TESTING OF HYPOTHESIS

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Stating the Problem: — The problem of testing hypothesis is posed as follows:

The decision is to be based on the value of a certain RV X, the distribution of which is known to belong to a class $\{f(x,\theta): \theta \in \Omega\}$. Take a r.s. ~~$\{f(x,\theta): \theta \in \Omega\}$~~ $(x_1, \dots, x_n) = \underset{\sim}{x}$ of size n from $\{f(x;\theta): \theta \in \Omega\}$. To test whether the data $\underset{\sim}{x}$ supports the hypothesis $H_0: \theta \in \Omega_0$ or $H_1: \theta \in \Omega_1$, where $\Omega = \Omega_0 \cup \Omega_1$.

Randomised Test: — We can slightly generalise the idea of a critical region by defining a test of the following structure:

For any given data $\underset{\sim}{x}$, a test chooses among the two decisions: rejection of H_0 or acceptance of H_0, with certain probabilities that depends on $\underset{\sim}{x}$ and are denoted by $\phi(\underset{\sim}{x})$ and $\{1-\phi(\underset{\sim}{x})\}$ respectively.

If the value of $\underset{\sim}{X}$ is $\underset{\sim}{x}$, a random experiment is constructed with two possible outcomes R and R^c with probabilities $\phi(\underset{\sim}{x})$ and $\{1-\phi(\underset{\sim}{x})\}$. Then perform the random experiment and if in this trial R occurs, the hypothesis H_0 is rejected.

A randomised test is therefore completely characterized by a function $\phi(\underset{\sim}{x})$ such that

(i) $0 \le \phi(\underset{\sim}{x}) \le 1$, $\forall\ \underset{\sim}{x} \in \mathscr{X}$

(ii) $\phi(\underset{\sim}{x}) = P[H_0 \text{ is rejected} \mid \underset{\sim}{x} \text{ is observed}]$ $\forall\ \underset{\sim}{x} \in \mathscr{X}$

The function $\phi(x)$ is called the critical function of the test.

Non-randomised test: — If a test given by a critical function $\phi(\underset{\sim}{x})$, which takes only the values 1 and 0, then the set of points for which $\phi(\underset{\sim}{x}) = 1$ is just the region of rejection or critical region, say W. Then

$$\phi(\underset{\sim}{x}) = \begin{cases} 1, & \underset{\sim}{x} \in W \\ 0, & \underset{\sim}{x} \in \mathscr{X} - W \end{cases}$$

Note that the test given by $\phi(\underset{\sim}{x})$ is then a non-randomised test.

A non-randomised test procedure assigns to each possible data $\underset{\sim}{x}$, one of the two decisions: rejection of H_0 or acceptance of H_0, with certainty and there by divides the sample space $\mathscr{X}$ into two complementary regions W and $\mathscr{X} - W$ such that if $\underset{\sim}{x}$ falls in W, the hypothesis H_0 is rejected; Otherwise it is accepted. The set W is called the critical region.

Power function and Testing Problem:–

If the distribution of $\underset{\sim}{X}$ is $L(\underset{\sim}{x}, \theta)$ and the critical function $\phi(\underset{\sim}{x})$ is used, then the power function of the test given by $\phi(\underset{\sim}{x})$ is

$$P[\text{reject } H_0] = \begin{cases} \sum_{\underset{\sim}{x} \in \mathcal{X}} P_\theta[\text{reject } H_0 / \underset{\sim}{x} \text{ is observed}]\, L(\underset{\sim}{x}, \theta), & \text{if } \underset{\sim}{X} \text{ is of discrete type.} \\ \int_{\mathcal{X}} P_\theta[\text{reject } H_0 / \underset{\sim}{x} \text{ is observed}]\, L(\underset{\sim}{x}, \theta)\, d\underset{\sim}{x}, & \text{if } \underset{\sim}{X} \text{ is of continuous type.} \end{cases}$$

$$= \begin{cases} \sum_{\underset{\sim}{x} \in \mathcal{X}} \phi(\underset{\sim}{x}) L(\underset{\sim}{x}, \theta), & \text{if } \underset{\sim}{X} \text{ is of discrete type.} \\ \int_{\mathcal{X}} \phi(\underset{\sim}{x}) \cdot L(\underset{\sim}{x}, \theta)\, d\underset{\sim}{x}, & \text{if } \underset{\sim}{X} \text{ is of continuous type.} \end{cases}$$

$$= E_\theta[\phi(\underset{\sim}{X})] = \beta_\phi(\theta), \text{ say.}$$

Let $\alpha \in (0,1)$ be a chosen level of significance. A test given by $\phi(\underset{\sim}{x})$ is called a <u>level α</u> test if $\beta_\phi(\theta) \le \alpha, \ \forall\ \theta \in \Omega_0$

$$\Leftrightarrow \sup_{\theta \in \Omega_0} \beta_\phi(\theta) \le \alpha.$$

▨ If for a test given by $\phi(\underset{\sim}{x})$,
$\sup_{\theta \in \Omega_0} \beta_\phi(\theta) = \alpha^*$, then the size of the test is α^* <u>or</u> $\phi(\underset{\sim}{x})$ is a size α^* test.

For a preassigned level α, consider those tests $\phi(\underset{\sim}{x})$ whose size is $\le \alpha$ that is, consider the class of level α tests. Then in the class of level α tests, find a test $\phi(\underset{\sim}{x})$ whose power $\beta_\phi(\theta)$ is maximum, $\theta \in \Omega_1$.

Therefore, the problem is to select a critical function $\phi(\underset{\sim}{x})$ so as to maximize the power.

$\beta_\phi(\theta) = E_\theta \phi(\underset{\sim}{x})$, $\forall\ \theta \in \Omega_1$, subject to the condition,

$$E\,\phi(\theta) \le \alpha, \ \forall\ \theta \in \Omega_0$$

$$\Leftrightarrow \sup_{\theta \in \Omega_0} E_\theta\, \phi(\underset{\sim}{x}) \le \alpha.$$

Testing a simple Null hypothesis against a simple alternative:

Let $(X_1, \dots, X_n)$ be a r.s. from one or other member of the parametric family $\{f_0(x), f_1(x)\}$. We wish to test $H_0: X_i \sim f_0(x)$ against $H_1: X_i \sim f_1(x)$.

[If the members of the parametric family $\{f_0(x), f_1(x)\}$ have the same probability law and $f_0(x) = f(x, \theta_0)$, $f_1(x) = f(x, \theta_1)$. Then the testing problem becomes $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$. Here $\Omega = \{\theta_0, \theta_1\}$ is the parameter space.]

Most Powerful test:— Let $C_\alpha = \left\{ \phi(\underset{\sim}{x}) : E_{f_0}\{\phi(x) \le \alpha\} \right\}$ be the class of all level α tests for testing H_0 against H_1. A test $\phi^*(\underset{\sim}{x}) \in C_\alpha$ is called most powerful test for testing H_0 against H_1 at level α iff

$$E_{f_1}[\phi^*(\underset{\sim}{x})] > E_{f_1}[\phi(x)], \ \forall \ \phi(\underset{\sim}{x}) \in C_\alpha .$$

Construction of MP test:— Let X be a r.s. from one or other member of the parametric family $\{f_0(x), f_1(x)\}$.

To test $H_0: X \sim f_0(x)$ against $H_1: X \sim f_1(x)$.

suppose this two distn are discrete. If at first, we restrict attention to non-randomised test, the optimum test is defined as a critical region W satisfying $\sum_{x \in W} f_0(x) \le \alpha$ and $\sum_{x \in W} f_1(x)$ is maximum. ——— (*)

To each point $x \in \mathscr{X}$, there are two attached value, its probabilities under $f_0(x)$ & $f_1(x)$.

The selected points in W are to have a total value not exceeding α under $f_0(x)$ and as large as possible under $f_1(x)$. The selected points in W should have $f_0(x)$ so that we can affort large no. of points under the restriction (*) and simultaneously should have large $f_1(x)$ so that $\sum_{x \in W} f_1(x)$ is as large as possible.

Here the most valuable points are those with the highest value of $r(x) = \dfrac{f_1(x)}{f_0(x)}$.

The points x's are therefore rated according to the value of the ratios $r(x)$ and select for W in this order, as many as one can afford under the restriction (*).

Formally, this means W is the set of all points x for which $r(x) > c$, where c is determined from $P_{H_0}[x \in W] = \sum_{x: r(x) > c} f_0(x) = \alpha$.

Here a difficulty may arise, it may happen that a certain point is included, the value α has not yet being reached, but it would be exceeded if the next point was also included. The next value α can be achieved by permitting randomisation.

Ex.(1):- Let X be a RV with PMF under H_0 and under H_1 are given by

x	1	2	3	4	5	6
$f_0(x)$	0.01	0.01	0.01	0.01	0.01	0.95
$f_1(x)$	0.05	0.01	0.03	0.04	0.02	0.85

Find a MP test for testing $H_0: X \sim f_0(x)$ against $H_1: X \sim f_1(x)$ at level $\alpha = 0.03$.

Solution:-

x	1	2	3	4	5	6
$r(x) = \frac{f_1(x)}{f_0(x)}$	5	1	3	4	2	0.89

Here $r(1) > r(4) > r(3) > r(5) > r(2) > r(6)$.

Here $\alpha = 0.03$

Then $x = 1$ is the first point to fall in the critical region W; $x = 4$ is the second point, $x = 3$ is the 3rd to go, etc., such that $\sum_{x \in W} f_0(x) = \alpha = 0.03$

Note that, $f_0(1) + f_0(4) + f_0(3)$

$$= 0.01 + 0.01 + 0.01$$
$$= 0.03$$

Hence, $W = \{1, 4, 3\}$ is a most powerful (MP) critical region for testing H_0 against H_1 at level $\alpha = 0.03$,

The MP critical region can be expressed as

$$W = \{x : r(x) > 2\}.$$

■ Neyman-Pearson MP Test :- The above consideration are formulated in the following lemma :

Fundamental Lemma of Neyman-Pearson :-

(I) Sufficiency Part :- For testing $H_0: X \sim f_0(x)$ against $H_1: X \sim f_1(x)$, any test $\phi(x)$ satisfying $E[\phi(x)/H_0] = \alpha$ .... (*)

and
$$\phi(x) = \begin{cases} 1 & \text{if } f_1(x) > k.f_0(x) \\ \gamma(x) & \text{if } f_1(x) = kf_0(x) \\ 0 & \text{if } f_1(x) < k.f_0(x) \end{cases} \quad \text{..........(**)}$$

for some $k \geq 0$, $0 \leq \gamma(x) \leq 1$ is MP for testing H_0 against H_1, at level α.

