x) = a^x$, $f(y) = a^y \ \forall x, y \in \mathbb{R}$.

$f(x+y) = a^{x+y}$

$f(0) = 1 \neq 0$.

39. If A and B are real orthogonal matrices of the same order and $|B| + |A| = 0$. Prove that $|A+B| = 0$.

ANS:-

$|A| + |B| = 0$

$\Rightarrow |A| = -|B|$

$|A| \cdot |B| = -1$ $[\because |B| = |B^{-1}|$ as they are orthogonal$]$

Let, $C = A(A^T + B^T)B$

$\Rightarrow |C| = |AA^T B + AB^T B| = |B + A|$ ——— (i)

and $|C| = |A| \, |A^T + B^T| \, |B| = -|A^T + B^T|$

$\Rightarrow -|(A+B)^T| = -|A+B|$ ——— (ii)

$|A+B| = -|A+B|$

$\Rightarrow 2|A+B| = 0$

$\Rightarrow |A+B| = 0$.

— By Tanujit Chakraborty
M: 8420253573
Mail: tanujitisi@gmail.com

40. $\cos^8\theta - \sin^8\theta = 1$, How many roots are there between $[0, 2\pi]$?

ANS:- Note that, $\cos^8\theta = 1+\sin^8\theta$ is possible only if,

$\cos^8\theta = 1$ and $\sin^8\theta = 0$

$\therefore \theta = 0, \pi, 2\pi$,

Hence 3 roots are there between $[0, 2\pi]$.

41. Find the value of $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$

Soln.:-

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \prod_{n=2}^{\infty}\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n}\right)$$

$$= \prod_{n=2}^{\infty} \frac{n+1}{n} \cdot \frac{n-1}{n}$$

$$= \left(\frac{2+1}{2}\cdot\frac{2-1}{2}\right)\left(\frac{3+1}{3}\cdot\frac{3-1}{3}\right)\cdots\cdots$$

$$= \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{4}{3}\cdot\frac{2}{3}\cdots\cdots$$

$$= \frac{1}{2}$$

42. Determine whether there is a one-to-one function $f: \mathbb{R}\to\mathbb{R} \ni f(x^2) - [f(x)]^2 \geq \frac{1}{4}\ \forall x$

Ans:- Take $x=0$, then $f(0) - (f(0))^2 \geq \frac{1}{4}$

$$\Rightarrow (f(0))^2 + \left(\tfrac{1}{2}\right)^2 - 2\cdot\tfrac{1}{2}\cdot f(0) \leq 0$$

$$\Rightarrow \left(f(0)-\tfrac{1}{2}\right)^2 \leq 0$$

$$\Rightarrow f(0) - \tfrac{1}{2} = 0$$

$$\Rightarrow f(0) = \tfrac{1}{2}$$

Also, taking $x=1$ we have $f(1) - \frac{1}{2} = 0$

$\therefore f(0) = f(1) = \frac{1}{2}$

$\therefore$ This is not one-to-one function.

43. If $u_n = \frac{1}{1\cdot n} + \frac{1}{2(n-1)} + \frac{1}{3(n-2)} + \cdots\cdots + \frac{1}{(n-1)}$; Find $\lim_{n\to\infty} u_n$.

Soln.:-

$$u_n = \frac{1}{(n+1)}\left[\left(1+\frac{1}{n}\right) + \left(\frac{1}{2}+\frac{1}{n-1}\right) + \left(\frac{1}{3}+\frac{1}{n-2}\right) + \cdots\cdots + \left(\frac{1}{n}+1\right)\right]$$

$$= \frac{1}{(n+1)}\, 2\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$\therefore \lim_{n\to\infty} u_n = 2\lim_{n\to\infty}\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{n}\cdot\lim_{n\to\infty}\frac{n}{n+1}$$

$$= 2\cdot 0\cdot 1 = 0$$

44. If $0<u<1$ and $u_{n+1}=1-\sqrt{1-u_n}$ $\forall\ n>1$, then Prove that (i) $\{u_n\}$ converges to zero

(ii) $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}=\frac{1}{2}$.

ANS:- (i) $0<u_1<1$

$\Rightarrow 0<\sqrt{1-u_1}<1$

$\Rightarrow 0<1-\sqrt{1-u_1}<1$

i.e. $0<u_2<1$

Similarly, $0<u_3<1$ ..... and so on.

Let $0<u_n<1$, then $0<1-\sqrt{1-u_n}<1$, i.e. $0<u_{n+1}<1$

thus $\{u_n\}$ is bounded.

Again, $u_{n+1}-u_n = 1-\sqrt{1-u_n}-u_n$

$= (1-u_n)-\sqrt{1-u_n}$

$= (\sqrt{1-u_n})^2-\sqrt{1-u_n}$

$= \sqrt{1-u_n}\,(\sqrt{1-u_n}-1)$

<0 as $0<\sqrt{1-u_n}<1$

so, $\sqrt{1-u_n}-1<0$

$\therefore u_{n+1}<u_n$

$\therefore \{u_n\}$ is monotonically decreasing.

$\therefore \{u_n\}$ converges to zero.

(ii) Let $\lim_{n\to\infty} u_n = l$, then $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}$

$\therefore \lim_{n\to\infty}\frac{1-\sqrt{1-l}}{l} = \lim_{n\to\infty}\frac{l}{l(1+\sqrt{1-l})} = \frac{1}{1+\sqrt{1-0}} = \frac{1}{2}$

Since u_n converges to zero.

45. Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous function $\ni$ $g(x)=g\left(\frac{x-1}{2}\right)\forall x$.

S.T. g must be a constant function.

Soln.:→ $g(x)=g\left(\frac{x-1}{2}\right)$

$\Rightarrow g\left(\frac{x-1}{2}\right) = g\left(\frac{\frac{x-1}{2}-1}{2}\right) = g\left(\frac{x-3}{4}\right)$

again putting $x=\frac{x-1}{2}$,

$g\left(\frac{x-1}{2}\right)=g\left(\frac{x-7}{8}\right)$ and so on.

generally we have, $g\left(\frac{x-1}{2}\right)=g\left(\frac{x-(2^n-1)}{2^n}\right)$

$\therefore g(x)=g\left(\frac{x}{2^n}-1+\frac{1}{2^n}\right)$

$\therefore \lim_{n\to\infty} g(x) = g(-1) \Rightarrow g(x)=g(-1)=\text{constant}\ \forall x$.

46. $\lim_{x\to 0} (\cos x)^{1/x^2} = ?$

Soln.:→ $(\cos x)^{1/x^2} = k$, say

$\therefore \ln k = \frac{1}{x^2}\ln(\cos x)$

$\therefore \lim_{x\to 0}(\ln k) = \lim_{x\to 0} \frac{\ln\cos x}{x^2} \quad \left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{-\tan x}{2x} \quad \left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{-\operatorname{cosec}^2 x}{2}$

$= -\frac{1}{2}$

$\therefore \lim_{x\to 0} k = e^{-1/2}$

47. Find the greatest and least value of the function $f(x) = x^3 - 3x^2 + 2x + 1$ in $[2,3]$.

Soln.:– $f(x) = x^3 - 3x^2 + 2x + 1$

$f'(x) = 3x^2 - 6x + 2$

$f'(2) = 2 > 0$

$f'(3) = 11 > f'(2)$

$\therefore f(x)$ is an increasing function.

Note that $f''(x) = 6x - 6 > 0 \;\; \forall x \in [2,3]$

$\therefore f(x)$ is concave.

Thus the function has min. value at $x = 2$ and max. value at $x = 3$.

$\therefore$ min. value $= f(2) = 1$

$\therefore$ max. value $= f(3) = 7$.

48. S.T. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$ can never be an integer value.

Soln.:→
We are to show :- $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{\text{an odd number}}{\text{an even number}} \;\; \forall n > 1$

Let; $P(n) : 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{\text{an odd number}}{\text{an even number}} \;\; \forall n > 1$

When $n = 2$, LHS $= 1 + \frac{1}{2} = \frac{3}{2} = \frac{\text{an odd number}}{\text{an even number}}$

$\therefore P(2)$ is true. Let $P(m)$ be true

$\Rightarrow 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} = \frac{\text{an odd no.}}{\text{an even no.}} = \frac{k}{r}$ (say)

Now, $P(m+1) = \frac{k}{r} + \frac{1}{m+1}$

m is odd or even,
but in case, it can be shown that

$P(m) = \frac{\text{an odd no.}}{\text{an even no.}}$

$\therefore P(n)$ is true for all $n \in \mathbb{N}$

49. Suppose $\{a_K\}$ is bounded sequence $(a_K \geq 0)$. S.T. $\frac{1}{n}\sum_{K=1}^{n} a_K \to 0$ as $n \to \infty$ iff $\frac{1}{n}\sum_{K=1}^{n} a_K^2 \to 0$ as $n \to \infty$.

ANS:-

50. Give an example of a function $f: [a,b] \to \mathbb{R} \ni |f(x)-f(y)| < (x-y)$ $\forall\ x, y \in [a,b]$. Prove that any function satisfying the above condition also satisfies

$$\left| \int_a^b f(x)\,dx - (b-a)f(a) \right| \leq \frac{1}{2}(b-a)^2, \text{ provided}$$

$f(\cdot)$ is integrable on $[a,b]$.

Soln.:→ $f(x) = \sin x$

$$|f(x)-f(y)| = |\sin x - \sin y| = \left| 2\cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right) \right|$$

$$\leq 2\left| \sin\left(\frac{x-y}{2}\right) \right|$$

$$\leq 2\left| \frac{x-y}{2} \right| = |x-y|$$

[$\because$ for $x>0$, $\sin x \leq x$]

[$\left| \frac{f(x)-f(y)}{x-y} \right| \leq 1 \quad \therefore \left| \lim_{x \to y} \frac{f(x)-f(y)}{x-y} \right| \leq 1 \Rightarrow |f'(y)| \leq 1$]

Let $\int_{a}^{u} f(x)\,dx = F(u)$

By Taylor's theorem:—

$$F(b) = F(a) + (b-a)F'(a) + \frac{(b-a)^2}{2} F''(c^*)$$

$$\Rightarrow F(b) - F(a) = (b-a)f(a) + \frac{(b-a)^2}{\lfloor 2} f'(c^*)$$

$$\Rightarrow \left| \int_a^b f(x)\,dx - (b-a)f(a) \right| = \frac{(b-a)^2}{2} |f'(c^*)|$$

$$\leq \frac{(b-a)^2}{2}$$

51. Let $F(x) = \sum_{k=0}^{n} a_k x^k$, where a_k satisfy $\sum_{k=0}^{n} \frac{a_k}{k+1} = 0$ so that there exists a real root of $f(x)=0$ in the interval $(0,1)$.

Soln.: $\Rightarrow$ $$F(x) = \int_0^x f(t)\,dt = \int_0^x \left(\sum_{k=0}^{n} a_k t^k\right) dt$$
$$= \sum_{k=0}^{n} a_k \cdot \frac{x^{k+1}}{k+1}$$

Clearly $F(x)$ satisfies the conditions of Rolle's theorem as $F(0)=0$ and $F(1) = \sum_{k=0}^{n} \frac{a_k}{k+1} = 0$

Hence $\exists$ a 'c' $\in (0,1) \ni F'(c) = 0 \Rightarrow F(c) = 0$.

52. Find the maximum value of $\iint_\Delta dx\,dy$ as a function of m, $0<m<1$, where,
$$\Delta = \left\{(x,y) : \frac{x^2}{m} + \frac{y^2}{1-m} \le 1\right\}$$

Soln.: $\rightarrow$
$$\iint_\Delta dx\,dy$$
$$= \sqrt{m}\sqrt{1-m} \int_0^{2\pi} \left(\int_0^1 r\,dr\right) d\theta \quad ; \text{ where } \frac{x}{\sqrt{m}} = r\cos\theta, \quad \frac{y}{\sqrt{1-m}} = r\sin\theta$$
$$= \pi\sqrt{m}\sqrt{1-m}$$
$$\le \pi \cdot \frac{m+1-m}{2} \quad (\because AM \ge GM)$$
$$= \pi/2.$$

53. Find max (xyz) subject to $x^2+2y^2+9z^2 = 6$

Soln.: $\rightarrow$ $$\frac{x^2+2y^2+9z^2}{3} \geq \sqrt[3]{x^2 \cdot 2y^2 \cdot 9z^2}$$
$$\Rightarrow \left(\frac{x^2+2y^2+9z^2}{3}\right)^3 \geq 18x^2y^2z^2$$
$$\Rightarrow 2^3 \geq 18x^2y^2z^2$$
$$\Rightarrow (xyz)^2 \le \frac{8}{18}$$
$$\Rightarrow xyz \le \frac{2}{3}.$$

54. Maximize $x+y$ sub. to the condition that $2x^2+3y^2 \le 1$.

Soln.:→

$$\frac{x^2}{1/2}+\frac{y^2}{1/3} \le 1$$

let, $z = x+y$

Now, $4x + 6y\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x}{3y}$$

at the touching pnt. $-\frac{2x}{3y} = -1$

$\Rightarrow 2x = 3y$ and $2x^2+3y^2=1$

$\Rightarrow 2\left(\frac{3y}{2}\right)^2+(3y^2)=1$

$\Rightarrow 15y^2 = 2$

$\Rightarrow y = \pm\sqrt{\frac{2}{15}}$

$\therefore x = \frac{3}{2}\left(\pm\frac{\sqrt{2}}{\sqrt{15}}\right) = \pm\sqrt{\frac{3}{10}}$

$\therefore$ $\text{Max}(z) = \sqrt{\frac{3}{10}}+\sqrt{\frac{2}{15}} = \frac{5}{\sqrt{30}}$.

55. If $a_1, a_2, \ldots, a_n$ are positive real nos. then

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1}$$ is always

i) $\ge n$, ii) $\le n$, iii) $n^{1/n}$, iv) none of these.

Soln.:→ AM $\ge$ GM gives

$$\frac{\frac{a_1}{a_2}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1}}{n} \ge \sqrt[n]{\frac{a_1}{a_2}\cdots\cdots\frac{a_{n-1}}{a_n}\cdot\frac{a_n}{a_1}}$$

$$= 1$$

$$\therefore \frac{a_1}{a_2}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1} \ge n$$

56. The maximum possible value of xy^2z^3 subject to the conditions $xyz \ge 0$ and $x+y+z=3$ is

i) 1, ii) $1/8$, iii) $1/4$, iv) $27/16$.

Soln.:→ $x+y+z=3 \Rightarrow x.\frac{2y}{2}+3.\frac{z}{3}=3$

Applying AM $\ge$ GM,

$$\frac{x+2.\frac{y}{2}+3.\frac{z}{3}}{1+2+3} \ge \sqrt[6]{x\left(\frac{y}{2}\right)^2\left(\frac{z}{3}\right)^3}$$

$$\Rightarrow \left(\frac{3}{6}\right)^6 \ge \frac{xy^2z^3}{2^2.3^3}$$

$$\Rightarrow xy^2z^3 \le \frac{27}{16}.$$

— By Tanujit Chakraborty
M: 8420253573
Mail: tanujitisi@gmail.com

57. Let A & B be two invertible $n\times n$ real matrices. Assume that $A+B$ is invertible. S.T. $A^{-1}+B^{-1}$ is also invertible.

Soln. :→ A, B invertible
$A+B$ "

$$|A|\,|A^{-1}+B^{-1}|\,|B| = |B+A| \neq 0$$

$$\Rightarrow |A^{-1}+B^{-1}| \neq 0 \text{ as } |A|, |B| \neq 0$$

$\Rightarrow A^{-1}+B^{-1}$ is invertible.

58. Let A be $n\times n$ orthogonal mtx where A is even and suppose $|A| = -1$, S.T. $|I-A| = 0$, where I denotes $n\times n$ identity mtx.

Soln. :→ $A^{-1} = A^T$ $\quad |A| = -1$

$\Rightarrow \frac{1}{\lambda} = \lambda$ $\quad \Rightarrow \prod_{i=1}^{n} \lambda_i = -1$ then at least one $\lambda_i = -1$

$\Rightarrow \lambda = \pm 1$

∴ Ch. equation is $|\lambda I_n - A| = 0$

$\Rightarrow |I_n - A| = 0$ for $\lambda_i = +1$.

59. a) Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 1+x_1y_1 & 1+x_1y_2 & \cdots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \cdots & 1+x_2y_n \\ \vdots & \vdots & & \vdots \\ 1+x_ny_1 & 1+x_ny_2 & \cdots & 1+x_ny_n \end{bmatrix}$$

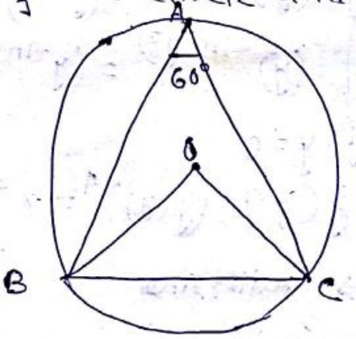
(b) Find the inverse of the matrix $A+\alpha\alpha'$ where A is the diagonal matrix $=$ dia$(A_1, A_2, \ldots, A_K)$ & $\alpha' = \left(\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_K}\right)$; $\alpha_i > 0 \ \forall\, i = 1(1)K$.

Soln. :→

(a) $$|A| = \begin{vmatrix} 1 & x_1 & 0 & \cdots & 0 \\ 1 & x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & 0 & \cdots & 0 \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix} = 0$$

(b) We know, $$(A+uv^T)^{-1} = A^{-1} - \frac{A^{-1}u\,v^T A^{-1}}{1+v^T A^{-1} v}$$

60. Evaluate: which of the following is necessarily true about the $\angle ABO$? Given, ABC be an inscribe triangle, where O be the centre of the circle and $\angle BAC = 60°$.



Soln: →

Let $\angle ABO = \theta$,

$\angle BOC = 2 \times \angle BAC = 120°$

Then $\angle OBC = \angle OCB = 30°$

In $\triangle ABC$, we have,

$$(\theta + 30°) + 30° + \angle ABO = 120°$$

$$\Rightarrow \theta + \angle ACO = 60°$$

$$\Rightarrow 0 < \theta < 60°$$

61) Do there exist function $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R} \ni$ $f(g(x)) = x^2$ and $g(f(x)) = x^3 \ \forall x \in \mathbb{R}$

Ans:- $f(x^3) = f(g(f(x))) = \{f(x)\}^2$

Now $x \in \{-1, 0\} \Rightarrow x^3 = x \Rightarrow f(x) = \{f(x)\}^2$

$\Rightarrow x \in \{0, 1\}$

Hence $\exists$ different $a, b \in \{-1, 0, 1\}$ such that $f(a) = f(b)$.

But then $a^3 = g(f(a)) = g(f(b)) = b^3$, a contradiction.

Thus the function f and g satisfying the given conditions don't exist.

62) Do there exist functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R} \ni$ $f(g(x)) = x^2$ and $g(f(x)) = x^4 \ \forall x \in \mathbb{R}$?

ANS:- Counter example:-

Define,
$$g(x) = \begin{cases} |x|^{\ln|x|} & \text{if } |x| \geq 1 \\ |x|^{-\ln|x|} & \text{if } 0 < |x| < 1 \\ 0 & \text{if } x = 0 \end{cases}$$

g is even and $|a| = |b|$ whenever $g(a) = g(b)$.

So, we can define f as an even function $\ni$.

$f(x) = y^2$, where y satisfies $g(\pm y) = x$

$f(g(x)) = x^2$ is clearly verified by the definition of f.

$$g(f(x)) = g(y^2) = \begin{cases} (y^2)^{\ln(y^2)} = (y^{\ln y})^4 & ; y \geq 1 \\ (y^2)^{-\ln(y^2)} = (y^{-\ln y})^4 & ; 0 < y < 1 \end{cases}$$

and $g(f(x)) = 0$ if $y = 0$

thus $g(f(x)) = g(y^2) = (g(y))^4 = x^4$.

63) Find all real numbers satisfying

$$6^x + 2^{2x} + 24^x - 36^x - 16^x = 1$$

Soln:- Rewrite the given relation as:

$$6^x + 4^x - 36^x + 24^x - 16^x = 1$$

Let $6^x = a$, $4^x = b$, we have

$$a + b - a^2 + ab - b^2 = 1$$

$$\Rightarrow a^2 - ab + b^2 - a - b + 1 = 0$$

$$\Rightarrow 2a^2 - 2ab + 2b^2 - 2a - 2b + 2 = 0$$

$$\Rightarrow (a^2 - 2ab + b^2) + (a^2 - 2a + 1) + (b^2 - 2b + 1) = 0$$

$$\Rightarrow (a - b)^2 + (a - 1)^2 + (b - 1)^2 = 0$$

$\therefore$ $a = 1$ and $b = 1$ when $a = b$.

$\Rightarrow$ $4^x = 1$ and $6^x = 1$, giving $x = 0$ only.

• **64)** Two boxes contain between them 65 balls of several different sizes. Each ball is white, black, red or yellow. If you take any five balls of the same colour, at least two of them will always be of the same size (radius). Prove that there are at least three balls which lie in the same box, have the same colour and are of the same size.

Soln.:→ We will make repeated use of pigon-hole-principle (PHP). As there are 65 balls and 2 boxes, one of these boxes must contain at least $\left[\frac{65}{2}\right]+1=33$ balls.

Consider that box, now we have four colours (white, black, red, yellow) and hence there must be at least $\left(\frac{33}{4}\right)+1=9$ balls of the same colour.

There can be at most 4 different sizes available for these 9 balls of the same colour. For if there were 5 (or more) different sizes, then collection of 5 balls, all of different sizes, could not satisfy the given property.

Thus of these 9 balls (of the same colour and in the same box) there must be at least 3 balls of the same size.

• **65)** Find all continuous function $f:(0,\infty)\to(0,\infty) \ni f(1)=1$ and

$$\frac{1}{2}\int_0^x (f(t))^2 dt = \frac{1}{x}\left(\int_0^x f(t)\,dt\right)^2.$$

Soln.:→ Define, $F(x)=\int_0^x f(t)dt$ and $G(x)=\int_0^x (f(t))^2 dt$

since $f:(0,\infty)\to(0,\infty)$ we have $F(x)>0 \;\forall x>0$

Also, $\frac{1}{2}G(x)=\frac{1}{x}\{F(x)\}^2$, from the given condition on differentiation, we have

$$\frac{1}{2}G'(x)=\frac{1}{x}\cdot 2F(x)\cdot F'(x)-\frac{1}{x^2}(F(x))^2$$

this means that $\frac{1}{2}(F(x))^2=\frac{2}{x}F(x)F'(x)-\frac{1}{x^2(F(x))^2}$

or, $\frac{1}{2}\left(\frac{xF'(x)}{F(x)}\right)^2=2\frac{xF'(x)}{F(x)}-1$

Solving this equation as a quadratic in $\frac{xF'(x)}{F(x)}$ we have

$$\frac{xF'(x)}{F(x)}=2\pm\sqrt{2}=k(\text{say})$$

on integration, we obtain $\int\frac{dF(x)}{F(x)}=k\int\frac{dx}{x}$

$\Rightarrow \ln F(x)=k\ln x+\ln\lambda$

$\Rightarrow F(x)=\lambda x^k$

$\Rightarrow f(x)=\lambda k x^{k-1} \Rightarrow f(1)=1 \Rightarrow \lambda k=1$

$\therefore f(x)=x^{k-1}=x^{1+\sqrt{2}}/x^{1-\sqrt{2}}$

66> If the quadratic equation $x^2+ax+b+1=0$ has non-zero integer solutions, then

(a) a^2+b^2 is a prime number

(b) ab is a prime number

(c) both a and (b)

(d) neither (a) nor (b)

Ans:- (d) $\alpha+\beta=-a$, $\alpha\beta=(b+1)$

$$\therefore a^2+b^2=(\alpha+\beta)^2+(\alpha\beta-1)^2$$
$$=(\alpha^2+1)(\beta^2+1)$$

67> Let $u=(\sqrt{5}-2)^{1/3}-(\sqrt{5}+2)^{1/3}$ and $v=(\sqrt{189}-8)^{1/3}-(\sqrt{189}+8)^{1/3}$, then for each positive integer n, $u^n+v^{n+1}=?$

(a) -1, (b) 0, (c) 1, (d) 2.

Soln.:→ (b) $u^3=(\sqrt{5}-2)-(\sqrt{5}+2)-3(\sqrt{5}-2)^{1/3}(\sqrt{5}+2)^{1/3}\cdot(u)$

i.e. $u^3=-4-3u$

$\Rightarrow (u-1)(u^2-u+4)=0$

u^2-u+4 is always +ve. So, $u=1$

Similarly $v^3+15v+16=0$

$\Rightarrow (v+1)(v^2-v+16)=0$

$\Rightarrow v=-1$.

So, for each n, $u^n+v^{n+1}=0$.

68. The number of real values of x satisfying the equation $x\cdot2^{1/x}+\frac{1}{x}\cdot2^x=4$ is/are

(a) 1, (b) 2, (c) 3, (d) 4.

Soln.:→ (a) If $x<0$, LHS = −ve but RHS = +ve.

If $x=0$, LHS = not defined.

If $x>0$, use AM ≥ GM inequality

$$x\cdot2^{1/x}+\frac{1}{x}\cdot2^x\geq 2\sqrt{2^{1/x+x}}$$
$$\geq 2\cdot\sqrt{2^2}=4$$

$\Rightarrow x\cdot2^{1/x}=\frac{1}{x}\cdot2^x$, So, $x=1$.

69. If $a+b+c=3$, $a^2+b^2+c^2=1$ and $a^3+b^3+c^3=3$, then $abc=?$

(a) 1 , (b) 2 , (c) 3 , (d) 4

Soln.:→ (d) $a^3+b^3+c^3-3abc$

$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

70. Let a, b, c be real numbers satisfying the equations, $ab-a=b+119$, $bc-b=c+59$, and $ca-c=a+71$, then the number of possible values of $a+b+c$ is/are

(a) 2 , (b) 4 , (c) 8 , (d) 16.

Soln.:→ (a) $ab-a-b=119$,

$\Rightarrow (a-1)(b-1)=120$ etc.

71. The numbers of distinct real roots of the equation

$x^4+8x^2+16=4x^2-12x+9$ is

(a) 1 , (b) 2 , (c) 3 , (d) 4

Soln.:→ (a) $(x^2+4)^2=(2x-3)^2$

$\Rightarrow x^2+4=\pm(2x-3)$

72. Let $f(x)$ and $g(x)$ be functions, which take integers as arguments. Let $f(x+y)=f(x)+g(y)+8$ for all integers x and y. Let $f(x)=x$ for all negative numbers x and let $g(8)=17$, then $f(0)=?$

(a) 8 , (b) 9 , (c) 17 , (d) 72

Soln.:→ (c) Put $x=-8$, $y=8$ in the given functional equation.

Q73. Let $x = \left[\dfrac{2007\cdot 2006\cdot 2004\cdot 2003}{\frac{1}{3}\times(2005)^4}\right]$, where $[x]$ denotes the greatest integer integer less than or equal to x. Then $\dfrac{((x^2+1).x^2)+1}{(x^2+1)}$ is

(a) 80 (b) 80·2 (c) 80·5 (d) 81

Soln.:→ (b)

$$x = \left[3\cdot\frac{2007}{2005}\cdot\frac{2006}{2005}\cdot\frac{2004}{2005}\cdot\frac{2003}{2005}\right]$$

$$= \left[3\left(1+\frac{2}{2005}\right)\left(1+\frac{1}{2005}\right)\left(1-\frac{1}{2005}\right)\left(1-\frac{2}{2005}\right)\right]$$

$$= \left[3\left(1-\frac{4}{(2005)^2}\right)\left(1-\frac{1}{(2005)^2}\right)\right]$$

$\Rightarrow x = 2.$

Q74) A graph is defined in polar co-ordinates by

$$r(\theta) = \cos\theta + \frac{1}{2}$$

The smallest x-co-ordinates of any point on this graph is

(a) $1/16$ (b) $-1/16$ (c) $1/8$, (d) $-1/8$

Soln.:→ (b) $x = r\cos\theta$

$= \cos^2\theta + \frac{1}{2}\cos\theta$

$= \left(\cos\theta + \frac{1}{4}\right)^2 - \frac{1}{16}$

75) A monic polynomial is one in which the coefficient of the highest order term is 1. The monic polynomial $P(x)$ (with integer coefficient) of least degree that satisfies

$$P\left(\sqrt{2}+\sqrt{5}\right)=0 \text{ is}$$

(a) $x^4-x^3-14x^2+9=0$ (b) $x^4-14x^2+9=0$

(c) $x^4+x^3-14x^2+9=0$ (d) $x^4+14x^2-9=0$

Soln.: (b) Let $x=\sqrt{2}+\sqrt{5}$

squaring, $x^2=7+2\sqrt{10}$

$\Rightarrow x^2-7=2\sqrt{10}$

squaring again, $x^4-14x^2+9=0$

76. Let $x\geq 1$, $f(x)=\dfrac{\sqrt{[x]}+\sqrt{\{x\}}}{\sqrt{x}}$ where $[\cdot]$ denotes G.I.F. and $\{\ \}$ denotes fractional part. Determine the smallest number $k \ni f(x)\leq k$ for each $x\geq 1$.

Ans:- Let $x=a+b$ where $a=[x]$, $b=\{x\}$

$$f(x)=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}}$$

$$\left(f(x)\right)^2=\frac{a+b+2\sqrt{ab}}{a+b}=1+\frac{2\sqrt{ab}}{a+b}.$$

using $(AM\geq GM)\leq 1+1 \Rightarrow f(x)\leq\sqrt{2}$.

77. Solve the equation $\left(\sqrt{2+\sqrt{2}}\right)^x+\left(\sqrt{2-\sqrt{2}}\right)^x=2^x$.

Ans:-

$$1+\frac{\sqrt{2}}{2}=1+\cos\frac{\pi}{4}=2\cos^2\frac{\pi}{8}$$

$$\left(\frac{2+\sqrt{2}}{4}\right)^{x/2}+\left(\frac{2-\sqrt{2}}{4}\right)^{x/2}=\left(\cos\frac{\pi}{8}\right)^x+\left(\sin\frac{\pi}{8}\right)^x$$

$$\Rightarrow x=2.$$

78. Let $f(x)$ be a polynomial with real coefficient for which the equation $f(x)=x$ has no real solution. Prove that the equation $f(f(x))=x$ has no real solution, either.

