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Statistics & Economics.
Kolkata, W. B.,8420253573
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SOLVED PROBLEMS ON MATHEMATICS

OBJECTIVE TYPE QUESTIONS

Topics :

Theory Of Equations

Number Theory

Linear Algebra

Abstract Algebra

Real Analysis

Set , Combinatorics & Probability

Differential Equations

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THEORY OF EQUATIONS

OBJECTIVE TYPE

MULTIPLE CHOICE TYPE :-

1. The sum of roots of equation $x^7+9x^6-2=0$ is
(a) 0 (b) 3 (c) -9 (d) 7.

Solution:- (c) Sum of the roots $= \left(-\dfrac{\text{coefficient of } x^6}{\text{coefficient of } x^7}\right)$

$= -9.$

2. The sum of all the real roots of the equation $|x-2|^2+|x-2|-2=0$ is
(a) 4 (b) 2 (c) 6 (d) none.

Solution:- (a) $|x-2|=t$

$t^2+t-2=0$

$\therefore$ $t=1,-1$, but -1 is not acceptable.

so, $x=3,1$.

Sum of the roots $=4$.

3. The equation $3x^{10}+7x^6+5x^4+2x^2+1=0$ has
(a) 10 (b) 6 (d) 2 (c) zero
real roots.

Solution:- Let $f(x)=3x^{10}+7x^6+5x^4+2x^2+1$

$f(-x)=3x^{10}+7x^6+5x^4+2x^2+1$

Here $f(x)$ and $f(-x)$ has no sign change.

By Sign Rule, $f(x)=0$ has no real roots.

4. The equations $x^2-kx-21=0$ and $x^2-3kx+35=0$ have a common root then the value of k is equal to
(a) -6 (b) 4 (c) 5 (d) 6

Solution:- (b) From the above two equations, we have

$$\frac{x^2}{-35k-63k}=\frac{-x}{35+21}=\frac{1}{-3k+k}$$

$\therefore$ $x=\frac{7k}{4}$, $x=\frac{28}{k}$

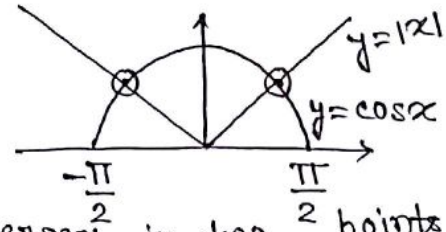
So, $k^2=16$

$\Rightarrow k=\pm 4$.

5. The number of solutions of the equation $|x| = \cos x$ is

(a) 1 (b) 2 (c) 3 (d) none

Solution:- (b)



Two graphs intersect in two points which are the required two real roots.

6. The equation $(x-a)^3 + (x-b)^3 + (x-c)^3 = 0$ has

(a) all real roots (b) one real, two imaginary roots

(c) three real roots (d) none of the above.

Solution:- (b) Let $f(x) = (x-a)^3 + (x-b)^3 + (x-c)^3$

$$f'(x) = 3\{(x-a)^2 + (x-b)^2 + (x-c)^2\} > 0 \ \forall x$$

So, $f'(x)=0$ has no repeated roots.

7. If $\sqrt{a}+\sqrt{b}$ be one of the roots of the given equation (a and b are not perfect squares) with rational coefficients, then lowest degree of such an equation must be

(a) 2 (b) 3 (c) 4 (d) none.

Solution:- (c) Irrational roots occur in pairs.

If one root is $\sqrt{a}+\sqrt{b}$, then the other roots are:

$$\sqrt{a}-\sqrt{b}, \ -\sqrt{a}+\sqrt{b}, \ \sqrt{a}-\sqrt{b}.$$

$\therefore$ Number of roots are $= 4$.

So, the lowest degree is 4.

8. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $x^n - nax - b = 0$ and then $(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdots\cdots(\alpha_1-\alpha_n)$ equals

(a) $n\alpha$ (b) $n(\alpha_1^{n-1}+a)$ (c) $na.\alpha_1^{n-1}$ (d) none

Solution:- (b) $(x-\alpha_1)(x-\alpha_2)\cdots\cdot(x-\alpha_n) = x^n + nax - b$

Differentiating w.r.t. x,

$$(x-\alpha_2)(x-\alpha_3)\cdots\cdot(x-\alpha_n) + (x-\alpha_1)(x-\alpha_3)\cdots\cdot(x-\alpha_n) + \cdots\cdots + (x-\alpha_1)(x-\alpha_2)\cdots\cdot(x-\alpha_{n-1}) = nx^{n-1} + na$$

Putting $x=\alpha_1$, we get —

$$(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdots\cdots(\alpha_1-\alpha_n) = \alpha_1^{n-1}.n + na.$$

9. Let $p, q \in \{1, 2, 3, 4\}$, the number of equations of $px^2 + qx + 1 = 0$ having real roots are

(a) 7 (b) 8 (c) 9 (d) none

Solution:- (a)

$\Delta = q^2 - 4p \geq 0$ for real roots.

i.e. $q^2 \geq 4p$

Now, if $p = 1$, then $q^2 \geq 4$, i.e. $q = 2, 3, 4$.

if $p = 2$, then $q^2 \geq 8$, i.e. $q = 3, 4$.

if $p = 3$, then $q^2 \geq 12$, i.e. $q = 4$

if $p = 4$, then $q^2 \geq 16$, i.e. $q = 4$

There are 7 such favourable cases.

10. If $0 < a < b < c$ and the roots α, β of the equation $ax^2 + bx + c = 0$ are imaginary, then

(a) $|\alpha| < 1$ (b) $|\beta| < 1$ (c) $|\alpha| = |\beta|$ (d) none

Solution:- (c)

Since roots are imaginary, so $\Delta = b^2 - 4ac < 0$

$\therefore$ The roots α and β are given by, $\alpha = \dfrac{-b + i\sqrt{4ac - b^2}}{2a}$

and $\beta = \dfrac{-b - i\sqrt{4ac - b^2}}{2a}$.

and $\alpha = \bar{\beta}$, so $|\alpha| = |\beta|$.

Moreover, $|\alpha| = \sqrt{\dfrac{b^2}{4a^2} + \dfrac{4ac - b^2}{4a^2}} = \sqrt{\dfrac{c}{a}}$

$\Rightarrow |\alpha| > 1 \quad (\because c > a)$.

$\therefore |\alpha| = |\beta|$ & $|\alpha| > 1$.

MULTIPLE CHOICE TYPE:-

11. The congruence $35x \equiv 14 \pmod{21}$ has
(a) 7 solutions (b) 6 solutions (c) Unique solution (d) No solution

Solution:- (a) $35x - 14 \mid 21$
$\gcd(35, 21) = 7$
and 7 divides 14; hence the given congruence has 7 solutions

12. The maximum value of $f(x) = (x-2)^n (3-x)^n$ for a natural number $n \geq 1$ and $2 \leq x \leq 3$ is
(a) $\frac{1}{2^n}$ (b) $\frac{1}{4^n}$ (c) $\frac{1}{8^n}$ (d) $\frac{1}{16^n}$.

Solution:- [If $a+b = \lambda$ (>0) is given then ab is maximum when $a = b = \frac{\lambda}{2}$.]

(b) Here $(x-2)+(3-x) = 1$ then $(x-2) = \frac{1}{2} = (3-x)$

$$\therefore [f(x)]_{max} = \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n = \left(\frac{1}{4}\right)^n.$$

13. If $a \equiv b \pmod{n}$. Prove that $\gcd(a, n)$ is
(a) $\gcd(b, n)$ (b) b (c) n (d) none

Solution:- (a) $\gcd(a, n) = d$
$\Rightarrow d \mid a$, $d \mid n$ but $n \mid (a-b)$
$\Rightarrow d \mid a-b$, $d \mid a$
$\Rightarrow d \mid a-(a-b) = b$
$\Rightarrow d \mid a$, $d \mid b$.
Same as $\gcd(b, n) = d$. So, $\gcd(a, n) = \gcd(b, n)$.

14. The highest power of 3 contained in 1000! is
(a) 493 (b) 494 (c) 495 (d) 496

Sol. (d) $p = 3$, $n = 1000$.
The highest power of a prime no. contained in $n!$ is given by

$$k(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$$

$$\text{So, Ans} = \left[\frac{1000}{3}\right] + \left[\frac{1000}{9}\right] + \left[\frac{1000}{3^3}\right] + \left[\frac{1000}{3^4}\right] + \left[\frac{1000}{3^5}\right] + \left[\frac{1000}{3^6}\right]$$

$$= 496.$$

15. The sum of the series

$$\frac{1}{5\cdot6}+\frac{1}{6\cdot7}+\cdots+\frac{1}{104\cdot105} \text{ is}$$

(a) $\frac{22}{21}$ (b) $\frac{20}{21}$ (c) 1 (d) None

Solution:- (b)

$$\frac{1}{5\cdot6}+\frac{1}{6\cdot7}+\cdots+\frac{1}{104\cdot105}$$

$$=\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{104}-\frac{1}{105}\right)$$

$$=\frac{1}{5}-\frac{1}{105}$$

$$=\frac{100}{105}=\frac{20}{21}.$$

16. Which of the following is false:

(a) $\exists$ a natural number which when divided by 5 leaves remainder 2 and which when divided by 4 leaves remainder 0.