Proof :- Suppose that $\phi(x)$ is a test satisfying (*) and (**) and $\phi^*(x)$ is any test with $E[\phi^*(x) | H_0] \leq \alpha$. Denote by

$\mathscr{X}^+ = \{x : f_1(x) - kf_0(x) > 0\}$ and $\mathscr{X}^- = \{x : f_1(x) - kf_0(x) < 0\}$, the two subsets of the sample space $\mathscr{X}$.

Assume that X is a continuous R.V.

Note that
$$\int_{\mathscr{X}} \{\phi(x) - \phi^*(x)\}\{f_1(x) - kf_0(x)\} dx$$
$$= \int_{\mathscr{X}^+ \cup \mathscr{X}^-} \{\phi(x) - \phi^*(x)\}\{f_1(x) - kf_0(x)\} dx$$
$$\geq 0 \quad \text{.......... (1)}$$

[For $x \in \mathscr{X}^+$, $f_1(x) - kf_0(x) > 0$ and $\phi(x) - \phi^*(x) = \{1 - \phi^*(x)\} \geq 0$

For $x \in \mathscr{X}^-$, $f_1(x) - kf_0(x) < 0$ and $\phi(x) - \phi^*(x) = -\phi^*(x) \leq 0$.]

The difference between the power of $\phi(x)$ and $\phi^*(x)$ is

$$E[\phi(x)/H_1] - E[\phi^*(x)/H_1] = \int_{\mathscr{X}} \{\phi(x) - \phi^*(x)\} f_1(x) dx$$
$$> k. \int_{\mathscr{X}} \{\phi(x) - \phi^*(x)\} f_0(x) dx \quad \text{from (1).}$$
$$= k. \{E[\phi(x)/H_0] - E[\phi^*(x)/H_0]\} \geq 0$$

[$\because$ $E[\phi(x)/H_0] = \alpha$ and $E[\phi^*(x)/H_0] \leq \alpha$]

Hence, $E[\phi(X) | H_1] \geq E[\phi^*(X) / H_1]$, for any level α test $\phi^*(x)$.

Ex. (2):- Let $(X_1, \ldots, X_n)$ be a r.s. from
$$f(x,\theta) = \begin{cases} \theta e^{-\theta x} & , \text{if } x>0 \\ 0 & , \text{ow} \end{cases}$$
where, $\theta \in \Omega = \{\theta_0, \theta_1\}$, $\theta_0 < \theta_1$ are unknowns. Find an MP test for testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$ at level α.

Solution:- The PDF of $\underset{\sim}{X} = (X_1, \ldots, X_n)$ is
$$L(\underset{\sim}{x}, \theta) = \prod_{i=1}^{n} f(x_i, \theta) = \theta^n . e^{-\theta \sum_{i=1}^{n} x_i}, \text{ if } x_i > 0 \ \forall\, i = 1(1)n.$$
where, $\theta \in \Omega = \{\theta_0, \theta_1\}$

[To test: $H_0: \underset{\sim}{X} \sim L(\underset{\sim}{x}, \theta_0)$ against $H_1: \underset{\sim}{X} \sim L(\underset{\sim}{x}, \theta_1)$]

By N-P lemma, MP test for testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$, $(\theta_0 < \theta_1)$ is given by the critical function
$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & , \dfrac{L(\underset{\sim}{x}, \theta_1)}{L(\underset{\sim}{x}, \theta_0)} = K^* > K \\ \gamma & , K^* = K \\ 0 & , K^* < K \end{cases}$$
such that $E[\phi(\underset{\sim}{X}) \mid H_0] = \alpha$.

Note that $\dfrac{L(\underset{\sim}{x}, \theta_1)}{L(\underset{\sim}{x}, \theta_0)} > k$
$$\Rightarrow \left(\frac{\theta_1}{\theta_0}\right)^n . e^{-(\theta_1 - \theta_0) \sum_{i=1}^{n} x_i} > k$$
$$\Rightarrow -(\theta_1 - \theta_0) \sum_{i=1}^{n} x_i > k_1 \text{ , say}$$
$$\Rightarrow \sum_{i=1}^{n} x_i < c \text{ , say, as } \theta_0 < \theta_1$$
Here $\sum_{i=1}^{n} x_i \sim \text{Gamma}(n, \theta)$ is a continuous R.V. and
$$P_\theta\left[\frac{L(\underset{\sim}{x}, \theta_1)}{L(\underset{\sim}{x}, \theta_0)} = k\right] = P_\theta\left[\sum_{i=1}^{n} X_i = c\right] = 0 \ \forall\, \theta$$
Hence, $\phi(\underset{\sim}{x})$ reduces to
$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & , \text{if } \sum_{i=1}^{n} x_i < c \\ 0 & , \text{ow} \end{cases}$$
where c is such that
$$\begin{aligned} \alpha &= E[\phi(\underset{\sim}{X}) / H_0] \\ &= 1. P_{H_0}\left[\sum_{i=1}^{n} X_i < c\right] \\ &= P_{\theta = \theta_0}\left[2\theta_0 \sum_{i=1}^{n} X_i < 2\theta_0 c\right] \\ &= P_{\theta = \theta_0}\left[\chi^2_{2n} < 2\theta_0 c\right] \end{aligned}$$

$$\Rightarrow 2\theta_0 c = \chi^2_{1-\alpha, 2n}$$

$$\Rightarrow c = \frac{1}{2\theta_0} \cdot \chi^2_{1-\alpha, 2n}$$

[Here $X_i \sim$ Exp. with mean $\frac{1}{\theta}$, $i = 1(1)n$.

$$\Rightarrow 2\theta X_i \overset{iid}{\sim} \chi_2^2, \; i = 1(1)n$$

$$\Rightarrow 2\theta \sum_{i=1}^{n} X_i \sim \chi^2_{2n} \,.]$$

Hence, an MP test for testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$, $\theta_0 < \theta_1$ at level α is given by
$$\phi(\underset{\sim}{x}) = \begin{cases} 1, & \Sigma x_i < \frac{1}{2\theta_0} \chi^2_{1-\alpha, 2n} \\ 0, & \text{ow} \end{cases}$$

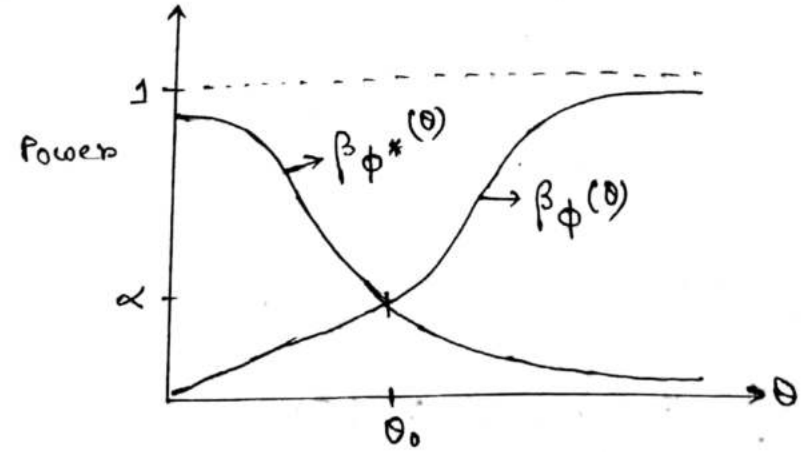
Remark:-

(1) Power function of the test is given by $\phi(\underset{\sim}{x})$ is
$$\beta_\phi(\theta) = E_\theta\{\phi(X)\}$$
$$= P\left[\sum_{i=1}^{n} x_i < \frac{\chi^2_{1-\alpha, 2n}}{2\theta_0}\right]$$
$$= P_\theta\left[2\theta \sum_{i=1}^{n} x_i < \left(\frac{\theta}{\theta_0}\right) \chi^2_{1-\alpha, 2n}\right]$$
$$= P_\theta\left[\chi^2_{2n} < \left(\frac{\theta}{\theta_0}\right) \cdot \chi^2_{1-\alpha, 2n}\right]$$
$$= F_{\chi^2_{2n}}\left(\frac{\theta}{\theta_0} \cdot \chi^2_{1-\alpha, 2n}\right)$$
which is increasing in θ.

(2) It can be shown that an MP test of $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$, $\theta_1 < \theta_0$, at level α is given by
$$\phi^*(\underset{\sim}{x}) = \begin{cases} 1, & \sum_{i=1}^{n} x_i > \frac{\chi^2_{\alpha, 2n}}{2\theta_0} \\ 0, & \text{ow} \end{cases}$$

The power function of the test $\phi^*(\underset{\sim}{x})$ is $\beta_{\phi^*}(\theta) = 1 - F_{\chi^2_{2n}}\left(\frac{\theta}{\theta_0} \chi^2_{\alpha, 2n}\right)$

which decreases as θ increases.



(3) Note that the critical ~~region~~ point $\frac{1}{2\theta_0}\chi^2_{1-\alpha,2n}$ and the corresponding test $\phi(x)$ depends only on θ_0 and the relative position of θ_1 w.r.t. θ_0 but not on the exact value of θ_1. That is, the MP test $\phi(x)$ is independent of θ_1 as long as $\theta_1 > \theta_0$. Hence, we could get the same most powerful test $\phi(x)$ for testing $H_0: \theta = \theta_0$ against any alternative $\theta_1 (> \theta_0)$. That is, the test $\phi(x)$ remains MP for testing $H_0: \theta = \theta_0$ for any $\theta_1 (> \theta_0)$. Therefore $\phi(x)$ is a uniformly MP test for testing $H_0: \theta = \theta_0$ against the composite alternative $H_1: \theta > \theta_0$.

Uniformly Most Powerful Test [UMP Test]:— We now define an optimum test for testing $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$.

Definition:— Let $C_\alpha = \{\phi(x) : E_{\theta_0}[\phi(X)] \leq \alpha\}$ be the class of all level α tests for testing $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$. A test $\phi^*(x) \in C_\alpha$ is called UMP for testing $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$ if

$$E_\theta[\phi^*(x)] \geq E_\theta[\phi(x)], \ \forall\ \theta > \theta_0 \text{ for all } \phi(x) \in C_\alpha.$$

Use of N-P lemma in finding UMP test for testing simple null hypothesis against composite alternative:—

Suppose to find a UMP test for testing $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$. By N-P lemma, find an MP test for testing the simple null $H_0: \theta = \theta_0$ against the simple alternative $H_1: \theta = \theta_1$ where $\theta_1 > \theta_0$. If the MP test, obtained is independent of the exact value of $\theta_1 (> \theta_0)$, then the MP test remains a most powerful test for testing $H_0: \theta = \theta_0$ against any $\theta > \theta_0$ and is therefore a UMP test for testing $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$.

Ex. (3):- Let $X_1, \ldots, X_n$ be a r.s. from $B(1, p)$, $p \in \Omega = \{p_0, p_1\}$. Find an MP test for testing $H_0: p = p_0$ against $H_1: p = p_1$, $p_1 > p_0$ at level α. Describe how randomization is applied to attain the exact size α.