Soln. Suppose, if possible that $f(f(a))=a$, let $b=f(a)$, then $f(b)=a$. By hypothesis $b\neq a$. Assume that $a<b$ then $f(a)-a>0$ and $f(b)-b<0$. So, by intermediate value theorem $f(x)-x=0$ should be a root between (a,b). But this contradicts our assumption. Hence, $f(f(x))$ can have no real solution.

79. Let $a \in [0,4]$. Prove that the area bounded by the curves $y = 1 - |x-1|$ and $y = |2x-a|$ can't exceed $1/3$.

Soln.: When $a \in [0,1]$, the area is a triangle formed by $(0,0)$, $(1/2, 0)$ and $(1,1)$ with area equals $1/4$.

When $a \in [1,3]$, the area is a quadrilateral with vertices at $\left(\frac{a}{3}, \frac{a}{3}\right)$, $\left(\frac{a}{2}, 0\right)$, $\left(\frac{a+2}{3}, \frac{4-a}{3}\right)$ and $(1,1)$.

So, the net area is $\frac{1}{3} - \frac{(a-2)^2}{6}$ which also does not exceed $1/3$.

When $a \in [3,4]$, the area is same as when $a \in [0,1]$.

80. Determine a value of the parameter θ so that $f(x) = \cos^2 x + \cos^2(x+\theta) - \cos x \cos(x+\theta)$ is a constant function of x.

Ans:– $f(x) = \sin^2\theta + (2\cos\theta - 1)(\cos^2 x \cos\theta - \sin 2x \sin\theta)$

The function $f(x)$ is costant when $(2\cos\theta - 1) = 0$

i.e. $\theta = \pi/3$.

and the constant value is $3/4$.

81. Find the maximum value of $(1+2x)^2(3-2x)$.

Ans:– AM $\geq$ GM

$$\Rightarrow \frac{2 \cdot \frac{1+2x}{2} + (3-2x)}{2+1} \geq \sqrt[3]{\left(\frac{1+2x}{2}\right)^2(3-2x)}$$

$$\Rightarrow (1+2x)^2(3-2x) \leq 2^2 \cdot \left(\frac{4}{3}\right)^3$$

82. Minimize $3x + 4y$ subject to $x^2y^3 = 6$.

Ans:– AM $\geq$ GM

$$\Rightarrow \frac{\frac{3x}{2} + \frac{3x}{2} + \frac{4y}{3} + \frac{4y}{3} + \frac{4y}{3}}{5} \geq \sqrt[5]{\left(\frac{3x}{2}\right)^2\left(\frac{4y}{3}\right)^3}$$

$$\Rightarrow \frac{3x+4y}{5} \geq 2, \text{ as } x^2y^3 = 6$$

$$\Rightarrow 3x + 4y \geq 10$$

$(3x+4y)$ attain its maximum sub. to $x^2y^3 = 6$ when '=' holds in AM $\geq$ GM, i.e. iff $\frac{3x}{2} = \frac{4y}{3} \Rightarrow x = \frac{8}{9}y$

$\therefore x^2y^3 = 6 \Rightarrow y = \frac{3}{2}, x = \frac{4}{3}$.

83. If in an isosceles triangle with base 'a', vertical angle $20°$ and lateral side of each with length 'b' is given then the value of $a^3 + b^3$ equals

(a) $3ab$ (b) $3ab^2$ (c) $3a^2b$ (d) 3

ANS:- (b) $\sin 10° = \frac{a}{2b} \Rightarrow \sin 30° = 3\sin 10° - 4\sin^3 10°$

$$\Rightarrow \frac{1}{2} = \frac{3a}{2b} - \frac{4a^3}{8b^3}$$

$$\Rightarrow 1 = \frac{3a}{b} - \frac{4a^3}{4b^3}$$

$$\Rightarrow a^3 + b^3 = 3ab^2$$

84. If $a^2 + b^2 + c^2 - 2ab = 0$, then the point of concurrency of family of lines $ax + by + c = 0$ lies on the line

(a) $y = x$ (b) $y = x+1$ (c) $y = -x$ (d) $3x = y$

ANS:- (c) $(a-b)^2 - c^2 = 0$

$\Rightarrow (a-b-c)(a-b+c) = 0$

If $a - b = c \Rightarrow ax + by + (a-b) = 0$

$\Rightarrow a(x+1) + b(y-1) = 0 \Rightarrow x = -1, y = 1$

If $a - b = -c \Rightarrow ax + by + (b-a) = 0$

$\Rightarrow a(x-1) + b(y+1) = 0 \Rightarrow x = 1, y = -1.$

85. The values of k for which the inequality $k\cos^2 x - k\cos x + 1 \geq 0$ $\forall\, x \in (-\infty, \infty)$ holds is

(a) $k < -\frac{1}{2}$ (b) $k > 4$ (c) $-\frac{1}{2} \leq k \leq 4$ (d) $\frac{1}{2} \leq k \leq 5$

ANS:- (c) $k\cos^2 x - k\cos x + 1 \geq 0 \quad \forall\, x \in (-\infty, \infty)$

$\Rightarrow k(\cos^2 x - \cos x) + 1 \geq 0$ ......(i)

But $\cos^2 x - \cos x = \left(\cos x - \frac{1}{2}\right)^2 - \frac{1}{4}$

$\Rightarrow -\frac{1}{4} \leq \cos^2 x - \cos x \leq 2$

From (i), we get $2k + 1 \geq 0 \Rightarrow k \geq -\frac{1}{2}$

$\Rightarrow -\frac{k}{4} + 1 \geq 0$

$\Rightarrow k \leq 4 \Rightarrow -\frac{1}{2} \leq k \leq 4$

86. Number of triangles with each side having integral length and the longest side is of 11 unit is equal to k^2. Then the value of k is equal to

(a) 3 (b) 4 (c) 5 (d) 6.

ANS: (d) Let the three sides be $a \leq b \leq c = 11$

Then $6 \leq b < 11$ and $c - b < a \leq b$. As b decreased by 1 with the range of a decrease by 2.

When $b = 11$, we have $1 \leq a \leq 11$

Hence the total no. of triangles is $11 + 9 + 7 + 5 + 3 + 1 = 36$.

$\therefore k^2 = 36$

$\therefore k = 6$.

87) Prove that $1 < \frac{1}{1001} + \frac{1}{1002} + \cdots + \frac{1}{3001} < \frac{4}{3}$.

Soln.: →
Consider 2001 numbers $\frac{1}{k}$, $1001 \le k \le 3001$.
Using AM–HM inequality, we get

$$\left(\sum_{k=1001}^{3001} k\right)\left(\sum_{k=1001}^{3001} \frac{1}{k}\right) \geq (2001)^2$$

But $\sum_{k=1001}^{3001} k = (2001)^2$

Hence we get the inequality $\sum_{k=1001}^{3001} \frac{1}{k} > 1$.

On the other hand grouping 500 terms at a time, we also have

$$S = \sum_{k=1001}^{3001} \frac{1}{k} < \frac{500}{1000} + \frac{500}{1500} + \frac{500}{2000} + \frac{500}{2500} + \frac{1}{3001}$$

$$< \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{3000} = \frac{3851}{3000} < \frac{4}{3}$$

[**Remark:**– If $S = \sum_{k=n+1}^{3n+1} \frac{1}{k}$, there are $(2n+1)$ terms in the sum and the middle term is $\frac{1}{2n+1}$; then $\frac{29}{27} < S < \frac{7}{6}$.]

88. If ${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \cdots + (-1)^r\,{}^nC_r = 28$, then n is equal to (a) 6 (b) 7 (c) 8 (d) 9

Solution:–
(d) ${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \cdots\cdots + (-1)^r\,{}^nC_r$

$$= {}^{n-1}C_0 - \left({}^{n-1}C_0 + {}^{n-1}C_1\right) + \left({}^{n-1}C_1 + {}^{n-1}C_2\right) - \left({}^{n-1}C_2 + {}^{n-1}C_3\right) + \cdots + (-1)^r\left({}^{n-1}C_{r-1} + {}^{n-1}C_r\right);$$

$= (-1)^r \cdot {}^{n-1}C_r$

$\therefore (-1)^r\,{}^{n-1}C_r = 28 \Rightarrow r$ is even

$\therefore {}^{n-1}C_r = 28 = 7 \times 4 = \frac{7 \times 8}{2} = {}^8C_2$

$\Rightarrow n = 9$

— By Tanujit Chakraborty
M: 8420253573
Mail: tanujitisi@gmail.com

89. If $x \in \left(0, \frac{\pi}{2}\right)$ and $\cos x = \frac{1}{2}$ then the value of $\sum_{n=0}^{\infty} \frac{\cos nx}{3^n}$ is equal to

(a) 1 (b) -1 (c) 2 (d) -2

Soln: (a) Let $C = 1 + \frac{\cos x}{3} + \frac{\cos 2x}{3^2} + \cdots\cdots$

and $S = \frac{\sin x}{3} + \frac{\sin 2x}{3^2} + \cdots\cdots$

$\Rightarrow C + iS = 1 + \frac{e^{ix}}{3} + \frac{e^{2ix}}{3^2} + \cdots\cdots$

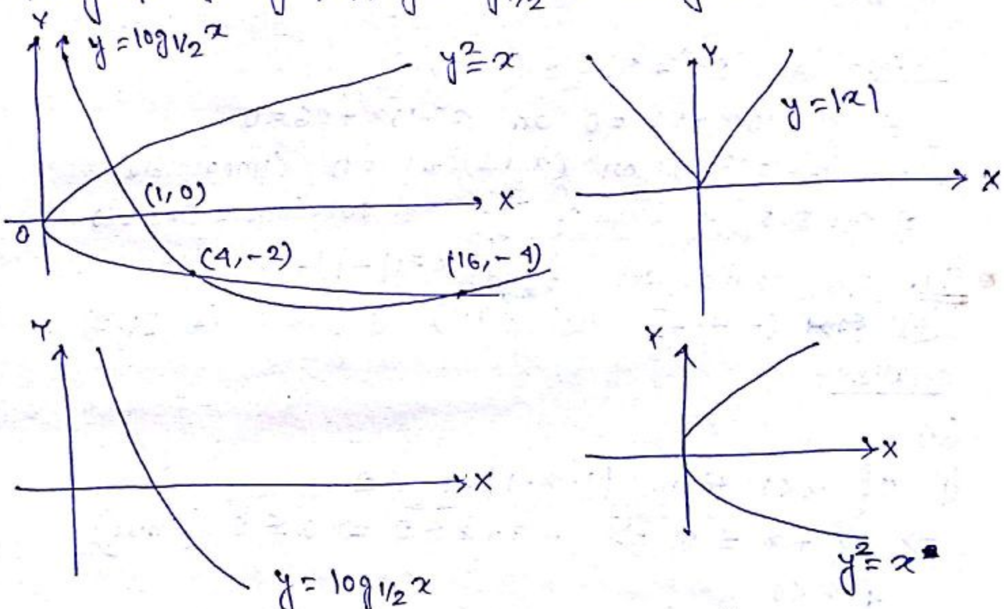
$= \frac{1}{1 - \frac{e^{ix}}{3}} = \frac{3}{3 - \cos x - i\sin x}$

comparing real parts

$C = \frac{3(3-\cos x)}{(3-\cos x)^2 + \sin^2 x} \Rightarrow C = 1 \left(\because \cos x = \frac{1}{2} \right)$

90. Draw the graph of $y = |x|$, $y = \log_{1/2} x$ and $y^2 = x$.

Soln:



91. How many solutions are possible in $0 \le x \le 99$ for the equation $|3 - 3^x| + |1 - 3^x| = 1 - 3^x - \frac{3^{-x}}{4}$.

Solution:- LHS $= |3 - 3^x| + |3^x - 1|$

$\geq |(3 - 3^x) + (3^x - 1)| \geq 2$

But RHS $= 1 - \left(3^x + \frac{3^{-x}}{4}\right)$

$= 1 - \left\{ \left(3^{x/2} + \frac{3^{-x/2}}{2}\right)^2 - 2 \cdot 3^{x/2} \cdot \frac{3^{-x/2}}{2} \right\}$

$= 2 - \left(3^{x/2} + \frac{3^{-x/2}}{2}\right)^2 < 2$

$\therefore$ Given equation is not solvable for any real x.

Ans:- 0

Q92. If $f(x) = \log_e(6 - |x^2+x-6|)$, then domain of $f(x)$ has how many integral values of x?

(a) 5 (b) 4 (c) infinite (d) none of these

Soln.:→

(b) $f(x)$ is defined only when $6 - |x^2+x-6| > 0$

i.e. $|x^2+x-6| < 6 \Rightarrow -6 < x^2+x-6 < 6$

$\Rightarrow x^2+x > 0$ and $x^2+x-12 < 0$

$\Rightarrow x(x+1) > 0$ and $(x+4)(x-3) < 0$

$\Rightarrow (x < -1$ or $x > 0)$ and $(-4 < x < 3)$

$\Rightarrow x \in (-4,-1) \cup (0,3)$

$\Rightarrow x = -3, -2, 1, 2$ as integral values.

Q93. The sum of the real solutions of $2|x|^2 + 51 = |1+20x|$ is

(a) 5 (b) 0 (c) 24 (d) none of these

Soln.:→ (d) $2x^2+51 = \pm(1+20x)$

$\Rightarrow x^2-10x+25 = 0$ or $x^2+10x+26 = 0$

$\Rightarrow (x-5)^2 = 0$ or $(x+5)^2+1 = 0$ (impossible)

$\Rightarrow x = 5, 5$ ∴ sum of the real solutions $= 5+5 = 10$.

Q94. The solution set of $||x-1|-1| + x \le 2$ is

(a) $(-\infty, 2]$, (b) $[0,1)$ (c) $[0,2)$ (d) $[1,2)$

Solution:-

(a)

(i) If $x < 0$, then $|1-x-1| + x \le 2$

$\Rightarrow |x| + x \le 2 \Rightarrow -x+x \le 2 \Rightarrow 0 \le 2$ (True)

∴ $x < 0$

(ii) If $0 \le x < 1$, then $|1-x-1| + x \le 2$

$\Rightarrow |x| + x \le 2 \Rightarrow 2x \le 2 \Rightarrow x \le 1$, ∴ $0 \le x < 1$

(iii) If $1 \le x < 2$, then $|x-1-1| + x \le 2$

$\Rightarrow |x-2| + x \le 2 \Rightarrow 2-x+x \le 2 \Rightarrow 2 \le 2$ (True)

(iv) If $x \ge 2$, then $|x-1-1| + x \le 2$

$\Rightarrow x-2+x \le 2 \Rightarrow x \le 2$ ∴ $x = 2$ [∵ $x \ge 2$]

∴ Required solution set is $(-\infty, 2]$

95. If domain of $f(x) = \sqrt{\dfrac{1}{|x-1| + [x]}}$ be (a, b), then

($[\cdot]$ denotes greatest integer function)

(a) $a = 1, b = \infty$ (b) $a = -\infty, b = 0$ (c) $a = -\infty, b = 1$

(d) none of these.

Solution:- (c) we must have, $|x-1| > [x]$ ......(1)

$\because x-1 < [x] \le x$, i.e. $[x] \geq x-1$ ....(2)

$\therefore$ On combining (1) and (2), we have $|x-1| > x-1$

This is true only if $x-1 < 0$, i.e. if $x < 1$, i.e. if $x \in (-\infty, 1)$

$\therefore D_f = (-\infty, 1) \Rightarrow a = -\infty, b = 1$

96. If there are 4 distinct solutions of $\left||x-2012| + \log_2 a\right| = 3$, then $a \in$

(a) $(-\infty, -6)$ (b) $\left(-\infty, \frac{1}{8}\right)$, (c) $\left(-\infty, -\frac{1}{8}\right)$, (d) none of these.

Solution:- (b) We have $|x-2012| + \log_2 a = \pm 3$

$\Rightarrow |x-2012| = -\log_2 a + 3, \ -\log_2 a - 3$

$\therefore$ If there are 4 distinct solutions of the above equation, then we must have $-\log_2 a + 3 > 0$ and $-\log_2 a - 3 > 0$

i.e. $\log_2 a < 3$ and $\log_2 a < -3 \Rightarrow a < 2^{-3}$

$\therefore a \in \left(-\infty, \frac{1}{8}\right)$

97. The number of values of k for which the equation $x^3 - 3x + k = 0$ has two distinct roots lying in the interval $(0, 1)$ are

(a) 3 (b) 2 (c) infinitely many

(d) no value of k satisfies the requirement

Soln.:-

(d) Let there be a value of k for which $x^3 - 3x + k = 0$ has two distinct roots between 0 and 1. Let, a, b are two distinct roots of $x^3 - 3x + k = 0$ lying between 0 and 1 such that $a < b$

Let $f(a) = f(b) = 0$. Since between any two roots of a polynomial $f(x)$ there exists at least one roots of its derivative $f'(x)$. Therefore $f'(x) = 3x^3 - 3$ has at least one root between a and b. But $f'(x) = 0$ has two roots equal to ± 1 which don't lie between a and b. Hence $f(x) = 0$ has no real roots lying between 0 and 1 for any value of k.

98. If $\frac{dy}{dx} = f(x) + \int_0^1 f(x)\,dx$, then the equation of the curve $y = f(x)$ passing through $(0,1)$ is

(a) $f(x) = \frac{2e^x - e + 1}{3 - e}$ (b) $f(x) = \frac{3e^x - 2e + 1}{2(2-e)}$

(c) $f(x) = \frac{e^x - 2e + 1}{e + 1}$ (d) none of these

Soln.: → (a) $f''(x) = f'(x)$

$$\Rightarrow \frac{f''(x)}{f'(x)} = 1$$

On integrating $f'(x) = ce^x$

which gives $f(x) = ce^x + D$

But $f(0) = 1 \Rightarrow c + D = 1$

$\therefore f(x) = ce^x + 1 - c$

So, $f'(x) = ce^x$. Putting it in $f'(x) = f(x) + \int_0^1 f(x)\,dx$

$$\Rightarrow ce^x = ce^x + 1 - c + \int_0^1 (ce^x + 1 - c)\,dx$$

$$\Rightarrow c = \frac{2}{3-e}.$$

So, $f(x) = \frac{2e^x - e + 1}{3 - e}$.

99. A staircase has 10 steps. A person can go up the steps one at a time, two at a time, or any combination of 1's and 2's. the number of ways in which the person can go up the stairs is

(a) 89 (b) 144 (c) 132 (d) 211

Soln.: → (a) $x + 2y = 10$, where x is the number of times he takes single steps, and y is the number of times he takes two steps

	Case	Total no. of ways
1	$x=0, y=5$	$5!/5! = 1$
2	$x=2, y=4$	$6!/2!\,4! = 15$
3	$x=4, y=3$	$7!/3!\,4! = 35$
4	$x=6, y=2$	$8!/2!\,6! = 28$
5	$x=8, y=1$	$9!/8! = 9$
6	$x=10, y=0$	$10!/10! = 1$

$\therefore\ p = 89$

100. Find the remainder when $1690^{2608} + 2608^{1690}$ is divided by 7?

Soln.:→ $1690 = 7\times 241 + 3$

$2608 = 7\times 372 + 4$

Let $S = 1690^{2608} + 2608^{1690}$

$= (7\times 241 + 3)^{2608} + (7\times 372 + 4)^{1690}$

$=$ a number multiple of $7 + 3^{2608} + 4^{1690}$.

Let $S' = 3^{2608} + 4^{1690}$

Clearly remainder in S and S' will be same when divided by 7.

$S' = 3\times 3^{3\times 867} + 4\times 4^{3\times 563}$

$= 3\times 27^{867} + 4\times 64^{563}$

$= 3(28-1)^{867} + 4(63+1)^{563}$

$= 3[\text{multiple of } 7 - 1] + 4[\text{multiple of } 7 + 1]$

$=$ multiple of $7 + 1$

∴ Hence remainder is 1.

101. Find the value of $\displaystyle\sum_{\substack{i=0 \\ (i\neq j\neq k)}}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i 3^j 3^k}$.

Solution:– Let us first of all find the sum without any restriction i, j, k.

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i 3^j 3^k} = \left(\sum_{i=0}^{\infty}\frac{1}{3^i}\right)^3 = \frac{27}{8}.$$

For the requirement sum we have to remove the cases when $i=j=k$ or when any two of them are equal and not equal to other variable (say, $i=j\neq k$).

Case I:– When $i=j=k$

In this case $\displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i 3^j 3^k} = \sum_{i=0}^{\infty}\frac{1}{3^i} = \frac{27}{16}$.

Case II:– $i=j\neq k$

In this case, $\displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i 3^j 3^k} = \left(\sum_{i=0}^{\infty}\frac{1}{3^{2i}}\right)\left(\sum_{k=0}^{\infty}\frac{1}{3^k}\right)$

$$= \sum_{i=0}^{\infty}\frac{1}{3^{2i}}\left(\frac{3}{2} - \frac{1}{3^i}\right)$$

$$= \frac{3}{2}\cdot\frac{9}{8} - \frac{27}{26} = \frac{135}{8\cdot 26}$$

Hence required sum $= \dfrac{27}{8} - \dfrac{27}{26} - \left(\dfrac{135}{8\cdot 26}\right)\cdot 3 = \dfrac{81}{208}$.

102. Find the solution of the differential equation
$$f(x)\frac{dy}{dx} + f'(x)\,y = 1.$$

Solution:- $f(x)\,dy + f'(x)\,y\,dx = dx$

i.e. $d\left(f(x).y\right) = d(x)$

Integrating we get, $y.f(x) = x + c$

or, $f(x) = \frac{x+c}{y}$.

103. If $\int_0^x f(x)\sin t\,dt = \text{constant}$, $0 < x < 2\pi$ and $f(\pi) = 2$ then find the value of $f\left(\frac{\pi}{2}\right)$.

(a) 2 (b) 4 (c) 6 (d) 8

Solution:-

(b) Differentiate both sides, we get

$f'(x)(1-\cos x) + f(x)\sin x = 0$

$\Rightarrow \int \frac{f'(x)}{f(x)}dx = \int \frac{\sin x}{1-\cos x}dx$

$\Rightarrow \ln|f(x)| = -2\ln \sin\frac{x}{2} + \ln c$

$\Rightarrow f(x) = \frac{c}{\left(\sin\frac{x}{2}\right)^2}$

$\Rightarrow f(\pi) = 2$

$\Rightarrow c = 2$

$f\left(\frac{\pi}{2}\right) = 4.$

104. For $a \in R$ if $|x+a-3| + |x-2a| = |2x-a-3|$ is true for all $x \in R$, then exhaustive set of a is

(a) $a \in [-4, 4]$, (b) $a \in [-3, 2]$, (c) $a \in \{-2, 2\}$ (d) $a \in \{1\}$

Solution:- (d) $|x| + |y| = |x+y|$

$\Rightarrow xy \geq 0$, therefore $(x-(3-a))(x-2a) \geq 0 \;\forall x \in R$

$\Rightarrow x^2 - x(3+a) + 2a(3-a) \geq 0 \;\forall x \in R$

$\Rightarrow (a+3)^2 - 8a(3-a) \leq 0 \Rightarrow (a-1)^2 \leq 0 \Rightarrow a = 1$

which is true $\forall\; x \in R$

105. If A is a skew-symmetric matrix, then $B = (I-A)(I+A)^{-1}$ is (where I is the identity mtx of the same order as A)

(a) idempotent mtx (b) symmetric mtx,
(c) orthogonal " (d) none of these

Soln.: → (c)

$$B = (I-A)(I+A)^{-1}$$

$$\Rightarrow B^T = (I+A^T)^{-1}(I-A^T)$$

$$= (I-A)^{-1}(I+A)$$

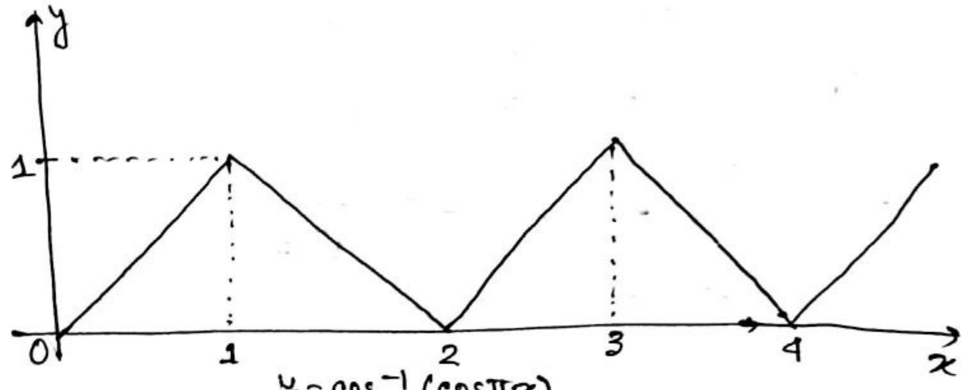
$BB^T = I$ as $(I-A)(I+A) = (I+A)(I-A)$.

106. If $f(x) = \max\left(\frac{1}{\pi}\cos^{-1}(\cos \pi x), \{x\}\right)$ and $g(x) = \min\left(\frac{1}{\pi}\cos^{-1}(\cos \pi x), \{x\}\right)$ (where $\{\cdot\}$ represents fractional part of x). Then find the value of

$$\int_1^2 f(x)\,dx \Big/ \int_1^2 g(x)\,dx$$

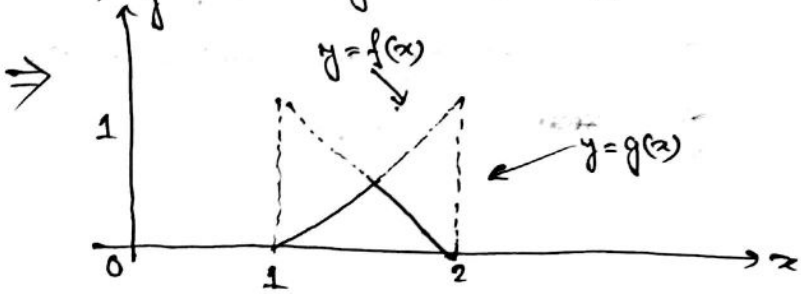
(a) 1 (b) 3 (c) 5 (d) 7

Solution:–
(b)



$y = \cos^{-1}(\cos \pi x)$

⇒



$$\Rightarrow \int_1^2 f(x)\,dx = \frac{3}{4} \text{ and } \int_1^2 g(x)\,dx = \frac{1}{4}$$

$\Rightarrow$ Ratio = 3

107. If $\sin(\sin x + \cos x) = \cos(\cos x - \sin x)$ and largest possible value of $\sin x$ is $\frac{\pi}{k}$, then find the value of K.

Solution:–

$$\sin(\sin x + \cos x) = \cos(\cos x - \sin x)$$
$$\cos(\cos x - \sin x) = \cos\left(\frac{\pi}{2} - (\sin x + \cos x)\right)$$
$$\therefore \cos x - \sin x = 2n\pi \pm \left(\frac{\pi}{2} - (\sin x + \cos x)\right)$$

Taking +ve sign

$$\cos x - \sin x = 2n\pi + \frac{\pi}{2} - \sin x - \cos x$$

$\cos x = n\pi + \frac{\pi}{4}$, for $n = 0$, $\cos x = \frac{\pi}{4}$, which is the only possible value

$$\Rightarrow \sin x = \frac{\sqrt{16 - \pi^2}}{4} \quad \text{-----} (i)$$

Taking –ve sign

$$\sin x = \frac{\pi}{4} \quad \text{--------} (ii)$$

From (i) & (ii), we get $\frac{\pi}{4}$ as the largest value.

Hence $K = 4$.

108. The no. of solution(s) of the equation

$$z^2 - z - |z|^2 - \frac{64}{|z|^5} = 0 \text{ is/are}$$

(a) 0, (b) 1, (c) 2, (d) 3

Soln:→ (b)

$$z^2 - z = \cancel{|z|^2} + \frac{64}{|z|^5}$$
$$\Rightarrow z^2 - \cancel{z} = \bar{z}^2 - \bar{z}$$

$z = 2$ is the only solution.

So, there is only one solution of the given equation.

109. If function $f(x) = \cos(nx)\sin\left(\frac{5x}{n}\right)$, satisfies $f(x+3\pi) = f(x)$, then find the number of integral values of n.

(a) 8 (b) 9 (c) 10 (d) 11

Solution:-

(a) $f(x+\lambda) = f(x)$

$\Rightarrow \cos n(x+\lambda)\sin\left(\frac{5(x+\lambda)}{n}\right) = \cos(nx)\sin\left(\frac{5x}{n}\right)$

At $x=0$, $\cos n\lambda \sin\left(\frac{5\lambda}{n}\right) = 0$

if $\cos n\lambda = 0$, $n\lambda = r\pi + \frac{\pi}{2}$, $r \in I$

$n(3\pi) = r\pi + \frac{\pi}{2}$ $(\because \lambda = 3\pi)$

$(3n - r) = \frac{1}{2}$ [not possible]

$\therefore \cos n\lambda \neq 0$ $\therefore \sin\left(\frac{5\lambda}{n}\right) = 0 \Rightarrow \frac{5\lambda}{n} = p\pi$ $(p \in I) \Rightarrow n = \frac{15}{p}$

For $p = \pm 1, \pm 3, \pm 5, \pm 15$

$n = \pm 15, \pm 5, \pm 3, \pm 1$

110. If $f(x) = x + \sin x$, then find

$$\frac{2}{\pi^2} \cdot \int_{\pi}^{2\pi} \left(f^{-1}(x) + \sin x\right) dx$$

(a) 2 (b) 3 (c) 6 (d) 9

Solution:-

(b) Let $x = f(t) \Rightarrow dx = f'(t)\,dt$

$$\Rightarrow \int_{\pi}^{2\pi} f^{-1}(x)\,dx = \int_{\pi}^{2\pi} t\, f'(t)\,dt = \left(t\,[f(t)]\right)_{\pi}^{2\pi} - \int_{\pi}^{2\pi} f(t)\,dt$$

$$= (4\pi^2 - \pi^2) - \int_{\pi}^{2\pi} f(t)\,dt$$

$$I = \int_{\pi}^{2\pi} \left(f^{-1}(x) + \sin x\right) dx = \int_{\pi}^{2\pi} f^{-1}(x)\,dx + \int_{\pi}^{2\pi} \sin x\,dx$$

$$= 3\pi^2 - \int_{\pi}^{2\pi} f(t)\,dt + \int_{\pi}^{2\pi} \sin x\,dx$$

$$= 3\pi^2 - \int_{\pi}^{2\pi} \left(f(x) - \sin x\right) dx$$

$$= 3\pi^2 - \int_{\pi}^{2\pi} x\,dx = 3\pi^2 - \frac{1}{2}(4\pi^2 - \pi^2)$$

$$= \frac{3}{2}\pi^2$$

$$\Rightarrow \frac{2}{\pi^2} I = 3$$

111. The maximum value of xyz for +ve x, y, z subject to condition that $xy+yz+zx=12$ is

(a) 9 , (b) 6 , (c) 8 , (d) none.