(b) $\exists$ a natural number which when divided by 7 leaves remainder 1 and which when divided by 6 leaves remainder 4.

(c) $\exists$ a natural number which when divided by 3 leaves remainder 2 and which when divided by 4 leaves remainder 1.

(d) $\exists$ a natural number which when divided by 6 leaves remainder 1 and which when divided by 5 leaves remainder 3.

Sol. (d)

(a) $\rightarrow$ e.g. 12

(b) $\rightarrow$ e.g. 22

(c) $\rightarrow$ e.g. 5

17. The sum of the series

$$\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{99}+\sqrt{100}} \text{ is}$$

(a) 1 (b) 9 (c) 10 (d) None

Solution:- (b)

$$S=\frac{1-\sqrt{2}}{1-2}+\frac{\sqrt{2}-\sqrt{3}}{2-3}+\frac{\sqrt{3}-\sqrt{4}}{3-4}+\cdots+\frac{\sqrt{99}-\sqrt{100}}{99-100}$$

$$=-1+10$$

$$=9.$$

18. The last digit of 43^{17} is

(a) 3 (b) 7 (c) 1 (d) None

Solution:- (a) $43 \equiv 3 \pmod{10}$

$(43)^{17} \equiv 3^{17} \pmod{10}$; i.e. last digit of 43^{17} is the last digit of 3^{17}.

Now, $3^4 \equiv 1 \pmod{10}$

$(3^4)^4 \equiv 1 \pmod{10}$

$3^{16} \cdot 3 \equiv 3 \pmod{10}$

So, the last is 3.

19. The remainder when $(2222)^{5555}$ is divisible by 7 is

(a) 3 (b) 5 (c) 7 (d) 9

Solution:- (b) $2222 \equiv 3 \mod 7$

$(2222)^3 \equiv 27 \mod 7$

$(2222)^3 \equiv (-1) \mod 7$

$(2222)^{5553} \equiv (-1)^{1851} \mod 7$

$(2222)^2 \equiv 9 \mod 7$

$(2222)^{5553} \equiv -9 \mod 7 -9$

$\equiv 5 \mod 7.$

20. The unit digit of $(2337)^{2337}$ is

(a) 3 (b) 5 (c) 7 (d) 9

Solution:- Last digit

(c) $(2337)^1 \equiv 7 \pmod{10}$

$(2337)^2 \equiv 9 \pmod{10}$

$(2337)^3 \equiv 3 \pmod{10}$

$(2337)^4 \equiv 1 \pmod{10}$

$(2337)^5 \equiv 7 \pmod{10}$

$2337 \equiv 1 \pmod 4$

$(((2337)^{584})^4 \cdot 2337 \equiv 7 \pmod{10}$

unit digit is 7.

21. The least positive residue in $2^{41} \pmod{23}$ is

(a) 3 (b) 5 (c) 7 (d) 9

Solution:-

If $a \equiv b \pmod m$ then b is said to be the residue of a modulo m.

If $a \equiv b \pmod m$ then $a^n \equiv b^n \pmod m$ $\forall\ n \in I^+$ But the converse is not true.

23 is a prime & 2 is a prime to 23.

By Fermat's theorem,
$$2^{22} \equiv 1 \pmod{23}$$
$$2^{44} \equiv 1 \pmod{23}$$
$$2^{44} \equiv 24 \pmod{23}$$
$$2^{41}.8 \equiv 3 \times 8 \pmod{23}$$
$$2^{41} \equiv 3 \pmod{23}$$

So, the least positive residue is 3.

22. The remainder when $4(29)! + 5!$ is divisible by 31 is

(a) 3 (b) 5 (c) 7 (d) None

Solution:- (d)

Wilson's theorem states that "If p is a prime then $(p-1)! + 1 \equiv 0 \pmod p$. The converse of this theorem is also true.

So, By Wilson's theorem, $(30)! + 1 \equiv 0 \pmod{31}$, since 31 is prime.
$$(31-1)(29)! + 1 \equiv 0 \pmod{31}$$
$$\Rightarrow -29! + 1 \equiv 0 \pmod{31}$$
$$\Rightarrow 29! - 1 \equiv 0 \pmod{31}$$
$$\Rightarrow 4(29)! - 4 \equiv 0 \pmod{31}$$
$$\Rightarrow 4(29)! - 4 + 124 \equiv 0 \pmod{31}$$
$$\Rightarrow 4(29)! + 120 \equiv 0 \pmod{31}$$
$$\Rightarrow 4(29)! + 5! \equiv 0 \pmod{31}$$

23. The smallest positive inte-ger that has remainder 4, 3 and 1 when divided by 5, 7 and 9, respectively, is

(a) 211 (b) 201 (c) 199 (d) 189

Solution:-

$$x \equiv 4 \pmod 5$$
$$x \equiv 3 \pmod 7$$
$$x \equiv 1 \pmod 9$$

let $x = 4+5t$, $t \in \mathbb{Z}$

So, $4+5t \equiv 3 \pmod 7$

$\Rightarrow 5t+1 \equiv 0 \pmod 7$

let, $t = 4+7u$, $u \in \mathbb{Z}$

So, $x = 4+5(4+7u) = 24+35u$.

$24+35u \equiv 1 \pmod 9$

$\Rightarrow 35u+23 \equiv 0 \pmod 9$

$\Rightarrow 36u-u \equiv -23 \pmod 9$

$\Rightarrow -u \equiv -23 \pmod 9$

$\Rightarrow 23-u \equiv 0 \pmod 9$

$\therefore u = 5+9v$ (let), $v \in \mathbb{Z}$

$\therefore x = 24+35(5+9v)$
$= 199+315v$.

So, the smallest positive solution is 199.

24. The last digit of 3^{80} is

(a) 3 (b) 9 (c) 7 (d) 1

Solution:- (d)

$$3^1 \equiv 3 \pmod{10}$$
$$3^2 \equiv 9 \pmod{10}$$
$$3^3 \equiv 7 \pmod{10}$$
$$3^4 \equiv 1 \pmod{10}$$
$$3^5 \equiv 3 \pmod{10}$$

$$3^{80} = (3^{16})^5$$
$$= (3^{3\times5}\cdot 3)^5$$
$$= (3^{25})^3 \cdot 3^5$$
$$\equiv 3^3 \cdot 3 \pmod{10}$$
$$\equiv 1 \pmod{10}$$

More explicitly, we can write:-

Ends in 9: $3^{10}, 3^{30}, 3^{50}, 3^{70}, 3^{90}, \ldots$

Ends in 1: $3^{20}, 3^{40}, 3^{60}, 3^{80}, 3^{100}, \ldots$

25. The last digit of 9^{9^9} is

(a) 1 (b) 7 (c) 9 (d) None.

Solution:- (c) $9^1 = 9$, last digit

$9^2 = 1$. last digit

$9^3 = 9$, last digit

$9^9 = 9$, last digit

$9^{9^9} = 9$, last digit.

So, the answer is 9.

MULTIPLE CHOICE TYPE :-

26. Which of the following sets is not LIN?

a) $\{1, x, 1+x+x^2\}$ in a vector space of all polynomials over the field of real numbers.

b) $\{1, x, x^2, \ldots\}$ in a vector space of all polynomials over the field of real numbers.

c) $\{(1,1,0,0), (0,1,-1,0), (0,0,0,3)\}$ in $V_4(R)$.

d) $\{(1,2,1), (3,1,5), (3,-4,7)\}$ in $V_3(R)$.

Sol a) Let a, b, c be scalars such that

$$a(1) + bx + c(1+x+x^2) = 0$$
$$\Rightarrow (a+c) + (b+c)x + cx^2 = 0$$
$$\Rightarrow a+c=0, \quad b+c=0, \quad c=0$$
$$\Rightarrow a=0, \quad b=0, \quad c=0$$

$\therefore$ the vectors $1, x, 1+x+x^2$ are LIN over the field of real no's.

b) $a_0 1 + a_1 x + a_2 x^2 + \ldots = 0$

By definition of equality of two polynomials

$$a_0 = a_1 = a_2 = \ldots = 0$$

$\therefore$ the vectors are LIN.

c) $a(1,1,0,0) + b(0,1,-1,0) + c(0,0,0,3) = 0$

$$\Rightarrow \begin{cases} a=0 \\ a+b=0 \\ -b=0 \\ 3c=0 \end{cases}$$

$\Rightarrow a=b=c=0$ is the only sol^n.

$\therefore$ the vectors are LIN.

d) $$\begin{vmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{vmatrix} = 0 \Rightarrow \text{rank}(A) < 3$$

$\Rightarrow$ the set of vectors are linearly dependent.