Solution:— The PMF of $\underset{\sim}{X} = (X_1, \ldots, X_n)$ is

$$L(\underset{\sim}{x}, p) = p^{\sum_{i=1}^n x_i}(1-p)^{n - \sum_{i=1}^n x_i} \ ; x_i = 0, 1.$$

where, $p \in \Omega = \{p_0, p_1\}$.

By N-P lemma, an MP test for testing $H_0: p = p_0$ against $H_1: p = p_1$, $p_1 > p_0$, at level α is given by

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \dfrac{L(\underset{\sim}{x}, p_1)}{L(\underset{\sim}{x}, p_0)} = k^* > k \\ \gamma & \text{if } k^* = k \\ 0 & \text{if } k^* < k \end{cases}$$

such that $E[\phi(\underset{\sim}{x}) / H_0] = \alpha$

Now, $\dfrac{L(\underset{\sim}{x}, p_1)}{L(\underset{\sim}{x}, p_0)} > k$

$$\Rightarrow \left\{\frac{p_1(1-p_0)}{p_0(1-p_1)}\right\}^{\sum_{i=1}^{n} x_i} \left(\frac{1-p_1}{1-p_0}\right)^n > k$$

$$\Rightarrow \left(\sum_{i=1}^{n} x_i\right) \cdot \ln\left\{\frac{p_1(1-p_1)}{p_0(1-p_0)}\right\} > k_1$$

$$\Rightarrow \sum_{i=1}^{n} x_i > c \left[\text{Here } p_1 > p_0 \text{ and } 1-p_0 > 1-p_1 \right.$$

$$\Rightarrow \frac{p_1(1-p_0)}{p_0(1-p_1)} > 1$$

$$\left. \Rightarrow \ln\left\{\frac{p_1(1-p_1)}{p_0(1-p_0)}\right\} > 0 \right]$$

Hence, $\phi(\underset{\sim}{x}) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} x_i > c \\ \gamma, & \text{if } \sum x_i = c \\ 0, & \text{if } \sum x_i < c \end{cases}$

where c and γ are determined from

$$\alpha = E[\phi(\underset{\sim}{x}) / H_0] = 1 \cdot P_{H_0}\left[\sum_{i=1}^{n} x_i > c\right] + \gamma \cdot P_{H_0}\left[\sum_{i=1}^{n} x_i = c\right]$$

Since, $\sum_{i=1}^{n} x_i \sim Bin(n, p)$ is a discrete RV, there may not exist a 'c' $\ni$ $P_{H_0}\left[\sum_{i=1}^{n} x_i > c\right] = \alpha$ is true. Then, randomization is required on the boundary set $\left\{\underset{\sim}{x} : \sum_{i=1}^{n} x_i = c\right\}$.

Then there exists c_1 such that $P_{p_0}\left[\sum_{i=1}^{n} x_i > c_1\right] = \alpha_1 < \alpha < \alpha_2$,

where, $\alpha_2 = P_{p_0}\left[\sum_{i=1}^{n} x_i > c_1 - 1\right]$

In this case, we have $c = c_1$,

and $\alpha = P_{p_0}\left[\sum_{i=1}^{n} x_i > c_1\right] + \gamma \cdot P_{p_0}\left[\sum_{i=1}^{n} x_i = c_1\right]$

$$\therefore \alpha = \alpha_1 + \gamma(\alpha_2 - \alpha_1)$$

$$\Rightarrow \gamma = \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1}.$$

Hence, MP test for testing $H_0 : p = p_0$ against $H_1 : p = p_1$, $p_1 > p_0$, at level α is given by

$$\phi(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i > c \\ \dfrac{\alpha - \alpha_1}{\alpha_2 - \alpha_1}, & \text{if } \sum_{i=1}^{n} x_i = c_1 \\ 0, & \text{if } \sum_{i=1}^{n} x_i < c. \end{cases}$$

where, $P_{p_0}\left[\sum_{i=1}^{n} x_i > c\right] = \alpha_1 < \alpha < \alpha_2 = P_{p_0}\left[\sum_{i=1}^{n} x_i > c - 1\right]$

Remark:-

①. Power function of the test is given by $\phi(\underset{\sim}{x})$ is

$$\beta_\phi(p) = E_p\left[\phi(\underset{\sim}{X})\right]$$
$$= P_p\left[\sum_{i=1}^{n} X_i > c_1\right] + \frac{\alpha-\alpha_1}{\alpha_2-\alpha_1} P_p\left[\sum_{i=1}^{n} X_i = c_1\right]$$
$$= \frac{\int_0^p u^{c_1-1}(1-u)^{n-c_1}\,du}{B(c_1, n-c_1+1)} + \left(\frac{\alpha-\alpha_1}{\alpha_2-\alpha_1}\right) P_p\left[\sum_{i=1}^{n} X_i = c_1\right]$$

which is increasing in p.

②. Note that the MP test is given by $\phi(\underset{\sim}{x})$ depends only on the relative position of p_1 w.r.t. p_0 but not on the exact value of p_1, that is $\phi(\underset{\sim}{x})$ is independent of p_1 as long as $p_1 > p_0$. Therefore it remains MP at level α test for testing $H_0: p = p_0$ against any alternative $p_1 > p_0$ and is therefore a UMP test for testing $H_0: p = p_0$ against $H_1: p > p_0$.

Ex.(4):- Let $X_1, \ldots, X_n$ be r.s. from $N(0, \sigma^2)$. Find an MP test for testing $H_0: \sigma = \sigma_0$ against $H_1: \sigma = \sigma_1$, $\sigma_1 > \sigma_0$, at level α. Also suggest a UMP test for testing $H_0: \sigma = \sigma_0$ against $H_1: \sigma > \sigma_0$.

Solution:- The PDF of $\underset{\sim}{X}$ is

$$L(\underset{\sim}{x}; \sigma^2) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2} \quad ; x_i \in \mathbb{R}$$

By N-P lemma, an MP test at level α of $H_0: \sigma = \sigma_0$ against $H_1: \sigma = \sigma_1$, $\sigma_1 > \sigma_0$ is given by

$$\phi(\underset{\sim}{x}) = \begin{cases} 1, & \text{if } \frac{L(\underset{\sim}{x}, \sigma_1^2)}{L(\underset{\sim}{x}, \sigma_0^2)} = k^* > k \\ \gamma, & \text{if } k^* = k \\ 0, & \text{if } k^* < k \end{cases}$$

such that $E\left[\phi(\underset{\sim}{x}) / H_0\right] = \alpha$

Now, $\frac{L(\underset{\sim}{x}, \sigma_1^2)}{L(\underset{\sim}{x}, \sigma_0^2)} > k$

$$\Rightarrow \left(\frac{\sigma_0}{\sigma_1}\right)^n \cdot e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2 \cdot \left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)} > k$$

$$\Rightarrow -\frac{1}{2}\sum_{i=1}^{n} x_i^2 \left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right) > k_1$$

$$\Rightarrow \sum_{i=1}^{n} x_i^2 > c.$$

$\left[\begin{array}{l} \text{Here } \sigma_1 > \sigma_0 \\ \Rightarrow \frac{1}{\sigma_1^2} < \frac{1}{\sigma_0^2} \\ \Rightarrow -\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right) > 0 \end{array}\right]$

Also, $P_{\sigma}\left[\dfrac{L(\underset{\sim}{x},\sigma_1^2)}{L(\underset{\sim}{x},\sigma_0^2)}=k\right]$

$$=P_{\sigma}\left[\sum_{i=1}^{n}x_i^2=c\right]=0$$

Since, $\sum_{i=1}^{n}x_i^2\sim\sigma^2\chi^2_n$, a continuous distn..

Hence, $\phi(\underset{\sim}{x})=\begin{cases}1 & ,\text{if } \sum x_i^2>c\\ 0 & \text{OW}\end{cases}$

where, c is determined from

$$\alpha=E\left[\phi(\underset{\sim}{x})/H_0\right]=P_{\sigma_0}\left[\sum_{i=1}^{n}x_i^2>c\right]$$

$$=P_{\sigma_0}\left[\frac{\sum_{i=1}^{n}x_i^2}{\sigma^2}>\frac{c}{\sigma_0^2}\right]$$

$$=P\left[\chi^2_n>\frac{c}{\sigma_0^2}\right]$$

$$\Rightarrow c=\sigma_0^2\chi^2_{\alpha,n}$$

Hence, an MP test at level α of $H_0:\sigma=\sigma_0$ against $H_1:\sigma=\sigma_1$, $\sigma_1>\sigma_0$, is given by

$$\phi(\underset{\sim}{x})=\begin{cases}1, & \text{if } \sum_{i=1}^{n}x_i^2>\sigma_0^2\chi^2_{\alpha,n}\\ 0, & \text{OW}\end{cases}$$

Hence, $\phi(\underset{\sim}{x})$ depends on σ_0 and the relative position σ_1 w.r.t. σ_0 but not on the exact value of σ_1. Hence, $\phi(\underset{\sim}{x})$ remains MP for testing H_0 such that $\sigma=\sigma_0$ against any alternative $\sigma_1(>\sigma_0)$. Hence, $\phi(\underset{\sim}{x})$ is UMP test for testing $\sigma=\sigma_0$ against $H_1:\sigma>\sigma_0$ at level α.

<u>Ex.(5)</u>:- Let $X_1,X_2,\dots,X_n$ be r.s. from $N(\mu,\sigma_0^2)$, $\mu\in\Omega=\{\mu_0,\mu_1\}$ and σ_0 is known. Find an MP level α test of $H_0:\mu=\mu_0$ against $\mu=\mu_1(>\mu_0)$. Hence suggest an UMP test at level α for testing $H_0:\mu=\mu_0$ against $H_1:\mu>\mu_0$.

Ex.(6):- Suppose the no. of system failure in each month has a $P(\lambda)$ distribution. The no. of such failure was observed for n months. Find an MP level α test for testing $H_0: \lambda = 2$ against $H_1: \lambda = 4$. Also suggest a UMP level α test for testing $H_0: \lambda = 2$ against $H_1: \lambda > 2$.

Ex.7. Let $(X_1, \ldots, X_5)$ be a r.s. from the distn with PMF

$$f_0(x) = \begin{cases} \theta^x (1-\theta)^{1-x} & ; x = 0,1 \\ 0 & ; \text{ow} \end{cases} \quad (0<\theta<1)$$

Show that there does not exist an MP CR of size $\alpha = 0.10$ for testing $H_0: \theta = 0.6$ Vs. $H_1: \theta = \theta_1 \ (>0.6)$.