Solution:-

$$\frac{xy+yz+zx}{3} \geq (xy \cdot yz \cdot zx)^{1/3}$$

$$\Rightarrow (xyz) \leq 8$$

112. Let a, b, c be any real numbers such that $a^2+b^2+c^2=1$ then the quantity $ab+bc+ca$ satisfies the conditions

(a) $ab+bc+ac = \text{constant}$

(b) $-\frac{1}{2} \leq ab+bc+ca \leq 1$

(c) $-\frac{1}{4} \leq ab+bc+ca \leq 1$

(d) $-1 \leq ab+bc+ca \leq \frac{1}{2}$.

Solution:-

$$(a+b+c)^2 \geq 0$$

$$\Rightarrow a^2+b^2+c^2 \geq -2(ab+bc+ca)$$

$$\Rightarrow \frac{1}{2} \geq -(ab+bc+ca)$$

$$\Rightarrow (ab+bc+ca) \geq -\frac{1}{2}.$$

113. $x^2-y^2=666$; Does X and Y have integral solution?

Ans:-

$$x^2-y^2=666$$

$$(x+y)(x-y)=666$$

$$\Rightarrow x+y=k, \quad x-y=\frac{666}{k}$$

$$\therefore k+\frac{666}{k}=2x$$

$$\Rightarrow$$

There will be no solution.

114. a, b, c are 3 positive real numbers such that $a+b+c=2$, then prove that $\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} > 8$.

Soln.:- Let $1-a=x, \quad 1-b=y, \quad 1-c=z$

$$3-(a+b+c) = x+y+z = 1 \quad (\because a+b+c=2)$$

Now, $\frac{1-x}{x} \cdot \frac{1-y}{y} \cdot \frac{1-z}{z}$

$$= \frac{y+z}{x} \cdot \frac{z+x}{y} \cdot \frac{y+x}{z}$$

$$= \left(\frac{y+z}{2}\right)\left(\frac{z+x}{2}\right)\left(\frac{y+x}{2}\right) \cdot \frac{8}{xyz}$$

$$> \sqrt{yz}\,\sqrt{zx}\,\sqrt{zy}\,\frac{8}{xyz} \quad (\text{By AM} > \text{GM inequality})$$

$$\Rightarrow \frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} > 8 \quad (\text{Proved})$$

115. $a+b+c=1$ then prove that $\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1} \le 21$.

Solution:-

$$4a+4b+4c=4$$

$$\Rightarrow (4a+1)+(4b+1)+(4c+1)=7$$

Applying c-s inequality :-

$$a_1=\sqrt{4a+1}, \quad a_2=\sqrt{4b+1}$$

$$a_3=\sqrt{4c+1}$$

$$b_i=1$$

$$\therefore \left(\sum_{i=1}^{3} a_i \cdot 1\right)^2 \le \left(\sum_{i=1}^{3} a_i^2\right)\left(\sum_{i=1}^{3} 1\right) \text{ ; where } a_i = a_1, a_2, a_3$$

$$\Rightarrow \left(\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1}\right)^2 \le (4a+1+4b+1+4c+1) \times (1+1+1)$$

$$= 3\times 7$$

$$= 21$$

$\therefore$ (Proved).

~~**116.** $ax^2+bx+c=0$; $bx^2+cx+a=0$; $cx^2+ax+b=0$; Is it possible that each of equation has 2 real roots?~~

~~Soln. :-~~

116. If $f(x)$ is a polynomial function satisfying $f(x)f\left(\frac{1}{x}\right) = f(x)+f\left(\frac{1}{x}\right)$ and $f(3)=28$ then $f(4)=$?

Solution:-

$$f(x)\cdot f\left(\tfrac{1}{x}\right) = f(x)+f\left(\tfrac{1}{x}\right)$$

is satisfied by $f(x) = \pm x^n + 1$.

$$f(3) = +3^3+1 = 28$$

Hence, $n=3$.

So, $f(4) = 4^3+1$

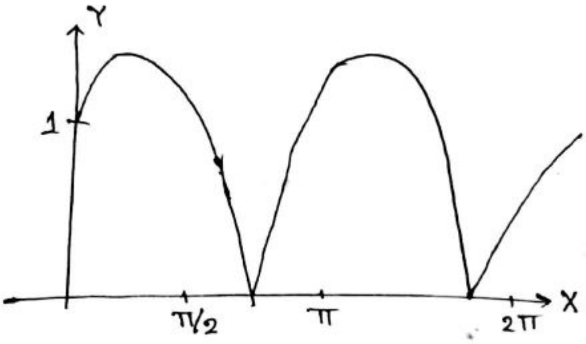
$$= 65.$$

— By Tanujit Chakraborty

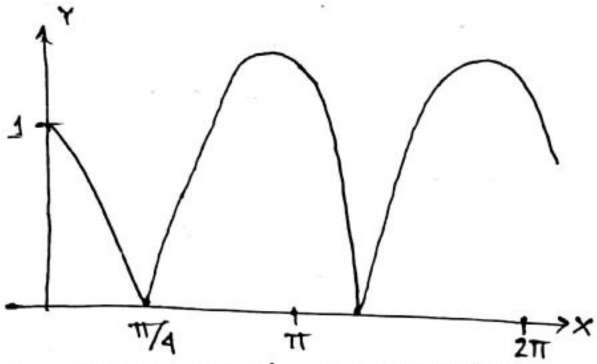
M: 8420253573

Mail: tanujitisi@gmail.com

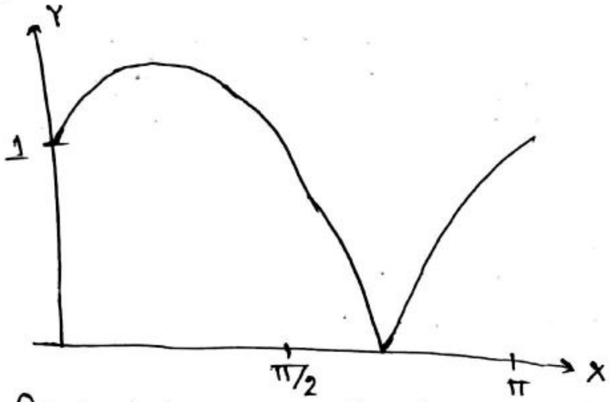
—: Some Important Graphs:-



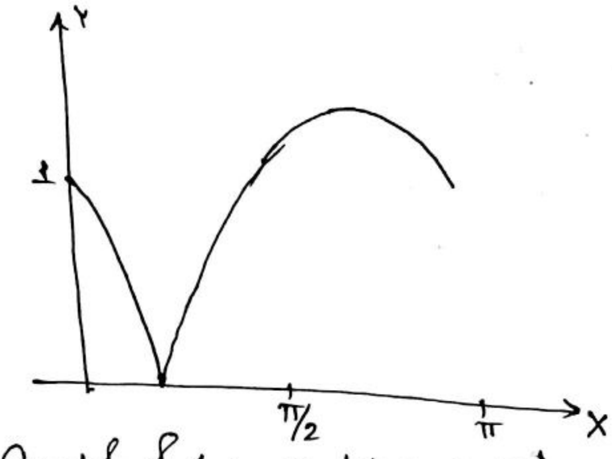
Graph 1: $y = |\sin x + \cos x|$ $(0 \leq x \leq 2\pi)$



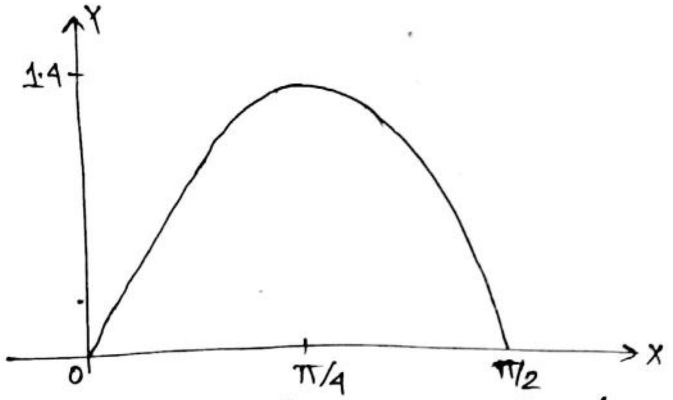
Graph2: $y = |\sin x - \cos x|$ $(0 \leq x \leq 2\pi)$



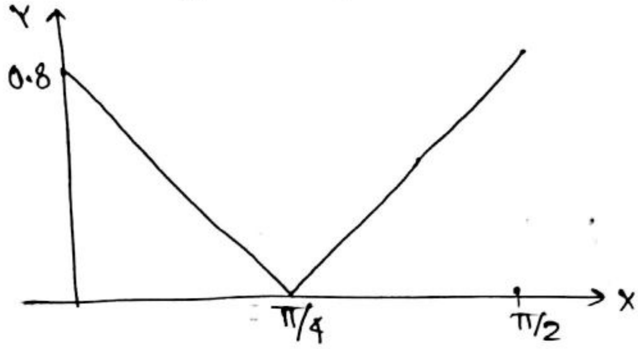
Graph of 3. $y = |\sin x + \cos x|$ $(0 \leq x \leq \pi)$



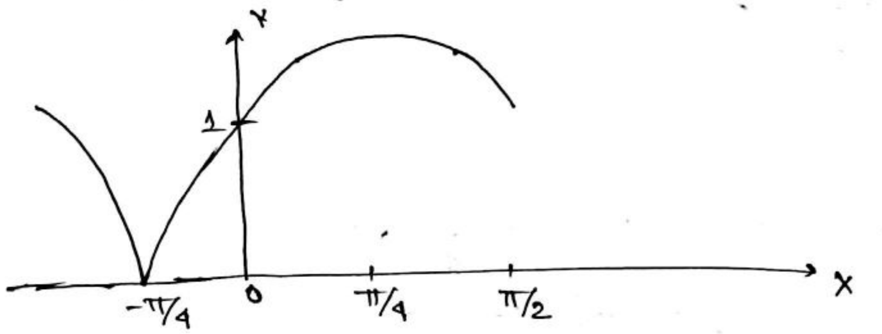
Graph of 4 :- $y = |\sin x - \cos x|$



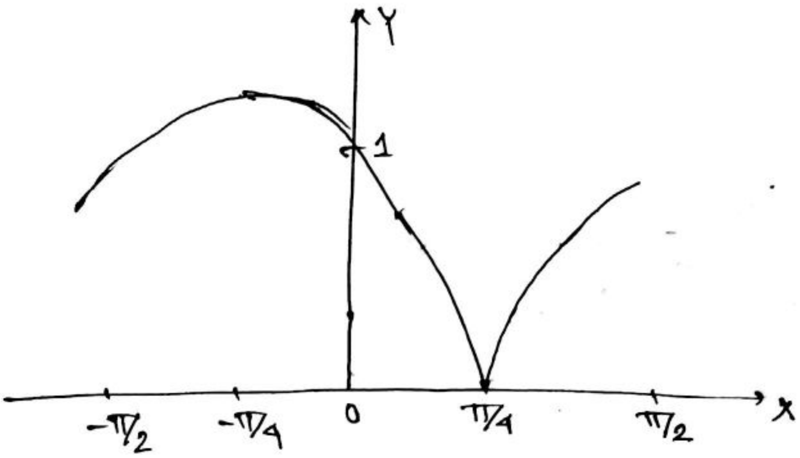
Graph 5:- $y = |\sin x + \cos x|$ $(0 \le x \le \pi/2)$.



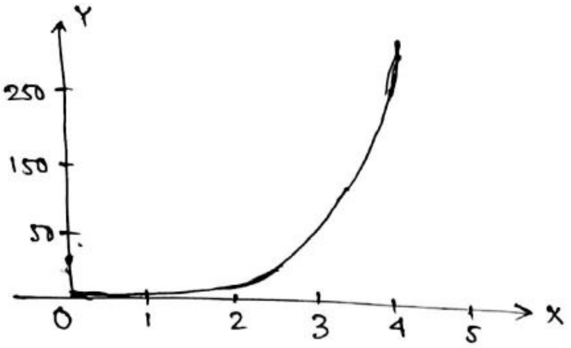
Graph 6:- $y = |\sin x - \cos x|$ $(0 \le x \le \pi/2)$



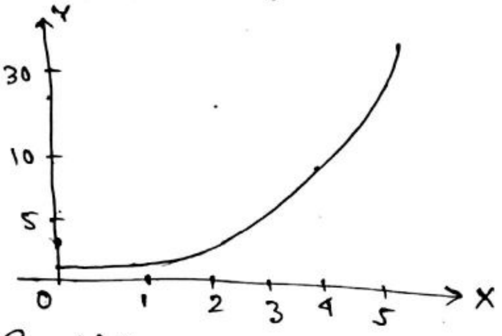
Graph 7:- $y = |\sin x + \cos x|$ $(-\pi/2 \le x \le \pi/2)$



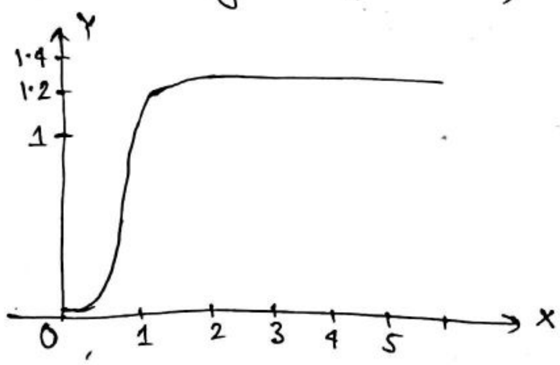
Graph 8:- $y = |\sin x - \cos x|$ $(-\pi/2 \le x \le \pi/2)$



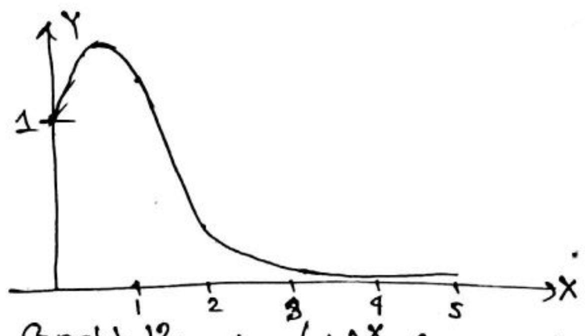
Graph 9:- $y = x^{x}$ $(0 \le x \le 5)$



Graph 10:- $y = 2^{x}$ $(0 \le x \le 5)$



Graph:- 11. $y = x^{\frac{1}{x}}$ $(0 \le x \le 5)$



Graph 12. $y = \left(\frac{1}{x}\right)^{x}$ $(0 \le x \le 5)$

117. If $2x+4y=1$, then prove that $x^2+y^2 \geq \frac{1}{20}$.

Solution:- maximize x^2+y^2 subject to $2x+4y-1=0$ by method of Lagrange multiplier. $\longrightarrow$

$$F = x^2+y^2+\lambda(2x+4y-1)$$

$$\frac{\partial F}{\partial x} = 2x+2\lambda = 0 \quad ; \quad \frac{\partial F}{\partial y} = 2y+4\lambda = 0$$

$$\therefore x = -\lambda \qquad \therefore y = -2\lambda$$

$$2x+4y=1 \qquad x_{max} = +\frac{1}{10},\ y_{max} = \frac{1}{5}$$

$$\Rightarrow \lambda = \frac{-1}{10} \ ;$$

$$\therefore x^2+y^2 \geq \frac{1}{100}+\frac{4}{100} = \frac{5}{100} = \frac{1}{20}.$$

(Proved)

118. If a, b, c are positive real numbers $\ni a+b+c=1$, prove that $a^2+b^2+c^2 \geq \frac{1}{3}$.

soln.:- Using C-S inequality,

$$\left(\sum_{i=1}^{3} x_i y_i\right)^2 \leq \left(\sum x_i^2\right)\left(\sum y_i^2\right)$$

Taking $y_i = 1$

$$\Rightarrow (a+b+c)^2 \leq (a^2+b^2+c^2) \cdot 3$$

$$\Rightarrow a^2+b^2+c^2 \geq \frac{1}{3}.$$

119. If $x+\left(\frac{1}{x}\right) = -1$, find $x^{99}+\left(1/x^{99}\right)$.

[Ans. 2]

Soln.:- If, $a_n = x^n + \frac{1}{x^n}$

then, $a_{n+1} = a_n \cdot a_1 - a_{n-1}$ for $n \geq 1$.

$a_0 = 2$, $a_2 = -a_1 - a_0 = -1$, $a_3 = 2$, $a_4 = -1$, $a_5 = 2$.

$a_1 = -1$; $a_n = -1$, $a_{n-1} = -1$. So, $a_{99} = 2$.

$$\therefore a_{n+1} = x^{99}+\frac{1}{x^{99}} = \left(x^{98}+\frac{1}{x^{98}}\right)\left(x+\frac{1}{x}\right) - \left(x^{97}+\frac{1}{x^{97}}\right)$$
$$= -a_n - a_{n-1} = +1+1 = 2.$$

120. If a, b, c, x are real numbers such that $abc \neq 0$ and

$$\frac{xb+(1-x)c}{a} = \frac{xc+(1-x)a}{b} = \frac{xa+(1-x)b}{c},$$

then prove that $a=b=c$ or $a+b+c=0$.

Soln.:-

$$\frac{xb+(1-x)c}{a} = \frac{xc+(1-x)a}{b} = \frac{xa+(1-x)b}{c} = 1$$

$$\therefore x = \frac{a-c}{b-c},\quad x = \frac{b-a}{c-a},\quad x = \frac{c-b}{a-b}$$

The only solution of these are: $a=b=c$ or $a+b+c=0$.

• **121.** If $f: R \to R$ is given by $f(x) = \dfrac{4^x}{4^x+2}$ $\forall x \in R$. Show that $f(x) + f(1-x) = 1$. Hence show that

$$f\left(\frac{1}{1997}\right) + f\left(\frac{2}{1997}\right) + \cdots + f\left(\frac{1996}{1997}\right) = 998.$$

Solution:- $f(1-x) = \dfrac{4^{1-x}}{4^{-x+1}+2} = \dfrac{4/4^x}{4/4^x+2} =$

$$f(x) + f(1-x) = 1.$$

Now, Putting $x = \dfrac{1}{1997}, \dfrac{2}{1997}, \dfrac{3}{1997}, \ldots, \dfrac{998}{1997}$

so, $f\left(\frac{1}{1997}\right) + f\left(\frac{2}{1997}\right) + \cdots + f\left(\frac{1996}{1997}\right)$

$$= \underbrace{(1+1+\cdots+1)}_{998 \text{ terms}}$$

$$= 998.$$

• **122.** If $\gcd(a,b) = 1$, then $\gcd(a+b, a-b)$ is
(a) a or b (b) 1 or 2 (c) 1 or 3 (d) none.

Soln:- (b) Let $d = \gcd(a+b, a-b)$ then
$d \mid (a+b)$ and $d \mid (a-b)$.
$\therefore$ $d \mid (a+b+a-b)$, $\Rightarrow d \mid 2a$ and
$\therefore$ $d \mid (a+b-a+b)$, $\Rightarrow d \mid 2b$.
thus $d \mid (2a, 2b)$, $\Rightarrow d \mid 2(a,b)$
Hence $d = 1$ or 2, because $\gcd(a,b) = 1$.

• **123.** Let $p > 1$ and $2^p - 1$ is prime, then p is
(a) prime (b) even (c) odd (d) none.

Soln.:- (a) suppose p = composite $= mn$, where $m, n > 1$ ...

$\therefore$ $2^p - 1 = (2^m)^n - 1 = a^n - 1^n$ ($2^m = a$, say)
$= (a-1)(a^{n-1} + a^{n-2} + \cdots + 1^{n-1})$
> 1 = composite number,

but which contradict our assumption,
so, p is a prime number.

• **124.** The number of solutions (positive integers) of the equation $3x + 5y = 1008$ is
(a) 61 (b) 67 (c) 79 (d) none.

Soln.:- (b) $x, y \in \mathbb{N}$, then $3 \mid 5y \Rightarrow 3 \mid y$, $y = 3k$ $\forall k \in \mathbb{N}$

Thus $3x + 15k = 1008$
$\Rightarrow x + 5k = 336$
$\Rightarrow 5k \le 335$
$\Rightarrow k \le 67$.

125. When 2^m+3^n is a perfect square, then $(m,n)=$
(a) 1, 4 (b) 1, 2 (c) 4, 2 (d) none.

Solution:- Let $2^m+3^n=k^2$, then $(-1)^n \equiv 2^m \equiv k^2 \equiv 1 \pmod 3$
$(\because 3 \nmid k)$ where m is even, say $2p$.

$\therefore (k-2^p)(k+2^p)=3^n \Rightarrow k-2^p=1$ and $k+2^p=3^n$
$\Rightarrow 2^{p+1}+1=3^n.$

Again, $(-1)^n \equiv 3^n \pmod 4 = 2^{p+1}+1=1$, where n is even, say $2q$.

$\therefore (3^q-1)(3^q+1)=2^{p+1}$, $\Rightarrow 3^q-1=2$, $\Rightarrow 3^q=3$, $\Rightarrow q=1$
and so $p=2$.
Hence, we get the only solution is $(4,2)$.

126. The number of values of n for which $3^9+3^{12}+3^{15}+3^n$ a perfect cube, is (a) 2 (b) 6 (c) 8 (d) none.

Solution:- (d) $3^9+3^{12}+3^{15}+3^n$

$$= 3^9(1+3^4+3^6+3^{n-9})$$
$$= (3^3)^3\{1+3.3^2+3.(3^2)^2+(3^2)^3+3^{n-9}-3(3^2)^2\}$$
$$= (3^3)^3(1+3^2)^3,$$

$\Rightarrow 3^{n-9}-3^5=0$
$\Rightarrow n-9=5$
$\Rightarrow n=14.$

127. The number of integral solutions of $xy=2^2.3^4.5^7(x+y)$ is
(a) 675 (b) 680 (c) 685 (d) none.

Soln.:- (a) Let $N=2^2.3^4.5^7$
then, $xy=N(x+y)$
$\Rightarrow xy-Nx-Ny=0$
$\Rightarrow (x-N)(y-N)=N^2=2^4.3^8.5^{14}$

$\therefore$ The number of integral solutions $=(4+1)(8+1)(14+1)$
$=675.$

128. A printer numbers the pages of a book starting with 1 and uses 3189 digit in all, then the number of pages are
(a) 200 (b) 300 (c) 400 (d) none.

Soln.:- (d) No. of digits used for numbering pages 1 to 9
$=9\times1=9.$
Similarly, 10 to 99 $=90\times2=180$, 100 to 999 $=900\times3=2700$.
Number of digits will remain after using 2889 $(=9+180+2700)$ digits $=3189-2889=300$. The digits can be used for numbering $300\div4=75$ pages, i.e. from 1000 to 1074. Hence the book has 1074 pages.

129. The unit's digit of $3^{1001}.7^{1002}.13^{1003}$ is
(a) 1 (b) 3 (c) 5 (d) none.

Sol. (d) unit digit in 3^{1001} is 3;
7^{1002} is 9;
and 13^{1003} is 7;

$\left[\begin{array}{l} 3^1 = (3),\ 7^2 = 4(9) \\ 13^3 = 219(7) \end{array} \right]$

$\therefore$ Ans. is $= 3 \times 9 \times 7 = 18(9)$;
$\therefore$ 9 is in unit place.

130. How many ordered triplet (x, y, z) of non zero real numbers have the property that each number is the product of the other two?

Sol.
$$x = yz,\quad y = zx,\quad z = xy$$
$$\therefore xyz = (xyz)^2$$
i.e. $xyz = 0$ or 1.
Now, $xyz = x^2 = y^2 = z^2$
$\Rightarrow |x| = |y| = |z| = 1$.
However the remaining 4 cases are: $(1,1,1)$, $(-1,-1,-1)$, $(-1,1,-1)$ or $(1,-1,-1)$; i.e. it has 4 solutions.

131. The number of pairs of positive integers (x, y) which satisfy the equation $x^2 + y^2 = x^3$ is
(a) 0 (b) 1 (c) 2 (d) none.

Soln.:- (d) $y^2 = x^2(x-1)$
so, if k is an integer satisfying $x - 1 = k^2$
$$\Rightarrow x = k^2 + 1$$
Thus there are infinitely many solutions.

132. The remainder on dividing $1234^{567} + 89^{1011}$ by 12 is
(a) 1 (b) 7 (c) 9 (d) none.

Sol. (c) $1234 \equiv 1 \pmod 3 \Rightarrow 1234^{567} \equiv 1 \pmod 3$ and $89 \equiv -1 \pmod 3$
$\Rightarrow 89^{1011} \equiv -1 \pmod 3$
$\therefore 1234^{567} + 89^{1011} \equiv 0 \pmod 3$.
Here 1234 is even, so $1234^{567} \equiv 0 \pmod 4$ and $89 \equiv 1 \pmod 4$,
$\Rightarrow 89^{1011} \equiv 1 \pmod 4$
Thus $1234^{567} + 89^{1011} \equiv 1 \pmod 4$.
Hence it is $9 \pmod{12}$.

133. Consider the equation of the form $x^2+bx+c=0$, the number of such equations that have real roots and have coefficients b and c in the set $\{1,2,3,4,5,6\}$, (b may be equal to c), is

(a) 16 (b) 19 (c) 21 (d) none.

soln.:- (b) Let $x^2+bx+c=0$ has real roots, then $b^2-4c \geq 0$, and also, $S=\{1,2,3,4,5,6\}$. Now, $S_1=\{4,8,12,16,20,24\}$ = set of possible values of 4c.

Thus the number of equations will be same as the number of pairs of elements (a_1, a_2), $a_1 \in S$, $a_2 \in S_1$ such that

$a_1^2 - 4a_2 \geq 0$, i.e. $1+2+4+6+6=19$.

134. If $16-x^2 > |x-a|$ is to be satisfied by atleast one non-negative values of x, then complete set of values of 'a' is

(a) $(-8, 8)$, (b) $\left(-16, \frac{65}{4}\right)$, (c) $\left(-8, \frac{65}{4}\right)$, (d) none.

soln.:- (b) $16-x^2 > |x-a|$

$$\Rightarrow x^2-16 < x-a < 16-x^2$$

$$\Rightarrow x^2-16-x < -a < 16-x^2-x$$

$$\Rightarrow x+16-x^2 > a > -16+x^2+x$$

$$\Rightarrow \frac{65}{4}-\left(x-\frac{1}{2}\right)^2 > a > -16+x^2+x;\ x \leq 0$$

$$\therefore a \in \left(-16, \frac{65}{4}\right) \quad (\because x \in \mathbb{R})$$

135. Number of solutions for $x^2-2-2[x]=0$, where $[\cdot]$ = greatest integer, is

(a) 0 (b) 1 (c) 2 (d) none.

Sol. (b) $x^2 = 2+2[x] \geq 0 \Rightarrow [x] = -1$

when $[x]=-1$, then $x^2-2=-2 \Rightarrow x=0$, which is not possible.

when $[x]=1 \Rightarrow x^2=4 \Rightarrow x=\pm 2$ (impossible)

when $[x]=2 \Rightarrow x^2=6 \Rightarrow x=\pm\sqrt{6}$, i.e. only one possible value, i.e. $\sqrt{6}$.

136. The solution set of $\left|\frac{x+1}{x}\right| + |x+1| = \frac{(x+1)^2}{|x|}$ is

(a) $\{x: x \geq 0\}$ (b) $\{x: x>0\} \cup \{-1\}$ (c) $\{-1, 1\}$ (d) none.

Sol.

(b) $$\frac{|x+1|}{|x|} + |x+1| \pm \frac{|x+1|^2}{|x|}$$

$$\Rightarrow |x+1|\left(\frac{1}{|x|} + 1 - \frac{|x+1|}{|x|}\right) = 0$$

$$\Rightarrow |x+1| = 0 \quad \text{or,} \quad 1+|x|-|x+1| = 0$$

$$\Rightarrow |x+1| = 0$$

$$\Rightarrow x+1>0 \text{ and } x \neq 0.$$

i.e. $x=-1$, or >0

i.e. $\{x: x>0\} \cup \{-1\}$.

137. The sum of the cubes of the root of equation $x^4+ax^3+bx^2+cx+d=0$, is
(a) a^3-3c (b) $3ab-a^3$ (c) $3ab-c$ (d) none.