27. The eigen value of A^4, where $A = \begin{pmatrix} 1 & 0 & -1 \\ 9 & 4 & 1 \\ 3 & 1 & 1 \end{pmatrix}$, is

a) 3, 4, 5, b) 1, 2, 3, c) 5, 6, 7, d) none.

Sol d) $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 9 & 4-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

[Theorem: If eigen values of a mtx. A be $\lambda_1, \lambda_2, \dots, \lambda_n$ then eigen value of the mtx. A^m be $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$.

$\therefore$ For the mtx A^4, the eigen values are:

$(1)^4 = 1$, $(2)^4 = 16$, $(3)^4 = 81$.]

28. Let, $M_{2\times2}(R)$ be the vector space of all 2x2 matrices over R and

Let $W_1 = \left\{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} : x, y \in \mathbb{R} \right\}$ & $W_2 = \left\{ \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$

then dim $(W_1 \cap W_2)$ is

a) 2 b) 3 c) 4 d) 1

Sol d) $W_1 \cap W_2 = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$

dim $(W_1 \cap W_2) = 1$.

29. Let $T : R^n \to R^m$, $m > n$, be a linear transformation. consider the following statements about T

(i) T can be one to one (ii) T can onto (iii) dim $(T(R^n)) \geq n$

(A) Only (i) is true (B) Only (ii) is false

(C) Only (ii) is true (D) Only (iii) is true

Sol. (B) $T : R^n \to R^m$, $m > n$.

i.e. no. of elements in domain < no. of elements in range.

i.e. T can be one to one is true statement.

But T can be onto is False as $m > n$.

$d\left(T(R^n)\right) \geq n$ is also a true statement as $n < m$.

30. If $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in K^{2\times 2}$, $b \neq 0$ then A has eigen values if K is

(A) R (B) C (C) Q (D) All above

Sol (D) Ch. equation $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 2a\lambda + (a^2+b^2) = 0$$

$$\Rightarrow \lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2} = a \pm b$$

Hence the eigen values of A be $a+b$ and $a-b$.

Then A has eigen values when $K = R$, C or Q

31. The no. of solution of the system of equation

$$2x + y - z = 7$$
$$x - 3y + 2z = 1$$
$$x + 4y - 3z = 5$$

a) Unique sol^n b) no sol^n c) many sol^n d) exactly two sol^n

Sol^n: b)
$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 5 \end{pmatrix}$$

$$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{b}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 \\ -1 & -4 & 3 \\ 1 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

rank$(A) = 2$

$$\begin{bmatrix} 2 & 1 & -1 & : & 7 \\ 1 & -3 & 2 & : & 1 \\ 1 & 4 & -3 & : & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 & : & 7 \\ -1 & -4 & 3 & : & -6 \\ 1 & 4 & -3 & : & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 & : & 7 \\ -1 & -4 & 3 & : & -6 \\ 0 & 0 & 0 & : & -1 \end{bmatrix}$$

rank$(A:b) = 3$ $\therefore$ rank$(A) \neq$ rank$(A:b)$

$\therefore$ the system has no solution.

32. The equations: $x-y+2z=4$
$3x+y+4z=6$
$x+y+z=1$

have

a) unique solution, b) infinite solution, c) no solution, d) none of them

Sol. b) $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

$\therefore |A| = 0$

$\text{rank}(A) < 3$

Since, $\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 4 \neq 0$

$\therefore \text{rank}(A) = 2$

$[A:B] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 3 & 1 & 4 & 6 \\ 1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 4 & -2 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\therefore \text{rank}(A:B) = 2 = \text{rank}(A) < 3$

$\therefore$ the given system of equations are consistent & have infinite no. of solutions.

33. If the matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then find the matrix A^{n+1}?

(a) $2A^n$ (b) 2^nA (c) $2^{n-1}A$ (d) A

Sol (b) $A^2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 2A$

$A^3 = A^2 \cdot A = (2A) \cdot A = 2A^2 = 4A$

$\therefore A^{n+1} = 2^n A$ [By induction]

$= 2^n \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Altern: $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda = 0$

So $A^2 - 2A = 0 \quad \therefore A^{n+1} = 2^n A$

$\Rightarrow A^3 = 2A^2$

$= 4A$

34. The following system of linear equation

$$x+3y+z=3$$
$$2x+3y+5z=4$$
$$4x+9y+\alpha z=\beta$$

is consistent if α, β don't equal to

(a) $(1,3)$ (b) $(5,10)$ (c) $(7,10)$ (d) None

Sol: (c) Given system of equation can be expressed as $Ax=B$

Argumented matrix $[A|B] = \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 2 & 3 & 5 & 4 \\ 4 & 9 & \alpha & \beta \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & -3 & 3 & -2 \\ 0 & -3 & \alpha-4 & \beta-12 \end{array}\right] \begin{array}{l} R_2 \leftrightarrow R_2-2R_1 \\ R_3 \leftrightarrow R_3-4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & -3 & 3 & -2 \\ 0 & 0 & \alpha-7 & \beta-10 \end{array}\right] R_3 \leftrightarrow R_3-R_2$$

Given system of linear equation be consistent

i.e. $\text{Rank}(A) = \text{Rank}(A|B) = 3$

i.e. $\alpha-7 \neq 0$, $\beta-10 \neq 0$

$\Rightarrow \alpha \neq 7$ & $\beta \neq 10$.

35. Let $\underset{\sim}{x}$ & $\underset{\sim}{y}$ in $\mathbb{R}^n$ be non-zero row vectors from the matrix $A = xy^T$, where y^T denotes the transpose of y. Then the rank of A is

a) 0 or 1 b) 2 c) at least $n/2$ d) none

Sol a) $A = xy^T$

Then A is a matrix of order 1×1.

if A is non-zero then $\text{rank}(A) = 1$

if A is zero mtx. then $\text{rank}(A) = 0$

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36. Which of the following is false ?

a) The eigen values of Hermition matrix are real.

b) The eigen values of real symmetric matrix are real.

c) the eigen values of skew hermition matrix are real.

d) the eigen values of unitary matrix may be real.

Sol: (c) As the eigen values of skew -hermition matrix are either pure imaginary or zero.

37. Let $M_n(R)$ be the set of $n \times m$ matrices with real entries, if all $A \in M_n(R)$ have both negative and positive eigen values then the set is having

a) positive semi definite matrices only

b) positive & negative sami definite matrices

c) negative definite matrices only

d) indefinite matrices

Sol : d) A positive and positive semi definite matrices have +ve eigen values only.

negative definite matrices have -ve eigen values only.

But indefinite matrices have both +ve and -ve eigen values.

38. Which of the following is true ?

a) The matrix $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ is diagonalisable

b) The matrix $\begin{bmatrix} 1 & 0 \\ 1 & 5 \end{bmatrix}$ is diagonalisable

c) The matrix $\begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$ is not diagonalisable

d) The matrix $\begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$ is not diagonalisable

$\underline{\text{Sol}^n}$: b) We know that any matrix (2x2) is said to be diagonalisable if it has two different eigen values.

$\begin{bmatrix} 1 & 0 \\ 1 & 5 \end{bmatrix}$ is lower triangular matrix, its diagonal entries are its eigen values since they are distinct.

$\Rightarrow$ the matrix is said to be diagonalisable.

39. If V is vector space on the field $Z/3Z$ and

$\dim_{Z/3Z}(V) = 3$ then V has

a) 27 elements, b) 9 elements, c) 30 elements, d) 15 elements

$\underline{\text{Sol}}$: a) Since, $\dim V = 3$ and $Z/3Z = \{0, 1, 2\}$

So, they are 3^3 elements in V which can be expressed as the linear combination of elements of basis.

40. If V is a vector space over the field $Z/7Z$ and $\dim_{Z/7Z}(V) = 2$ then V has

a) 49 elements b) 14 elements c) 128 elements d) None

$\underline{\text{Sol}}$ a) Since, $\dim V = 2$ and $Z/7Z = \{0, 1, 2, 3, 4, 5, 6\}$

So, there may be 7^2 elements which can be expressed which can be expressed as the linear combination of elements of basis & their coefficient could only be $\{0, 1, 2, 3, 4, 5, 6\}$.

41. If V is the real vector space of all mapping from $\mathbb{R}$ to $\mathbb{R}$, $V_1 = \{f \in V \mid f(-x) = f(x)\}$ and $V_2 = \{f \in V \mid f(-x) = -f(x)\}$, then which one of the following is correct?

A) Neither V_1 nor V_2 is a subspace of V.

B) V_1 is a subspace of V, but not V_2.

C) V_2 is a subspace of V, but not V_1.

D) Both V_1, V_2 are subspaces of V.

Solⁿ : D) The necessary and sufficient condition for a non-empty subset W of a vector space $V(F)$ to be subspace of V is

$a, b \in F$ and for all $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

$\lambda a \in W$

Here both V_1, V_2 satisfied these conditions.