Solution:-

$$T = \sum_{i=1}^{5} X_i, \quad \theta = 0.6$$

t	$P_{\theta_0}[T=t]$	$P_{\theta_0}[T \geq t]$
0	0.01024	1.00000
1	0.07680	0.98976
2	0.23040	0.91296
3	0.34560	0.68256
4	0.25920	0.33696
5	0.07776	0.07776

$T_{H_0} \sim \text{Bin}(5, 0.6)$ $\quad H_0: \theta = \theta_0 = 0.6$

$\quad H_1: \theta = 0.7 \ (>0.6)$

$$f_\theta(x_1, \ldots, x_5) = \begin{cases} \theta^{\sum x_i} (1-\theta)^{n - \sum x_i} & ; x_i = (0,1) ; i = 1(1)5 \\ 0 & ; \text{ow} \end{cases}$$

$$\therefore \quad f_1(\underset{\sim}{x}) > k f_0(\underset{\sim}{x})$$

$$\Rightarrow \frac{\theta_1^{\sum x_i} (1-\theta_0)^{\sum x_i}}{\theta_0^{\sum x_i} (1-\theta_1)^{\sum x_i}} > k'$$

$$\Rightarrow \sum_{i=1}^{5} x_i > k'', \text{ say.}$$

i.e. $T \geq c$, where c is determined as $P_{H_0}[T \geq c] = 0.10$

But $P_{H_0}[T \geq 5] = 0.07776$, $P_{H_0}[T \geq 4] = 0.33696$

$\therefore \nexists$ any $c \ni P_{H_0}[T \geq c] = 0.10$.

To get an MP test of exact size $\alpha = 0.10$, we randomize when $T = 4$ is observed and the corresponding test is

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } T > 4 \\ \gamma & \text{if } T = 4 \\ 0 & \text{if } T < 4 \end{cases}$$

where γ is $\ni$

$$E_{H_0}[\phi(\underset{\sim}{X})] = 0.10$$

i.e. $P_{H_0}[T > 4] + \gamma . P_{H_0}[T = 4] = 0.10$

i.e. $0.68256 + \gamma . 0.25920 = 0.10$

$\Rightarrow \gamma =$ (Ans)

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Ex.(8):- Let X be a single observation from one or other member of the family $\{p_0(x), p_1(x)\}$; where

$$p_0(x) = \begin{cases} \left(\frac{1}{2}\right)^{x+1} & , x = 0,1,2,\ldots \\ 0 & , \text{ow} \end{cases}$$

and $$p_1(x) = \begin{cases} \frac{1}{4}\left(\frac{3}{4}\right)^{x} & , x = 0,1,2,\ldots \\ 0 & , \text{ow} \end{cases}$$

Find an MP test of $H_0 : X \sim p_0(x)$ against $H_1 : X \sim p_1(x)$ at level $\alpha = 0.05$.

Solution:- Note that the testing problem reduces to $H_0 : p = \frac{1}{2}$ against $H_1 : p = \frac{1}{4}$.

By N-P lemma, an MP test of $H_0 : X \sim p_0(x)$ against $H_1 : X \sim p_1(x)$ at level α is given by

$$\phi(x) = \begin{cases} 1 & , \text{if } \frac{p_1(x)}{p_0(x)} = k^* > k \\ \gamma & , \text{if } k^* = k \\ 0 & , \text{if } k^* < k \end{cases}$$

such that $E[\phi(X)/H_0] = \alpha$.

Note that $\frac{p_1(x)}{p_0(x)} > k$

$$\Rightarrow \frac{3^x}{2^{x+1}} > k$$

$$\Rightarrow \left(\frac{3}{2}\right)^x > 2k$$

$$\Rightarrow x > c \text{ as } \ln\left(\frac{3}{2}\right) > 0.$$

Hence, $$\phi(x) = \begin{cases} 1 & \text{if } x > c \\ \gamma & \text{if } x = c \\ 0 & \text{if } x < c \end{cases}$$

where c and γ are such that

$$\alpha = E[\phi(X) \mid H_0]$$
$$= 1 . P_{p_0}[X > c] + \gamma P_{p_0}[X = c]$$

Now, $$P_{p_0}[X > c] = \sum_{x > c} p_0(x)$$
$$= \sum_{x=c+1}^{\infty} \left(\frac{1}{2}\right)^{x+1} = \frac{\left(\frac{1}{2}\right)^{c+2}}{1 - \frac{1}{2}} = \left(\frac{1}{2}\right)^{c+1}$$

Note that, $P_{p_0}[X > 4] = \frac{1}{2^5} < \alpha = \frac{1}{20} < \frac{1}{24} = P_p[X > 3]$

Thus, select $c=4$, and then

$$P_{p_0}[X>4] + \gamma . P_{p_0}[X=4] = \alpha = \frac{1}{20}$$

$$\Rightarrow \frac{1}{32} + \gamma . \frac{1}{32} = \frac{1}{20}$$

$$\Rightarrow \gamma = \frac{3}{5}.$$

Hence an MP test of H_0 against H_1 at level $\alpha = 0.05$ is

$$\phi(x) = \begin{cases} 1 & , x>4 \\ 3/5 & , x=4 \\ 0 & , x<4 \end{cases} = \begin{cases} 1 & , x=5,6,\ldots \\ 0.6 & , x=4 \\ 0 & , x=0,1,2,3. \end{cases}$$

Remark:- Consider a test for testing

$H_0 : X \sim f_0(x)$ against $H_1 : X \sim f_1(x)$, say

$$\phi(x) = \begin{cases} 1 & , \frac{f_1(x)}{f_0(x)} > k \\ 0 & , \text{ow} \end{cases}$$

for a given k and then $E[\phi(x)/H_0] = \alpha$, say.

By a sufficient part of NP lemma, $\phi(x)$ is an MP test for testing $H_0 : X \sim f_0(x)$ against $H_1 : X \sim f_1(x)$ at level

$E[\phi(X)/H_0] = \alpha$, say.

Then $\phi_1(x) = \begin{cases} 1, & \frac{f_1(x)}{f_0(x)} > k, \\ 0, & \text{ow}. \end{cases}$

is an MP test for testing $H_0 : X \sim f_0(x)$ against $H_1 : X \sim f_1(x)$ at level $E[\phi_1(x)/H_0] = \alpha$, say and we say that $\phi_1(x)$ is an MP test of H_0 against H_1 of its size.

Ex.(9):-

(a) Find an MP test for testing $H_0 : X \sim f_0(x)$ against $H_1 : X \sim f_1(x)$ based on a sample of size one from $\{f_0(x), f_1(x)\}$ where

$$f_0(x) = \begin{cases} \frac{e^{-1}}{x!} & , x=0,1,2,\ldots \\ 0 & , \text{ow} \end{cases}$$

and $$f_1(x) = \begin{cases} \frac{1}{2^{x+1}} & , x=0,1,2,\ldots \\ 0 & , \text{ow} \end{cases}$$

Solution:- By N-P lemma, for a given value of k,

the test $\phi(x) = \begin{cases} 1, & \text{if } \frac{f_1(x)}{f_0(x)} > k \\ 0, & \text{ow} \end{cases}$

is an MP test for testing $H_0 : X \sim f_0(x)$ against $H_1 : X \sim f_1(x)$ of its size.

Note that, $\gamma(x) = \frac{f_1(x)}{f_0(x)} = \frac{\lfloor x}{2^x} \cdot \frac{e}{2}.$

and $\frac{\gamma(x)}{\gamma(x-1)} = \frac{x}{2} \gtreqless 1$ according as $x \gtreqless 2$.

Clearly, $\gamma(0) > \gamma(1) = \gamma(2) < \gamma(3) < \gamma(4) < \cdots\cdots$

Then $x = 1$ or 2, are the last point to fall into the critical region, and $\gamma(0) > \gamma(3)$, $x = 3$ is the 3rd last point to go; as $\gamma(0) < \gamma(4)$, $x = 0$ is the 4th last point, etc.

Hence, $\phi(x) = \begin{cases} 1, & x \neq 1,2,3,0 \quad \text{or} \quad x \in \mathscr{X} - \{1,2,3,0\} \\ 0, & \text{OW} \end{cases}$

is an MP test of H_0 against H_1 of its size $= E[\phi(X)/H_0]$

$$= 1 - \left(\sum_{x=0}^{3} \frac{e^{-1}}{x!}\right) = 1 - \frac{8}{3}e^{-1}.$$

Again, $\phi_1(x) = \begin{cases} 1, & x \neq 1,2,3,0,4 \\ 0, & \text{OW} \end{cases}$

is an MP test of H_0 against H_1 of its size $= E[\phi_1(x)/H_0]$

$$= 1 - \left(\sum_{x=0}^{4} \frac{e^{-1}}{x!}\right)$$

$$= 1 - \frac{65}{24}e^{-1}.$$

(b) [Continuation]

Show that $W = \left\{x : \frac{2^x}{x!} < \frac{e}{2}\right\}$

is an MP critical region for testing H_0 against H_1. Also, show that the power of the test is greater than the size.

Hints:-

$W = \{x : r(x) > 1\}$

$= \left\{x : \frac{x!}{2^x} \cdot \frac{e}{2} > 1\right\}$

$= \left\{x : \frac{2^x}{x!} < \frac{2}{e}\right\}$

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Ex. (10) :- Suppose our problem is to test $H_0: X \sim P_0(x)$ against $H_1: X \sim P_1(x)$, where

x	$P_0(x)$	$P_1(x)$
0	1/40	4/5
1	15/40	1/10
2	1/5	1/20
3	2/5	1/40
4	0	1/40

Find an MP test for testing H_0 against H_1 of its size.

Ex. (11):- Let $X_1, \ldots, X_n$ be a r.s. from geometric distribution with p.m.f.

$$f_X(x) = \begin{cases} p(1-p)^x & , x = 0, 1, 2, \ldots \\ 0 & , \text{ow} \end{cases}$$

where, $p \in \Omega = \{p_0, p_1\}$, $p_0 < p_1$.

Find an MP test of $H_0: p = p_0$ against $H_1: p = p_1$, $p_1 > p_0$ at level α. Also, show that the test can be carried out using binomial distribution.

Ex.(12):- Let X be a single observation from the PDF

$$f(x;\theta) = \left\{ \frac{1}{\pi\{1+(x-\theta)^2\}} \right. , x \in \mathbb{R}$$

Show that the test $\phi(x) = \begin{cases} 1, & \text{if } 1<x<3 \\ 0 & \text{OW} \end{cases}$

is an MP test for testing $H_0: \theta=0$ against $H_1: \theta=1$ of its size.

Solution:- For a particular value of K, the test

$$\phi(x) = \begin{cases} 1, & \frac{f(x,1)}{f(x,0)} > K \\ 0, & \text{OW} \end{cases}$$

is an MP test of $H_0: \theta=0$ against $H_1: \theta=1$ of its size, by NP lemma,

Now, $\frac{f(x,1)}{f(x,0)} > K \Rightarrow \frac{1+x^2}{1+(x-1)^2} > K$

$$\Rightarrow x^2(K-1) - 2Kx + (2K-1) < 0$$

[If $(K-1)>0$, $x^2 - \frac{2K}{(K-1)}x + \frac{2K-1}{K-1} < 0$

$$\Rightarrow (x-\alpha)(x-\beta) < 0$$

where, $\alpha+\beta = \frac{2K}{K-1}$, and $\alpha\beta = \frac{2K-1}{K-1}$

$$\Rightarrow \alpha < x < \beta$$

In the given MP test $\alpha=1, \beta=3$.