Soln:- (d) Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the root of the equation.
Here $a_0=1$, $a_1=a$, $a_2=b$, $a_3=c$, $a_4=d$.

$$\therefore a_0 s_1 + a_1 = 0 \Rightarrow s_1 + a = 0 \Rightarrow s_1 = -a,$$
$$\Rightarrow \alpha_1+\alpha_2+\alpha_3+\alpha_4 = -a.$$

Now, $a_0 s_2 + a_1 s_1 + 2a_2 = 0$
$$\Rightarrow s_2 + a(-a) + 2b = 0$$
$$\Rightarrow s_2 = \alpha^2 - 2b,$$
$$\Rightarrow \alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2 = a-2b,$$

$$\therefore a_0 s_3 + a_1 s_2 + a_2 s_1 + 3a_3 = 0$$
$$\Rightarrow s_3 = -a^3+3ab-3c$$
$$\Rightarrow \alpha_1^3+\alpha_2^3+\alpha_3^3+\alpha_4^3 = 3ab-a^3-3c.$$

138. The integral roots of $5x^3-11x^2+12x-2=0$ are (a) $(1,2,3)$ (b) $(-1,-2,-3)$ (c) $(0,1,2)$ (d) none.

Soln:- (d) $f(x) = 5x^3-11x^2+12x-2=0$
where constant term $=-2$, and divisisors of constant term are $\pm 1, \pm 2$, i.e. the possible value of integral roots are $\pm 1, \pm 2$.
Now, $f(1) \neq 0$, $f(-1) \neq 0$; $f(2) \neq 0$; $f(-2) \neq 0$. So it has no integral roots.

139. If $f(x) = x^2+2bx+2c^2$ and $g(x) = -x^2-2cx+b^2$ are such that $\min f(x) > \max. g(x)$, then relation between b and c, is
(a) $|c| > \sqrt{2}$ (b) $0 < c < \frac{b}{2}$ (c) $|c| < \sqrt{2}|b|$ (d) none.

Soln:- (a) $f(x) = (x+b)^2 + 2c^2 - b^2$
$$\Rightarrow \min f(x) = 2c^2 - b^2$$
$$g(x) = b^2+c^2-(x+c)^2$$
$$\Rightarrow \max g(x) = b^2 + c^2$$
Thus, $\min f(x) > \max g(x)$
$$\Rightarrow 2c^2-b^2 > b^2+c^2$$
$$\Rightarrow c^2 > 2b^2$$
$$\Rightarrow |c| > \sqrt{2}|b|.$$

140. If $x^2 - x + a - 3 = 0$ has at least one negative value of x, then complete set of values of 'a' is

(a) $(-\infty, 1)$ (b) $(-\infty, 2)$ (c) $(-\infty, 3)$ (d) none.

Soln:- (c) $x^2 - x + a - 3 = 0$ has at leas one negative root and

for real roots, $1 - 4(a-3) \geq 0$

$$\Rightarrow a \leq \frac{13}{4}$$

$$\Rightarrow a \in \left(-\infty, \frac{13}{4}\right)$$

Now, both root will be non-negative if $D \geq 0$, $\Rightarrow a - 3 \geq 0$

$$\Rightarrow a \geq 3$$

$$\therefore a \in \left[3, \frac{13}{4}\right]$$

$$\therefore a \in \left[-\infty, \frac{13}{4}\right] \cup \left[3, \frac{13}{4}\right]$$

$$\therefore a \in (-\infty, 3).$$

141. Let α, β are the roots of the equation $x^2 + ax + b = 0$, then max. value of the expression $-(x^2 + ax + b) - \left(\frac{\alpha - \beta}{2}\right)^2$ will be

(a) $\frac{1}{4}(a^2 - 4b)$ (b) 0 (c) 1 (d) none.

Soln:- (b) Let, $Z = -(x^2 - ax + b)$

Now, $Z_{max.} = -\frac{D}{4a} = -\frac{a^2 - 4b}{4} = \frac{4b - a^2}{4} = +\left(\frac{\alpha - \beta}{2}\right)^2$

$\therefore$ Thus the max. value of the given equation is 0.

142. Let $p(x) = x^2 + bx + c$, where b and c are integers and $p(x)$ is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, then $p(1)$ is

(a) 4 (b) 8 (c) 24 (d) none.

Soln:- (a) $\because p(x)$ is a factor of $3(x^4 + 6x^2 + 25) - (3x^4 + 4x^2 + 28x + 5)$

$$= 14(x^2 - 2x + 5)$$

$$\therefore p(x) = x^2 - 2x + 5$$

$$\Rightarrow p(1) = 4.$$

143. The value of a for which $(a^2-1)x^2 + 2(a-1)x + 2 > 0$ $\forall x$ are

(a) $a \geq 1$ (b) $a \leq 1$ (c) $a > -3$ (d) none.

Soln:- (d) we know, $px^2 + qx + c > 0$ if $p > 0$, and $q^2 - 4pc < 0$,

$\therefore (a^2-1)x^2 + 2(a-1)x + 2 > 0$ $\forall x$

Now, $a^2 - 1 > 0$ and $4(a-1)^2 - 8(a^2-1) \leq 0$

$\Rightarrow a^2 - 1 \geq 0$ and $-4(a-1)(a+3) \leq 0$

$\Rightarrow a \leq -1$ or $a \geq 1$ and $a \leq -3$ or $a \geq 1$

i.e. $a \leq -3$ or $a \geq 1$.

• **144.** The sum of real roots of the equation
$x^2 - 2^{2007} \cdot x + |x - 2^{2006}| + 2(2^{4011} - 1) = 0$ is
(a) 2^{2006} (b) 2^{2007} (c) $2^{2006} + 2^{2007}$ (d) none.

sol. (b) $\therefore (x - 2^{2006})^2 + |x - 2^{2006}| - 2 = 0$
$\Rightarrow |x - 2^{2006}|^2 + |x - 2^{2006}| - 2 = 0$
$\Rightarrow x = 2^{2006} + 1,\ 2^{2006} - 1$
$\therefore$ the sum of real roots are $= 2^{2007}$.

• **145.** Consider an expression $x^2 + y^2 + 2x + y =$ const. If for two constants α, β, the conditions $x > \alpha$ and $x < \beta$ imply the same limits for the values of y, then $\alpha + \beta$ is
(a) -2 (b) -4 (c) 1 (d) none.

sol. (a) $x^2 + y^2 + 2x + y = k$
$\Rightarrow (x+1)^2 + \left(y + \frac{1}{2}\right)^2 = k + \frac{5}{4}$
$\Rightarrow x = -1 \pm \sqrt{\left(k + \frac{5}{4}\right)^2 - \left(y + \frac{1}{2}\right)^2}$

Now, the two values of x corresponds to α and β as y takes the same limits of values.
Hence $\alpha + \beta = -2$.

• **146.** $\frac{a^4 + b^4}{a^2 + b^2} + \frac{b^4 + c^4}{b^2 + c^2} + \frac{c^4 + a^4}{c^2 + a^2} \geq$
(a) $a + b + c$ (b) $a^2 + b^2 + c^2$ (c) $ab + bc + ca$ (d) none.

Soln:- $(a^2 - b^2)^2 \geq 0$
$\Rightarrow a^4 + b^4 \geq 2a^2 b^2$
$\Rightarrow 2a^4 + 2b^4 \geq a^4 + b^4 + 2a^2 b^2 = (a^2 + b^2)^2$
$\Rightarrow \frac{a^4 + b^4}{a^2 + b^2} \geq \frac{a^2 + b^2}{2}$ ------- (1)
similarly, $\frac{b^4 + c^4}{b^2 + c^2} \geq \frac{b^2 + c^2}{2}$ ------- (2)
and, $\frac{c^4 + a^4}{c^2 + a^2} \geq \frac{c^2 + a^2}{2}$ ------- (3)
(1) + (2) + (3) implies
$\frac{a^4 + b^4}{a^2 + b^2} + \frac{b^4 + c^4}{b^2 + c^2} + \frac{c^4 + a^4}{c^2 + a^2} \geq a^2 + b^2 + c^2$.

—— By Tanujit Chakraborty
M: 8420253573
Mail: tanujitisi@gmail.com

147. Let $m > 1$, $n \in \mathbb{N}$, then $1^m + 2^m + 2^{2m} + 2^{3m} + \cdots + 2^{nm-m} >$

(a) n^{1-m} (b) $(1-m)^n$ (c) $n^{1-m}(2^n-1)$ (d) none.

Soln. :- (c) $\dfrac{1^m + 2^m + 4^m + 8^m + \cdots + (2^{n-1})^m}{n} > \left(\dfrac{1+2+4+\cdots+2^{n-1}}{n}\right)^m$

[$\because m > 0$ and AM of mth power > mth power of AM]

$\Rightarrow 1^m + 2^m + 4^m + \cdots + 2^{(n-1)m} > n\left(\dfrac{2^n-1}{n}\right)^m$

$> n^{1-m}(2^n-1)^m$.

148. Let $x^2+y^2=c^2$, then the least value of $x^{-2}+y^{-2}$ is

(a) c (b) c^2 (c) c^3 (d) none.

Sol. (d) Let $Z = x^{-2}+y^{-2} = \dfrac{x^2+y^2}{x^2y^2} = \dfrac{c^2}{x^2y^2}$ and

it will be min., when x^2y^2 will be maximum.

As $x^2+y^2=c^2$, then x^2y^2 is max. when $x^2=y^2=\dfrac{c^2}{2}$

$\therefore Z_{min.} = \dfrac{c^2}{\frac{c^4}{4}} = \dfrac{4}{c^2}$.

149. $n^n\left(\dfrac{n+1}{2}\right)^{2n} >$

(a) $n!$ (b) $(n!)^2$ (c) $(n!)^3$ (d) none.

Sol. (c) $\dfrac{1^3+2^3+\cdots+n^3}{n} > \left(1^3 2^3 \ldots n^3\right)^{1/n}$ [$\because$ AM > GM]

$\Rightarrow \dfrac{n(n+1)^2}{4} > \{(n!)^3\}^{1/n}$

$\Rightarrow n^n\left(\dfrac{n+1}{2}\right)^{2n} > \{(n!)^3\}$.

150. If $a_1, a_2, a_3, \ldots, a_n$ are non-negative and $a_1a_2a_3\cdots a_n = 1$, then $(1+a_1)(1+a_2)\cdots(1+a_n) \geq$

(a) 2^n (b) 3^n (c) 4^n (d) none.

Sol. :- (a) $\left(\dfrac{1+a_i}{2}\right) \geq \sqrt{a_i}$, where $i = 1(1)n$. (AM $\geq$ GM)

Putting the all i values and then multiplies the inequalities,

$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 2^n\sqrt{a_1a_2\cdots a_n}$

$\Rightarrow (1+a_1)(1+a_2)\cdots(1+a_n) \geq 2^n$ ($\because a_1a_2\cdots a_n = 1$).

151. If $a_1, \ldots, a_n$ are positive real nos. whose product is a fixed number c, then the minimum value of $a_1+a_2+\cdots+a_{n-1}+2a_n$ is

(a) $n(2c)^{1/n}$ (b) $(n+1)c^{1/n}$ (c) $2nc^{1/n}$ (d) none.

Sol. (a) AM $\geq$ GM

so, LHS $\geq n(a_1 \cdots 2a_n)^{1/n} = n(2c)^{1/n}$.

152. Let x_i are non-negative reals and $S = x_1 + x_2 + \cdots + x_n$, then $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n \leq$

(a) $\frac{S^2}{2}$ (b) $\frac{S^2}{3}$ (c) $\frac{S^2}{4}$ (d) none.

Sol.:- (c) $(x_1 + x_3 + x_5 + \cdots)(x_2 + x_4 + x_6 + \cdots) \geq x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n$

when expanding as LHS, we must get RHS and many additional non-negative terms since $x_i \neq 0$.

Thus maximum acheived by taking $x_1 = x$, $x_2 = S - x$ and all other terms 0, But

$$x(S-x) \leq \frac{S^2}{4} \text{ with equality when } x = \frac{S}{2}$$

(Using AM $\geq$ GM)

153. For any positive reals x, y, z and a is the arithmatic mean of x, y, z then $x^x y^y z^z$ is

(a) $\geq (xyz)^a$ (b) $< (xyz)^a$ (c) $> (xyz)^a$ (d) none.

Sol. (a) Let $x \geq y \geq z$, then $x^x y^y \geq x^y y^x$, as $\left(\frac{x}{y}\right)^x \geq \left(\frac{x}{y}\right)^y$ is obviously true.

Similarly, $y^y z^z \geq y^z z^y$ and $z^z x^x \geq z^x x^z$.

Multiplying all these,

$$(x^x y^y z^z)^2 \geq x^{y+z} \cdot y^{z+x} \cdot z^{x+y}$$

$$\Rightarrow x^x y^y z^z \times (x^x y^y z^z)^2 \geq x^{x+y+z} y^{x+y+z} z^{x+y+z}$$

$$\Rightarrow (x^x y^y z^z)^3 \geq (xyz)^{3a}$$

$$\Rightarrow x^x y^y z^z \geq (xyz)^a$$

154. The number of integers between 1 and 567 are divisible by either 3 or 5, is

(a) 200 (b) 250 (c) 300 (d) none.

Sol. Let $Z = \{1, 2, 3, \ldots, 566, 567\}$

$P = \{x \in \frac{Z}{3} \text{ devides } x\}$

and $Q = \{x \in \frac{Z}{5} \text{ devides } x\}$

Here, $|P| = 189$ [$\because 567 = 189 \times 3$]

and $|Q| = 113$ [$\because 567 = 113 \times 5 + 2$]

$P \cap Q$ = set of multiple of both 3 and 5,

$|P \cap Q| = 37$; $|P \cup Q| = 189 + 113 - 37 = 265$.

155. Sets A and B have 3 and 6 elements respectively. What can be the minimum number of elements in $A \cup B$?

Sol. $n(A \cup B) \geq \max\{n(A), n(B)\}$

Thus $n(A \cup B) \geq \max\{3, 6\} = 6$.

156. A has n elements. How many (B, C) are such that $\phi \neq B \subseteq C \subseteq A$?

(a) 2^n (b) 3^n (c) 4^n (d) none.

Sol. (b) There are ${}^nC_m$ choices for a subset B with m elements. Then each of the remaining $n-m$ elements can be in C or not, so there are 2^{n-m} choices for C.

Thus the total no. of pairs (B, C) is $\sum 2^{n-m} \cdot {}^nC_m = \sum 2^m \cdot {}^nC_m$

$= (1+2)^n = 3^n$ (from Binomial theorem) $[\because {}^nC_m = {}^nC_{n-m}]$

157. If $X = \{n : n$ is a positive integer, $n \leq 50\}$, $A = \{n \in X : n$ is even$\}$ and $B = \{n \in X : n$ is a multiple of $7\}$, then what is the number of elements in the smallest subset of X containing both A and B?

Sol. :- The number of integers $\leq n$ and divisible by k is given by $\left[\frac{n}{k}\right]$, where $[\cdot]$ denotes the greatest integer function.

Accordingly, $n(A) = \left[\frac{50}{2}\right] = 25$, $n(B) = \left[\frac{50}{7}\right] = 7$

$n(A \cap B) = \left[\frac{50}{14}\right] = 3$.

$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 25 + 7 - 3 = 29$.

158. If $\dfrac{1}{x(x+1)(x+2)\cdots(x+n)} = \displaystyle\sum_{r=0}^{n} \frac{A_r}{x+r}$, then $A_r = ?$

Sol. By method of representation for partial fraction A_r, is obtained by putting $x+r=0$, i.e. $x=-r$ in all factors, except $(x+r)$.

$$\therefore A_r = \frac{1}{-r(-r+1)\cdots(-r+r+1)(-r+r+1)(-r+r+2)\cdots(-r+n)}$$

$$= \frac{1}{(-1)^r \{r!\}\{(n-r)!\}}$$

159. When m, n are positive integers, then $(m+n)!$ is divisible by

(a) $m!\,n!$ (b) $m!+n!$ (c) $m!-n!$ (d) none.

Sol. :- $(m+n)! = 1 \cdot 2 \cdots m(m+1)(m+2)\cdots(m+n)$

$= \dfrac{m! \cdot (m+1)(m+2)\cdots(m+n)\,n!}{n!}$

$= m!\,n! \times$ an integer

$\Rightarrow \dfrac{(m+n)!}{m!\,n!} =$ an integer.

• 160. If n and r are positive integers such that $0 < r < n$ then the roots of the quadratic equation
$${}^nC_{r-1}x^2 + 2 \cdot {}^nC_r x + {}^nC_{r+1} = 0 \text{ are}$$
(a) rational (b) imaginary (c) real & distinct (d) none.

Sol. (c) Discriminant $(D) = 4\{({}^nC_r)^2 - {}^nC_{r-1} \cdot {}^nC_{r+1}\}$
$= 4(a-b)$, where $a = ({}^nC_r)^2$,
$b = {}^nC_{r-1} \cdot {}^nC_{r+1}$.

$$\therefore \frac{a}{b} = \frac{r+1}{r} \cdot \frac{n-r+1}{n-r} = \left(1+\frac{1}{r}\right)\left(1+\frac{1}{n-r}\right) > 1 \text{ as, } n > r.$$

$\therefore a > b$
$\Rightarrow D > 0$.

• 161. A bag contains unlimited number of white, red, black and blue balls. The number of ways of selecting 10 balls so that there is at least one ball of each colour is
(a) 180 (b) 270 (c) 192 (d) none.

Sol. (d) No. of ways = coeff. of x^{10} in $(x+x^2+x^3+\cdots)^4$
$=$ coeff. of x^{10} in $x^4(1-x)^{-4}$
$=$ coeff. of x^6 in $(1-x)^{-4}$
$= \frac{(6+1)(6+2)(6+3)}{1\cdot2\cdot3}$ $\left[\because \text{coeff. of } x^r \text{ in } (1-x)^{-4} = \frac{(r+1)(r+2)(r+3)}{1\cdot2\cdot3}\right]$
$= \frac{7\times8\times9}{1\times2\times3} = 84$

• 162. The number of ways of selecting r balls with replacement out of n balls numbered 1, 2, 3, ..., 100 such that the largest numbered selected is 10 is 271, then $r =$
(a) 3 (b) 4 (c) 5 (d) none.

Sol. (a) From the given condition, we can write
$10^r - 9^r = 271$,
Applying Trial and Error method;—

$r = 1$, $10 - 9 = 1$
$r = 2$, $10^2 - 9^2 = 19$
$r = 3$, $10^3 - 9^3 = 271$

$\therefore r = 3$

163. n men and n women sit along a line alternatively in x ways and along a circle in y ways such that $x = 10y$, then the number of ways in which n men can sit at a round table so that all shall not have same neighbours is

(a) 6 (b) 12 (c) 36 (d) none.

Sol. (b) $\frac{x}{y} = \frac{2 \cdot \lfloor n \lfloor n}{\lfloor n-1 \lfloor n} = 2n$

$\Rightarrow x = 2ny = 10y$

$\Rightarrow n = 5.$

Hence the required number $= \frac{1}{2} \times \lfloor 4 = 12.$

164. A contest consists of predicting the result (win, draw or defeat) of 10 matches. The number of ways in which one entry contains at least 6 incorrect result is

(a) $\sum_{r=6}^{10} {}^{10}C_r \cdot 3^r$, (b) $\sum_{r=1}^{5} {}^{10}C_r \cdot 2^r$, (c) $\sum_{r=6}^{10} {}^{10}C_r$, (d) none.

Sol. (d) Since total number of ways of predicting the results of one match is 3, so results of 10 matches is 3^{10}. Now, number of ways that the result of one match is correct is 1 and also number of ways to predict wrongly of one match is 2.

No. of ways to predict wrongly exactly r matches $= {}^{10}C_r 2^r 1^{10-r}$

∴ the required number is $3^{10} - \sum_{r=1}^{4} {}^{10}C_r \cdot 2^r$.

165. Let 1 to 20 are placed in any around a circle. Then the sum of some 3 consecutive numbers must be at least

(a) 30 (b) 31 (c) 32 (d) none.

Sol. (c) Suppose $x_1, x_2, \ldots, x_{20}$ be the numbers placed around the circle. Now the mean of the 20 sums of 3 consecutive numbers such as $(x_1+x_2+x_3)$, $(x_2+x_3+x_4)$, ……, $(x_{19}+x_{20}+x_1)$, $(x_{20}+x_1+x_2)$ is $\frac{1}{20}\{3(x_1+x_2+\cdots+x_{20})\} = \frac{3 \times 20 \times 21}{2 \times 20}$
$= 31.5$

Thus from Pigon Hole Principle that at least one of the sums must be ≥ 32.

166. The number of different seven-digit numbers can be written using only three digits 1, 2, 3 under the condition that the digit 2 occurs twice in each number is

(a) 512 (b) 640 (c) 672 (d) none.

Sol. (c) We have to put 2 twice in each numbers. So, any 2 out of the 7 places can be chosen in ${}^7C_2$ ways. The remaining 5 places can be filled with the other two numbers in 2^5 ways. The required number of numbers are ${}^7C_2 \times 2^5 = 672$.

• **167.** The value of $\left\{\sum_{i=0}^{100}\binom{K}{i}\binom{M-K}{100-i}\left(\frac{K-i}{M-100}\right)\right\}\Big/\binom{M}{100}$, where $M-K>100$, $K>100$ and $\binom{m}{n}=\frac{m!}{n!(m-n)!}$, is

(a) $\frac{K}{M}$ (b) $\frac{M}{K}$ (c) $\frac{K}{M^2}$ (d) none.

Sol. (a) $\sum_{i=0}^{100}\binom{K}{i}\binom{M-K}{100-i}\left(\frac{K-i}{M-100}\right)\Big/\binom{M}{100}$

$$= \left(\frac{K}{M-100}\right)\sum_{i=0}^{100}\left[\frac{\binom{K}{i}\binom{M-K}{100-i}}{\binom{M}{100}} - \sum_{i=0}^{100}\frac{i\binom{K}{i}\binom{M-K}{100-i}}{(M-100)\binom{M}{100}}\right]$$

$$= \frac{K}{M-100}\cdot\frac{\binom{M}{100}}{\binom{M}{100}} - \frac{\frac{K}{M}\cdot 100\binom{M}{100}}{(M-100)\binom{M}{100}} = \frac{K}{M}.$$

• **168.** Let n be an odd positive integer. If $i_1, i_2, \dots, i_n$ is a permutation of $1, 2, 3, \dots, n$. Then $(1-i_1)(2-i_2)\cdots(n-i_n)$ is

(a) odd (b) even (c) prime (d) none.

Sol. (b) Since n is odd, let $n=2m+1$, where m is a non-negative integer. Then set $S=\{1,2,\dots,n\}$ contains $m+1$ odd nos., namely $1, 3, 5, \dots, 2m+1$, but only m even nos, namely $2, 4, \dots, 2m$.

This is also true for the permutation $i_1, \dots, i_n$ of S. Consider $m+1$ numbers $1-i_1, 3-i_3, \dots, n-i_n$ which are of the form $r-i_r$, where r is odd. Since i_s is even for only m values of S, by P.H.P., one of the $m+1$, numbers, $i_1, i_2, \dots, i_n$, say i_t is odd, where t is also odd. Hence $t-i_t$ is even and the product $(1-i_1)(2-i_2)\cdots(n-i_n)$ is even.

• **169.** The ~~total~~ number of integral solutions for (u,v,w) such that $uvw=24$, is

(a) 120 (b) 122 (c) 124 d) none.

Sol. (a) $24 = (1\times1\times24) = (1\times2\times12) = (1\times3\times8) = (1\times4\times6)$
$= (2\times3\times4) = (2\times2\times6)$

Each bracket in which all the different numbers will give $3!\times{}^3C_2+3! = 24$ values of (u,v,w) and also those bracket in which two numbers are same will give $\frac{3!}{2!}\times{}^3C_2\times\frac{3!}{2!} = 12$ values of (u,v,w). The required number of integral sol. for $(u,v,w) = 24\times4+2\times12$
$= 120$.

170. The number of ways to give 16 different things to 3 persons according as A<B<C so that B gets 1 more than A and C gets 2 more than B, is

(a) 4!5!7! (b) $\frac{4!5!7!}{16!}$ (c) $\frac{16!}{4!5!7!}$ (d) none

Sol. (c) Here $x+y+z=16$, $x=y+1$, $y=z+2$

$\therefore x=4, y=5, z=7$

$\therefore$ Required number of ways $= {}^{16}C_4 \times {}^{12}C_5 \times {}^{7}C_7 = \frac{16!}{4!5!7!}$.

171. For how many positive integers n less than 17, is $\lfloor n + \lfloor n+1 + \lfloor n+2$ an integral multiple of 49?

(a) 4 (b) 5 (c) 6 (d) none.

Sol.:- (b) $\lfloor n + \lfloor n+1 + \lfloor n+2 = \lfloor n \{1+(n+1)+(n+2)(n+1)\}$

$= \lfloor n \, (n+2)^2$.

Since 49 devides $(n+2)^2 \lfloor n$, so either 7 devides $(n+2)$ or 49 divides $\lfloor n$. Thus $n = 5, 12, 14, 15, 16$, i.e. number of integers are 5.

172. Given, $f(x+y) = f(x) + f(y) \; \forall \; x, y \in \mathbb{R}$ and $f(1)=3$, then the value of $\sum_{r=1}^{n} f(r)$ is

(a) $3^n - 1$ (b) $\frac{3}{2}(3^n-1)$ (c) $\frac{3}{2} \cdot 3^{n-1}$ (d) none.

Sol. (b) Let $f(x) = a^{\lambda x}$, λ is a constant,

$f(1) = a^{\lambda} = 3$

$\therefore \sum_{r=1}^{n} f(r) = \sum_{r=1}^{n} a^{\lambda r} = a^{\lambda} + a^{2\lambda} + a^{3\lambda} + \cdots + a^{n\lambda}$

$= \frac{a^{\lambda}(a^{n\lambda}-1)}{(a^{\lambda}-1)} = \frac{3(3^n-1)}{3-1} = \frac{3}{2}(3^n-1)$.

173. If $f: R \to R$, satisfies $f(x+y) = f(x) + f(y) \; \forall \; x, y \in \mathbb{R}$ and $f(1)=7$ then $\sum_{r=1}^{n} f(r)$ is

(a) $\frac{7(n+1)}{2}$ (b) $7n(n+1)$ (c) $\frac{7n(n+1)}{2}$ (d) none.

Sol. (c) Putting $x=1, y=0$, then $f(1) = f(1) + f(0) \Rightarrow f(0) = 0, \Rightarrow f(1) = 7$.

Again, putting $x=1, y=1$, then $f(2) = 2f(1) = 14$, Similarly,

$f(3) = 21$ and so on.

$\sum_{r=1}^{n} f(r) = 7\{1+2+3+\cdots+n\} = \frac{7n(n+1)}{2}$.

• **174.** If $f(x).f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and $f(4) = 65$, then $f(6)$ is
(a) 201 (b) 205 (c) 215 (d) none.

Sol. (d) $f(x) = x^n \pm 1$, where $n \in I$
1st case: $f(4) = 4^n + 1 = 65 \Rightarrow 4^n = 64 \Rightarrow n = 3$.
2nd case: $f(4) = 4^n - 1 = 65 \Rightarrow 4^n = 66$, impossible as $n \in I$.
$\therefore f(x) = x^3 + 1$
& $f(6) = 6^3 + 1 = 217$

• **175.** Let f be a function of a real variable such that it satisfies $f(x+y) = f(x) + f(y)$ $\forall x, y \in I$, then
$f\left(\frac{m}{n}\right) =$
(a) $\frac{m}{n}$ (b) $\frac{f(m)}{f(n)}$ (c) $\frac{m}{n}f(1)$ (d) none.

Sol. :— Since $f(x+y) = f(x) + f(y)$ $\forall x, y$.
So, $f(x) = ax$ is the functional form, where $a =$ constant.
$\therefore f(1) = a$
i.e. $f\left(\frac{m}{n}\right) = a\left(\frac{m}{n}\right) = f(1).\frac{m}{n}$.

• **176.** For any +ve a, b, prove that $\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \geq 8$.

Sol. AM $\geq$ GM

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \geq 2\sqrt{\left(a+\frac{1}{a}\right)^2\left(b+\frac{1}{b}\right)^2}$$
$$\geq 2\left(ab + \frac{1}{ab} + \frac{a}{b} + \frac{b}{a}\right)$$
$$\geq 2(2+2) \quad \left[\begin{array}{l} \because ab + \frac{1}{ab} \geq 2 \\ \frac{a}{b} + \frac{b}{a} \geq 2 \end{array}\right]$$
$$\geq 8.$$

• **177.** If $a_1, a_2, \ldots, a_n$ are non-ve and $a_1 a_2 \cdots a_n = 1$,
S.T. $(1+a_1)(1+a_2)\cdots(1+a_n) \geq 2^n$.

Sol. By AM $\geq$ GM
$$\left(\frac{1+a_i}{2}\right) \geq \sqrt{a_i} \quad \forall i = 1(1)n$$
Multiplying all these
$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 2^n\sqrt{a_1 \cdots a_n} = 2^n.$$

178. If $a_i > 0 \ \forall i = 1(1)n$. Prove that
$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \geq n^2.$$

Sol. AM > GM
$$\frac{a_1+\cdots+a_n}{n} > (a_1 a_2 \cdots a_n)^{1/n}$$
and
$$\frac{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}{n} > \left(\frac{1}{a_1}\cdots\frac{1}{a_n}\right)^{1/n}$$
$$\Rightarrow (a_1+\cdots+a_n) > n(a_1\cdots a_n)^{1/n}$$
and
$$\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) > n\left(\frac{1}{a_1\cdots a_n}\right)^{1/n}$$
$$\Rightarrow (a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) > n^2.$$

179. S.T. $\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1} > n$.

Sol. AM > GM
$$\frac{1}{n}\left(\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_n}{a_1}\right) > \left(\frac{a_1}{a_2}\cdot\frac{a_2}{a_3}\cdots\cdots\frac{a_n}{a_1}\right)$$
or,
$$\left(\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_n}{a_1}\right) > n.$$

180. If $a_1, a_2, \ldots, a_n$ be non-negative real numbers such that $a_1+a_2+a_3+\cdots+a_n = m$, then prove that $\sum\limits_{i<j} a_i a_j \leq \frac{m^2}{2}$.