42. Let P be an $n \times n$ idempotent mtx, i.e. $P^2 = P$. Which of the following is FALSE?

a) P^T is idempotent.

b) the possible eigen values of P can be zero.

c) the non-diagonal entries of P can be zero.

d) there may be infinite no. of $n \times n$ non-singular matrices that are idempotent.

Solⁿ : d) Given P be an $n \times n$ idempotent mtx.

s.t $P^2 = P$

if P is idempotent then P^T is also idempotent.

$P(P - I) = 0$

$\Rightarrow$ the possible eigen values of P are 0 and 1.

$\Rightarrow$ non-diagonal entries of P can be zero.

So, d) is false.

ABSTRACT ALGEBRA

WORLD OF MATHEMATICS
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MULTIPLE CHOICE TYPE:-

43. Let G be the set of cube roots of unity. Then under multiplication of complex numbers
(a) G is a group of finite order
(b) G is an abelian group
(c) G is a cyclic group
(d) None of the above.

Sol. (c) Here $G = \{1, w, w^2\}$
So, G is an abelian cyclic group of order 3, since

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

44. The irreducible polynomials in $C[X]$ are the polynomials of degree (a) 0 (b) 1 (c) 2 (d) None.

Sol. (b)
The polynomials of degree 0 are the invertible element of $C[X]$. By the fundamental theorem of algebra, any polynomial of positive degree has a root in C and hence a linear factor. Therefore, any polynomials of degree greater than 1 are reducible and those of degree 1 are irreducible.

45. Which of the following statement is false:
(a) The polynomial $x^3 - x + 1$ is irreducible in $\mathbb{Z}/2\mathbb{Z}[X]$
(b) The polynomial $x^2 - 3$ is irreducible in $\mathbb{Q}[X]$.
(c) The polynomial $x^2 + 1$ is irreducible in $\mathbb{Z}/5\mathbb{Z}[X]$.
(d) The polynomial $x^2 + 1$ is irreducible in $\mathbb{Z}/7\mathbb{Z}[X]$.

Sol. (c)
The polynomial x^2+1 is reducible in $\mathbb{Z}/5\mathbb{Z}[X]$.
As $Z_5 = \{0, 1, 2, 3, 4\}$ and $f(x) = x^2 + 1$.
$$f(3) = 10 \equiv 0 \pmod 5$$
$\Rightarrow x^2 + 1$ is reducible in $\mathbb{Z}/5\mathbb{Z}[X]$.

46. Let $f: G \to H$ be a group homomorphism from a group G into a group H with Kernel K. If $O(G) = 75$, $O(H) = 45$, $O(K) = 15$. Then the order of the image $f(G)$ is :

(a) 3 (b) 5 (c) 15 (d) 45

Sol. (b) $f(G) \cong \frac{G}{K}$.

$$o\{f(G)\} = O\left(\frac{G}{K}\right) = \frac{O(G)}{O(k)} = \frac{75}{15} = 5.$$

47. Which of the following is a cyclic group?

(i) $\mathbb{Z}_{12} \times \mathbb{Z}_9$ (ii) $\mathbb{Z}_{10} \times \mathbb{Z}_{85}$ (iii) $\mathbb{Z}_{22} \times \mathbb{Z}_{21} \times \mathbb{Z}_{65}$ (iv) None.

Sol. (iii)

Any group $\mathbb{Z}_p \times \mathbb{Z}_r$ is said to be cyclic if the greatest common divisor (GCD) of p and r is equal to 1. Similarly, for $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ is called cyclic if GCD between any two is equal to one.

(i) GCD of $(12, 9) = 3$

(ii) GCD of $(10, 85) = 5$

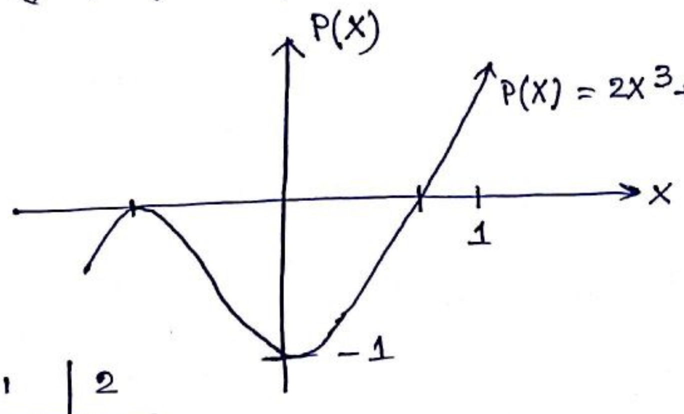
(iii) GCD of $(22, 21)$ & $(21, 65)$ & $(22, 65) = 1$.

$\Rightarrow \mathbb{Z}_{22} \times \mathbb{Z}_{21} \times \mathbb{Z}_{65}$ is a cyclic group.

48. Factor $P(X) = 2X^3 + 3X^2 - 1$ in $\mathbb{Q}[X]$

(i) $(x-1)^2(2x+1)$ (ii) $(x+1)^2(2x-1)$ (iii) $(x-1)^2(2x-1)$ (iv) $(x+1)^2(2x+1)$.

Sol. (ii)



X	−1	0	½	1	2
P(X)	0	−1	0	4	27

$\therefore P(X) = (x+1)^2(2x-1)$.

49. Any group G of order $2p$, where p is a prime number has a normal subgroup of order p, then the index of subgroup H in G is

(a) p (b) 2 (c) $p^2/2$ (d) none.

Solution:- (b) Given that $O(G) = 2p$.

Since p is prime, and

By Cauchy's theorem, G has an element of order p then the cyclic group, $H = \{a, a^2, \ldots, a^p\}$ is a subgroup of order p.

$\therefore$ The index of H in G is $= \dfrac{O(G)}{O(H)} = \dfrac{2p}{p} = 2$.

50. If $1, Z_1, Z_2, \ldots, Z_{11}$ are the 12 roots of unity forming the cyclic group under multiplication. Then Z_9 generates a cyclic sub-group of the above containing:

(a) 12 elements (b) 9 elements (c) 8 elements (d) none

Solution:- (d) The integral divisors of 9 are 1, 3, 9.

$\therefore$ All the elements of order 1, 3, 9 will give subgroups.

So, $\{Z_9\}$ has the subgroups $\{e\}$, (Z_1), (Z_3), (Z_9).

So, there is 4 elements,

51. The n equal rotations of a regular polygon of n sides

(a) form an abelian but not cyclic group

(b) form a cyclic group

(c) don't form a cyclic group

(d) form non-abelian non-cyclic group.

Solution:- (b) The rotations are the generators of the group G.

Hence, G must be a cyclic group.

52. Let $G = \{Z \in C : |Z| = 1\}$. Then under multiplication of complex numbers

(a) G is a group of order (finite) (b) G is a group of infinite order

(c) G is a cyclic group (d) None of the above.

Solution:- (c) Let $G = \{Z \in C : |Z| = 1\}$ and $Z_1 Z_2 \in G$.

Then $Z_1 Z_2 \in G \Rightarrow |Z_1| = 1, |Z_2| = 1$.

$\Rightarrow |Z_1 Z_2| = |Z_1||Z_2| = 1$.

$\therefore$ G is closed for multiplication.

And $\exists$ inverse of every element in G. Hence, G is multiplicative group.

MULTIPLE CHOICE TYPE :- ANALYSIS

53. If f and F be both continuous in $[a,b]$ and are derivable in (a,b) and if $F'(x) = f'(x)\ \forall\ x$ in (a,b), then $f(x)$ and $F(x)$ differ by

(A) 1 in $[a,b]$ (B) x in $[a,b]$ (C) constant in $[a,b]$ (d) none

Solⁿ: (c) Since they are continuous

$$f'(x) = F'(x)$$

Let $\phi(x) = f(x) - F(x)$

$$\Rightarrow \phi'(x) = f'(x) - F'(x) = 0 \quad \text{i.e. } \phi(x) = \text{Constant.}$$

54. Let $f_x(x) = n \sin^{2n+1} x \cos x$, then the value of

$$\lim_{n\to\infty} \int_0^{\pi/2} f_n(x)\,dx - \int_0^{\pi/2} \lim_{n\to\infty} \left(f_n(x)\right) dx \text{ is}$$

(a) $1/2$ (b) 0 (c) $-1/2$ (d) $-\infty$

Solⁿ: (d) value $= \lim_{n\to\infty} \left[\dfrac{n \Gamma\left(\frac{2n+1+1}{2}\right) \Gamma\left(\frac{1+1}{2}\right)}{2\,\Gamma\left(\frac{2n+1+1+2}{2}\right)} \right]$

$$- \int_0^{\pi/2} \lim_{n\to\infty} n \sin^{2n+1} x \cos x\, dx$$

$$= \lim_{n\to\infty} \frac{n}{2} \frac{\Gamma(n+1)}{\Gamma(n+2)} - \int_0^{\pi/2} \infty = \lim_{n\to\infty} \frac{n}{2} \frac{1}{n+1} - \infty = -\infty$$