Hence, $1+3 = \frac{2K}{K-1} \Rightarrow K=2$]

Set, $K=2$, $\frac{f(x,1)}{f(x,0)} > 2$

$$\Rightarrow 1<x<3.$$

For $K=2$, the test $\phi(x) = \begin{cases} 1, & 1<x<3 \\ 0, & \text{OW} \end{cases}$

is an MP test of H_0 against H_1 of its size

$$= E[\phi(X)/H_0] = P[1<X<3/\theta=0]$$

$$= \int_1^3 \frac{1}{\pi(1+x^2)}dx = \frac{1}{\pi}[\tan^{-1}x]_1^3$$

$$= \frac{1}{\pi}\left[\tan^{-1}3 - \tan^{-1}1\right]$$

$$= \frac{1}{\pi}\tan^{-1}\left(\frac{3-1}{1+3.1}\right)$$

$$= \frac{1}{\pi}\tan^{-1}\left(\frac{1}{2}\right).$$

Ex. (13):– Find an MP test of testing H_0 such that $H_0: X \sim f_0(x)$ against $H_1: X \sim f_1(x)$ of its size, where

$$f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \; x \in \mathbb{R}$$

$$f_1(x) = \frac{1}{2} e^{-|x|}, \; x \in \mathbb{R}$$

S.T. the power of the test is greater than its size.

Solution:– By N-P lemma, for a particular value of K, the test

$$\phi(x) = \begin{cases} 1, & \frac{f_1(x)}{f_0(x)} > K \\ 0, & \text{ow} \end{cases}$$

is an MP test of H_0 against H_1 of its size.

Now, $\frac{f_1(x)}{f_0(x)} > K$

$$\Rightarrow e^{\frac{1}{2}\{x^2 - 2|x|\}} > K_1$$

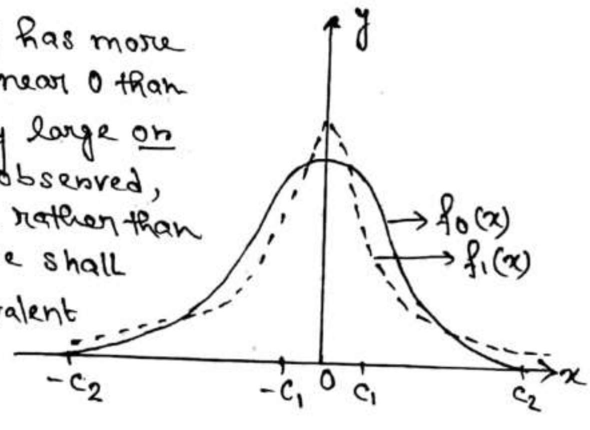
$$\Rightarrow e^{\frac{1}{2}\{(|x|-1)^2 - 1\}} > K_1$$

$$\Rightarrow (|x|-1)^2 > K_2^2, \; K_2 > 0$$

$$\Rightarrow |x|-1 < -K_2 \text{ or } |x|-1 > K_2$$

$$\Rightarrow |x| < c_1 \text{ or } |x| > c_2$$

[Alternative:– Note that $f_1(x)$ has more probability in its tails and near 0 than $f_0(x)$ has. If either a very large or very small value of x is observed, we suspect that H_1 is true rather than H_0. For some c_1 and c_2, we shall reject H_0 iff $\frac{f_1(x)}{f_0(x)} > K$ equivalent to $|x| < c_1$ or $|x| > c_2$.]



Hence, for some c_1 and c_2, the test

$$\phi(x) = \begin{cases} 1, & |x| < c_1 \text{ or } |x| > c_2 \\ 0, & \text{ow} \end{cases}$$

is an MP test of H_0 against H_1 of its size.

Note that,
$$\beta_\phi(f_1) = P_{f_1}[|X| < c_1 \text{ or } |X| > c_2]$$
$$= \int_W f_1(x)\,dx, \quad W = \{x : |x| < c_1 \text{ or } |x| > c_2\}$$
$$> \int_W f_0(x)\,dx, \quad \text{as } f_1(x) > f_0(x) \; \forall x \in W$$
$$= P_{f_0}[|X| < c_1 \text{ or } |X| > c_2]$$
$$= \beta_\phi(f_0). \quad \text{(Proved)}$$

Ex.(14):- Find an MP test of $H_0: X \sim N(0, \frac{1}{2})$ against $H_1: X \sim C(0,1)$ of its size.

Solution:- For a given K, the test

$$\phi(x) = \begin{cases} 1, & \frac{f_1(x)}{f_0(x)} > K \\ 0, & \text{ow} \end{cases}$$

is an MP test of H_0 against H_1 of its size, By N-P lemma,

Note that, $\frac{f_1(x)}{f_0(x)} > K$

$$\Rightarrow \frac{e^{x^2}}{1+x^2} > K_1, \text{ say}$$

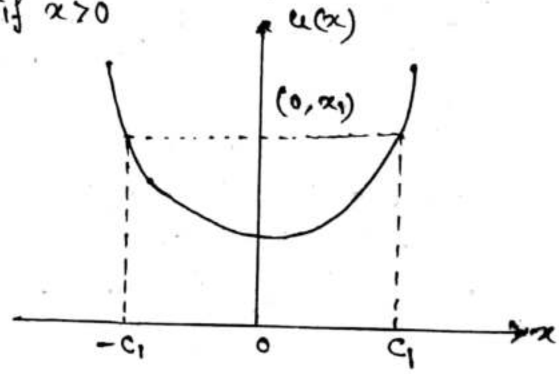
Let $u(x) = \frac{e^{x^2}}{1+x^2}$

Now, $u'(x) = \frac{(1+x^2)e^{x^2}.2x - e^{x^2}.2x}{(1+x^2)^2}$

$$= \frac{2x^3.e^{x^2}}{(1+x^2)^2}$$

$$\left[u'(0) = 0 \Rightarrow 2x^3.e^{x^2} = 0 \Rightarrow x = 0 \text{ or } e^{x^2} = 0 \Rightarrow x^2 = -\infty \right]$$

$$= \begin{cases} < 0, & \text{if } x < 0 \\ > 0, & \text{if } x > 0 \end{cases}$$



From the graph, $u(x) > K_1$

$$\Leftrightarrow |x| > c_1$$

Hence, for a particular value of c_1, the test

$$\phi(x) = \begin{cases} 1, & |x| > c_1 \\ 0, & \text{ow} \end{cases}$$

is an MP test of H_0 against H_1 of its size.

Ex.(15):- Find an MP test at level $\alpha = 0.05$ for testing $H_0: X \sim N(0,1)$ against $H_1: X \sim C(0,1)$.

Solution:- For a given K, the test $\phi(x) = \begin{cases} 1, & \frac{f_1(x)}{f_0(x)} > K \\ 0, & \text{ow} \end{cases}$

is an MP test of H_0 against H_1 of its size, by NP lemma.

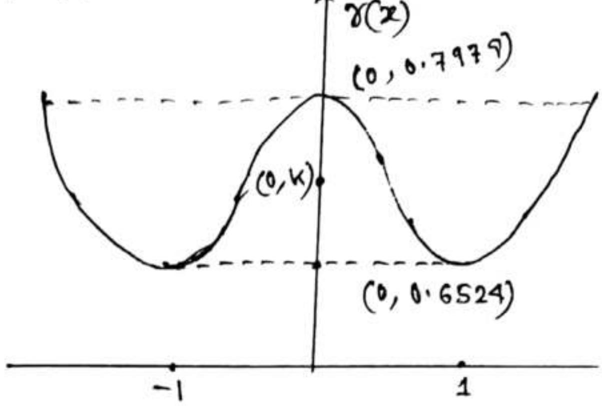
Note that, $r(x) = \frac{f_1(x)}{f_0(x)} > K$

$$\Rightarrow \frac{e^{x^2/2}}{1+x^2} > K_1, \text{ say}.$$

Let $u(x) = \dfrac{e^{x^2/2}}{1+x^2}$

Note that, $u'(x) = \dfrac{x(x^2-1)\, e^{x^2/2}}{(1+x^2)^2}$

Now, $u'(x) = \begin{cases} <0, & x<-1 \\ >0, & -1<x<0 \\ <0, & 0<x<1 \\ >0, & x>1 \end{cases}$



[For $K > 0.7979$, then the critical region:
$|x| > c_2$ with size < 0.1118.
For $0.6524 \le K \le 0.7979$,
then critical region:
$|x| < c_1$ or $|x| > c_2$ with
size $\in (0.1118, 0.3913)$
For, $K < 0.6524$, the critical region: $x \in \mathbb{R}$ with size $= 1$.]

For $\alpha = 0.05$, a small quantity, then $u(x) > K_1$, where K_1 is such that $P[u(x) > K_1 / H_0] = 0.05$ and from the graph

$u(x) > K_1 \Leftrightarrow |x| > c_2$.

Hence, $\phi(x) = \begin{cases} 1, & |x| > c_2 \\ 0, & \text{ow} \end{cases}$

is an MP test of H_0 against H_1 at level $\alpha = 0.05$, where

$$0.05 = P\left[|X| > c_2 / H_0\right]$$
$$= P\left[|X| > c_2 / X \sim N(0,1)\right]$$
$$= 2\left[1 - \Phi(c_2)\right]$$
$$\Rightarrow 1 - \Phi(c_2) = 0.025 = 1 - \Phi(\tau_{0.025})$$
$$\Rightarrow c_2 = \tau_{0.025} = 1.96.$$

Hence, $\phi(x) = \begin{cases} 1, & |x| > 1.96 \\ 0, & \text{ow} \end{cases}$

is an MP test for testing $H_0: X \sim N(0,1)$ against $H_1: X \sim C(0,1)$ at level $\alpha = 0.05$.

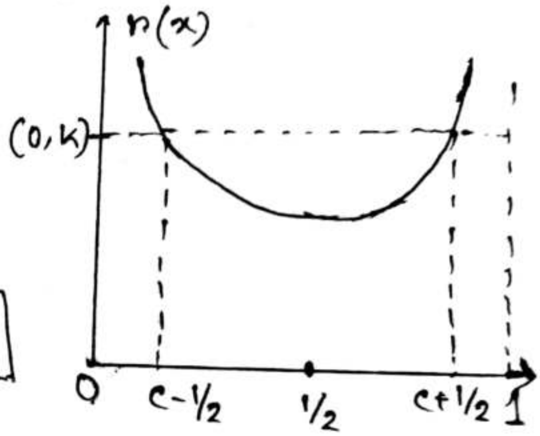
Ex.(18):- Let $f_0(x) = \begin{cases} 4x & , 0 < x < \frac{1}{2} \\ 4(1-x) & , \frac{1}{2} \le x < 1 \\ 0 & , \text{OW} \end{cases}$

and $f_1(x) = \begin{cases} 1 & , 0 < x < 1 \\ 0 & , \text{OW} \end{cases}$

Find an MP test of level α of $H_0 : X \sim f_0(x)$ against $H_1 : X \sim f_1(x)$. Find the power of this test.