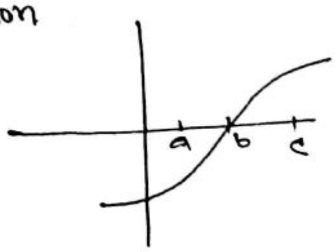
Sol.
$$m^2 = (a_1+\cdots+a_n)^2$$
$$m^2 = a_1^2+\cdots+a_n^2 + 2\sum_{i<j} a_i a_j$$
$$\Rightarrow \sum_{i<j} a_i a_j \leq \frac{m^2}{2}. \qquad [\because a_1^2+\cdots+a_n^2 \geq 0]$$

181. Let $f(x) = (x-a)^3 + (x-b)^3 + (x-c)^3$, $a<b<c$.
Then no. of real roots of $f(x)=0$ is
(a) 3 (b) 2 (c) 1 (d) none.

Sol. (c) $f'(x) = 3\{(x-a)^2+(x-b)^2+(x-c)^2\} > 0$

$\therefore f(x)$ is an increasing function

Note that $f(x) < 0$ if $x < a$
$f(x) > 0$ if $x > c$



$\therefore$ there is one root.

• **182.** If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $x^n + 1 = 0$, then $(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_n) = ?$

Sol. $x^n + 1 = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$

Putting $x=1$,

$\therefore\ 2 = (1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_n)$.

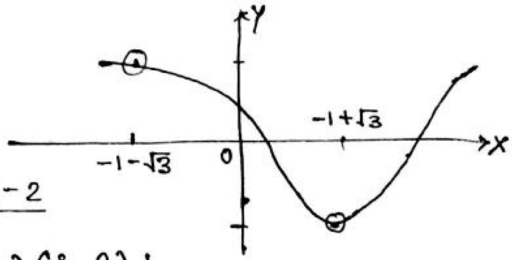
• **183.** The equation $\frac{1}{3} + \frac{1}{2}s^2 + \frac{1}{6}s^3 = s$ has exactly ______ solution(s) in $[0,1]$.

Solution:-

$f(s) = \frac{1}{3} + \frac{1}{2}s^2 + \frac{1}{6}s^3 - s$

$f(1) = 0$

$f(0) = \frac{1}{3}$



$f'(s) = s + \frac{1}{2}s^2 - 1 = \frac{s^2 + 2s - 2}{2}$

$= \frac{1}{2}(s-\alpha)(s-\beta)$;

here $\alpha = -1-\sqrt{3}$, $\beta = -1+\sqrt{3}$.

$$\therefore f'(s) = \begin{cases} >0 & \text{if } x<\alpha \text{ or } x>\beta \\ <0 & , \alpha < x < \beta. \end{cases}$$

There are two roots.

• **184.** Find the no. of real roots of the equation $2\cos\left(\frac{x^2+x}{6}\right) = 2^x + 2^{-x}$.

Sol. $\cos\left(\frac{x^2+x}{6}\right) = \frac{2^x + 2^{-x}}{2} \geq \sqrt{2^x \cdot 2^{-x}} = 1$, by AM $\geq$ GM.

But $\cos\left(\frac{x^2+x}{6}\right) \leq 1$.

$\therefore \cos\left(\frac{x^2+x}{6}\right) = 1 = \cos\left(\frac{n\pi}{2}\right)$

$\therefore \frac{x^2+x}{6} = \frac{n\pi}{2}$

$\Rightarrow x^2 + x - 3n\pi = 0$

Here $b^2 - 4ac = 1 - 4.1(-3n\pi)$

$= 1 + 12n\pi$

$\geq 0\ \forall\ n = 0, 1, 2, \ldots$

$\therefore$ There are infinitely many roots.

185. Find the no. of real roots of the polynomial $f(x) = x^5 + x^3 - 2x + 1$.

Sol. Descarte's Sign Rule:-

$f(x) = 0$ has two sign changes.

$\therefore$ No. of +ve roots ≤ 2.

$f(-x) = 0 = -x^5 - x^3 + 2x + 1$
$\quad\quad\quad\quad - \quad - \quad + \quad +$

$\therefore f(-x)$ has one sign change.

$\therefore$ No. of -ve root ≤ 1.

$\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 = 1$.

$\Rightarrow$ there is no negative roots.

$\therefore$ As complex roots occur in pair, so, there is one +ve root.

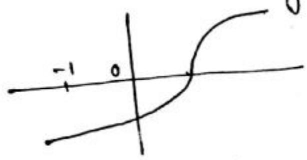
186. Let $f(x) = x^3 + 3x - 2$, $x \in \mathbb{R}$, S.T. $f(x) = 0$ has only one real root.

Sol.

$f'(x) = 3(x+1)^2 > 0$

$f(-1) = -6 < 0$, $f(2) > 0$

$\therefore$ It has one root.



187. If $\lim\limits_{x \to \infty} f(x) = 1$ and $\lim\limits_{x \to \infty} f'(x) = \alpha$, find α.

Sol. Let $f(x) = 1 + \frac{K}{x^p}$, $p > 0$

then $\lim\limits_{x \to \infty} f(x) = 1$.

$\therefore \alpha = \lim\limits_{x \to \infty} f'(x) = \lim\limits_{x \to \infty} \frac{K(-p)}{x^{p+1}} = 0$.

188. Let $f(0) = 1$, $\lim\limits_{x \to \infty} f''(x) = 4$ and $f(x) \geq f(1)$. Let $f(x)$ is polynomial $\forall x \in \mathbb{R}$. Find $f(2)$.

Sol.

$f''(x) = 4 =$ constant.

$\Rightarrow f(x) = 2x^2 + ax + b$

$f(0) = 1 \Rightarrow b = 1$.

$f(1) = 3 + a$

$f(x) \geq f(1) \Rightarrow f'(1) = 0$

$\Rightarrow 4 + a = 0$

$\Rightarrow a = -4$

$\therefore f(x) = 2x^2 - 4x + 1$

$\therefore f(2) = 1$.

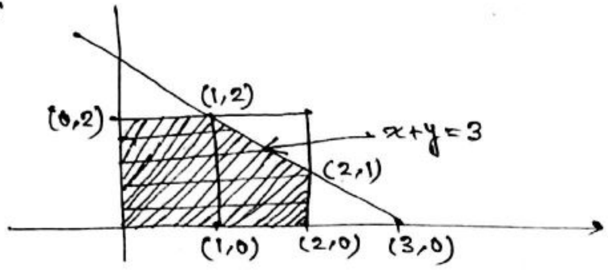
189. Let $f(x)=\begin{cases} x & \text{if } x\in[0,2] \\ 0 & \text{if } x\notin[0,2]\end{cases}$

and $g(x)=\begin{cases} 1 & \text{if } x\in[0,2] \\ 0 & \text{if } x\notin[0,2]\end{cases}$

Let $A=\{(x,y): x+y\le 3\}$

Then find the value of the integral $\iint_A f(x)g(y)\,dx\,dy$.

Sol.



$$\iint_A f(x)g(y)\,dx\,dy$$

$$= \iint_{\{(x,y):\ 0<x<y<2\ \&\ x+y\le 3\}} x\cdot 1\cdot dx\,dy$$

$0<x<1$ $0<y<2$
& $1\le x<2$, $0\le y<3-x$.

$$= \int_0^1\int_0^2 x\,dy\,dx + \int_1^2\left(\int_0^{3-x} x\,dy\right)dx$$

$$= 1+\left[\frac{3}{2}x^2-\frac{x^3}{3}\right]_1^2$$

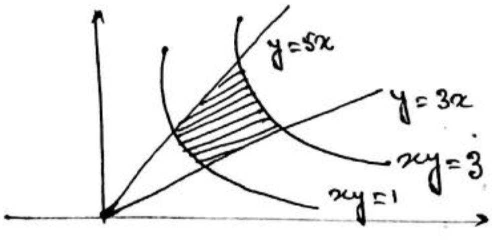
$$= \frac{19}{6}.$$

190. Using the change of variables evaluate $\iint_R xy\,dx\,dy$, when the region R is bounded by the curves

$xy=1$, $xy=3$;
$y=3x$, $y=5x$

in the 1st quardinate.

Sol.



$u=xy$
$v=\frac{x}{y}$

Then transformation domain is $A=\{(u,v): 1\le u\le 3,\ 3\le v\le 5\}$

$\therefore y=\sqrt{uv},\ x=\sqrt{\frac{u}{v}}$

Jacobian of the transformation is

$$J = \begin{vmatrix} \frac{1}{\sqrt{v}} \cdot \frac{1}{2\sqrt{u}} & \sqrt{u}\left(-\frac{1}{2} \cdot \frac{1}{v\sqrt{v}}\right) \\ \sqrt{v} \cdot \frac{1}{2\sqrt{u}} & \sqrt{u} \cdot \frac{1}{2\sqrt{v}} \end{vmatrix}$$

$$= \frac{1}{4v} + \frac{1}{4} \cdot \frac{1}{v} = \frac{1}{2v}.$$

$$I = \int_3^5 \int_1^3 u \cdot \frac{1}{2v} \, du dv$$

$$= \left[\frac{u^2}{4}\right]_1^3 \left[\ln v\right]_3^5$$

$$= \log\left(\frac{25}{9}\right).$$

Q191. Given that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, then the value of

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+xy+y^2)} dx dy$, here $D = \{(x,y) \in \mathbb{R}^2 \,;\, x^2+y^2 \le 1\}$

Sol.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+yx+y^2)} dx dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left\{(x-y/2)^2 + \frac{3}{4}y^2\right\}} dx dy$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} e^{-(x-y/2)^2} dx \right\} e^{-\frac{3}{4}y^2} dy$$

$$= \int_{-\infty}^{\infty} \sqrt{\pi}\, e^{-\frac{3}{4}y^2} dy = \frac{2\sqrt{\pi}}{\sqrt{3}} \int_{-\infty}^{\infty} e^{-u^2} du \qquad \left[\text{let } \frac{\sqrt{3}}{2} y = u\right]$$

$$= 2\sqrt{\frac{\pi}{3}} \cdot \sqrt{\pi}$$

$$= \frac{2\pi}{\sqrt{3}}.$$

— By TANUJIT CHAKRABORTY.
M: 8420253573
Mail: tanujitisi@gmail.com

192. Evaluate $\int_1^2\int_1^2\int_1^2\int_1^2 \frac{x_1+x_2+x_3-x_4}{x_1+x_2+x_3+x_4}\,dx_1dx_2dx_3dx_4$

Sol. $\int_1^2\int_1^2\int_1^2\int_1^2 \frac{x_i\,dx_1dx_2dx_3dx_4}{x_1+x_2+x_3+x_4} = \frac{1}{4}$ as $\int_1^2\int_1^2\int_1^2\int_1^2 \frac{x_1+x_2+x_3+x_4}{x_1+x_2+x_3+x_4}\,dx_1dx_2dx_3dx_4 = 1.$

$\therefore I = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.$

193. Find the value of $\iint\limits_{x^2+y^2\le 1} \frac{2}{1+x^2+y^2}\,dx\,dy$

Sol. Let $x = r\cos\theta$, $y = r\sin\theta$

as $x^2+y^2 \le 1$

$\therefore 0 < r < 1$

& $0 < \theta < 2\pi$

$J = \begin{vmatrix} \cos\theta & \sin\theta \\ r\sin\theta & r\cos\theta \end{vmatrix} = r$

$\therefore I = \int_0^{2\pi}\int_0^1 \frac{2}{1+r^2}\, r\,dr\,d\theta$

$= \left[\ln(1+r^2)\right]_0^1 \left[\theta\right]_0^{2\pi}$

$= 2\pi \log_e 2.$

194. Give an example of each of the types of functions

(a) The function which is continuous but not differentiable at only (i) one point (ii) two points (iii) ten points.

(b) The function which is discontinuous at (i) four points (ii) 15 points.

(c) The function which is differentiable once but not twice at (i) one point (ii) three points.

Solution:– (a) (i) $f(x) = |x|$ (ii) $f(x) = |x| + |x-1|$

(iii) $f(x) = \sum_{k=1}^{10} |x-k|$

(b) (i) $f(x) = [x]$, $0 < x < 5$

(ii) $f(x) = [x]$, $0 < x < 16$

(c) (i) $f(x) = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$

$= x^2\,\text{sign}(x)$, where $\text{sign}(x) = \begin{cases} -1, & \text{if } x \le 0 \\ 1, & \text{if } x > 0 \end{cases}$

(ii) $f(x) = x^2\,\text{sign}(x) + (x-1)^2\,\text{sign}(x-1) + (x-2)^2\,\text{sign}(x-2)$.

195. A, B are two square matrices such that $AB = A$ and $BA = B$, then
(a) A, B both are idempotent (b) only A is idempotent.
(c) only B is idempotent (d) none.

Sol. (a) $(AB)A = A \times A = A^2$
$\Rightarrow A(BA) = A^2$
$\Rightarrow AB = A^2$
$\Rightarrow A = A^2$
Hence A is idempotent.
similarly, B is idempotent.

196. Let B is non-singular matrix and A is a square matrix, then $\det(B^{-1}AB) =$
(a) $\det B$ (b) $\det A$ (c) none.

Sol. (b) $\det(B^{-1}AB) = \det(B^{-1}) \det(A) \det(B)$
$= \dfrac{1}{\det(B)} \cdot \det(A) \cdot \det(B)$
$= \det A$.

197. Let $\frac{1-3p}{2}, \frac{1+4p}{3}, \frac{1+p}{6}$ are the probabilities of 3 mutually exclusive and exhaustive events, then the set of all values of p is
(a) $[-1/4, 1/3]$ (b) $(0,1)$ (c) $(0,\infty)$ (d) none.

Sol. (a) $\frac{1-3p}{2} \geq 0$, $\frac{1+4p}{3} \geq 0$ $\frac{1+p}{6} \geq 0$ and
$\frac{1-3p}{2} + \frac{1+4p}{3} + \frac{1+p}{6} = 1$
$\Rightarrow -\frac{1}{4} \leq p \leq \frac{1}{3} \Rightarrow p \in [-1/4, 1/3]$

198. A subset A of the set $X = \{1, 2, 3, \ldots, 100\}$ is chosen at random. The set X is reconstructed by replacing the elements of A and another subset B of X is chosen at random. the probability that $A \cap B$ contains exactly 10 elements is
(a) ${}^{100}C_{10}\left(\frac{3}{4}\right)^{90}$ (b) ${}^{100}C_{10}\left(\frac{1}{2}\right)^{100}$ (c) ${}^{100}C_{10}\left(\frac{2}{3}\right)^{100}$ (d) none.

Sol. (d) A and B can be chosen in general
in $= \left\{\sum_{r=0}^{90} {}^{90}C_r 2^{90-r}\right\} \cdot {}^{100}C_{10}$ ways
$= (1+2)^{90} \cdot {}^{100}C_{10} = {}^{100}C_{10} \cdot 3^{90}$ ways.

199. Let A be a 2X2 matrix to be written down using the numbers 1, −1 as elements. The probability that the matrix is non-singular is

(a) 1/2 (b) 3/8 (c) 5/8 (d) none.

Sol. (a)

A 2x2 matrix has 4 elements each of which can be chosen in 2 ways. So, total number of 2X2 square matrices with elements 1, −1 is $2^4 = 16$.

Out of these 16 matrices, following matrices are singular

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Thus the number of non-singular matrices $= 16 - 8 = 8$.

$\therefore$ Required probability is $= \frac{8}{16} = \frac{1}{2}$.

200. Let $A_1, A_2, \ldots, A_n$ are n independent events with $P(A_j) = \frac{1}{1+j}$ $(1 \le j \le n)$. The probability that not one of $A_1, A_2, \ldots, A_n$ occur is

(a) $\frac{1}{(n+1)!}$ (b) $\frac{2}{n+1}$ (c) $\frac{n!}{(n+1)!}$ (d) none.

Sol. (c) Required probability $= P(\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n)$

$$= P(\bar{A}_1) P(\bar{A}_2) \ldots P(\bar{A}_n)$$

$$= \frac{1}{2} \times \frac{2}{3} \times \ldots \times \frac{n}{n+1} = \frac{n!}{(n+1)!} \quad \text{(Ans)}$$

$$\left[\because P(\bar{A}_j) = 1 - P(A_j) = 1 - \frac{1}{1+j} = \frac{j}{1+j} \right]$$

For more: www.ctanujit.in www.ctanujit.org www.ctanujit.com

● 201. The last two digits of $2^{2^{73}}$ are

(a) 2, 4 ; (b) 6, 8 ; (c) 9, 6 ; (d) None

Solution:– (c)

$$2^{10} = 1024 \equiv 24 \pmod{100}$$
$$2^{20} \equiv 24^2 \pmod{100}$$
$$\equiv 76 \pmod{100}$$
$$2^{22} \equiv 4 \pmod{100}$$

so, in general, $2^{20+k} \equiv 2^k \pmod{100}$, $k > 1$.

so, we shall find $2^{70+3} = 2^{73} \pmod{20}$,

$$2^{73} \equiv 2^{13} \pmod{20} \equiv 12 \pmod{20}$$

$$\therefore\ 2^{2^{73}} = 2^{20m+12} \equiv 2^{12} \pmod{100} \equiv 96 \pmod{10}$$

i.e. the last two digits are 9 and 6.

● 202. Solution set of $\dfrac{(x-1)(x-2)^2(x+4)}{(x+2)(x-3)} \geq 0$ is

(a) $(-\infty, 4] \cup (-2, 1] \cup \{2\} \cup (3, \infty)$; (b) $[-4, -2) \cup [1, 2] \cup (3, \infty)$;

(c) $(-\infty, -4] \cup \{2\} \cup (3, \infty)$; (d) none.

Sol. (a) $G(x) = \dfrac{(x-1)(x-2)^2(x+4)}{(x+2)(x-3)} \geq 0$

So, sign change is like this:–

$$\underset{}{+}\ \underset{-4}{\cdot}\ -\ \underset{-2}{\cdot}\ +\ \underset{1}{\cdot}\ -\ \underset{2}{\cdot}\ -\ \underset{3}{\cdot}\ +$$

$\Rightarrow x \in (-\infty, 4] \cup (-2, 1] \cup \{2\} \cup (3, \infty)$.

● 203. If $x_1, x_2, \ldots, x_n$ are the roots of $x^n + ax + b = 0$, then the value of $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)\ldots =$

(a) $nx_1 + b$; (b) nx_1^{n-1} ; (c) $nx_1^{n-1} + a$; (d) none.

Sol. (c) $x^n + ax + b = (x - x_1)(x - x_2)(x - x_3)\ldots(x - x_n)$

$$\Rightarrow (x - x_2)(x - x_3)\ldots(x - x_n) = \frac{x^n + ax + b}{(x - x_1)}$$

$$\Rightarrow (x_1 - x_2)(x_1 - x_3)\ldots(x_1 - x_n) = \lim_{x \to x_1} \frac{x^n + ax + b}{x - x_1}, \quad \text{[By L'Hospital Rule]}$$
$$= nx_1^{n-1} + a.$$

● 204. Let there three values of x such that $x, [x], \{x\}$ are in H.P., then the number of values of x are

(a) 1 ; (b) 2 ; (c) 3 ; (d) None

Sol. (a) $[x] = \dfrac{2\{x\}x}{\{x\}+x}$; Now, we know $x = [x] + \{x\}$, putting value of x, we get

$$\Rightarrow [x]^2 = 2\{x\}^2$$

$$\Rightarrow \{x\}^2 = \frac{1}{2}[x]^2$$

$$\Rightarrow 0 < \frac{[x]^2}{2} < 1,$$

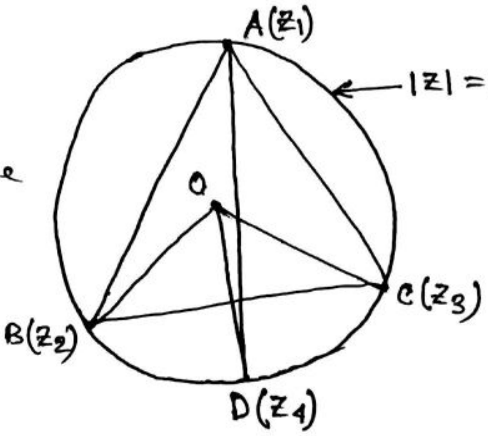
$$\Rightarrow 0 < [x]^2 < 2, \Rightarrow 0 < [x] < \sqrt{2} \Rightarrow [x] = 1 \Rightarrow \{x\} = \frac{1}{\sqrt{2}},$$

so, $x = 1 + \dfrac{1}{\sqrt{2}}$ is the only value.

● 205. $A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of a ΔABC inscribed in the circle $|z| = 2$. Internal angle bisector of $\angle A$, meet the circumcircle again at $D(z_4)$, then

(a) $z_4^2 = z_2 z_3$; (b) $z_4 = \dfrac{z_2 z_3}{z_1}$ (c) $z_4 = \dfrac{z_1 z_3}{z_2}$ (d) None

Sol. (a)
$\angle BOD = 2\angle BAD = A$ and
$\angle COD = 2\angle CAD = A$.
From rotation about the point 'O', we have



$$\frac{z_4}{z_2} = e^{iA}, \quad \frac{z_3}{z_4} = e^{iA}$$

$$\therefore \quad z_4^2 = z_2 z_3.$$

● 206. Number of integral terms in the expansion of $(\sqrt{6}+\sqrt{7})^{32} =$

(a) 15 ; (b) 17 ; (c) 19 ; (d) none.

Sol. (b) $$(\sqrt{6}+\sqrt{7})^{32} = \sum_{r=0}^{32} {}^{32}C_r \cdot 6^{r/2} \cdot 7^{\frac{32-r}{2}}.$$

For integral terms $\frac{r}{2}$ and $\frac{32-r}{2}$ both are integers and which is in turn possible if $\frac{r}{2}$ is an integer.

$\therefore$ $r = 0, 2, 4, \ldots, 32$ means r can take 17 different values.

● 207. Let p is an odd prime and $n = 1 + p!$, then total number of prime in the list $n+1, n+2, n+3, \ldots, n+p-1$ is eaual to

(a) $p-3$; (b) $p-5$; (c) 0; (d) none.

Sol. (c)

$$\because \; n = 1 + p!$$

$$\therefore \; n + r = (r+1) + p!$$

If $1 \le r \le p-1$, then $2 \le r+1 \le p$ and clearly, $(n+r)$ is divisible by $r+1$. $\therefore$ $n+r$ can't be a prime

Hence, there is no prime in the given list.

● 208. The remainder obtained when $1! + 2! + 3! + \cdots + 95!$ is divided by 15, is

(a) 3; (b) 5; (c) 7; (d) none.

Sol. (a) Here $1! + 2! + 3! + 4! = 33$ and $15 \mid n!$ where $n \ge 5$.

The remainder is same as the remainder obtained by dividing 33 with 15., i.e., 3.

● 209. The value of $\sum\limits_{0 \le i,} \sum\limits_{j \le n} \left({}^nC_i + {}^nC_j \right)$ is

(a) $n.2^{n-1}$; (b) $n.2^n$; (c) $(n+1)2^n$; (d) none.

Sol. (b)

$$\sum\limits_{0 \le i,} \sum\limits_{j \le n} {}^nC_i = \sum_{r=0}^{n} {}^nC_r \underbrace{(1+1+\cdots+1)}_{(n-r)\text{ times}} = \sum_{r=0}^{n} (n-r).{}^nC_r$$

$$= \sum_{r=0}^{n} r.{}^nC_r = n.2^{n-1}.$$

$$\therefore \sum\limits_{0 \le i,} \sum\limits_{j \le n} \left({}^nC_i + {}^nC_j \right) = n.2^{n-1} + n.2^{n-1} = n.2^n,$$ which is the req^d. result.

● 210. Let $f : [4,4] \sim \{-\pi, 0, \pi\} \to R$ such that $f(x) = \cos(\sin x) + \left[\dfrac{x^2}{|a|} \right]$, where $[\cdot] = g.i.$, is an odd function. Complete set of values of 'a' is

(a) $(-\infty, -16) \cup [16, \infty)$; (b) $(-\infty, -16) \cup (16, \infty)$;

(c) $(-16, 16) \cup \{0\}$ (d) none.

Sol. (b) since $f(x)$ is odd, so $\left[\dfrac{x^2}{|a|} \right]$ doesnot depend upon the value of x. Also, since $x \in [-4, 4]$, so, $0 \le x^2 \le 16$, $\Rightarrow \left[\dfrac{x^2}{|a|} \right] = 0$ if $|a| > 16$. So, $a \in (-\infty, -16) \cup (16, \infty)$.

211. If $\sqrt{5x-x^2-6} + \frac{\pi}{2}\int_0^x dt > x\int_0^{\pi} \sin^2 t\,dt$, then $x \in$

(a) $(2,3)$; (b) $(-\infty,2)\cup(3,\infty)$; (c) $\left(\frac{5}{2},3\right)$; (d) none.

Sol. (a)

$$\sqrt{5x-x^2-6} + \frac{\pi x}{2} > x\left\{\frac{1}{2}\int_0^{\pi}(1-\cos 2t)\,dt\right\}$$

$$\Rightarrow \sqrt{5x-x^2-6} + \frac{\pi x}{2} > x\left\{\frac{1}{2}\left(t-\frac{1}{2}\sin 2t\right)_0^{\pi}\right\}$$

$$\Rightarrow \sqrt{5x-x^2-6} + \frac{\pi x}{2} > \frac{\pi x}{2}$$

$$\Rightarrow \sqrt{5x-x^2-6} > 0$$

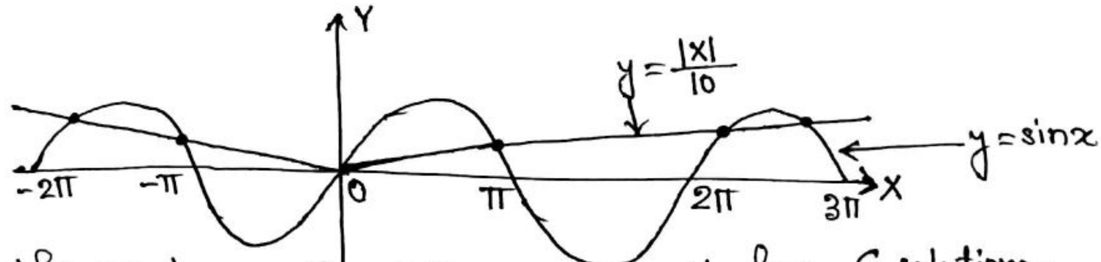
$\Rightarrow x^2-5x+6<0,$

$\Rightarrow (x-2)(x-3)<0$, i.e., $x \in (2,3)$.

212. Total number of solutions of $\sin x = \frac{|x|}{10}$ is

(a) 0; (b) 3; (c) 4; (d) none.

Sol. (d)



Two graphs meet exactly 6 times, hence, it has 6 solutions.

213. Let $f(x) = \cos x\left(\sin x + \sqrt{\sin^2 x + \sin^2\theta}\right)$, where '$\theta$' is a given constant, then maximum value of $f(x)$ is

(a) $\sqrt{1+\cos^2\theta}$; (b) $\sqrt{1+\sin^2\theta}$; (c) $|\cos\theta|$; (d) none.

Sol. (b)

$$\{f(x)\sec x - \sin x\}^2 = \sin^2 x + \sin^2\theta,$$

$$\Rightarrow f^2(x)(1+\tan^2 x) - 2f(x)\tan x = \sin^2\theta$$

$$\Rightarrow f^2(x)\tan^2 x - 2f(x)\tan x + f^2(x) - \sin^2\theta = 0$$

$$\Rightarrow 4f^2(x) \geq 4f^2(x)\{f^2(x)-\sin^2\theta\}$$

$$\Rightarrow f^2(x) \leq 1+\sin^2\theta,$$

i.e., $|f(x)| \leq \sqrt{1+\sin^2\theta}$.

214. If l_1 and l_2 are the side lengths of two variable squares S_1 and S_2, respectively. If $l_1 = l_2 + l_2^3 + 6$, then the rate of change of the area of S_2 with respect to rate of change of the area of S_1 when $l_2 = 1$ is

(a) $\frac{3}{2}$; (b) $\frac{2}{3}$; (c) $\frac{4}{3}$; (d) none.

Sol. (d) Let A_1 and A_2 be the area of the sequences S_1 and S_2,

$A_1 = l_1^2$ and $A_2 = l_2^2$.

$\therefore \frac{dA_1}{dl_1} = 2l_1$ and $\frac{dA_2}{dl_2} = 2l_2$.

$\Rightarrow \frac{dA_2}{dA_1} = \frac{l_2}{l_1} \cdot \frac{dl_2}{dl_1} = \frac{l_2}{l_1} \cdot \frac{1}{1+3l_2}$

When $l_2 = 1$, $l_1 = 8$, then $\frac{dA_2}{dA_1} = \frac{1}{32}$.

215. If $f(x) = (4+x)^n$, $n \in N$ and $f^r(0)$ represents the rth derivative of $f(x)$ at $x=0$, then the value of $\sum_{r=0}^{\infty} \frac{f^r(0)}{r!} =$

(a) 2^n; (b) e^n; (c) 5^n (d) none.

Sol. (c) $f'(x) = n\,(4+x)^{n-1}$

$f''(x) = n(n-1)(4+x)^{n-2}$

$\vdots$

$f^r(x) = n(n-1)\ldots(n-r+1)\cdot(4+x)^{n-r}, \quad r \le n.$

$f^r(0) = \frac{n!}{(n-r)!} \cdot 4^{n-r} \quad , r \le n$

$= 0 \quad , r > n$

$\therefore \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} = \sum_{r=0}^{n} \binom{n}{r} \cdot 4^{n-r} = (1+4)^n = 5^n.$

216. Let $f(x)$ be a differentiable function such that

$f'(x) = \frac{1}{(\log_3(\log_{1/4}(\cos x + a)))}$. If $f(x)$ is increasing $\forall x$, then $a \in$

(a) $(5, \infty)$; (b) $\left(1, \frac{5}{4}\right)$; (c) $\left(\frac{5}{4}, 5\right)$; (d) none.

Sol. (d) $\because$ $f(x)$ is increasing, so, $f'(x) > 0 \ \forall x$.

$\therefore \log_3(\log_{1/4}(\cos x + a)) > 0 \Rightarrow \cos x + a > 0$ $< \frac{1}{4}$.