55. If $\{s_n\}$ be a convergent sequence of positive numbers $\ni$ $s_n = \frac{1}{2}\left(s_{n-1} + s_{n-2}\right)\ \forall\ n \geq 2$ then $\lim_{n\to\infty} S_n = ?$

(a) $\frac{1}{3}(s_2 + 2s_3)$ (b) $\frac{1}{3}(s_3 + \frac{1}{2}s_1)$ (c) $\frac{1}{3}(s_1 + 2s_2)$ (d) None

Solⁿ: (c) Let $\lim_{n\to\infty} S_n = l$

$$s_n = \frac{1}{2}\left(s_{n-1} + s_{n-2}\right)$$

$$s_3 = \frac{1}{2}(s_2 + s_1)$$

$$s_4 = \frac{1}{2}(s_3 + s_2)$$

$$\vdots \qquad \vdots$$

$$s_{k-1} = \frac{1}{2}(s_{k-2} + s_{k-3})$$

$$s_k = \frac{1}{2}(s_{k-1} + s_{k-2})$$

Adding all these, $S_K + \frac{1}{2} S_{K-1} = \frac{1}{2}(S_1 + 2S_2)$

$$\Rightarrow l + \frac{1}{2} l = \frac{1}{2}(S_1 + 2S_2)$$

$$\Rightarrow l = \frac{1}{3}(S_1 + 2S_2).$$

56. For every function $f : [0,1] \to \mathbb{R}$, which is twice differentiable and satisfies $f'(x) \geq 1 \ \forall \ x \in [0,1]$, we must have

a) $f''(x) \geq 0 \ \forall \ x \in [0,1]$

b) $f(x) \geq x \ \forall \ x \in [0,1]$

c) $f(x_2) - x_2 \leq f(x_1) - x_1 \ \forall \ x_1, x_2 \in [0,1]$ with $x_2 \geq x_1$

d) $f(x_2) - x_2 \geq f(x_1) - x_1 \ \forall \ x_1 x_2 \in [0,1]$ with $x_2 \geq x_1$

Solⁿ: d) Taylor's formula gives

$$f(x_2) - f(x_1) > (x_2 - x_1) f'\left(\frac{x_1 + x_2}{2}\right) \ \forall \ x_1, x_2 \in [0,1]$$
with $x_2 > x_1$

if $x_2 \geq x_1$, then $f'\left(\frac{x_1+x_2}{2}\right) = f'(x_1) \geq 1 \ \forall \ x_1 \in [0,1]$

$$f(x_2) - f(x_1) > (x_2 - x_1) f'(x_1)$$
$$\geq (x_2 - x_1) \ \forall \ x_2 \geq x_1 \ \forall \ x_1, x_2 \in [0,1]$$

$$\Rightarrow f(x_2) - x_2 \geq f(x_1) - x_1$$

57. A function f is defined as $\{0,1\}$, by $f(x) = 1/x \ \forall \frac{1}{x} > x > \frac{1}{n+1}$ for all $n = 1, 2, 3, \ldots$ If given that $f \in R\{0,1\}$ then evaluate $\int_0^1 f(x)\,dx$

A) $\frac{\pi^2}{6}$, b) $\frac{\pi^2}{6} + 1$, c) $\frac{\pi^2}{6} - 1$, d) none

Solⁿ: c) $\int_0^1 f(x)\,dx = \lim_{n \to \infty} \sum_{r=1}^{n} \int_{\frac{1}{r+1}}^{1/r} \frac{1}{r}\,dx$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{r}\left\{\frac{1}{r} - \frac{1}{r+1}\right\}$$

$$= \lim_{n \to \infty} \left\{(1 - 1/2) + 1/2\,(1/2 - 1/3) + \cdots + 1/n\left(1/n - \frac{1}{n+1}\right)\right\}$$

$$= \lim_{n \to \infty} \left\{\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right) - \left(1 - \frac{1}{n+1}\right)\right\}$$

$$= \frac{\pi^2}{6} - 1.$$

58. Let f be a differentiable function defined on $[0,1]$,

$y \in (0,1) \ni f(x) < f(y) = f(0) \ \forall x \in [0,1], x \neq y$,

then

a) $f'(y)=0$ and $f'(0)=0$ b) $f'(y)=0$ and $f(0)>0$

c) $f'(y)=0$ and $f'(0) \leq 0$ d) $f'(y)>0$ and $f'(0) \leq 0$.

Solⁿ: c) $f(y) = f(0)$ is maximum of f in $[0,1]$

$\therefore f'(y) = 0$

$$f'(0) = \lim_{h \to 0} \frac{f(h)-f(0)}{h}$$

Since $f(h) < f(0) \ \forall h \in [0,1]$

$\Rightarrow f(h) - f(0) < 0$

$\Rightarrow \dfrac{f(h)-f(0)}{h} < 0 \quad [\because h>0]$

i.e. $f'(0) \leq 0$

59. Let $\{a_n\}$ and $\{b_n\}$ be sequence of real nos. defined as $a_1 = 1$ and for $n \geq 1$, $a_{n+1} = a_n + (-1)^n 2^{-n}$, $b_n = \dfrac{2a_{n+1} - a_n}{a_n}$ then

a) $\{a_n\}$ converges to zero and $\{b_n\}$ is a cauchy sequence

b) $\{a_n\}$ converges to non-zero and $\{b_n\}$ is a cauchy sequence.

Solⁿ: b) $a_1 = 1$

$a_2 = a_1 + \frac{(-1)}{2} = \frac{1}{2}$

$a_3 = a_2 + \frac{1}{2^2} = \frac{3}{4} \quad [a_2 = 1/2]$

$a_4 = a_3 - \frac{1}{2^3} = \frac{5}{8} \quad [a_3 = 3/4]$

$a_5 = a_4 + \frac{1}{2^4} = \frac{11}{16} \quad [a_4 = 5/8]$

$\vdots \qquad \vdots$

and so on.

Since $a_1, a_3, a_5, \ldots$ is a decreasing sequence and $a_2, a_4, a_6, \ldots$ is a increasing sequence.

$\{a_n\}$ converges to $\frac{1}{3}(a_1 + 2a_2) = \frac{1}{3}\left(1 + \frac{2}{2}\right) = \frac{2}{3}$

Again, $b_n = \dfrac{2a_{n+1} - a_n}{a_n}$

$b_1 = 0$

$b_2 = 2$

$b_3 = 2/3$

$b_4 = 6/5$

$b_5 = 10/11$

$\vdots \quad \vdots$

and so on.

$b_1, b_3, b_5 \ldots$ are increasing seq.

& $b_2, b_4, b_6, \ldots$ are decreasing seq.

the $\{b_n\}$ converges to limit

$\frac{1}{3}(b_1 + 2b_2) = \frac{1}{3}(0 + 2\cdot 2) = \frac{4}{3}$

$\therefore \{b_n\}$ is a cauchy sequence.

60. On $x = c[0,1]$ define $T : x \to x$ by $T(f(x)) = \int_0^x f(t)\partial t$

$\forall f$ in x then

a) T is one-one and onto

b) T is one-one but not onto

c) T is not one-one but onto

d) T is neither one-one nor onto

Sol^n: a) $T(f(x)) = \int_0^x f(t)\partial t \quad \forall f$ in x

$\Rightarrow$ Let $x_1, x_2 \in X$

s.t, $T(f(x_1)) = T(f(x_2))$

$\Rightarrow \int_0^{x_1} f(t)\partial t = \int_0^{x_2} f(t)\partial t$.

$\Rightarrow x_1 = x_2$

i.e. T is one-one

for each $T(f(x))$, $\exists$ only one $x \in X$,

s.t, $T(f(x)) = \int_0^x f(t)\partial t$

$\Rightarrow$ T is onto

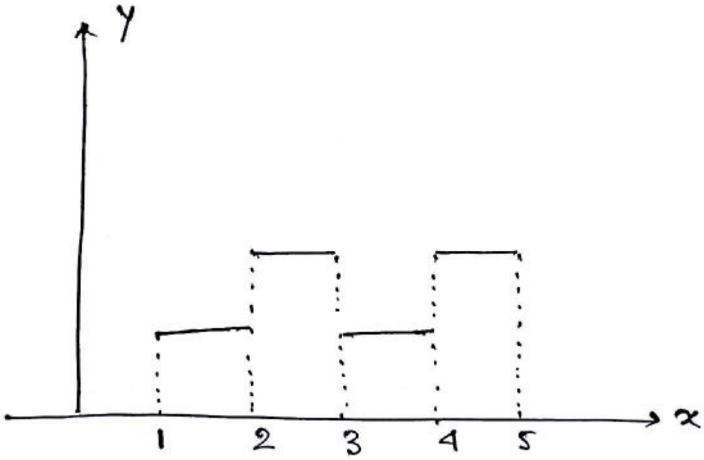
61. Evalute, $\int_0^5 f(x)\,dx$, if $f(x)=\begin{cases} 0, & \text{if } 0\le x\le 1 \\ 1, & \{1\le x<2\}\cup\{3\le x<4\} \\ 2, & \{2\le x<3\}\cup\{4\le x<5\} \end{cases}$

by using Reimann & Lebegue definition of the integral

a) $R\int_0^5 f(x)\,dx > L\int_0^5 f(x)\,dx$ b) $R\int_0^5 f(x)\,dx < L\int_0^5 f(x)\,dx$

c) $R\int_0^5 f(x)\,dx = L\int_0^5 f(x)\,dx$ d) None

Solⁿ: c) Using Reimann definition of the integral (where the subdivision is taken of the segment $[0,5]$ by the division points $x_0, x_1, \ldots, x_n$ on X-axis) the upper & lower Riemann sums tend to the common value