Hints:- Note that

$$r(x) = \frac{f_1(x)}{f_0(x)} = \begin{cases} \frac{1}{4x} & , 0 < x < \frac{1}{2} \\ \frac{1}{4(1-x)} & , \frac{1}{2} \le x < 1 \end{cases}$$



Now, $r(x) > k$

$\Rightarrow x < \frac{1}{2} - c$ or $x > \frac{1}{2} + c$

$\Rightarrow |x - \frac{1}{2}| > c$.

[Note that, $r(x) = \frac{1}{4\{\frac{1}{2} - |x - \frac{1}{2}|\}}$, $0 < x < 1$]

Ex. (17):- Let $f(x,\theta) = \begin{cases} 2\theta . x + 2(1-\theta)(1-x) & , 0 < x < 1 \\ 0 & , \text{ow} \end{cases}$

where, $\theta \in \Omega = \{\theta_0, \theta_1\}$, $\theta_0 < \theta_1$.
Find an MP test for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ of its size.

Ex. (18):- Let $X_1, \dots, X_n$ be a r.s. from the PDF

$$f(x;\lambda) = \begin{cases} \dfrac{\lambda}{x^{\lambda+1}} & , x > 1 \\ 0 & , \text{OW} \end{cases}$$

Find an MP test for testing $H_0: \lambda = \lambda_0$ against $H_1: \lambda < \lambda_0$ at level α.

Ex. (19):- Let $H_0 : X \in f_0(x)$ against $H_1 : X \in f_1(x)$; where

x	1	2	3	4	5
$f_0(x)$	$1/5$	$1/5$	$1/5$	$1/5$	$1/5$
$f_1(x)$	$1/6$	$1/4$	$1/6$	$1/4$	$1/6$

Obtain an MP test of its size. For a given size, is it unique?

Devising the best test :-

In a non-sequential testing procedure usually the best test is obtained by maximizing the probability of rejecting a false hypothesis (i.e. power) subject to the condition that the probability of rejecting a hypothesis when it is true (i.e. size) lies below a certain level or equal to some $\alpha \in (0,1)$, called level of significance specified value.

Simple Vs. Simple Hypothesis & Simple Vs Composite Hypothesis $\longrightarrow$ NP test

Composite Vs. Composite Hypothesis $\longrightarrow$ LR test

If NP test & LR test both exist then they will be identical.

Limitations of Neyman Pearson Lemma :-

1) sample size is pre-determined (i.e. non-sequential).
2) It gives optimum tests for testing simple Vs Simple alternative hypothesis.

However, optimum tests for testing Simple Vs. Composite alternative may be obtained by using this lemma.

Result :- If a sufficient statistic T exists, then the NP test will be a function of the sufficient statistic.

Proof :- Since T is sufficient,

i.e. we can write $f_\theta(x_1, \ldots, x_n) = g_\theta(t)\, h(x_1, \ldots, x_n)$

for $\theta \in \Theta = \{\theta_0, \theta_1\}$.

Consequently, we write,

$$f_1(x_1, \ldots, x_n) > k.\, f_0(x_1, \ldots, x_n)$$

$$\Leftrightarrow g_{\theta_1}(t)\, h(\underset{\sim}{x}) > k\, g_{\theta_0}(t)\, h(\underset{\sim}{x})$$

$$\Leftrightarrow \frac{g_{\theta_1}(t)}{g_{\theta_0}(t)} > k.$$

For testing $H_0: \theta = \theta_0$ Vs. $H_1: \theta = \theta_1$ the NP lemma has the BCR

$$W_0 = \left\{ \underset{\sim}{x} \;\middle|\; f_1(\underset{\sim}{x}) > k f_0(\underset{\sim}{x}) \right\}$$

$$= \left\{ t \;\middle|\; \frac{g_{\theta_1}(t)}{g_{\theta_0}(t)} > k \right\} \longrightarrow \text{a function of the statistic (sufficient) } T.$$

Hence, the result is proved.

(II) Necessary Part of NP lemma :—

If $\phi(\underset{\sim}{x})$ is an MP test at level α, for testing $H_0: X \sim f_0(x)$ against $H_1: X \sim f_1(x)$, then for some $K \geq 0$, it satisfies

$$\phi(x) = \begin{cases} 1, & f_1(x) > K \cdot f_0(x) \\ 0, & f_1(x) < K f_0(x) \end{cases} \quad \text{———} (*)$$

It also satisfies $E\left[\phi(X)/H_0\right] = \alpha$ ———— (**)

unless there exists a test of size $< \alpha$ with power 1.

In the process of contructing an MP test, it is possible to reach at a critical region with power = 1, before reaching at size = α. Then a test of size less than α, will be an MP test. An MP level α test may not be unique but it is always possible to find an MP test with size = α.

It is important to note that MP test is uniquely determined by (*) and (**) except on the set $\{x : f_1(x) = K f_0(x)\}$. On this set, $\phi(\cdot)$ can be defined arbitarily provided the resulting test has size α and consequently $\phi(x)$ may not be unique. Actually it is always possible to define ϕ to be constant over this boundary set $\{x : f_1(x) = K f_0(x)\}$.

It follows that the MP test is determined uniquely by (*) and (**) whenever the set $\{x : f_1(x) = K f_0(x)\}$ has measures '0' (in particular probability 0). This unique test is then clearly non-randomized. [see Ex.(2)]. More generally, it is seen that randomization is not required except possible on the boundary set where it may be necessary to randomize in order to get the size = α [see Ex.(3)].

Theorem :— NP lemma and Sufficient Statistic :

If a non-trivial sufficient statistic T exists for the family $\{f_{\theta_0}(x), f_{\theta_1}(x)\}$, $\theta \in \Omega = \{\theta_0, \theta_1\}$, then MP test for testing $H_0: X \sim f_0(x)$ against $H_1: X \sim f_1(x)$ is a function of T.

Solution :— By factorization theorem,

$$f_\theta(x) = g(t, \theta) \cdot h(x) \quad ; \quad \theta \in \Omega = \{\theta_0, \theta_1\}$$

By necessary part of NP lemma, an MP test of $H_0: X \sim f_{\theta_0}(x)$ against $H_1: X \sim f_{\theta_1}(x)$ must be in the form:

$$\phi(x) = \begin{cases} 1, & f_{\theta_1}(x)/f_{\theta_0}(x) > K \\ 0, & f_{\theta_1}(x)/f_{\theta_0}(x) < K \end{cases}$$

for some $K \geq 0$.

Note that, $\frac{f_{\theta_1}(x)}{f_{\theta_0}(x)} = \frac{g(t,\theta_1)\cdot h(\underset{\sim}{x})}{g(t,\theta_0)\cdot h(\underset{\sim}{x})} = \frac{g(t,\theta_1)}{g(t,\theta_0)}$

Hence, the form of MP test reduces to

$$\phi(x) = \begin{cases} 1 & , \frac{g(t,\theta_1)}{g(t,\theta_0)} > K \\ 0 & , \frac{g(t,\theta_1)}{g(t,\theta_0)} < K \end{cases}$$

for some K.

that is, MP test can be defined in terms of T only.

<u>Alternative:</u>— If $\phi(x)$ is any test of Ho against H_1, then we define $\psi(t) = E[\phi(X)/T=t]$ which is free from θ, as T is sufficient.

Note that, as $0 \le \phi(x) \le 1$,

$0 \le \psi(t) \le 1$.

and $E_\theta[\phi(X)] = E_\theta\{E[\phi(X)/T]\}$

$= E_\theta\{\psi(T)\}, \forall \theta$

Hence, for any test function $\phi(x)$, there is an equivalent test function $\psi(t)$ which depends on x only through 't'.

If a family of distn. admits a non-trivial sufficient statistic, then to find MP test one can restrict attention to tests based on the sufficient statistic.

<u>Hence an MP test is a function of a sufficient statistic.</u>

<u>Ex.(1)</u>:- Let $\phi(x)$ be an MP test of $H_0: X \sim f_0(x)$ against $H_1: X \sim f_1(x)$ at level α. Let $\beta = E[\phi(X)/H_1] < 1$. Show that $\{1-\phi(x)\}$ is an MP test for testing the null hypothesis H_1 against the alternative H_0 at level $(1-\beta)$.

<u>Solution</u>:— As $\phi(x)$ is an MP test of $H_0: X \sim f_0(x)$ against $H_1: X \sim f_1(x)$ at level α, be necessity ~~part of~~ N-P lemma, we must have

$$\phi(x) = \begin{cases} 1 & , \frac{f_1(x)}{f_0(x)} > k \\ 0 & , f_1(x)/f_0(x) < k \end{cases}$$

with $E[\phi(X)/H_0] = \alpha$ and $\beta = E[\phi(X)/H_1] < 1$

Note that $1-\phi(x) = \begin{cases} 0 , & \frac{f_1(x)}{f_0(x)} > k \\ 1 , & \frac{f_1(x)}{f_0(x)} < k \end{cases}$

$$= \begin{cases} 1 , & \frac{f_0(x)}{f_1(x)} > \frac{1}{k} = k^* \\ 0 , & \frac{f_0(x)}{f_1(x)} < k^* \end{cases}$$

By sufficient part of N-P lemma, $\{1-\phi(x)\}$ is an MP test of $H_1: X \sim f_1(x)$ against $H_0: X \sim f_0(x)$ of its size

$$= E\left[(1-\phi(x))/H_1\right]$$
$$= 1-\beta,$$

with power $= E\left[\{1-\phi(x)\}/H_0\right]$

$$= 1-\alpha.$$

Non-existance of UMP tests:-

If NP MP test of $H_0: \theta=\theta_0$ against $H_1: \theta=\theta_1$ is independent of $\theta_1\ (>\theta_0)$, then MP test becomes UMP test for testing $H_0: \theta=\theta_0$ against $H_1: \theta>\theta_0$.

In general, MP test of $H_0: \theta=\theta_0$ against $H_1: \theta=\theta_1, \theta_1\neq\theta_0$, depends on θ_1, then there is no UMP test for testing $H_0: \theta=\theta_0$ against $H_1: \theta\neq\theta_0$. This means that a UMP test of $H_0: \theta=\theta_0$ against $H_1: \theta\neq\theta_0$ usually does not exist.