$\Rightarrow -\cos x < a < \frac{1}{4} - \cos x \ \forall x \in R \Rightarrow 1 < a < -\frac{3}{4}$, which is not possible.

217. Let $f(x) = \begin{cases} x|x| & ; \quad x \le -1 \\ [x+1]+[1-x] & ; \quad -1 < x < 1, \\ -x|x| & ; \quad x \ge 1 \end{cases}$

then the value of $\int_{-2}^{2} f(x)\,dx$ is

(a) $-\frac{8}{3}$ (b) $-\frac{7}{3}$ (c) $\frac{7}{3}$ (d) none.

Sol. (a). $f(x) = \begin{cases} -x^2 & , \quad x \le -1 \\ 1 & , \quad -1 < x < 0 \\ 2 & , \quad x = 0 \\ 1 & , \quad 0 < x < 1 \\ x^2 & , \quad x \ge 1 \end{cases}$

$\therefore$ $f(x)$ is an even function, i.e. $\int_{-2}^{2} f(x)\,dx = 2\int_{0}^{2} f(x)\,dx$

$$= 2\left\{ \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx \right\}$$

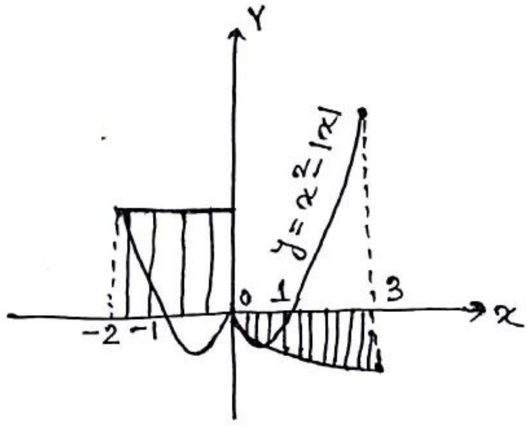
$$= 2\left(1 - \frac{x^3}{3}\right)_1^2 = -\frac{8}{3}.$$

218. Area bounded by $y = g(x)$, x-axis and the lines $x = -2, x = 3$, where $g(x) = \begin{cases} \max\{f(t) : -2 \le t \le x\}, \text{ where } -2 \le x < 0; \\ \min\{f(t) : 0 \le t \le x\}, \text{ where } 0 \le x \le 3 \text{ and} \end{cases}$

$f(x) = x^2 - |x|$, is equal to

(a) $\frac{113}{24}$ (b) $\frac{111}{24}$ (c) $\frac{117}{24}$ (d) none.

Sol. (a) $g(x) = \begin{cases} 2 & ; \quad -2 \le x < 0 \\ x^2 - x & ; \quad 0 \le x \le 1/2 \\ -\frac{1}{4} & ; \quad \frac{1}{2} < x \le 3 \end{cases}$



$\therefore$ Required area

$$= \int_{-2}^{0} 2\,dx + \int_{0}^{1/2} (x - x^2)\,dx + \int_{1/2}^{3} \frac{1}{4}\,dx$$

$$= \frac{113}{24} \text{ unit}^2.$$

● 219. Total number of positive integral values of n such that the equations $\cos^{-1}x + (\sin^{-1}y)^2 = \frac{n\pi^2}{4}$ and $(\sin^{-1}y)^2 - \cos^{-1}x = \frac{\pi^2}{16}$ are consistent, is equal to

(a) 1; (b) 2; (c) 3; (d) none.

Sol. (a) Here $2(\sin^{-1}y)^2 = \frac{4n+1}{16}\pi^2$

$$\Rightarrow 0 \le \frac{4n+1}{16}\pi^2 \le \frac{\pi^2}{4},$$

$$\Rightarrow -\frac{1}{4} \le n \le \frac{7}{4}.$$

Also, $2(\cos^{-1}x) = \frac{4n-1}{16}\pi^2$

$$\Rightarrow 0 \le \frac{4n-1}{16}\pi^2 \le \pi,$$

$$\Rightarrow \frac{1}{4} \le n \le \frac{8}{\pi} + 1.$$

Hence, the least positive integral value of n is 1.

● 220. Radius of bigger circle touching the circle $x^2+y^2-4x-4y+4=0$ and both the co-ordinate axis is

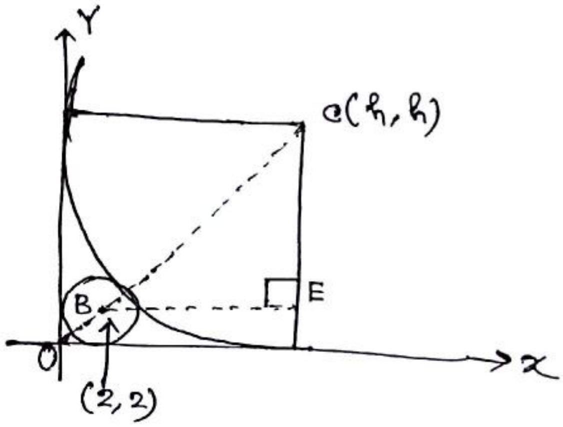
(a) $3+2\sqrt{2}$; (b) $2(3+2\sqrt{2})$; (c) $3-2\sqrt{2}$; (d) none.

Sol. (b) Let (h, h) be the centre of the required circle.

$\therefore \angle COD = \angle CBE = \frac{\pi}{4}$, $CB = h+2$ and $BD = h-2$.

$$\therefore \frac{h-2}{h+2} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

$$\Rightarrow h = \frac{2(\sqrt{2}+1)}{(\sqrt{2}-1)} = 2(3+2\sqrt{2}).$$



● 221. Tangents and normal drawn to parabola at $A(at^2, 2at)$, $t \ne 0$ meet the x-axis at point B and D, respectively. If the rectangle ABCD is

(a) $y-2a=0$; (b) $y+2a=0$; (c) $x-2a=0$; (d) none.

Sol. Equations of tangent & normal at A are $yt = x+at^2$, $y = -tx+2at+at^3$.

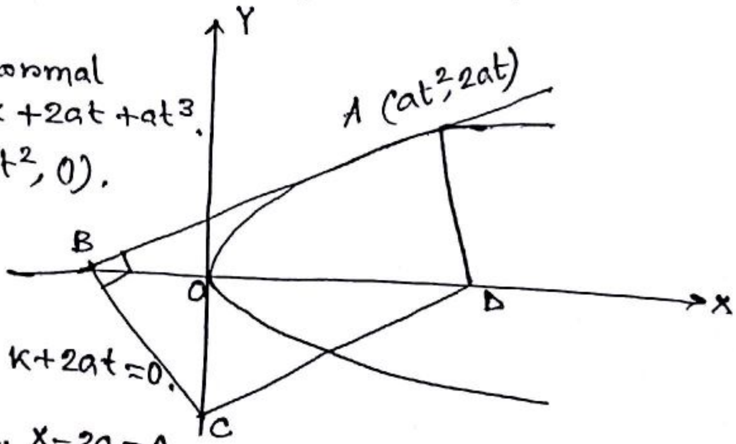
$\therefore B = (-at^2, 0)$ and $D = (2a+at^2, 0)$.

Suppose ABCD is rectangle, then midpoints of BD and AC will be coincident.

$\therefore h+at^2 = 2a+at^2-at^2$ and $k+2at=0$,

i.e. $h = 2a$, $k = -2at$.

Hence, the locus is $x = 2a$, i.e. $x-2a=0$.



222. The series $\sum_{K=2}^{\infty} \frac{1}{K(K-1)}$ converges to

(a) -1 (b) 1 (c) 0 (d) does not converge

Solution:- (b) $S_n = \sum_{K=2}^{n} \frac{1}{K(K-1)} = \sum_{K=2}^{n} \left(\frac{1}{K-1} - \frac{1}{K}\right)$

$$= \left(1 - \frac{1}{n}\right)$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1.$$

223. The limit $\lim_{x\to\infty} \left(\frac{3x-1}{3x+1}\right)^{4x}$ equals

(a) 1 (b) 0 (c) $e^{-8/3}$ (d) $e^{4/9}$.

Solution:- (c)

$$\lim_{x\to\infty} \left\{\left(\frac{1-\frac{1}{3x}}{1+\frac{1}{3x}}\right)^{x}\right\}^{4} = \left(\frac{e^{-1/3}}{e^{1/3}}\right)^{4} = e^{-8/3}.$$

$$\left[\text{Since} \lim_{x\to\infty}\left(1 + \frac{K}{x}\right)^{x} = e^{K}\right]$$

224. $\lim_{n\to\infty} \frac{1}{n}\left(\frac{n}{n+1} + \frac{n}{n+2} + \cdots + \frac{n}{2n}\right)$ equals

(a) ∞ (b) 0 (c) $\log_e 2$ (d) 1

Sol. (c) $\lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}\right)$

$$= \int_0^1 \frac{1}{1+x}\,dx = \left[\log(1+x)\right]_0^1 = \log_e 2.$$

225. Let K be an integer greater than 1. Then $\lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{Kn}\right]$ is

(a) $\log_e K$ (b) $(K-1)\log_e K$ (c) 0 (d) ∞

Sol. (a)

$$\lim_{n\to\infty}\left[\sum_{r=1}^{(K-1)n} \frac{1}{n+r}\right] = \int_0^{K-1} \frac{dx}{1+x} = \log(1+x)\Big]_0^{K-1}$$

$$= \log_e K,$$

Q. 226. Let S be the set of real numbers x for which the power series $\sum_{n=1}^{\infty} \left[1-(-2)^n\right] x^n$ converges. Then S equals

(a) $\{0\}$ (b) $(-1/2, 1/2)$ (c) $(-1/2, 1/2]$ (d) $(-1,1)$

Sol. (b) $\sum_{n=1}^{\infty} a_n x^n$ converges if $|x| < \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$.

So, $\sum_{n=1}^{\infty} \left[1-(-2)^n\right] x^n$ " " $|x| < \frac{1}{\lim_{n\to\infty} \left[1-(-2)^n\right]^{1/n}}$

i.e. $|x| < -\frac{1}{2}$

Q. 227. The maximum value of
$$\begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$
is

(a) 0 (b) 2 (c) 4 (d) 6.

Sol. (d)
$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix} \quad \begin{matrix} [R_1' = R_1 - R_2] \\ [R_2' = R_2 - R_3] \end{matrix}$$

$$= \left(1+4\sin 2x + \cos^2 x\right) + \left(\sin^2 x\right)$$
$$= 2 + 4\sin 2x$$

Since $|\sin 2x| \le 1$, so, $2 + 4\sin 2x \le 2+4 = 6$.

Q. 228. The limit $\lim_{x\to\infty} \left(\frac{4x}{5+4x}\right)^{2x}$ is

(a) $e^{-5/2}$ (b) $e^{-2/5}$ (c) e^{-5} (d) none.

Sol. (a) $\lim_{x\to\infty} \left(\frac{1}{1+\frac{5}{4x}}\right)^{2x} = \frac{1}{\left\{\lim_{x\to\infty}\left(1+\frac{5}{4x}\right)^x\right\}^2} = e^{-\frac{5}{4}\times 2}$
$= e^{-\frac{5}{2}}$.

229.

Q. The pages of a book are numbered consecutively starting from page 1. A total of 2989 digits was used to number the pages. Then the number of pages is divisible by

(a) 2 (b) 3 (c) 5 (d) 7.

Sol. (a) $2989 = 189 + 2800 = 189 + 2700 + 100$

$= 1\times9 + 20\times9 + 300\times9 + 25\times4$

From 1 to 9 pages, there are 9 digits
From 10 to 99 pages, there are 20×9 digits
from 100 to 999 pages, there are 300×9 " .
from 1000 to 1024 pages, there are 25×4 " .

So, there are total 1024 pages in the book,

Q. 230. There are 30 questions in a multiple choice test. A student gets 1 mark for each unattempted question, 0 mark for each wrong answer and 4 marks for each correct answer. A student has answered x questions correctly and has scored 60. Then the number of possible values of x is

(a) 15 (b) 10 (c) 6 (d) 5

Sol. (c)

$$\begin{aligned} 60 &= 10\times4 + 20\times1, & x=10 \\ &= 11\times4 + 16\times1 + 3\times0, & x=11 \\ &= 12\times4 + 12\times1 + 6\times0, & x=12 \\ &= 13\times4 + 8\times1 + 9\times0, & x=13 \\ &= 14\times4 + 4\times1 + 12\times0, & x=14 \\ &= 15\times4 + 0\times1 + 15\times0, & x=15. \end{aligned}$$

Total, 6 possible values are there, $10 \le x \le 15$,

Q. 231. The number of permutations of $\{1, 2, 3, 4, 5\}$ that keep at least one integer fixed is

(a) 81 (b) 76 (c) 120 (d) 60.

Sol. (c) There are total 5! permutations of these 5 digits.

Q.232. If $f(x)$ is a real valued function such that

$$2f(x) + 3f(-x) = 15 - 4x, \ \forall\, x \in \mathbb{R}, \text{ then } f(2) \text{ is}$$

(a) -15 (b) 22 (c) 11 (d) 0.

Sol. (c) Put $x = -x$, $2f(-x) + 3f(x) = 15 + 4x$

$$2f(x) + 3f(-x) = 15 - 4x$$

solving equations, we get $f(x) = 3 + 4x$, $f(2) = 11$.

Q.233. If M is a matrix of 3×3 order such that

$$[0 \ \ 1 \ \ 2]M = [1 \ \ 0 \ \ 0] \text{ and } [3 \ \ 4 \ \ 5]M = [0 \ \ 1 \ \ 0]$$

then $[6 \ \ 7 \ \ 8]M$ is equal to

(a) $[2 \ \ 1 \ \ -2]$ (b) $[0 \ \ 0 \ \ 1]$ (c) $[-1 \ \ 2 \ \ 0]$ (d) $[9 \ \ 10 \ \ 8]$

Sol. (b) Do yourself.

Q.234. For positive real numbers $a_1, a_2, \ldots, a_{100}$, let

$$p = \sum_{i=1}^{100} a_i \text{ and } q = \sum_{1\le i\le j\le 100} a_i a_j \text{ , then}$$

(a) $q = \frac{p^2}{2}$ (b) $q^2 \le \frac{p^2}{2}$ (c) $q < \frac{p^2}{2}$ (d) none.

Sol. (c)

$$a_1 + a_2 + \cdots + a_{100} = p,$$

$$p^2 = (a_1 + a_2 + \cdots + a_{100})^2 = \sum_{i=1}^{100} a_i^2 + 2\sum_{i<j}^{100} a_i a_j$$

$$\therefore p^2 - 2q \geq 0 \quad \left[\because \sum_{i=1}^{100} a_i^2 \geq 0\right]$$

Q.235. A club with x members is organised into four committees such that
(a) each member is in exactly two committees,
(b) any two committees have exactly one member in common.
Then x has
(a) exactly two values both between 4 and 8,
(b) exactly one value between 4 and 8,
(c) exactly two values both between 8 and 16,
(d) exactly one value between 8 and 16.

Sol. Four committees are there, let us denote members by A, B, C, D, E, F, ...

1st com :- A D E
2nd com :- A B F
3rd com :- B C E
4th com :- C D F

[(a) each member is exactly in two committees
(b) only two committees have exactly one member in common]

Q.236. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Then
(a) there exists a matrix C such that $A = BC = CB$.
(b) there is no matrix C such that $A = BC$.
(c) there exists a matrix C such that $A = BC$, but $A \neq CB$.
(d) there is no matrix C such that $A = CB$.

Sol. (c) $B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is an lower triangular mtx.

Take, $C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is an upper triangular mtx.

$\therefore BC = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$, $CB \neq \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

Q.237. If the matrix $A = \begin{bmatrix} a & 1 \\ 2 & 3 \end{bmatrix}$ has 1 as an eigenvalue, then trace(A) is
(a) 4 (b) 5 (c) 6 (d) 7

Sol. (b) $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a-1 & 1 \\ 2 & 3-1 \end{vmatrix} = 0 \Rightarrow a = 2.$

$\therefore$ trace$(A) = 2 + 3 = 5 =$ sum of diagonal elements.

Q.238. The eigenvalues of the matrix $X = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ are

(a) 1, 1, 4 (b) 1, 4, 4 (c) 0, 1, 4 (d) 0, 4, 4

Sol. (a) $|X - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$\Rightarrow (\lambda-1)(\lambda-1)(\lambda-4) = 0$$

$\Rightarrow \lambda = 1, 1, 4$ are the eigen values of X.

Q.239. A real 2X2 mtx. M such that

$$M^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1-\epsilon \end{pmatrix}$$

(a) exists for all $\epsilon > 0$ (b) does not exist for any $\epsilon > 0$

(c) exists for some $\epsilon > 0$ (d) none of the above.

Sol. (b)

Since M^2 is a diagonal matrix, so

$$M = \begin{bmatrix} i & 0 \\ 0 & \sqrt{1-\epsilon} \end{bmatrix}.$$

so, M is not a real matrix, for any values of ϵ

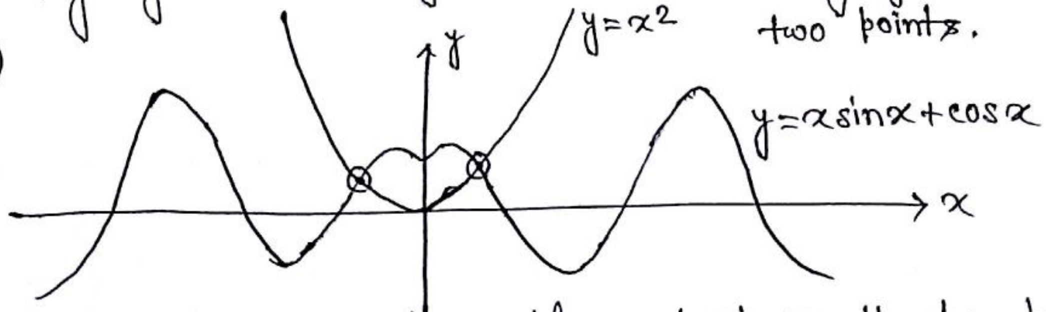
M is a non-real matrix.

Q.240. If $f(x) = x^2$ and $g(x) = x\sin x + \cos x$ then

(a) f and g agree at no points (b) f and g agree at exactly one point

(c) f and g agree at exactly two point (d) f and g agree at more than two points.

Sol. (c)



So, we can see two graphs meet at exactly two points.

Q.241. The value of $\left(\frac{1+i\sqrt{3}}{2}\right)^{2008}$ is

(a) $\frac{1+i\sqrt{3}}{2}$ (b) $\frac{1-i\sqrt{3}}{2}$ (c) $\frac{-1-i\sqrt{3}}{2}$ (d) $\frac{-1+i\sqrt{3}}{2}$

Solution:- (c) $A = \left(\frac{1+i\sqrt{3}}{2}\right)$, $A^2 = \frac{-1+i\sqrt{3}}{2}$, $A^4 = \frac{-1-i\sqrt{3}}{2}$.

$= -A$.

$$\therefore A^{2008} = \left(A^4\right)^{502} = A^4 = \frac{-1-i\sqrt{3}}{2}.$$

Q.242. The rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix}$ is less than 4 if and only if

(a) $a=b=c=d$ (b) at least two of a,b,c,d are equal

(c) at least three of a,b,c,d are equal (d) a,b,c,d are distinct real numbers.

Solution:- (b) rank $(A) < 4 \Rightarrow$ Det $(A) = 0$, i.e., $|A| = 0$.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & b(b-a) & c(c-a) & d(d-a) \\ 0 & b^2(b-a) & c^2(c-a) & d^2(d-a) \end{vmatrix}$$

[expanding by first column] $[R_i' = R_i - aR_{i-1}$ $\forall i>1]$.

$$= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & 1 & 1 \\ b & c & d \\ b^2 & c^2 & d^2 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & 1 & 1 \\ 0 & c-b & d-b \\ 0 & c(c-b) & d(d-b) \end{vmatrix}$$ $[R_i' = R_i - bR_{i-1}$ $\forall i>1]$

(expanding by first column)

$$= (a-b)(a-c)(b-c)(a-d)(d-b)\begin{vmatrix} 1 & 1 \\ c & d \end{vmatrix}$$

$$= (a-b)(a-c)(a-d)(b-c)(b-d)(c-d) = 0$$

$\Rightarrow a=b$

or, $b=c$, or, $a=d$, or, $a=c$,

or, $b=d$, or, $c=d$.

i.e. any two of a,b,c,d are equal.

Q. 243. A subset S of the set of numbers $\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is said to be good if it has exactly 4 elements and their gcd = 1. Then the number of good subset is

(a) 126 (b) 125 (c) 123 (d) 121.

Sol. (d) Total number of subsets containing exactly 4 elements from 9 elements are $= {}^9C_4 = \frac{9!}{4!\,5!} = 126$.

Now, gcd = 1, so we need not to take into count these subsets: $\{2, 4, 6, 8\}$, $\{2, 4, 6, 10\}$, $\{2, 4, 8, 10\}$, $\{2, 6, 8, 10\}$, $\{4, 6, 8, 10\}$.

So, there are total $(126 - 5) = 121$ good subsets.

Q. 244. In how many ways can three persons, each throwing a single die once, make a score of 11?

(a) 22 (b) 27 (c) 24 (d) 38

Sol. (b) We can use a tree diagram here:-

$1 \begin{cases} \to 6, 4 & : 3! \text{ ways} \\ \to 5, 5 & : 3 \text{ ways} \end{cases}$

$2 \begin{cases} \to 4, 5 & : 3! \text{ ways} \\ \to 3, 6 & : 3! \text{ ways} \end{cases}$

$3 \begin{cases} \to 4, 4 & : 3 \text{ ways} \\ \to 3, 5 & : 3 \text{ ways} \end{cases}$

So, there are total $(3! + 3 + 3! + 3! + 3 + 3) = 27$ ways.

Explanation:- For $\{1, 6, 4\}$ there are total 3! ways, since $\{1, 6, 4\}$, $\{4, 6, 1\}$, $\{6, 1, 4\}$, $\{1, 4, 6\}$, $\{4, 1, 6\}$, $\{6, 4, 1\}$ are six possibilities, but in case of $\{1, 5, 5\}$ there are total 3 ways, since there are $\{1, 5, 5\}$, $\{5, 1, 5\}$, $\{5, 5, 1\}$ only 3 such permutations.

245. x^2+x+1 is a factor of $(x+1)^n - x^n - 1$, whenever

(a) n is odd
(b) n is odd and a multiple of 3.
(c) n is even multiple of 3
(d) n is odd and not a multiple of 3.

Sol. (c) Let $n=1$, $(x+1)-x-1=0$

$n=2$, $(x+1)^2-x^2-1 = +2x$

$n=3$, $(x+1)^3-x^3-1 = 3x^2+3x$

$\vdots$

246. The map $f(x) = a_0\cos|x| + a_1\sin|x| + a_2|x|^3$ is differentiable at $x=0$ if and only if

(a) $a_1=0$ and $a_2=0$
(b) $a_0=0$ and $a_1=0$
(c) $a_1=0$
(d) a_0, a_1, a_2 can take any real value.

Sol. (c) $|x|^3$ is differentiable at $x=0$.

$\cos|x| = \cos x$ is differentiable at $x=0$.

But $\sin|x|$ is not differentiable at $x=0$

So, $a_1=0$ is the only criteria for $f(x)$ to be differentiable.

247. $f(x)$ is a differentiable function on the real line such that $\lim\limits_{x\to\infty} f(x)=1$ and $\lim\limits_{x\to\infty} f'(x)=\alpha$. Then

(a) α must be 0
(b) α need not to be 0, but $|\alpha|<1$.
(c) $\alpha>1$
(d) $\alpha<-1$.

Sol. (a) Let $f(x) = 1+\dfrac{k}{x^p}$, $p>0$

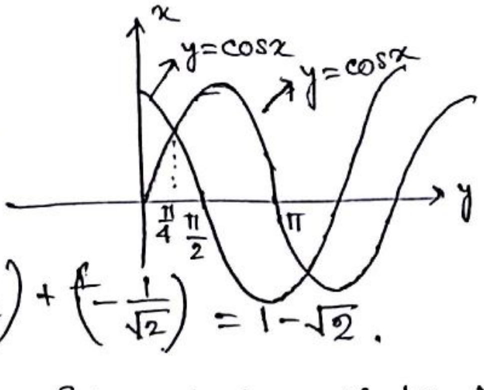
So, $\lim\limits_{x\to\infty} f(x) = 1$,

$\therefore \lim\limits_{x\to\infty} f'(x) = \lim\limits_{x\to\infty} \dfrac{k(-p)}{x^{p+1}} = 0 = \alpha$.

248. $\int_0^{\pi} \min(\sin x, \cos x)\,dx$ equals

(a) $1-2\sqrt{2}$ (b) 1 (c) 0 (d) $1-\sqrt{2}$.

<u>Sol.</u> (d) $f(x) = \begin{cases} \sin x, & 0 \le x \le \frac{\pi}{4} \\ \cos x, & \frac{\pi}{4} \le x \le \pi \end{cases}$



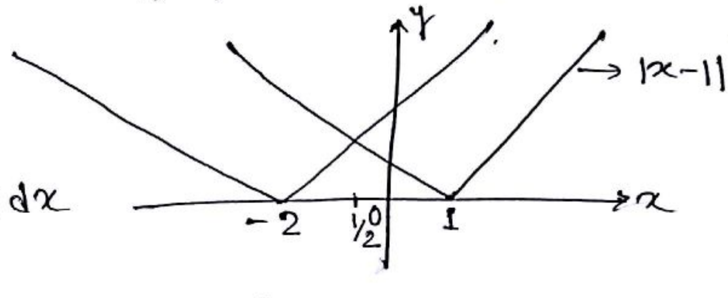
$\therefore \int_0^{\pi} \min(\sin x, \cos x)\,dx$

$= \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi} \cos x\,dx = \left(1-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) = 1-\sqrt{2}.$

249. The value of the integral $\int_{-2}^{2} \min\{|x-1|, |x+2|\}\,dx$, is

(a) $11/4$ (b) $9/4$ (c) $11/2$ (d) $9/2$.

<u>Sol.</u> $\int_{-2}^{2} \min\{|x-1|, |x+2|\}\,dx$



$= \int_{-2}^{-1/2} |x+2|\,dx + \int_{-1/2}^{2} |x-1|\,dx$

$= -\int_{-2}^{-1/2} (x+2)\,dx + \int_{-1/2}^{1} (1-x)\,dx + \int_{1}^{2} (x-1)\,dx = ?$

250. Let $f(x)$ be a given differentiable function. Consider the following differential equation in y $f(x)\frac{dy}{dx} = yf'(x) - y^2$.

The general solution of the equation is given by

(a) $y = -\frac{x+c}{f(x)}$ (b) $y^2 = \frac{f(x)}{x+c}$ (c) $y = \frac{f(x)}{x+c}$ (d) $y = \frac{[f(x)]^2}{x+c}$.

<u>Solution:-</u> (c) $\frac{dy}{dx} = \frac{yf'(x) - y^2}{f(x)}$

$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\frac{f'(x)}{f(x)} = \frac{1}{f(x)}$ $\left[\text{Put } \frac{1}{y} = v,\ -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}\right.$

$\Rightarrow \frac{dv}{dx} + v.\frac{f'(x)}{f(x)} = \frac{1}{f(x)}$ which is a linear eqn. in v.

$\therefore$ I.F. $= e^{\int \frac{f'(x)}{f(x)}dx} = e^{\log f(x)} = f(x)$. Hence solution is

$v.f(x) = \int \frac{1}{f(x)} \cdot f(x)\,dx + c = x + c$ i.e. $y = \frac{f(x)}{x+c}$.

251. The integral $\int_0^2 (|x-a|-|x-1|)\,dx$ is non-negative

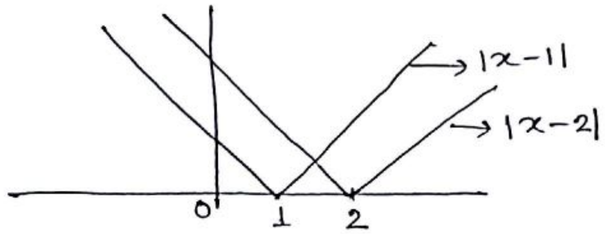
(a) only for a in $[0,2]$

(b) only for a outside $[0,2]$

(c) only for $a=1$.

(d) for any real number a.

Sol. (b) For $a=1$, $I=0$.



252. Let a and b be real numbers such that

$$\lim_{x\to\infty} \sqrt{x^2-x-1} - ax - b = 0.$$ Then the value of b is

(a) -1 (b) $-1/2$ (c) 0 (d) $1/2$

Sol. (b) $$\lim_{x\to\infty}\left(\sqrt{x^2-x-1}-ax-b\right) = \lim_{x\to\infty} \frac{(x^2-x-1)-(ax+b)^2}{\sqrt{x^2-x-1}+ax^2+b}$$

$$= \lim_{x\to\infty} \frac{(1-a^2)x^2-(1+2ab)x-(1+b^2)}{\sqrt{x^2-x-1}+ax^2+b} = 0$$

$$\Rightarrow 1-a^2=0 \Rightarrow a = 1, \quad 1+2ab=0$$

$$1+2b=0$$

$$\Rightarrow b = -1/2.$$

253. $\lim_{x\to 0} \sin \dfrac{e^x-x-1-x^2/2}{x^2}$ is eaual to

(a) 0 (b) $1/2$ (c) 1 (d) does not exist

Sol. (a)

For $x\to 0$, $e^x \simeq 1+x+\frac{x^2}{2!}$

So, $$\lim_{x\to 0} \sin \frac{\left(1+x+\frac{x^2}{2}\right)-\left(1+x+\frac{x^2}{2}\right)}{x^2}$$

$$= \lim_{x\to 0} \sin(0) = 0.$$

254. Let $f(x)$ be the function $f(x) = \begin{cases} \dfrac{x^p}{(\sin x)^q} & \text{if } x>0 \\ 0 & \text{if } x=0 \end{cases}$

Then $f(x)$ is continuous at $x=0$ if

(a) $p>q$ (b) $p>0$ (c) $q>0$ (d) $p<q$

Sol. (b)

$$\left|f(x)-f(0)\right| = \left|\frac{x^p}{(\sin x)^q} - 0\right|$$

$$\leq |x^p| < \epsilon$$

whenever $|x-0| < \epsilon^{1/p} = \delta$ if $p>0$.