$0(1-0)+1(2-1)+2(3-2)+1(4-3)+2(5-4)=6$

(Since the function is constant on each of the subintervals)

$\therefore R\int_0^5 f(x)\,dx = 6.$

Evaluating the Lebesgue integral where the subinterval is that of the interval $[0, 2+\delta[$, $\delta>0$, we get

$0[1-0]+1[(2-1)+(4-3)]+2[(3-2)+(5-4)]=6.$

$\therefore L\int_0^5 f(x)\,dx = 6.$

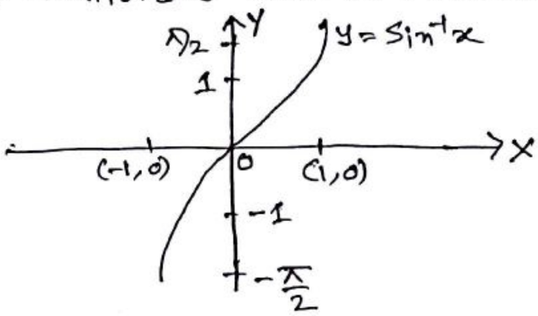
62. Let f be an one to one function from the closed interval $[-1, 1]$ to the set of real numbers $\mathbb{R}$, then

a) f must not be onto

b) range of f must contain a rational number.

c) range of f must contain an irrational no.

d) range of f must contain both rational and irrational nos.



Sol^n: d) $y = \sin^{-1}x$ (Let)

$x \in [-1, 1]$

$y \in [-\pi/2, \pi/2]$

$\Rightarrow$ range of f must contain rational & irrational nos.

f is onto here.

63. The sequence $\sqrt{11}$, $\sqrt{11+\sqrt{11}}$, $\sqrt{11+\sqrt{11+\sqrt{11}}}$, ... converges to

a) $\frac{1+\sqrt{43}}{2}$ b) $\frac{1+\sqrt{45}}{2}$ c) $\frac{1+\sqrt{23}}{2}$ d) $\frac{1+\sqrt{29}}{2}$

Sol^n: b) $s^2 = 11 + s \Rightarrow s^2 - s - 11 = 0 \Rightarrow s = \frac{1 \pm \sqrt{45}}{2}$

64. The sequence $\sqrt{2}$, $\sqrt{2+\sqrt{2}}$, $\sqrt{2+\sqrt{2+\sqrt{2}}}$, ... converges to

a) 2 b) 3 c) $\frac{2+\sqrt{2}}{3}$ d) $\frac{\sqrt{3}+1}{2}$

Sol^n: a) $s^2 = 2 + s \Rightarrow s^2 - s - 2 = 0 \Rightarrow (s+1)(s-2) = 0 \Rightarrow s = 2$

65. For $x > 0$, $\underset{x \to 0}{Lt} \left[(\sin x)^{1/x} + \left(\frac{1}{x}\right)^{\sin x} \right]$ is

a) 0 b) -1 c) 1 d) 2

Sol^n: c) $L = \underset{x\to 0}{Lt} (\sin x)^{1/x} + \underset{x\to 0}{Lt} \left(\frac{1}{x}\right)^{\sin x}$

$= 0 + \underset{x\to 0}{Lt}\, e^{\log\left(\frac{1}{x}\right)^{\sin x}}$ $\left[\because \underset{x\to 0}{Lt} (\text{decimal})^{\infty} = 0\right]$

$= e^{\underset{x\to 0}{Lt} \frac{\log(1/x)}{\operatorname{cosec} x}}$

Applying L'Hospital's rule, we get

$L = e^{\underset{x\to 0}{Lt} \frac{x(-1/x^2)}{-\operatorname{cosec} x \cot x}} = e^{\underset{x\to 0}{Lt} \frac{\sin x}{x} \cdot \tan x} = e^0 = 1.$

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66. The sequence $\{a^{1/n}\}$ converges for

a) $a<0$ b) $a>0$ c) $a \geq 0$ d) $a \leq 0$.

Sol^n: b) $a^{1/n} = 1 + bn$

for $a=0$, $\lt \left(a_n^{1/n}\right) = \lt(0) = 0$.

for $a>0$, $\lt\limits_{n\to\infty} a^{1/n} = 1$.

67. The function $f(x) = \dfrac{\log(1+ax) - \log(1-bx)}{x}$

is not defined at $x=0$. The value which should be assigned to f at $x=0$, so that $f(x)$ is continuous at $x=0$, is

a) $a-b$ b) $a+b$ c) $\log a + \log b$. d) none

Sol^n: b) $f(0) = \lt\limits_{x\to 0} f(x)$

$$= \lt_{x\to 0} \frac{\log(1+ax) - \log(1-bx)}{x}$$

$$= \lt_{x\to 0} \frac{a\log(1+ax)}{ax} + \lt_{x\to 0} \frac{-b\log(1-bx)}{-bx} \quad \left[\because \lt_{x\to 0} \frac{\log(1+x)}{x} = 1\right]$$

$= a.1 + b.1$

$= a+b$.

68. If a is a real number then

$$\lt_{n\to\infty} \begin{bmatrix} 1 & a/n \\ -a/n & 1 \end{bmatrix}^n$$ is equal to

a) $I_{2\times 2}$ b) $O_{2\times 2}$ c) $\underset{\sim}{1}$ d) None.

Sol^n: $A = \begin{bmatrix} 1 & a/n \\ -a/n & 1 \end{bmatrix} = \frac{1}{n}\begin{bmatrix} n & a \\ -a & n \end{bmatrix}$

Let $n = r\cos\theta$, $a = r\sin\theta$.

$\Rightarrow r = \sqrt{n^2+a^2}$; $\theta = \tan^{-1}(a/n)$

$A = \frac{r}{n}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \Rightarrow A^n = \frac{r^n}{n^n}\begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$

$$\Rightarrow A^n = \left(\sqrt{1+\frac{a^2}{n^2}}\right)^n \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$$

$$\Rightarrow \frac{A^n}{n} = \left(1+\frac{a^2}{n^2}\right)^{n/2} \begin{pmatrix} \frac{\cos n\theta}{n} & \frac{\sin n\theta}{n} \\ -\frac{\sin n\theta}{n} & \frac{\cos n\theta}{n} \end{pmatrix}$$

$$\Rightarrow \lim_{n\to\infty} \frac{A^n}{n} = (1+0) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \lim_{n\to\infty} A^n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O_{2\times 2}$$

69. The series $\sum \frac{(-1)^n}{(n+1)^p}$ is

a) Conditionally convergent if $0 < p \le 1$

b) absolutely convergent if $p > 1$

c) oscillatory if $p \le 0$

d) all above

Soln :- d) Case I: $p \le 0$

$p = -q$

then the given series becomes

$\sum (-1)^n (n+1)^q = -2^q + 3^q - 4^q + 5^q - \cdots$

This is an Oscillatory Series.

Case II: $0 < p < 1$.

The series is :

$$\sum u_n = -\frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \frac{1}{5^p} - \cdots$$

By Leibnitz's test this series is convergent.

Also, $|\sum u_n| = \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$

This series is p-series and $p \le 1$.

So, it is divergent.

Case III: $p > 1$, the series is

$$\sum_1 u_n = -\frac{1}{2^p} - \frac{1}{3^p} - \frac{1}{4^p} - \cdots$$

By Leibnitz's test the series is convergent

$$\left|\sum u_n\right| = \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

The series is also convergent.

So the series is absolutely convergent.

70. $\underset{x\to 0}{Lt}\left[\frac{\sin x}{x}\right]$ is equal to

a) 1 b) 0 c) does not exist d) none

Sol^n: b) Since, $\left|\frac{\sin x}{x}\right| < 1$

$\Rightarrow \frac{\sin x}{x}$ tends to 1 forms the values that are less than one as $x \to 0$

Thus, $\underset{x\to 0}{Lt}\left[\frac{\sin x}{x}\right] = 0$

71. The net profit of an industry in a year is given by $y = 2ax - x^2$, where x denotes the input. Then the profit increases in relation to x if

a) $0 < a < x$ b) $x = a$ c) $a < x < 2a$ d) $x < a$

Sol^n: d) $y = 2ax - x^2 \Rightarrow \frac{dy}{dx} = 2a - 2x$

Profit will increase if $2a - 2x > 0$ i.e. $x < a$

72. If $a_1 = 1$ and $a_{n+1} = \frac{4 + 3a_n}{3 + 2a_n}$, $n \geq 1$; then $\underset{n\to\infty}{Lt}\, a_n = l$. The l is equal to

a) $-\sqrt{2}$ b) $\sqrt{2}$ c) 2 d) none

Sol. (b) $l = \frac{4+3l}{3+2l} \Rightarrow l^2 = 2 \Rightarrow l = \sqrt{2}$, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = l$.