Ex.(2):- Let $X_1, \ldots, X_n$ be a r.s. from $N(\mu, \sigma_0^2)$, σ_0 known. Find a UMP test at level α for testing $H_0: \mu=\mu_0$ against $H_1: \mu>\mu_0$. Also find a UMP test at level α for testing $H_0: \mu=\mu_0$ against $H_1: \mu<\mu_0$. Hence, show that there does not exist a UMP test for testing $H_0: \mu=\mu_0$ against $H_1: \mu\neq\mu_0$.

Solution:- By N-P lemma, an MP level α test, for testing $H_0: \mu=\mu_0$ against $H_1: \mu=\mu_1, \mu_1>\mu_0$ is

$$\phi(x) = \begin{cases} 1, & \frac{L(\underset{\sim}{x}, \mu_1)}{L(\underset{\sim}{x}, \mu_0)} = k^* > k \\ \gamma, & k^* = k \\ 0, & k^* < k \end{cases}$$

with $E[\phi(\underset{\sim}{x})/H_0] = \alpha$

Here $L(\underset{\sim}{x}, \mu) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma_0^2}\cdot\sum_{i=1}^{n}(x_i-\mu)^2}$

Hence, $\frac{L(\underset{\sim}{x}, \mu_1)}{L(\underset{\sim}{x}, \mu_0)} > k$

$$\Rightarrow e^{-\frac{1}{2\sigma_0^2}\left[\Sigma(x_i-\mu_0)^2 - \Sigma(x_i-\mu_1)^2\right]} > k$$

$$\Rightarrow e^{\frac{1}{-2\sigma_0^2}\{(\mu_1-\mu_0)\Sigma x_i - (\mu_1^2-\mu_0^2)\}} > k$$

$$\Rightarrow (\mu_1-\mu_0)\cdot n\bar{x} > k_1$$

$$\Rightarrow \bar{x} > c \text{ as } \mu_1 > \mu_0$$

Also, $P_\mu\left[\dfrac{L(\underset{\sim}{x},\mu_1)}{L(\underset{\sim}{x},\mu_0)}=k\right]=P_\mu\left[\bar{x}=c\right]=0.$

Then, $\phi(\underset{\sim}{x})=\begin{cases}1 & ,\bar{x}>c\\ 0 & ,\text{OW}\end{cases}$

where c is such that
$$\alpha=E\left[\phi(\underset{\sim}{x})/H_0\right]$$
$$=1.P_{\mu=\mu_0}\left[\bar{X}>c\right]$$
$$=P_{\mu=\mu_0}\left[\frac{\bar{X}-\mu_0}{\sigma_0/\sqrt{n}}<\frac{c-\mu_0}{\sigma_0/\sqrt{n}}\right]$$
$$=P\left[Z<\frac{(c-\mu_0)\sqrt{n}}{\sigma_0}\right],\quad Z\sim N(0,1)$$

$$\Rightarrow \frac{(c-\mu_0)\sqrt{n}}{\sigma_0}=\tau_\alpha$$
$$\Rightarrow c=\mu_0+\frac{\sigma_0}{\sqrt{n}}\tau_\alpha.$$

Hence, an MP level α test, for testing $H_0:\mu=\mu_0$ against $H_1:\mu=\mu_1$, $\mu_1>\mu_0$ is
$$\phi(\underset{\sim}{x})=\begin{cases}1, & \bar{x}>\mu_0+\frac{\sigma_0}{\sqrt{n}}\tau_\alpha\\ 0 & ,\text{OW}\end{cases}$$

Similarly, an MP test at α-level for testing $H_0:\mu=\mu_0$ against $H_1:\mu=\mu_1$; $\mu_1<\mu_0$ is
$$\phi^*(x)=\begin{cases}1 & ,\bar{x}<\mu_0-\frac{\sigma_0}{\sqrt{n}}\tau_\alpha\\ 0 & ,\text{OW}\end{cases}$$

Clearly, $\phi(\underset{\sim}{x})$ [or, $\phi^*(\underset{\sim}{x})$] depends only on μ_0 and on the relative position of μ_1 w.r.t. μ_0 but not on the exact value of μ_1.

Hence, $\phi(\underset{\sim}{x})$ [or $\phi^*(\underset{\sim}{x})$] remains MP for testing $H_0:\mu=\mu_0$ against any alternative $\mu_1\,(>\mu_0)$ or $\mu_1\,(<\mu_0)$.

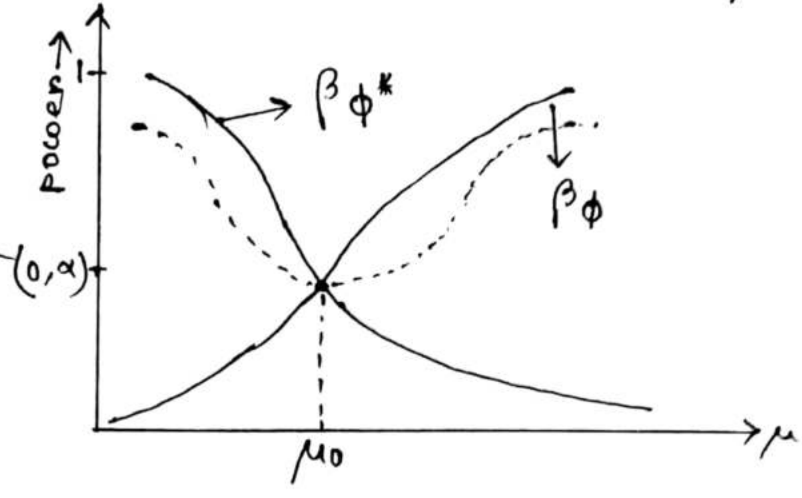
Therefore, $\phi(\underset{\sim}{x})$ [or, $\phi^*(\underset{\sim}{x})$] is a UMP level α test of $H_0:\mu=\mu_0$ against $H_1:\mu=\mu_1$ [or, against $H_1:\mu<\mu_0$].

Note that, $\beta_\phi(\mu)=E_\mu\left[\phi(\underset{\sim}{x})\right]$
$$=P_\mu\left[\bar{X}>\mu_0+\frac{\sigma_0}{\sqrt{n}}\tau_\alpha\right]$$
$$=P_\mu\left[\frac{\bar{X}-\mu_1}{\sigma_0/\sqrt{n}}>\frac{\mu_0-\mu_1}{\sigma_0/\sqrt{n}}+\tau_\alpha\right]$$
$$=1-\Phi\left(\tau_\alpha+\frac{\mu_0-\mu_1}{\sigma_0/\sqrt{n}}\right)\quad\uparrow\mu$$

and $\beta_{\phi^*}(\mu)=\Phi\left(\dfrac{\mu_0-\mu_1}{\sigma/\sqrt{n}}-\tau_\alpha\right)\quad\downarrow\mu$

For $\mu > \mu_0$, $\beta_\phi(\mu) > \beta_{\phi^*}(\mu)$ and for $\mu < \mu_0$, $\beta_\phi(\mu) < \beta_{\phi^*}(\mu)$

For any other test, for $\mu < \mu_0$, the power of ϕ^* is greater than that of the test. therefore, there is no UMP test for testing $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$.



<u>Ex.(3):-</u> Let $X_1, \ldots, X_n$ be a r.s. from $N(0, \sigma^2)$. Show that there does not exist a UMP test for testing $H_0: \sigma = \sigma_0$ against $H_1: \sigma \neq \sigma_0$.

Unbiasedness for hypothesis testing :-

A sample condition that one may wish to improve on tests of the hypothesis $H_0: \theta \in \Omega_0$ against $H_1: \theta \in \Omega_1$ is that 'a test rejects a false H_0 more often than a true H_0', that is, the probability of rejecting H_0 when it is false is at least as large as the probability of rejecting H_0 when it is true. This seems to be a reasonable requirement to place of a good test.

Definition:- A test $\phi(x)$ with the power functions

$$\beta_\phi(\theta) = E_\theta[\phi(X)] \text{ satisfies}$$

$$\sup_{\theta \in \Omega_0} \beta_\phi(\theta) = \alpha \text{ and}$$

$$\beta_\phi(\theta) \geq \alpha, \quad \theta \in \Omega_1,$$

is said to be an unbiased size α test for testing $H_0: \theta \in \Omega_0$ against $H_1: \theta \in \Omega_1$.

If, for a test $\phi(x)$, there exists a $\theta \in \Omega_1$, such that $\beta_\phi(\theta) < \alpha$, then $\phi(x)$ is called a Biased test.

* **Theorem:-** An MP test is necessarily unbiased.

[If β is the power of the MP test, $0 < \alpha < 1$, for testing $H_0: X \sim f_0(x)$ against $H_1: X \sim f_1(x)$, then $\beta > \alpha$ unless $f_0(x) = f_1(x) \; \forall x$]

Proof:- Consider a test given by $\phi(x) = \alpha, \; \forall x \in \mathscr{X}$

Note that $E[\phi(X)/H_0] = \alpha = E[\phi(X)/H_1]$

Hence, $\phi(x)$ is a size α test with power α.

As β is the power of an MP test among all level α tests,

$$\beta \geq E[\phi(X)/H_0] = \alpha$$

$$\Rightarrow \text{power} \geq \text{size}.$$

Hence, an MP test is unbiased.

[If $0 < \alpha = \beta < 1$, for $\alpha \in (0,1)$ then the power of an MP level α test $\phi(x)$ for testing $H_0: X \sim f_0(x)$ against $H_1: X \sim f_1(x)$ is α.

$$\Rightarrow E[\phi(X)/H_0] = E[\phi(X)/H_1], \; \forall \phi(x)$$

$$\Rightarrow \int_{\mathscr{X}} \phi(x) \cdot f_0(x)\,dx = \int_{\mathscr{X}} \phi(x) \cdot f_1(x)\,dx \quad \forall \phi(x)$$

$$\Rightarrow f_0(x) = f_1(x) \; \forall x \in \mathscr{X}$$].

Corollary :- A UMP level α test for testing $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$ is unbiased.

Proof :- Let $\phi(x)$ denotes a level α test for testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1, \theta_1 > \theta_0$.

Then $\phi(x)$ is unbiased. If $\phi(x)$ does not depend on $\theta_1 (> \theta_0)$, then $\phi(x)$ is a UMP level α test for testing $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$ and continuous to be unbiased. Hence, a UMP level α test for testing $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$ exists and it is unbiased.

Examples of Biased Tests :-

Let $X_1, \ldots, X_n$ be a r.s. from $N(\mu, \sigma_0^2)$, σ_0 is unknown. Show that the test

$$\phi(\underset{\sim}{x}) = \begin{cases} 1, & \bar{x} > \mu_0 + \frac{\sigma_0}{\sqrt{n}} \tau_\alpha \\ 0, & \text{OW} \end{cases}$$

or

$$\phi^*(\underset{\sim}{x}) = \begin{cases} 1, & \bar{x} < \mu_0 + \frac{\sigma_0}{\sqrt{n}} \tau_\alpha \\ 0, & \text{OW} \end{cases}$$

are biased tests for testing $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$.