So, $f(x)$ is continuous for $p>0$ at $x=0$.

255. The limit $\lim\limits_{x\to 0+} \log\left(\dfrac{1+x}{1-x}\right)^{1/x}$ equals

(a) 0 (b) 1 (c) 2 (d) does not exist.

Sol. (c)

$$L = \log\left(\frac{1+x}{1-x}\right)^{1/x}$$

$$\log L = \frac{1}{x}\log\left(\frac{1+x}{1-x}\right)$$

$$\lim_{x\to 0+} \log L = \lim_{x\to 0+} \frac{1}{x}\left\{2\left(x + \frac{x^3}{3} + \cdots\right)\right\}$$

$$= 2.$$

256. If $0<c<d$, then the sequence $a_n = (c^n + d^n)^{1/n}$ is

(a) bounded & monotone decreasing (b) bounded & monotone increasing

(c) monotone increasing & unbounded for $1<c<d$

(d) monotone decreasing & unbounded for $1<c<d$.

Sol. (b) $0<c<d \Rightarrow 0<c^n<d^n \ \forall\ n\in\mathbb{N}$

$$\therefore c^n + d^n < 2d^n$$

or, $a_n < 2^{1/n}.d \ \forall\ n\in\mathbb{N}$

or, $d < a_n < d.2^{1/n}$

By Squeeze theorem, $\lim(a_n) = d$. So, the sequence is bounded and monotonic increasing.

257. The limit $\lim_{x\to\infty}\left(1-\frac{1}{n^2}\right)^n$ equals

(a) e^{-1} (b) $e^{-1/2}$ (c) e^{-2} (d) 1

Sol. (d) $L=\left(1-\frac{1}{n^2}\right)^n$

$$\log L = n\log\left(1-\frac{1}{n^2}\right)$$

$$\lim_{x\to\infty}\log L = \lim_{x\to\infty}\left[-n\left\{\frac{1}{n^2}+\frac{1}{2n^4}+\cdots\infty\right\}\right]$$

$$=0$$

$$\therefore\ L=e^0=1.$$

258. The number of real solution of the equation $\left(\frac{9}{10}\right)^x = -3+x-x^2$ is

(a) 2 (b) 0 (c) 1 (d) none of the above

Sol. (b) $\left(\frac{9}{10}\right)^x = -\left(x-\frac{1}{2}\right)^2 + \frac{12-1}{4}$

LHS is always positive but RHS is negative, thus the equation does not have any real solution.

259. $\lim_{n\to\infty}\dfrac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n}$

(a) equals 0 (b) equals 1 (c) equals ∞ (d) none

Sol. (b) Cauchy's First limit theorem:- If $\lim_{n\to\infty}u_n=l$, then

$$\lim_{n\to\infty}\frac{u_1+u_2+\cdots+u_n}{n}=l.$$

Here $\lim_{n\to\infty}u_n=\lim_{n\to\infty}n^{1/n}=1$, so, by cauchy's first limit theorem $\lim_{n\to\infty}\dfrac{u_1+u_2+\cdots+u_n}{n}=\lim_{n\to\infty}\dfrac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n}=1.$

260. The sum of the series

$$1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\cdots\cdots \quad \text{is}$$

(a) e^2 (b) 3 (c) $\sqrt{5}$ (d) $\sqrt{8}$

Sol. (d) $\sqrt{8} = 2^{3/2} = \left(\frac{1}{2}\right)^{-3/2} = \left(1-\frac{1}{2}\right)^{-3/2}$

$$= 1+\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)+\frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{2!}\left(-\frac{1}{2}\right)^2+\cdots\cdots$$

$$= 1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\cdots\cdots$$

261. If $0<x<1$, then the sum of the infinite series

$$\frac{1}{2}x^2+\frac{2}{3}x^3+\frac{3}{4}x^4+\cdots\cdots \quad \text{is}$$

(a) $\log\frac{1+x}{1-x}$ (b) $\frac{x}{1-x}+\log(1+x)$ (c) $\frac{1}{1-x}+\log(1-x)$

(d) $\frac{x}{1-x}+\log(1-x)$

Sol. (b) $\frac{1}{2}x^2+\frac{2}{3}x^3+\frac{3}{4}x^4+\cdots\cdots$

$$=\left(1-\frac{1}{2}\right)x^2+\left(1-\frac{1}{3}\right)x^3+\left(1-\frac{1}{4}\right)x^4+\cdots\cdots$$

$$=\left\{x^2+x^3+x^4+\cdots\cdots\right\}-\left\{\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\cdots\right\}$$

$$=\left\{1+x+x^2+\cdots\cdots\right\}-\left\{x+\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots\right\}-1$$

$$=\frac{1}{(1-x)}+\log(1-x)-1$$

$$=\frac{x}{1-x}+\log(1-x).$$

262. The polar eqn. $r = a\cos\theta$ represents

(a) a spiral (b) a parabola (c) a circle (d) none.

Sol. (c) $r^2 = ar\cos\theta$

$\therefore x^2+y^2 = ax$ (since $r = x^2+y^2$, $x = a\cos\theta$)

$\therefore x^2+ax+y^2=0$

$\therefore \left(x+\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$

This is a circle of radius $\frac{a}{2}$ and centre $\left(-\frac{a}{2},0\right)$.

263. The value of the infinite product

$$P = \frac{7}{9} \times \frac{26}{28} \times \frac{63}{65} \times \dots \times \frac{n^3-1}{n^3+1} \times \dots\dots \text{ is}$$

(a) 1 (b) 2/3 (c) 7/3 (d) none

Sol. (b)

$$P = \frac{2^3-1}{2^3+1} \cdot \frac{3^3-1}{3^3+1} \dots\dots\dots$$

$$= \left(\frac{(2-1)(2^2+1+2)}{(2+1)(2^2+1-2)}\right)\left(\frac{(3-1)(3^2+1+3)}{(3+1)(3^2+1\;\;3)}\right)\dots\dots$$

$$= \left(\frac{1.2.3.4\dots}{3.4.5.6\dots}\right)\left(\frac{7.13.21\dots\dots}{3.7.13\dots\dots}\right)$$

$$= \frac{1.2}{3} = \frac{2}{3}.$$

264. The number of positive integers which are less than or equal to 1000 and are divisible by none of 17, 19 and 23 equals (a) 854 (b) 153 (c) 160 (d) none.

Sol. (a)

A : integers divisible by 17
B : " " " 19
C : " " " 23

$$n(A \cup B \cup C)^c = 1000 - n(A) - n(B) - n(C) + n(A \cap B) + n(B \cap C) + n(C \cap A) - n(A \cap B \cap C)$$

$$= 1000 - \left[\frac{1000}{17}\right] - \left[\frac{1000}{19}\right] - \left[\frac{1000}{23}\right] + \left[\frac{1000}{17 \times 19}\right] + \left[\frac{1000}{17 \times 23}\right] + \left[\frac{1000}{19 \times 23}\right] - \left[\frac{1000}{17 \times 19 \times 23}\right]$$

$$= 1000 - 58 - 52 - 43 + 3 + 2 + 2 - 0$$

$$= 854.$$

265. Let $\{a_n\}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n$ exists if and only if

(a) $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$ exist

(b) $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+2}$ exist

(c) $\lim_{n\to\infty} a_{2n}$, $\lim_{n\to\infty} a_{2n+1}$ and $\lim_{n\to\infty} a_{3n}$ exist

(d) none of the above.

Sol. (a) If a sequence converges then all of its subsequences converges.

$\therefore$ $\lim_{n\to\infty} a_n$ converges $\Rightarrow$ $\lim_{n\to\infty} a_{2n}$, $\lim_{n\to\infty} a_{2n+1}$ exist.

a_{2n} & a_{2n+1} cover all the terms in a_n.

So, Converse is also true.

266. Consider two series (i) $\sum_{n=1}^{\infty} \sin\frac{\pi}{n}$ (ii) $\sum_{n=1}^{\infty} (-1)^n \cos\frac{\pi}{n}$. Then

(a) both (i) and (ii) converge

(b) (i) converges, (ii) diverges

(c) (i) diverges, (ii) converges

(d) Both (i) and (ii) diverge.

Sol. (d) (i) $\sum_{n=}^{\infty} \sin\frac{\pi}{n} = \frac{\pi}{n} - \frac{1}{3!}\left(\frac{\pi}{n}\right)^3 + \cdots\cdots = U_n$

Let $V_n = \frac{1}{n}$, $\therefore \lim_{n\to\infty} \frac{U_n}{V_n} = \pi \neq 0$.

Since $\sum V_n$ diverges, so does $\sum U_n$.

(ii) $\sum_{n=1}^{\infty} (-1)^n \cos\frac{\pi}{n}$ converges if $\sum_{n=1}^{\infty} \left|(-1)^n \cos\frac{\pi}{n}\right| = \sum_{n=1}^{\infty} \cos\frac{\pi}{n}$ converges.

But $\lim_{n\to\infty} \cos\frac{\pi}{n} = \cos 0 = 1 \neq 0$,

So, $\sum_{n=1}^{\infty} \cos\frac{\pi}{n}$ diverges.

267. If $a = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$ and $b = \lim_{n\to\infty}\frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$, then

(a) both $a=\infty$ and $b=\infty$ (b) $a=\infty$ and $b=0$
(c) $a=\infty$ and $b=1$ (d) none.

Sol. (d) $\{u_n\}=\{\frac{1}{n}\}$, $\lim_{n\to\infty} u_n = 0$, $\therefore a=0$

So, By Cauchy's first theorem $\lim_{n\to\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n}=0$.

$\therefore b=0$.

268. For $x>0$, let $f(x)=\lim_{n\to\infty} n\left(x^{1/n}-1\right)$. Then

(a) $f(x)+f(1/x)=1$ (b) $f(xy)=f(x)+f(y)$
(c) $f(xy)=xf(y)+f(x)$ (d) none of the above is true.

Sol. $f(x)=\lim_{n\to\infty} n\left(x^{1/n}-1\right)=0$.

269. Let $\{a_n\}$ be a sequence of non-negative real numbers such that the series $\sum_{n=1}^{\infty} a_n$ is convergent. If p is a real number such that the series $\sum\frac{\sqrt{a_n}}{n^p}$ diverges, then

(a) $p<\frac{1}{2}$ (b) $p\le\frac{1}{2}$ (c) $\frac{1}{2}<p\le 1$ (d) $\frac{1}{2}\le p<1$.

Solution:- (a)

$\sum a_n$ is convergent.

$\Rightarrow \sum\frac{a_n}{n}$ is convergent.

$\sum\left(a_n+\frac{1}{n}\right)$ is convergent. By, AM $\geq$ GM

$\Rightarrow \sum\frac{\sqrt{a_n}}{n^{1/2}}$ is convergent

so, for $p<\frac{1}{2}$, the series is divergent.

● 270. In the Taylor expansion of the function $f(x) = e^{x/2}$ about $x=3$, the coefficient of $(x-3)^5$ is

(a) $e^{3/2} \cdot \frac{1}{5!}$ (b) $e^{3/2} \cdot \frac{1}{2^5 5!}$ (c) $e^{-3/2} \cdot \frac{1}{2^5 . 5!}$ (d) none.

Sol. (b) $$\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$= \frac{f^5(3)\,(x-3)^5}{5!} = \frac{e^{3/2}\,(x-3)^5}{2^5 . 5!}$$

● 271. Let $$f(x,y) = \begin{cases} e^{-1/(x^2+y^2)} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Then $f(x,y)$ is

(a) not continuous at $(0,0)$

(b) continuous at $(0,0)$ but doesnot have first order partial derivatives

(c) continuous at $(0,0)$ and has first order partial derivatives but not differentiable at $(0,0)$

(d) differentiable at $(0,0)$.

Sol. Check yourself.

272. The minimum value of the function $f(x,y) = 4x^2+9y^2-12x-12y+14$ is
(a) 1 (b) 3 (c) 14 (d) none

Sol. (a)
$$f(x,y) = 4x^2+9y^2-12x-12y+14$$
$$= (4x^2-12x+9)+(9y^2-12y+4)+1$$
$$= (2x-3)^2+(3y-2)^2+1$$
$$\geq 1$$
So, minimum value of $f(x,y)$ is 1.

273. If $M = \int_0^{\pi/2} \frac{\cos x}{x+2}\,dx$ and $N = \int_0^{4} \frac{\sin x \cos x}{(x+1)^2}\,dx$, then $M-N$ is
(a) π (b) $\pi/4$ (c) $\frac{2}{\pi-4}$ (d) $\frac{2}{\pi+4}$.

Sol. H.S. Level. Do yourself.

274. A permutation of $1, 2, \ldots, n$ is chosen at random. Then the probability that the numbers 1, 2 appear as neighbour equals
(a) $\frac{1}{n}$ (b) $\frac{2}{n}$ (c) $\frac{1}{n-1}$ (d) $\frac{1}{n-2}$.

Sol. (a) $P(A) = \dfrac{(n-1)!}{n!}$.

$= \frac{1}{n}$. Since there are $n!$ permutations in total. since 1, 2 appear as neighbour, so taking it as a group, so there are total $(n-1)!$ as no. of favourable cases.

275. Let n be a positive integer and $(1+x+x^2)^n = a_0 + a_1x + \dots + a_{2n}x^{2n}$, then the value of $a_0^2 - a_1^2 + a_2^2 - \dots + a_{2n}^2$ is

(a) 0 (b) a_0 (c) a_n (d) a_{2n}.

Sol. (c) Replacing x by $\left(-\frac{1}{x}\right)$, we get

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + \dots - a_{2n-1}\cdot\frac{1}{x^{2n-1}} + \frac{a_{2n}}{x^{2n}}.$$

or, $(1-x+x^2)^n = a_0x^{2n} - a_1x^{2n-1} + a_2x^{2n-2} + \dots + a_{2n} \dots\dots (1)$

And given $(1+x+x^2)^n = a_0 + a_1x + \dots\dots + a_{2n}x^{2n} \dots\dots (2)$

Multiplying corresponding sides of (1) and (2), we have

$$(1+x^2+x^4)^n = (a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}) \times (a_0x^{2n} - a_1x^{2n-1} + \dots - a_{2n-1}x + a_{2n}) \dots (3)$$

Replacing x by x^2 in (2), we get

$$(1+x^2+x^4)^n = (a_0 + a_1x^2 + a_2x^4 + \dots + a_nx^{2n} + \dots + a_{2n}x^{4n}) \dots\dots (4)$$

Equating coefficient of x^{2n} on both sides of (3) and (4)

$$a_0^2 - a_1^2 + a_2^2 - \dots + a_{2n}^2 = a_n.$$

276. The set of all real numbers x such that $\big||3-x| - |x+2|\big| = 5$ is

(a) $[3,\infty)$ (b) $(-\infty,-2]$ (c) $(-\infty,-2]\cup[3,\infty)$

(d) $(-\infty,-3]\cup[2,\infty)$.

Sol. (c) $\big(|3-x| - |x+2|\big)^2 = 25$

$\Rightarrow (3-x)^2 + (x+2)^2 - 2|3-x||x+2| = 25$

$\Rightarrow x^2 - x - |-x^2+x+6| = 6$

So, it is clear that $-x^2+x+6 < 0$, i.e., $x^2-x-6 \geq 0$

$(x-3)(x+2) \geq 0$

So, $x \leq -2$ & $x \geq 3$.

$\therefore x \in (-\infty,-2]\cup[3,\infty)$

277. The differential equation of the system of circles touching the y-axis at the origin is

(a) $x^2+y^2-2xy\frac{dy}{dx}=0$ (b) $x^2+y^2+2xy\frac{dy}{dx}=0$

(c) $x^2-y^2-2xy\frac{dy}{dx}=0$ (d) $x^2-y^2+2xy\frac{dy}{dx}=0$

Sol. (d)
$$x^2+y^2-2ax=0$$
$$2x+2y\frac{dy}{dx}-2a=0$$
$$\Rightarrow 2\left(x+y\frac{dy}{dx}\right)=2\left(\frac{x^2+y^2}{2x}\right)$$
$$\Rightarrow 2x^2+2xy\frac{dy}{dx}=x^2+y^2$$
$$\Rightarrow x^2-y^2+2xy\frac{dy}{dx}=0.$$

278. Let $y(x)$ be a non-trivial solution of the second order linear differential equation
$$\frac{d^2y}{dx^2}+2c\frac{dy}{dx}+ky=0,\text{ where } c<0,\ K>0,\ c^2-K>0,$$
then (a) $|y(x)|\to\infty$ as $x\to\infty$ (b) $|y(x)|\to 0$ as $x\to\infty$

(c) $\lim_{x\to\pm\infty}|y(x)|$ exists & is finite (d) none.

Sol. (a)
$$m^2+2cm+K=0$$
$$\therefore m=\frac{-2c\pm\sqrt{4c^2-4k}}{2}=\frac{-2c\pm\sqrt{4(c^2-K)}}{2}$$
$$=\frac{-2c\pm 2a}{2}\quad\left[\because c^2-K>0,\ a^2=c^2-K\right]$$
$$=\frac{-c-a}{2},\ \frac{-c+a}{2}$$

The general solution of the given L.D.E. is
$$y=c_1e^{m_1x}+c_2e^{m_2x}=c_1e^{-\left(\frac{c+a}{2}\right)x}+c_2e^{-\left(\frac{c-a}{2}\right)x}$$
so, $|y(x)|\to\infty$ as $x\to\infty$.

279. Let y be a function of x satisfying

$$\frac{dy}{dx} = 2x^3\sqrt{y} - 4xy$$

If $y(0)=0$ and then $y(1)$ equals

(a) $1/4e^2$ (b) $1/e$ (c) $e^{1/2}$ (d) $e^{3/2}$.

Sol. (a)

$$\frac{dy}{dx} + (4x)\,y = 2\sqrt{y}\,x^3 \quad \text{(Bernoulli's Equation)}$$

Putting $\sqrt{y} = z$, the equation reduces to

$$\frac{dz}{dx} + (2x)\,z = x^3 \quad \text{(linear in } z\text{)}$$

$$\therefore \text{ I.F.} = e^{\int 2x\,dx} = e^{x^2}.$$

Multiplying and integrating

$$z e^{x^2} = \int x^3 e^{x^2} dx \quad (\text{Put } x^2 = u)$$

$$= \frac{1}{2}(x^2-1)e^{x^2} + C.$$

$\therefore$ General solution is given by:- $\sqrt{y} = \frac{1}{2}(x^2-1) + Ce^{-x^2}$

Since $y(0)=0$, so, $C = \frac{1}{2}$.

$$\therefore\ y(1) = \left(\frac{1}{2e}\right)^2 = \frac{1}{4e^2}.$$

280. The differential equation of all the ellipses centred at the origin is

(a) $y^2 + x(y')^2 - yy' = 0$ (b) $xyy'' + x(y')^2 - yy' = 0$

(c) $yy'' + x(y')^2 - xy' = 0$ (d) none.

Sol.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, after differentiating w.r.t. x, we get

$$\Rightarrow \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \Rightarrow \frac{yy'}{b^2} = -\frac{x}{a^2}$$

$$\Rightarrow \frac{(y')^2}{b^2} + \frac{y(y'')}{b^2} = -\frac{1}{a^2}$$

$$\Rightarrow (y')^2 + y(y'')^2 = -\frac{b^2}{a^2}.$$

281. If $y(t)$ is a solution of $(1+t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$ then $y(1)$ equals

(a) $\frac{1}{2}$ (b) $e+\frac{1}{2}$ (c) $e-\frac{1}{2}$ (d) $-\frac{1}{2}$.

Sol. (d)

$$\frac{dy}{dt} - \frac{t}{1+t}y = \frac{1}{1+t} \quad (\text{linear in } y)$$

$$\therefore \text{I.F.} = e^{-\int \frac{t}{1+t}dt} = e^{-(t-\log(1+t))}$$
$$= e^{-t}.(1+t)$$

Multiplying and integrating

$$ye^{-t}(1+t) = \int e^{-t}(1+t)\frac{dt}{(1+t)} = -e^{-t}+c$$

when $y(0) = -1$, $\Rightarrow c = 0$.

$$\therefore ye^{-t}(1+t) = -e^{-t}$$

$$\therefore y = -\frac{1}{1+t}$$

$$\therefore y(1) = -\frac{1}{2}.$$

282. Consider the polynomial $x^5 + ax^4 + bx^3 + cx^2 + dx + 4$. If $(1+2i)$ and $(3-2i)$ are two roots of this polynomial then the value of a is

(a) $-524/65$ (b) $524/65$ (c) $-1/65$ (d) $1/65$.

Sol. (a) The polynomial has 5 roots. Since complex root occur in pairs, so there is one real root, taking it as m.

so, $m, 1+2i, 1-2i, 3+2i, 3-2i$ are the five roots.

Sum of the roots $= -\frac{a}{1} = 8+m$.

Product of the roots $= (1+4)(9+4)m = 65m = \frac{4}{1}$

$\therefore a = -8-\frac{4}{65} = -\frac{524}{65}$. $\therefore m = \frac{4}{65}$.

283. Show that the number 111......1 (91 digits) is not a prime number.

Solution:-

$$111......1 \text{ (91 digits)}$$

$$= 10^{90} + 10^{89} + ... + 10^2 + 10 + 1$$

$$= \frac{10^{91}-1}{10-1} = \left(\frac{10^{91}-1}{10^7-1}\right)\left(\frac{10^7-1}{10-1}\right)$$

$$= \left(10^{84} + 10^{77} + 10^{70} + ... + 1\right)\left(10^6 + 10^5 + ... + 10 + 1\right)$$

[$\because 91 = 13 \times 7$, therefore we divide and multiply by $(10^7 - 1)$ or $(10^{13} - 1)$]

Thus, 111......1 (91 digits) is not a prime number.

284. Find all the values of x for which the following inequality holds:
$$\frac{8x^2+16x-51}{(2x-3)(x+4)} > 3.$$

Solution:-

$$\frac{8x^2+16x-51}{(2x-3)(x+4)} - 3 > 0$$

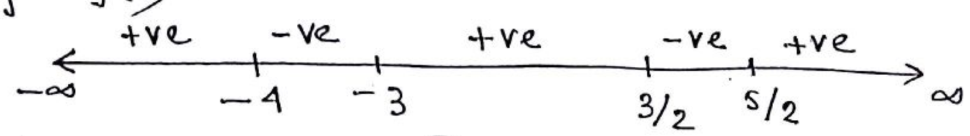
$$\Leftrightarrow \frac{2x^2+x-15}{2x^2+5x-12} > 0$$

Let $f(x) = \dfrac{2x^2+x-15}{2x^2+5x-12}$

Now, $2x^2+x-15 = 0 \Rightarrow x = -3, \frac{5}{2}$

and, $2x^2+5x-12 = 0 \Rightarrow x = -4, \frac{3}{2}$

Sign scheme for $f(x)$:-



$\therefore f(x) > 0 \Rightarrow x < -4$ or $x > \frac{5}{2}$ or, $-3 < x < \frac{3}{2}$

Hence, $x < -4$, or, $-3 < x < \frac{3}{2}$, or, $x > \frac{5}{2}$.

285. If x is real, prove that the expression $\frac{x^2+34x-71}{x^2+2x-7}$ has no value between 5 and 9.

Solution:- Let $y = \frac{x^2+34x-71}{x^2+2x-7}$

$\therefore y(x^2+2x-7) = x^2+34x-71$

or, $(y-1)x^2 + 2(y-17)x + (71-7y) = 0$

Since, x is real, so $D = b^2-4ac \geq 0$.

$\therefore 4(y-17)^2 - 4(y-1)(71-7y) \geq 0$

or, $8y^2 - 112y + 360 \geq 0$, or, $y^2 - 14y + 45 \geq 0$

Sign scheme for $y^2-14y+45$:-

$y^2 - 14y + 45 = 0$

$\therefore y = 5$ and 9 (real & unequal)

Since coefficient of y^2 in $y^2-14y+45$ is positive, therefore the sign scheme for $y^2-14y+45$ as is given below;

$$-\infty \longleftarrow \underset{+ve}{} \quad 5 \quad \underset{-ve}{} \quad 9 \quad \underset{+ve}{} \longrightarrow \infty$$

$\therefore y^2 - 14y + 45 \geq 0 \Rightarrow y \leq 5$ or $y \geq 9$

286. If $a<b<c<d$, then show that the quadratic equation $(x-a)(x-c) + \lambda(x-b)(x-d) = 0$ has real roots for all real values of λ.

Solution:- Let $f(x) = (x-a)(x-c) + \lambda(x-b)(x-d)$

Given $a<b<c<d$,

Now, $f(b) = (b-a)(b-c) < 0$ $[\because b-a>0$ & $b-c<0]$

and $f(d) = (d-a)(d-c) > 0$

Since, $f(b)$ and $f(d)$ have opposite signs therefore, equation $f(x)=0$ has one real root between b and d.

Since one root is real and a, b, c, d, λ are real, so, other roots will also be real.

287. Show that the equation $\sum_{i=1}^{3} \frac{a_i}{x-\lambda_i} = 0$, where $a_i > 0$ and $\lambda_1 < \lambda_2 < \lambda_3$ has two real roots lying in the interval (λ_1, λ_2) and (λ_2, λ_3).

Solution:- $f(x) = \frac{a_1}{x-\lambda_1} + \frac{a_2}{x-\lambda_2} + \frac{a_3}{x-\lambda_3}$

or, $f(x) = a_1(x-\lambda_2)(x-\lambda_3) + a_2(x-\lambda_1)(x-\lambda_3) + a_3(x-\lambda_1)(x-\lambda_2)$

Now, $f(\lambda_1) = a_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3) > 0$ $[\because a_1 > 0$ and $\lambda_1 < \lambda_2 < \lambda_3]$

$f(\lambda_2) = a_2(\lambda_2-\lambda_1)(\lambda_2-\lambda_3) < 0$

$f(\lambda_3) = a_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_2) > 0$

Since $f(\lambda_1) > 0$, $f(\lambda_2) < 0$, so $f(x)=0$ has one root between λ_1 and λ_2

And, $f(\lambda_2) < 0$, $f(\lambda_3) > 0$, so $f(x)=0$ has one root between λ_2 and λ_3.

288. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, then show that $1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \cdots + n^2 \cdot C_n = n(n+1)2^{n-2}$

Solution:- $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$

Differentiating both sides w.r.t. x, we get

$$n(1+x)^{n-1} = C_1 + C_2 \cdot 2x + \cdots + C_n \cdot nx^{n-1}$$

Multiplying by x in both sides, we get

$$nx(1+x)^{n-1} = C_1x + C_2 \cdot 2x^2 + \cdots + C_n \cdot n \cdot x^n$$

Differentiating w.r.t.x, bothsides, again, we get

$$n\left[(1+x)^{n-1} + x(n-1)(1+x)^{n-2}\right] = C_1 + C_2 \cdot 2^2 \cdot x + C_3 \cdot 3^2 \cdot x^2 + \cdots + C_n \cdot n^2 \cdot x^{n-1}$$

Putting x = 1, we get

$$C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \cdots + n^2 \cdot C_n = n\left[2^{n-1} + (n-1)2^{n-1}\right]$$
$$= n(n+1)2^{n-2}.$$

289. How many different words of 4 letters can be formed with the letters of the word "EXAMINATION"?

Solution:- Total number of letters in the given word = 11

'A' occurs twice, 'N' occurs twice and 'I' occurs twice.

Different letters are: E, X, A, M, I, N, T, O (eight)

Case I:- When two letters are identical and remaining two are different.

Letters used are:

(i) two A's and two out of E, X, A, M, I, N, T, O.

	No of selections	No of arrangements
	$1 \times {}^7C_2$	$1 \times {}^7C_2 \times \frac{7!}{2!} = 252$

(ii) two N's and two out of E, X, M, N, I, T, O.

	$1 \times {}^7C_2$	$1 \times {}^7C_2 \times \frac{7!}{2!} = 252$

(iii) two I's and two out of E, X, A, M, N, T, O.

	$1 \times {}^7C_2$	$1 \times {}^7C_2 \times \frac{7!}{2!} = 252$

Case II:- When two letters are identical and remaining two are identical.

Letters used are	No. of selection	No. of arrangements
(i) two A's & two N's	1×1	$\frac{4!}{2! \times 2!} = 6$
(ii) two A's & two I's	1×1	$\frac{4!}{2! \times 2!} = 6$
(iii) two N's & two I's	1×1	$\frac{4!}{2! \times 2!} = 6$

Case III:- When all four letters are different:

Letters used are	No. of selections	No. of arrangements
four out of E, X, A, M, I, N, T, O.	${}^8C_4$	${}^8C_4 \cdot 4! = 1680$

$\therefore$ Required number $= 756 + 18 + 1680 = 2454$.

SOLUTION TO SUBJECTIVE QUESTIONS

290. Let $S = \{(a_1, a_2, a_3, a_4) : a_i \in R, i = 1,2,3,4 ; a_1+a_2+a_3+a_4=0\}$ and $T = \{(a_1, a_2, a_3, a_4) : a_i \in R, i=1,2,3,4 ; a_1-a_2+a_3-a_4=0\}$
Find a basis for $S \cap T$. Also find its dimension.

Sol.

$S \cap T = \{(a_1, a_2, a_3, a_4) : a_1+a_2+a_3+a_4=0,\ a_1-a_2+a_3-a_4=0\}$

Let $\underset{\sim}{x} \in S \cap T$, then

$$a_1 + a_2 = -a_3 - a_4 \quad \text{——} (1)$$
$$a_1 - a_2 = -a_3 + a_4 \quad \text{——} (2)$$

$$a_1 = -a_3$$
$$a_2 = -2a_3 - a_4$$

Here, $\underset{\sim}{x} = \left(-a_3, -2a_3 - a_4, a_3, a_4\right)$

$= a_3(-1,-2,1,0) + a_4(0,-1,0,1)$

Here, $\{(-1,-2,1,0), (0,-1,0,1)\}$ forms a basis.
and also $\dim(S \cap T) = 2$.