73. $\underset{n\to\infty}{Lt} \dfrac{n^p \sin^2(n!)}{n+1}$, $0<p<1$, is equal to

a) 0 b) ∞ c) 1 d) none

Sol^n:- a) $\underset{n\to\infty}{Lt} \dfrac{\sin^2(n!)}{n^{1-p}(1+1/n)}$ $(\because 0<p<1)$

$= \dfrac{\text{some real no. in } [0,1]}{\infty}$ $[\because 1-p>0]$

$=0$

74. The series $\sum_{n=1}^{\infty} \sin\dfrac{\pi}{n^p}$ is

a) convergent for all values of p
b) convergent for $p\le 1$ and divergent for $p>1$
c) convergent for $p>1$ and divergent for $p\le 1$
d) divergent for all values of p.

Sol^n: c) $u_n = \sin\dfrac{\pi}{n^p} = \dfrac{\pi}{n^p} - \dfrac{\pi^3}{3!\,n^{3p}} + \cdots$

$v_n = \dfrac{1}{n^p}$

$\therefore \underset{n\to\infty}{Lt} \dfrac{u_n}{v_n} = \pi \neq 0$

$\sum v_n$ is convergent for $p>1$ and divergent for $p\le 1$

$\therefore \sum_1 u_n$ is convergent for $p>1$ and divergent for $p\le 1$

75. Let $u_n = \sin(1/n)$ and consider the series $\sum_1 u_n$.
Which of the following statement is true?
a) $\sum_1 u_n$ is convergent b) $u_n \to 0$ as $n\to\infty$
c) $\sum u_n$ is divergent d) $\sum u_n$ is absolutely convergent

Sol. $u_n = \sin 1/n$, $v_n = 1/n$ $\therefore \lim_{n\to\infty} \dfrac{u_n}{v_n} = \lim_{n\to\infty} \dfrac{\sin 1/n}{1/n} = 1$

So, $\sum v_n$ diverges.
$\therefore$ By Limit Comparison test, $\sum u_n$ also diverges.

76. If $f_n(x)$ be a function defined on $[0,1]$ and then the sequence $\{f_n(x)\}$, where $f_n(x) = x^n$, is —

(a) Uniformly convergent in $[0,1]$ (b) Uniformly convergent in $(0,1)$.

(c) Uniformly convergent in $\mathbb{R}$. (d) None.

Solution:-

(b) $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{, where } 0 \le x < 1 \\ 1 & \text{, where } x = 1. \end{cases}$$

Then the sequence is pointwise convergent in $[0,1]$ and uniformly convergent in $(0,1)$.

77. Which of the following functions is uniformly continuous on the domain as stated?

(a) $f(x) = x^2$, $x \in \mathbb{R}$ (b) $f(x) = \frac{1}{x}$, $x \in [1, \infty)$

(c) $f(x) = \tan x$, $x \in (-\pi/2, \pi/2)$ (d) $f(x) = [x]$, $x \in [0,1]$

Sol. (b)

$f(x) = \frac{1}{x}$ is uniformly continuous in $[1, \infty)$.

[See our study material for detailed solution]

MULTIPLE CHOICE QUESTIONS :-

78. The number of non-empty subset of a set consisting 6 elements is

(a) 63 (b) 64 (c) 65 (d) none.

Sol. (a)

The no. of non-empty subset of a set consisting n elements is $= 2^n - 1$

79. Let A and B be two sets having 7 common elements, then the number of elements common to $A \times B$ and $B \times A$ is

(a) 0 (b) 2^7 (c) 49 (d) none.

Sol. (c) The no. of common elements to $A \times B$ and $B \times A$ is $= n^2$.

80. The number of squares that can be formed on a chess board is

(a) 204 (b) 224 (c) 230 (d) None

Sol. (a) A chess board has 9 equispaced horizontal and vertical lines. We need to choose two consecutive horizontal and vertical lines to make a 1×1 square from among these which is done in $8 \times 8 = 8^2$ ways.

Similarly, 2×2 square needs 3 consecutive horizontal and vertical lines, i.e. in $7 \times 7 = 7^2$ ways.

$\therefore$ Total number of squares $= 8^2 + 7^2 + 6^2 + 5^2 + \cdots + 1^2$

$$= \sum_{i=1}^{8} i^2$$

$$= \frac{8(8+1)(16+1)}{6} = 204.$$

81. How many friends must you have to guarantee that at least five of them will have birthdays in the same month?

(a) $50 \le n \le 60$ (b) $49 \le n \le 59$ (c) $40 \le n \le 60$ (d) $49 \le n \le 60$

Sol. (d) No of friends $= n$

Months (Holes) $(m) = 12$.

By Extended Pigeon-Hole Principle,

$$\left[\frac{n-1}{m}\right] + 1 = 5$$

$$\Rightarrow \left[\frac{n-1}{12}\right] + 1 = 5$$

$$\Rightarrow 49 \le n \le 60.$$

82. Let U be the set of positive integers not exceeding 1000 then the number of ~~sets~~ of such integers which are not divisible by 3, 5, 7 is

(a) 255 (b) 456 (c) 457 (d) 256

Sol. (c) A : integers divisible by 3

B : " " " 5

C : " " " 7

$n(A) = \left[\frac{1000}{3}\right] = 333, \ n(B) = \left[\frac{1000}{5}\right] = 200, \ n(C) = \left[\frac{1000}{7}\right] = 142$

$n(A \cap B) = \left[\frac{1000}{15}\right] = 66, \ n(B \cap C) = \left[\frac{1000}{35}\right] = 28, \ n(A \cap C) = \left[\frac{1000}{21}\right] = 47$

$n(A \cap B \cap C) = \left[\frac{1000}{105}\right] = 9.$

By inclusion-exclusion principle,

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$= 543.$

So, required answer is $= n(A \cup B \cup C)^c$

$= 1000 - n(A \cup B \cup C) = 457.$

83. A and B toss a fair coin each simultaneously 50 times. The probability that both of them will not get tail in the same toss is

(a) $\left(\frac{3}{4}\right)^{50}$ (b) $\left(\frac{2}{7}\right)^{50}$ (c) $\left(\frac{1}{8}\right)^{50}$ (d) none.

Sol. (a) There are four possibilities in each toss, i.e.,

A = tail B = Head

A = Head B = tail

A = Head B = Head

A = tail B = tail

Total number of cases $= 4^{50}$.

In each case there are 3 posibilities of not getting tain in the same toss. ∴ favourable cases $= 3^{50}$.

Hence the required probability is $= \left(\frac{3}{4}\right)^{50}$.

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84. If the integers m and n are chosen at random between 1 to 100. Then the probability that a number of the form $7^m + 7^n$ is divisible by 5 is

(a) $1/4$ (b) $1/7$ (c) $1/8$ (d) None.

Sol. (a) The unit place of 7^k, where k is an integer will be 9, 3, 1, 7.

$7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2407, \dots, 7^7 \equiv 1$ (last digit)

Now, $7^m + 7^n$ is divisible by 5 if m = 3 or 7 and n = 7 or 3.

Also, $7^m + 7^n$ " " " " " m = 9 or 1 and n = 1 or 9.

Now, $7^m + 7^n$ is divisible by 5 only when the last digit in the unit place is zero.

$\therefore$ The required probability is $= \frac{4}{2^4} = \frac{1}{2^2} = \frac{1}{4}$.

85. The total number of subsets of a set of 12 elements are

(a) 144 (b) 12^{12} (c) 47900 (d) 4096

Sol. (d) Ans $= 2^{12}$

$= 4096$.

86. The total number of non-empty even subsets of a set having n elements is

(a) 2^{n-1} (b) $2^{n-1} - 1$ (c) 2^n (d) $2^{n+1} + 1$

Sol. (b) If a set having n elements then total no. of subsets is $= 2^n$

Total no. of even subsets is $= 2^{n-1}$

Excluding the empty set ϕ, we have $2^{n-1} - 1$ as total number of non-empty subsets.

87. A bar of unit length is broken into 3 parts x, y, z. The probability that a triangle can be formed from the resulting parts is

(a) $1/2$ (b) $1/3$ (c) $1/4$ (d) None

Sol. (c) Let $z = 1 - (x+y)$

$x > 0,\ y > 0,\ (x+y) < 1.$

The sample space is $= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$ = interior of a unit triangle with unit legs.

Then two conditions are needed to satisfy to form a triangle:

(a) The sum of the two sides is greater than the third side

(b) The difference between any two sides is smaller than the third one.

The area of the new triangle domain is $= \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$.

$\therefore$ Prob. is $= \frac{1/8}{1/2} = \frac{1}{4}$.

88. Total number of non-negative integer solutions of $x_1+x_2+x_3=10$ is

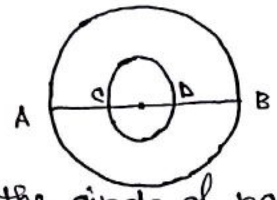
(a) ${}^{10}C_2$ (b) ${}^{10}C_3$ (c) ${}^{1}C_2$ (d) none.

Sol. (c) ${}^{n+r-1}C_{r-1} = {}^{10+3-1}C_{3-1} = {}^{12}C_2$

89. A point is selected at random from the interior of a circle. The probability that the point is closed to the centre than the boundary of the circle is

(a) $1/2$ (b) $1/4$ (c) $1/6$ (d) None

Sol. (b)



$AB = r$

$CD = r/2$

$n(S)$ = the area of the circle of radius $r = \pi(r^2)$

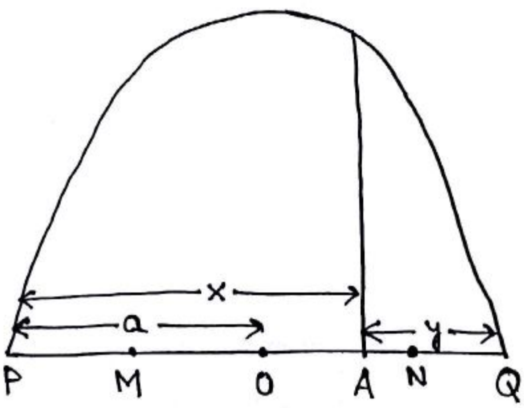
$n(E)$ = " " " " " " " $\frac{r}{2} = \pi\left(\frac{r}{2}\right)^2$.

$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{\pi\left(\frac{r}{2}\right)^2}{\pi r^2} = \frac{1}{4}$.

90. Let $x+y=2a$, where a is a constant, and all values of x lying between 0 and $2a$ are equally likely. Then the chance that $xy > \frac{3}{4}a^2$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) None.

Sol. (a)



$OP = a$, $AP = x$, $AQ = y$, $x+y = 2a$.

$AB^2 = AP \cdot PQ = xy$.

$MP = MO$, $NO = NQ$

If A lies in MN then $AB > a\sqrt{\frac{3}{4}}$.

$\therefore P\left(xy > \frac{3}{4}A^2\right)$

$= P\left(AB > a\frac{\sqrt{3}}{2}\right)$

$= \frac{MN}{AB} = \frac{1}{2}$.

91. Two finite sets have m and n elements. The total number of subsets of the first set is 56, more than the total number of subsets of the second set. The value of m, n are

(a) 7, 6 (b) 6. 3 (c) 5, 1 (d) none

Solution:- (b) We know that $2^m - 2^n = 56$.

By trial, $m = 6, n = 3$.

So, (b) is correct.

92. Total number of polynomials of the form $x^3 + ax^2 + bx + c$ which are divisible by x^2+1, where $a, b, c \in \{1, 2, \ldots, 10\}$ is

(a) 15 (b) 10 (c) 5 (d) None

Sol. (b) Here $i^3 + ai^2 + bi + c = 0$

and $(-i)^3 + b(-i)^2 + b(-i) + c = 0$

$\Rightarrow (b-1)i + (c-a) = 0$

$\therefore b = 1$ and $a = c$.

Hence, total number is $^{10}C_1 = 10$.

MULTIPLE CHOICE QUESTIONS :-

93. The differential equation $\left|\frac{dy}{dx}\right| = |y|$, $y(0) = 1$, $y \neq 0$ has

(a) unique solution (b) non-trivial solution
(c) finite number of solution (d) infinite number of solution

Solution:- (b)

The equation is $|y| = e^{x+c}$

$y(0) = 1$, gives $e^c = 1 = e^0 \Rightarrow c = 0$.

$\therefore y = e^x$, $y = -e^x$, there are these two solutions.
So, the ODE has non-trivial solution.

[Note:-
Trivial Solution:- A solution in which every variable has zero value is called trivial solution.
Infinite Solution:- If the constant(s) of the solution of the ODE remain undetermined then the equation has infinite number of solutions.]

94. Number of solution of the ODE $\frac{d^2y}{dx^2} + 4y = 0$, $y(0) = 0$, $y\left(\frac{\pi}{4}\right) = 1$ is

(a) 0 (b) 1 (c) 2 (d) None of the above

Solution:- (b)

$\frac{d^2y}{dx^2} + \lambda y = 0$ $(\lambda > 0)$ has the general solution

$$y = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x \text{; where } \lambda > 0.$$

So, $\frac{d^2y}{dx^2} + 4y = 0$ has the solution $y = c_1\cos 2x + c_2 \sin 2x$

$$y(0) = 0 \Rightarrow c_1 = 0$$
$$y\left(\frac{\pi}{4}\right) = 1 \Rightarrow c_2 = 1.$$

$\therefore y = \sin 2x$ is the unique solution of the given differential equation.

95. The solution of the ODE $\frac{dy}{dx} = x$, $y(0) = 0$ is

(a) unbound (b) positive (c) negative (d) zero

Solution:- (b)

$$\int dy = \int x\,dx$$

$$\Rightarrow y = x^2 + c$$

$y(0) = 0$ giving $c = 0$

$\therefore y = x^2$ is the solution which is always positive.

96. Number of solutions of the ODE $\frac{d^2y}{dx^2} = 0$, $y(0) = 1$ is

(a) 0 (b) 1 (c) infinite no. of solutions (d) none

Solution:- (c) $\frac{d^2y}{dx^2} = 0$ has the solution $y = c_1 x + c_2$

$y(0) = 1 \Rightarrow c_2 = 1$.

$\therefore y = c_1 x + 1$, c_1 is arbitrary constant.

$\Rightarrow$ The ODE has infinite number of solutions.

97. One of the integrating factors of the ODE

$(y^2 - 3xy)\,dx + (x^2 - xy)\,dy = 0$ is

(a) $1/(x^2y^2)$ (b) $1/(x^2y)$ (c) $1/(xy^2)$ (d) $1/(xy)$.

Solution:- (b) $M = y^2 - 3xy$

$N = x^2 - xy$

As $M\,dx + N\,dy = 0$ is homogeneous.

An I.F. is $= \dfrac{1}{Mx + Ny} = \dfrac{1}{(-2x^2y)}$.

So, $\dfrac{1}{x^2y}$ is an I.F. by ignoring the constant.

98. General solution of $(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)\,dy = 0$ is

(a) $y\sin(xy) = cx$ (b) $x\sec(xy) = cy$ (c) $y\tan(xy) = cx$ (d) none.

Solution:- (b) $xy\sin xy(y\,dx + x\,dy) + \cos xy(y\,dx - x\,dy) = 0$,

$\Rightarrow \tan xy \cdot d(xy) + \frac{dx}{x} - \frac{dy}{y} = 0$

$\Rightarrow \log|\sec xy| + \log\left|\frac{x}{y}\right| = c'$

$\Rightarrow x\sec xy = cy.$

99. The solution of the curve $y = f(x)$ satisfying the differential equation $\sqrt{(x-y)\frac{dy}{dx}} = |x^2 - y^2|$ and passing through the point $(1,0)$ is

(a) $(x-1) = y^2(x^2-y^2)$ (b) $y^2 = x-1$

(c) $(2x-3) + \frac{1}{(x^2-y^2)} = 0$ (d) none.

Solution:- (c) $x - y\frac{dy}{dx} = (x^2-y^2)^2$

$\Rightarrow \frac{d(x^2-y^2)}{2(x^2-y^2)^2} = dx \Rightarrow -\frac{1}{x^2-y^2} = 2x + c$ which passes through $(1,0)$.

$\therefore -1 = 2x + c \Rightarrow c = -3$. Hence curve is $(2x-3) + \frac{1}{x^2-y^2} = 0$.

100. The solution of the equation $x\,dy - y\,dx = \sqrt{x^2-y^2}\,dx$ subject to the condition $y(1) = 0$, is $y =$

(a) $x\sin(\log x)$ (b) $x^2\sin(\log x)$ (c) $x^2(x-1)$ (d) none.

Solution:- (a) $\frac{x\,dy - y\,dx}{x^2} = \frac{1}{x}\sqrt{1-\left(\frac{y}{x}\right)^2}\,dx \Rightarrow d\left(\frac{y}{x}\right) = \frac{1}{x}\sqrt{1-\left(\frac{y}{x}\right)^2}\,dx$

$\Rightarrow \frac{d\left(\frac{y}{x}\right)}{\sqrt{1-\left(\frac{y}{x}\right)^2}} = \frac{dx}{x} \Rightarrow \sin^{-1}\left(\frac{y}{x}\right) = \log x + c.$

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