291. Find the following limit:
$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right).$$

Solution:- Let $u_n = \dfrac{n}{\sqrt{n^2+n}}$

$$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}} = 1.$$

By Cauchy's first theorem:- $\displaystyle\lim_{n\to\infty}\left(\frac{u_1+\cdots+u_n}{n}\right) = 1.$

$$\text{So, } \lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right) = 1.$$

292. For any real number x and for any positive integer n show that $[x]+[x+\frac{1}{n}]+[x+\frac{2}{n}]+\cdots+[x+\frac{n-1}{n}] = [nx]$.

Solution:-

Let $x = [x]+y$, where $0 \le y < 1$,

Let p be an integer such that $p-1 \le ny < p$

Now, $x+\frac{K}{n} = [x]+y+\frac{K}{n}$

Also, $\frac{p+K-1}{n} < y+\frac{K}{n} < \frac{p+K}{n}$

so long as $\frac{p-1+K}{n} < 1$, i.e., $K < n-(p-1)$.

So, $y+\frac{K}{n} < 1$ and consequently

$$\left[x+\frac{K}{n}\right] = [x] \text{ for } K = 0,1,2,\ldots,n-p.$$

But $\left[x+\frac{K}{n}\right] = [x]+1$ for $K = n-p+1,\ldots,n-1$.

$$\therefore [x]+\left[x+\frac{1}{n}\right]+\cdots+\left[x+\frac{n-1}{n}\right]$$

$$=\Big(\underbrace{[x]+[x]+\cdots+[x]}_{(n-p+1 \text{ times})}\Big)+\Big(\underbrace{([x]+1)+([x]+1)+\cdots+([x]+1)}_{(p-1)\text{ times}}\Big)$$

$$= n[x]+(p-1) \quad \cdots\cdots\cdots (1)$$

Also, $[nx] = [n[x]+ny] = n[x]+(p-1)$

Since $p-1 \le ny < p$ $\cdots\cdots\cdots (2)$

From equation (1) & (2),

$$[x]+\left[x+\frac{1}{n}\right]+\cdots+\left[x+\frac{n-1}{n}\right] = [nx]$$

293. Prove that for $n > 1$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2} < 2-\frac{1}{n}.$$

Solution:- $P(2) = 1+\frac{1}{2^2} = \frac{5}{4} < 2-\frac{1}{2} = \frac{3}{2} = \frac{6}{4}.$

The statement is true for $n=2$.

Let, the statement is true for $n=m$.

$\therefore P(m) = 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{m^2} < 2-\frac{1}{m}.$

Now, we need to show that the statement is also true for $n=m+1$.

$$P(m+1) = 1+\frac{1}{2^2}+\cdots+\frac{1}{m^2}+\frac{1}{(m+1)^2}$$

$$< 2-\frac{1}{m}+\frac{1}{m(m+1)} \quad \left[\because \frac{1}{(m+1)^2} < \frac{1}{m(m+1)} \ \forall\, m>1\right]$$

$$< 2-\frac{1}{m+1}.$$

$\therefore$ The statement is true for $n=m+1$.

So, for all $n \in \mathbb{N}$ the statement is true. Hence proved.

294. Show that (a) $\phi(p) = p-1$ (b) $\phi(pq) = \phi(p)\phi(q)$, where p and q are prime numbers.

Solution:- (a) Let us take K as a positive integer and p be prime. The positive integers $\leq p^K$ which are not prime to p^K are $p, 2p, 3p, \ldots, (p^{K-1})p$. Therefore, the number of positive integers less than p^K and prime to p^K is $p^K - p^{K-1}$.

Hence $\phi(p^K) = p^K - p^{K-1} = p^K(1-1/p)$.

For $K=1$, $\phi(p) = p-1$.

(b) To prove this, we will use three lemmas:-

(i) a is prime to pq if and only if a is prime to p and a is prime to q.

(ii) If r be the residue of a modulo q and r is prime to q then a is prime to q.

(iii) If c be an integer and a is prime to q then the number of integers in the set $\{c, c+a, c+2a, \ldots, c+(n-1)a\}$ that are prime to q is $\phi(q)$

<u>Proof of $\phi(pq) = \phi(p)\phi(q)$</u> :- Since $\phi(1)=1$, the theorem is trivially true when p and q equals 1.

Let us assume $p>1$ and $q>1$. We arrange pq integers in q rows and p columns as follows:

$$\begin{array}{ccccc} 1 & 2 & \cdots & r & p \\ p+1 & p+2 & \cdots & p+r & 2p \\ 2p+1 & 2p+2 & \cdots & 2p+r & 3p \\ \cdots & \cdots & & \cdots & \cdots \\ (q-1)p+1 & (q-1)p+2 & \cdots & (q-1)p+r & qp \end{array}$$

The number of integers among these, that are prime to pq is $\phi(pq)$.

The number of integers in the first row that are prime to p is $\phi(p)$ [By Lemma-1]

Each column in the arrangements contain $\phi(q)$ integers prime to q [Lemma-2]

Hence $\phi(pq) = \phi(p)\phi(q)$. (By Lemma 3)

295. Determine x, y and z so that the 3×3 matrix with the following row vectors is orthogonal:

$$\left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right), \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0\right), (x, y, z).$$

<u>Solution:-</u>

<u>Ans:-</u> $\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}$ is an orthogonal matrix.

296. Solve: $dy/dx = (y+2)/(x-2)$

Solution:-
$$\int \frac{dy}{y+2} = \int \frac{dx}{x-2}$$

$\Rightarrow \log|y+2| = \log|x-2| + \log|c|$

$\Rightarrow (y+2)^2 = k(x-2)^2$ is the required solution.

297. If ω is a complex cube root of unity then show that
$a^3+b^3+c^3-3abc = (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$.

Solution:- $a^3+b^3+c^3-3abc$

$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= (a+b+c)\{a^2+b^2\omega^3+c^2\omega^3+(\omega+\omega^2)ab+(\omega+\omega^2)bc$
$+(\omega+\omega^2)ca\}$

$= (a+b+c)\{a^2+b^2\omega^3+c^2\omega^3+\omega ab+\omega^2 ab+\omega bc+\omega^2 bc$
$+\omega^2 bc+\omega ca+\omega^2 ca\}$

$= (a+b+c)\{a(a+b\omega^2+c\omega)+b\omega(a+b\omega^2+c\omega)+c\omega^2(a+b\omega^2+c\omega)$

$= (a+b+c)(a+b\omega^2+c\omega)(a+b\omega+c\omega^2)$.

298. Let f be a twice differentiable function such that
$f''(x) = -f(x)$; $f'(x) = g(x)$ and $h(x) = f^2(x)+g^2(x)$
Given that $h(5)=1$ and then find $h(10)$.

Sol. Take $f(x) = \sin x$
$f'(x) = -\cos x = g(x)$
$f''(x) = -\sin x = f(x)$

$h(x) = f^2(x)+g^2(x)$
$= \sin^2 x + \cos^2 x = 1$

$\therefore h(5) = h(10) = 10$.

299. Test the convergence of the series $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots$
[Assume $x>0$ and examine all possible cases]

Solution:— We have $u_n = \frac{n^n \cdot x^n}{n!}$,

$$u_{n+1} = \frac{(n+1)^{n+1} \cdot x^{n+1}}{(n+1)!}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{1}{x} \cdot \frac{n^n}{n!} \cdot \frac{(n+1)!}{(n+1)^{n+1}}$$

$$= \frac{1}{x} \cdot \frac{n^n}{(n+1)^n} = \frac{1}{x} \cdot \frac{n^n}{n^n\{1+(1/n)\}^n}$$

$$\Rightarrow \lim \frac{u_n}{u_{n+1}} = \frac{1}{x} \lim \frac{1}{\left[1+\frac{1}{n}\right]^n} = \frac{1}{ex}.$$

$\therefore$ By Ratio test, $\sum u_n$ is convergent if $\frac{1}{ex} > 1 \Rightarrow x < \frac{1}{e}$ and $\sum u_n$ is divergent if $x > \frac{1}{e}$.

But Ratio test fails when $x = \frac{1}{e}$. Here we can use Logarithmic test. When $x = \frac{1}{e}$, we have $\frac{u_n}{u_{n+1}} = e \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n}$.

$$\Rightarrow n \log \frac{u_n}{u_{n+1}} = n\left[1 - n\log\{1+1/n\}\right]$$

$$= n\left[1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right)\right]$$

$$= n\left[\frac{1}{2n} - \frac{1}{3n^2} + \cdots\right]$$

$$= \frac{1}{2} - \frac{1}{3n} + \cdots$$

$$\therefore \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \frac{1}{2} < 1,$$

$\therefore$ $\sum u_n$ is divergent at $x = \frac{1}{e}$.

300. Find $\frac{dy}{dx}$ if $x^{\cos y} + y^{\cos x} = 1$.

Solution:- $u = x^{\cos y}$

$\log u = \cos y \log x$

$\frac{1}{u} \cdot \frac{du}{dx} = \cos y \cdot \frac{1}{x} - \log x \cdot \sin y \cdot \frac{dy}{dx}$

$\therefore \frac{du}{dx} = x^{\cos y}\left(\frac{\cos y}{x} - \log x \cdot \sin y \cdot \frac{dy}{dx}\right) \cdots\cdots (1)$

$v = y^{\cos x}$

$\log v = \cos x \log y$

$\frac{dv}{dx} = y^{\cos x}\left\{-\log y \cdot \sin x + \cos x \cdot \frac{1}{y} \cdot \frac{dy}{dx}\right\} \cdots\cdots (2)$

Now, $u + v = 1$

$\frac{du}{dx} + \frac{dv}{dx} = 0.$

$\Rightarrow y^{\cos x} \cdot \cos x \cdot \frac{1}{y} \cdot \frac{dy}{dx} - x^{\cos y} \cdot \log x \sin y \frac{dy}{dx} = y^{\cos x} \cdot \log y \cdot \sin x - x^{\cos y} \cdot \frac{\cos y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y^{\cos x} \cdot \log y \cdot \sin x - x^{\cos y} \cdot \frac{\cos y}{x}}{y^{\cos x} \cdot \cos x \cdot \frac{1}{y} - x^{\cos y} \cdot \log x \cdot \sin y}.$

301. Find the inverse of the following mtx $\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_2 & c_3 & c_0 & c_1 \\ c_3 & -c_2 & c_1 & -c_0 \\ c_1 & -c_0 & c_3 & -c_2 \end{bmatrix}$,

where $c_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}$, $c_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}$, $c_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}$, $c_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$.

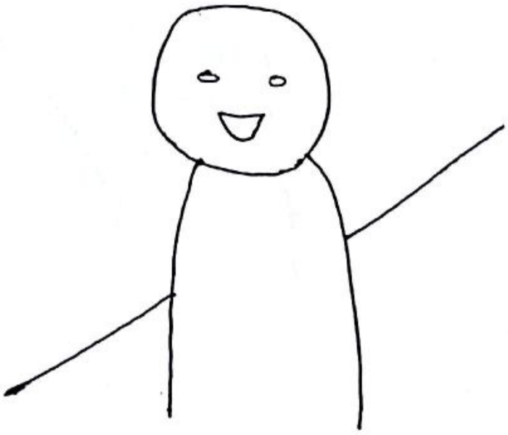
Solution:- $c_0^2 + c_1^2 + c_2^2 + c_3^2 = 1$

$c_0 c_3 = -\frac{1}{16}$, $c_2 c_1 = \frac{3}{16}$, $c_0 c_2 = \frac{-3}{16}$, $c_1 c_3 = \frac{-3}{16}$.

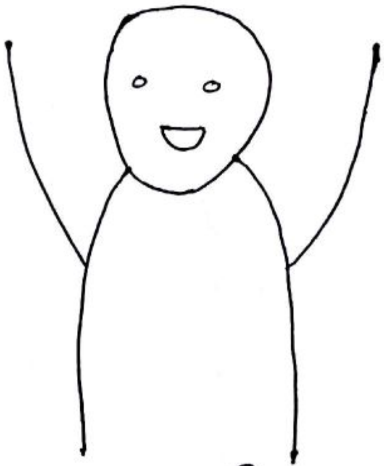
Here $A^2 = I \Rightarrow A = A^{-1}$. $A^2 = \left(\overrightarrow{\quad}\right) \times \left(\overrightarrow{\quad}\right)$

$= I$

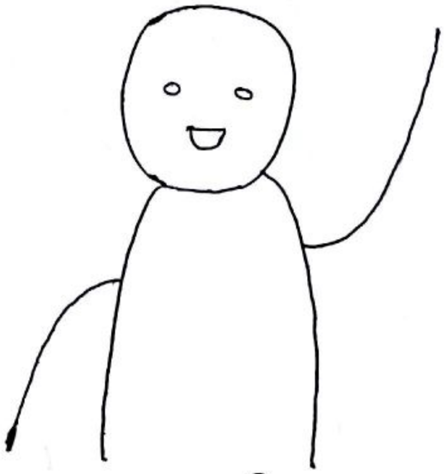
GRAPHS



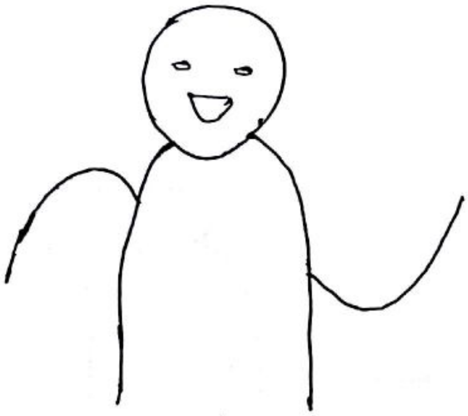
$y = x$



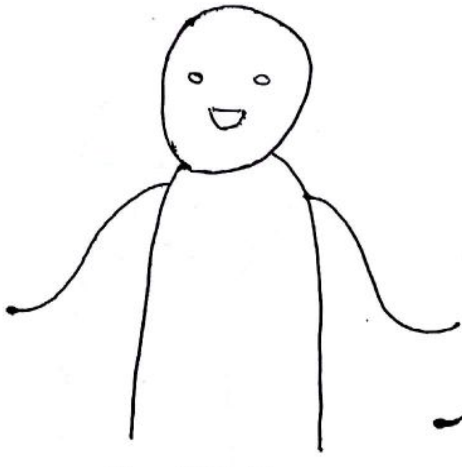
$y = x^2$



$y = x^3$

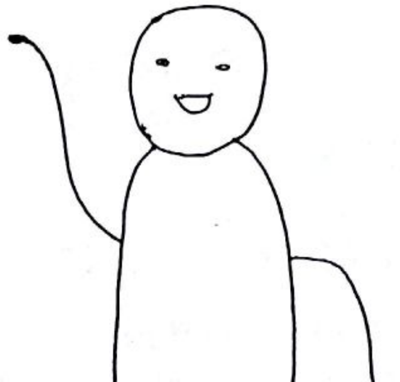


$y = \sin x$



$y = \cos x$

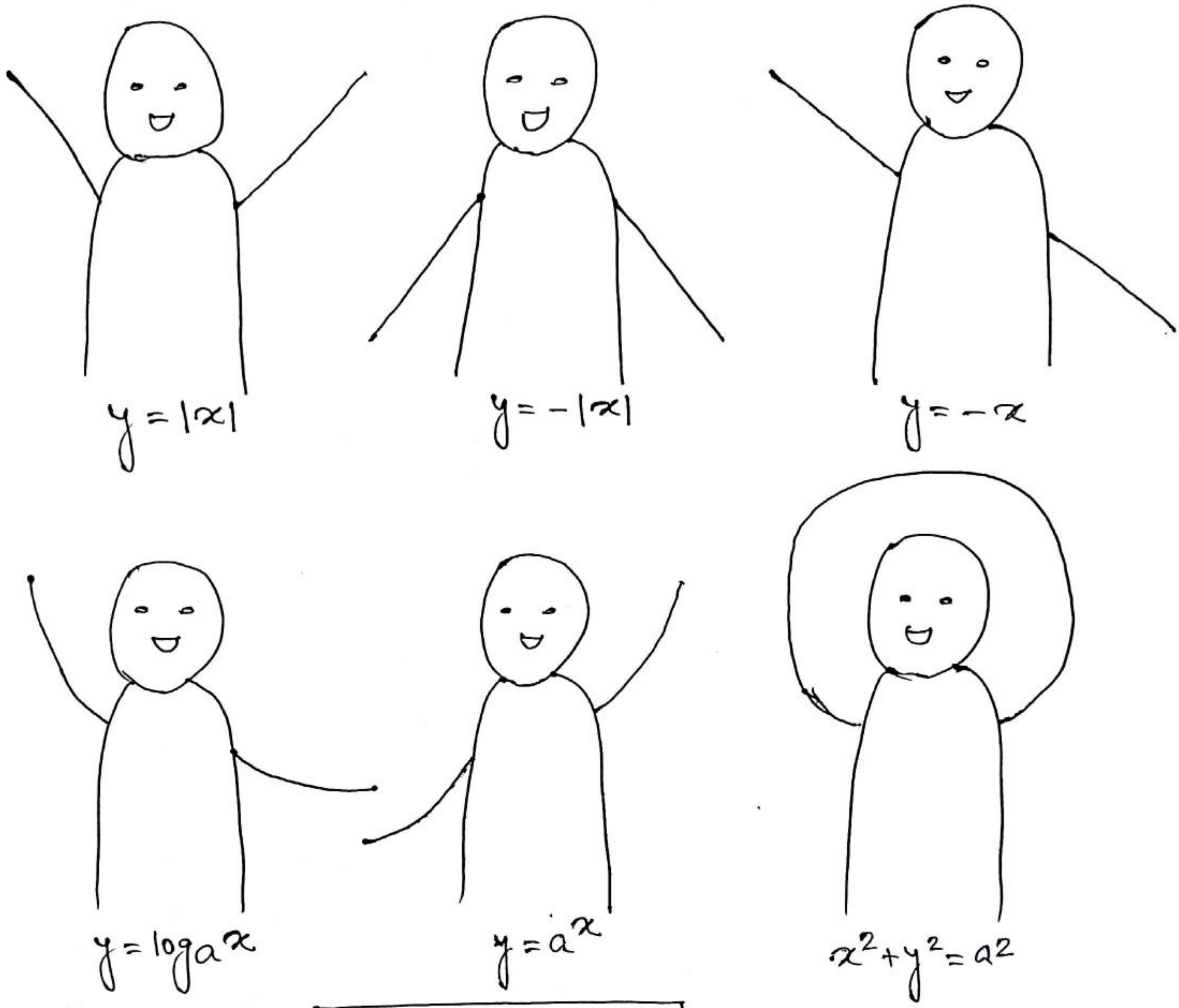
$y = \tan x$



$y = \cot x$

$y = \frac{1}{x}$

$y = -\frac{1}{x}$



VECTOR ALGEBRA

$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$.

Workdone = $\overrightarrow{Force} . \overrightarrow{AB}$ = Moment of the force.

$\vec{a}, \vec{b}$ are perpendicular if $\vec{a}.\vec{b} = \vec{0}$.

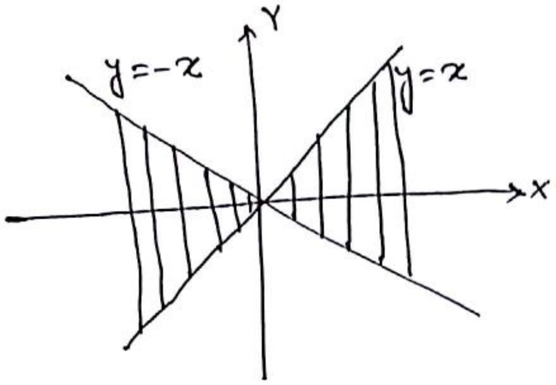
Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ $= \pm \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

Collinear :- $\vec{a} = t\vec{b}$ or $\vec{a} \times \vec{b} = 0 \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ - & - & - \\ - & - & - \end{vmatrix} = 0$.

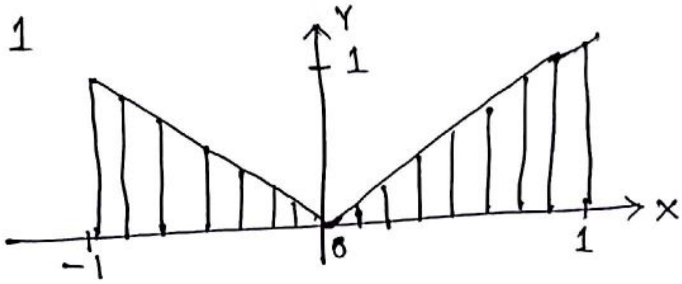
Coplaner :- $\vec{A}.(\vec{B} \times \vec{C}) = \vec{0}$.

Q. $|y| \leq |x| \leq 1$

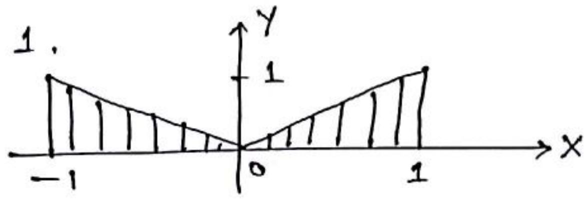
A:- $|y| \leq |x|$



$|x| \leq 1$



$|y| \leq |x| \leq 1$.



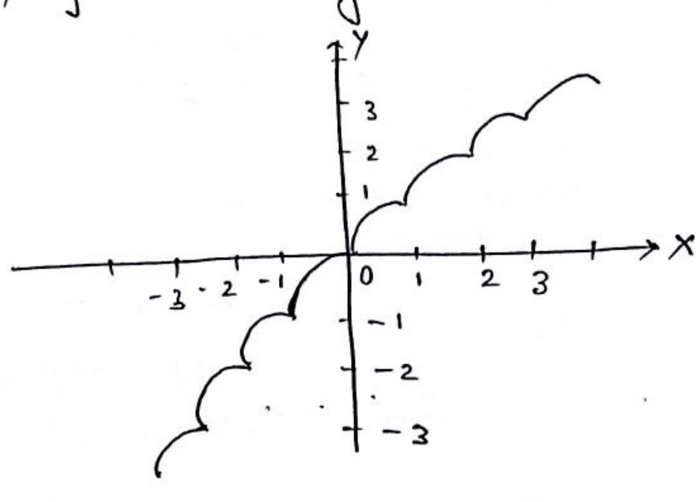
Q. $f(x) = [x] + \sqrt{\{x\}}$

$0 \leq \{x\} < 1$

$\Rightarrow \sqrt{\{x\}} \geq \{x\}$

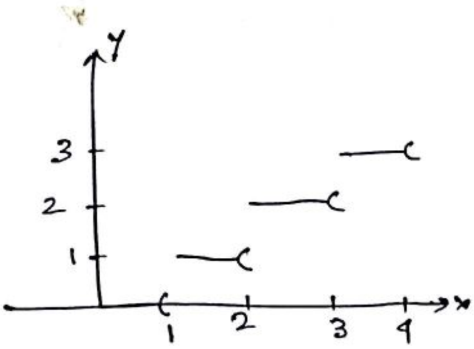
$\Rightarrow [x] + \sqrt{\{x\}} \geq [x] + \{x\} = x$.

'=' holds if x takes integral values.



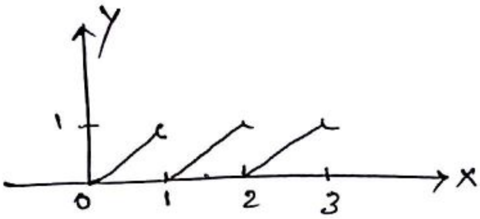
Q. $y = [x]$

x	y
$[0,1)$	0
$[1,2)$	1
$[2,3)$	2
$\vdots$	$\vdots$



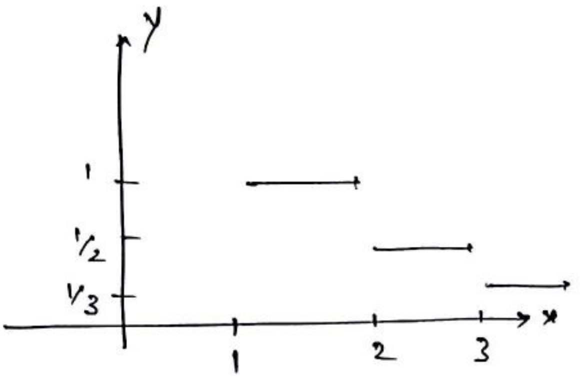
Q. $y = \{x\}$

x	y
0	0
1/4	1/4
1/2	1/2
3/4	3/4
1	0
$\vdots$	$\vdots$



Q. $y = \dfrac{1}{[x]}$

x	y
1	1
$1\frac{1}{4}$	1
$1\frac{1}{2}$	1
$1\frac{3}{4}$	1
2	$\frac{1}{2}$
$\vdots$	$\vdots$



Q. $y = \dfrac{x^2+1}{x^2-1} = \dfrac{(x^2-1)+2}{(x^2-1)} = 1 + \dfrac{2}{(x^2-1)}$

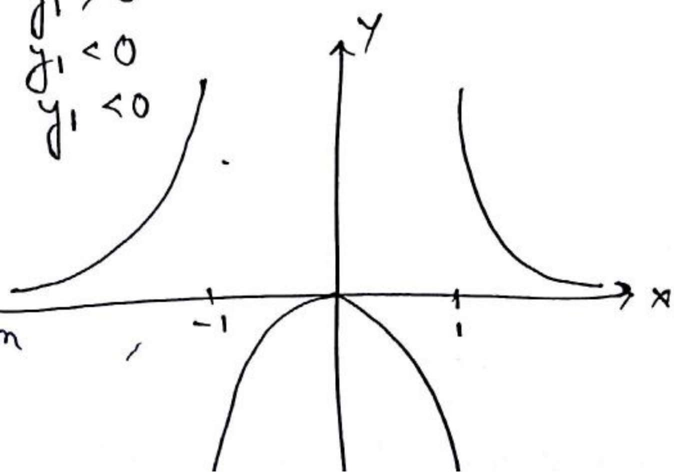
$$y_1 = \frac{dy}{dx} = \frac{-4x}{(x^2-1)^2}.$$

$-\infty \le x < -1, \quad y_1 > 0$

$-1 \le x \le 0, \quad y_1 > 0$

$0 \le x < 1, \quad y_1 < 0$

$1 \le x \le \infty, \quad y_1 < 0$

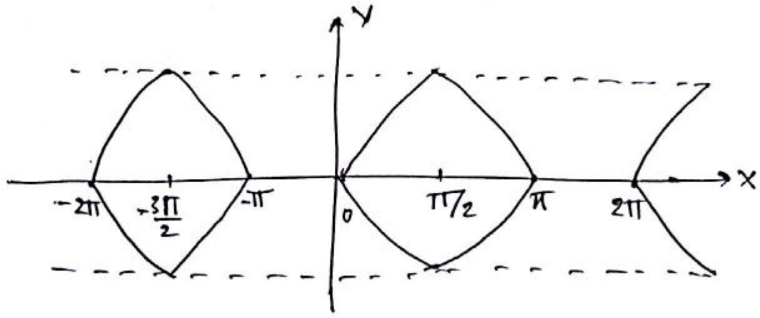


— By TANUJIT CHAKRABORTY

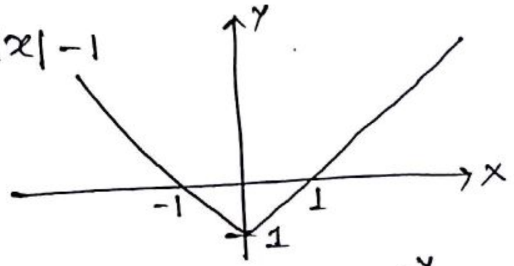
— MOB: 8420253573

— Mail: tanujitisi@gmail.com

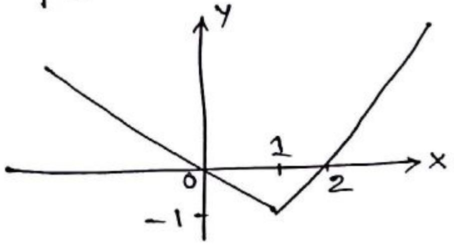
Q. $|y| = \sin x$



Q. $y = |x| - 1$



Q. $y = |x-1| - 1$



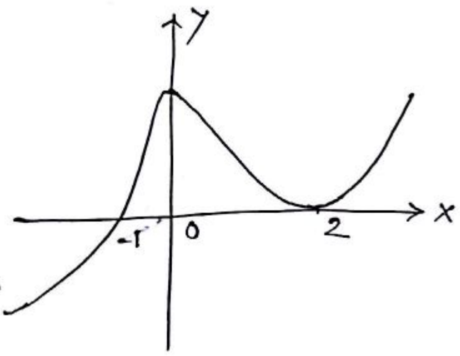
Q. $y = x^3 - 3x^2 + 4$

$f'(x) = 3x^2 - 6x = 3x(x-2)$

$f'(x) > 0$ for $-\infty < x < 0$

$f'(x) < 0$ for $0 < x < 2$

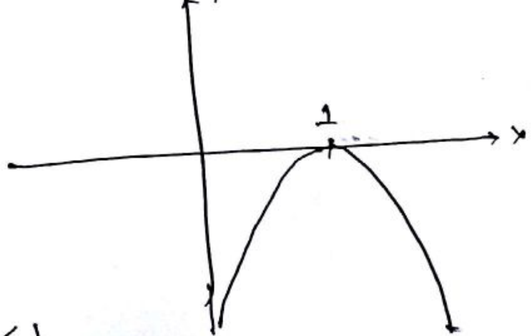
$f'(x) > 0$ for $2 < x < \infty$



x	-2	-1	0	1	2	3
$f(x)$	-16	0	4	2	0	4

Q. $y = \log_e x - (x-1),\ x > 0$

x	0.5	1	2	3
y	0.2	0	-0.7	-1.5



$f'(x) = \dfrac{1-x}{x}$

$f'(x) > 0$ for $0 < x < 1$

$f'(x) < 0$ for $1 < x < \infty$