Hints :- For $\mu < \mu_0$, $\beta_\phi(\mu) < \beta_\phi(\mu_0) = \alpha$

$\Rightarrow$ power > size

For $\mu > \mu_0$, $\beta_{\phi^*}(\mu) < \beta_{\phi^*}(\mu_0) = \alpha$

$\Rightarrow$ power < size

UMPU tests :- For a large class of problems, a UMP test does not exist, in this cases, it may be possible to restrict the class of all level α test to the class of all level α unbiased test and find a UMP test in the class of level α unbiased test.

Definition :- The U_α be the class of all unbiased level α test for testing $H_0: \theta \in \Omega_0$ against $H_1: \theta \in \Omega_1$.
If there exists a test $\phi \in U_\alpha$ that has maximum power at each $\theta \in \Omega_1$, we call $\phi(\cdot)$ a UMPU level α test.

Definition:- (UMP Critical Region):- Let $\underset{\sim}{X} = (X_1, \ldots, X_n)$ be an r.s. on an r.v. X having distn. belonging to the family

$$\mathcal{F} = \{ F_\theta : \theta \in \Theta \}$$

Then a critical region W_0 is called a uniformly most powerful (UMP) critical region of size α for testing $H_0: \theta = \theta_0$ Vs. $H_1: \theta \neq \theta_0$ if

$$P_{\theta_0}(W_0) = \alpha \quad \cdots\cdots (1)$$

$$\text{and} \quad P_\theta(W_0) \geq P_\theta(W) \ \forall \ \theta \neq \theta_0 \quad \cdots\cdots (2)$$

whatever the other region W, satisfying (1) may be.

Definition (MP Critical Region):- The critical region W_0 is called a Most Powerful (MP) critical region of size α for testing $H_0: \theta = \theta_0$ Vs. $H_1: \theta = \theta_1$ if

$$P_{\theta_0}(W) = \alpha \quad \cdots\cdots (1)$$

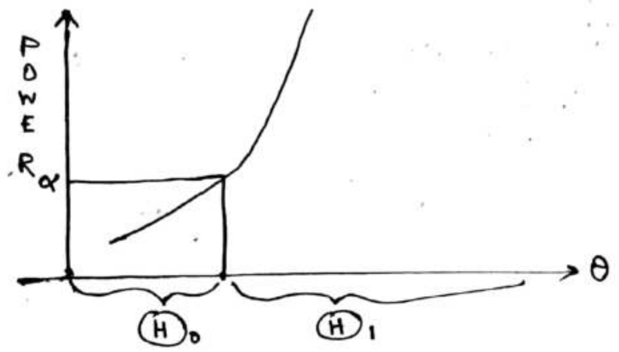
$$\text{and} \quad P_{\theta_1}(W_0) \geq P_{\theta_1}(W) \cdots\cdots (2)$$

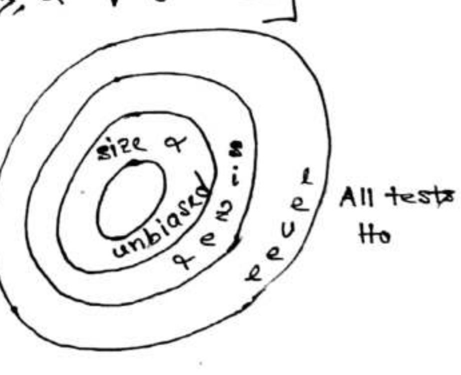
whatever the other CR W, satisfying (1), may be.

Definition (Unbiasedness of a test):- For testing $H_0: \theta \in \Theta_0$ Vs $H_1: \theta \in \Theta_1$, a size α test given by the critical region W (or, critical region ϕ) is said to be unbiased if

$$P_\theta(W) \geq \alpha \ \forall \ \theta \in \Theta_1$$

$$\left[\text{or}, \ E_\theta\left[\phi(\underset{\sim}{X}) \right] \geq \alpha \ \forall \ \theta \in \Theta_1 \right]$$





Definition (UMPU test):- A test given by the CR W_0 [a critical function ϕ] is said to be uniformly most powerful unbiased (UMPU) of size α for testing $H_0: \theta \in \Theta_0$ Vs $H_1: \theta \in \Theta_1$ if

(i) Size Condition:- $\sup_{\theta \in \Theta_0} P_\theta(W_0) = \alpha$

(ii) Unbiasedness condition:- $P_\theta(W_0) \geq \alpha \ \forall \ \theta \in \Theta_1$.

(iii) Power Condition:- $P_\theta(W_0) \geq P_\theta(W) \ \forall$ W satisfying (i) & (ii)

[uniformly for every $\theta \in \Theta_1$]

Further DESIDERATA for a Test of Hypothesis :–

1) Monotonicity of Power function :-

$$\beta_w(\theta) > \beta_w(\theta') \quad \forall \quad \theta \gtrless \theta'$$

i.e. when a null hypothesis is to be tested against a composite alternative one must like that the power of the test should increase with increase in the divergence of the true parameter point from the null hypothesis.

2) Consistency :- The sequence of tests corresponding to $\{W_n\}$ is consistent if for every value of θ lying in $(\Theta - \Theta_0)$, the power, $P_\theta(W_n) \longrightarrow 1$ as $n \to \infty$.

3) Invariance Property :- We shall say that the problem of testing $H_0 : \theta \in \Theta_0$ vs. $H : \theta \in (\Theta - \Theta_0)$ remains invariant under the transformation g of $\mathfrak{X}$ if the corresponding transformation $\bar{g}$ (one-to-one function) of Θ leaves Θ_0 (and hence $\Theta - \Theta_0$) unchanged.

— × —

The likelihood Ratio Test :

Let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_n)$ be a random vector with PDF or PMF $L(\underset{\sim}{x}; \theta)$, $\theta \in \Omega \subseteq R^k$.

Consider the problem of testing the null hypothesis $H_0: \theta \in \Omega_0$ against $H_1: \theta \in \Omega_1$, where $\Omega_0 \cup \Omega_1 = \Omega$.

Note that $\sup_{\theta \in \Omega_0} L(\underset{\sim}{x}; \theta)$ is the best possible explanation of the data $\underset{\sim}{x}$ in the sense of maximum likelihood that the null hypothesis H_0 can provide and $\sup_{\theta \in \Omega_1} L(\underset{\sim}{x}; \theta)$ is the best possible explanation of $\underset{\sim}{x}$ under Ω_1. The basic idea is to compare $\sup_{\theta \in \Omega_0} L(\underset{\sim}{x}; \theta)$ and $\sup_{\theta \in \Omega_1} L(\underset{\sim}{x}; \theta)$.

Note that the ratio $\dfrac{\sup_{\theta \in \Omega_0} L(\underset{\sim}{x}; \theta)}{\sup_{\theta \in \Omega_1} L(\underset{\sim}{x}; \theta)}$ is bounded, we make the comparison in a slightly different way by defining the likelihood ratio

$$\lambda(\underset{\sim}{x}) = \frac{\sup_{\theta \in \Omega_0} L(\underset{\sim}{x}; \theta)}{\sup_{\theta \in \Omega} L(\underset{\sim}{x}; \theta)}.$$

Here $\Omega_0 \in \Omega$, $\sup_{\theta \in \Omega_0} L(\underset{\sim}{x}; \theta) \leq \sup_{\theta \in \Omega} L(\underset{\sim}{x}; \theta)$.

Also note that $\lambda(\underset{\sim}{x})$ is a statistic and $0 \leq \lambda(\underset{\sim}{x}) \leq 1$.

Now, a small value of $\lambda(\underset{\sim}{x})$ near zero indicates that there is a much better explanation of the data $\underset{\sim}{x}$ under $\Omega = \Omega_0 \cup \Omega_1$, than the best one provided by H_0.

Hence, if $\lambda(\underset{\sim}{x})$ is small near zero, then the data supports H_1 and suspect H_0.

Definition: (1) For testing $H_0: \theta \in \Omega_0$ against $H_1: \theta \in \Omega_1$, a test of the form : reject H_0 iff $\lambda(\underset{\sim}{x}) < c$, where c is a contant which is determined by from the size restriction $\alpha = \sup_{\theta \in \Omega_0} P[\lambda(\underset{\sim}{x}) < c]$ is called a likelihood ratio test of size α.

(2) The likelihood ratio test statistic for testing $H_0: \theta \in \Omega_0$ versus $H_1: \theta \in \Omega_1$ is

$$\lambda(\underset{\sim}{x}) = \frac{\sup_{\Omega_0} L(\theta | \underset{\sim}{x})}{\sup_{\Omega} L(\theta | \underset{\sim}{x})}$$

A likelihood ratio test (LRT) is any test that has a rejection region of the form $\{\underset{\sim}{x} : \lambda(\underset{\sim}{x}) \leq c\}$, where c is any number satisfying $0 \leq c \leq 1$.

$$\boxed{\therefore\ \lambda = \lambda(\underset{\sim}{x}) = \frac{L(\hat{\Theta}_0)}{L(\hat{\Theta})} = \frac{\sup_{\theta \in \Theta_0} L(\underset{\sim}{x}, \theta)}{\sup_{\theta \in \Theta} L(\underset{\sim}{x}, \theta)};\quad 0 \leq \lambda \leq 1}$$

Ex.(1):- Let $X_1, \ldots, X_n$ be an r.s. from

$$f(x;\theta) = \begin{cases} \theta e^{-\theta x}, & \text{if } x > \theta \\ 0, & \text{ow} \end{cases}$$

Find the size α LRT of (i) $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$
(ii) $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$
(iii) $H_0: \theta \geq \theta_0$ against $H_1: \theta < \theta_0$

Solution:-

The likelihood function is

$$L(\underset{\sim}{x};\theta) = \begin{cases} \theta^n e^{-\theta \sum_{i=1}^n x_i}, & \text{if } x_i > \theta \\ 0, & \text{ow} \end{cases}$$

where, $\theta > 0$.

(i) To test $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$:-

Here $\Omega_0 = \{\theta_0\}$ and $\Omega = \{\theta, \theta > 0\}$

The likelihood ratio is

$\left[\because e^{-\theta_0 n\bar{x} + \frac{1}{\bar{x}} n\bar{x}} = e^{-\theta_0 n\bar{x} + n}\right]$

$$\lambda = \frac{\sup_{\theta \in \Omega_0} L(\underset{\sim}{x};\theta)}{\sup_{\theta \in \Omega} L(\underset{\sim}{x};\theta)} = \frac{L(\underset{\sim}{x};\theta_0)}{(\hat{\theta})^n \cdot e^{-\hat{\theta} \sum_{i=1}^n x_i}}$$

where, $\hat{\theta} = \frac{1}{\bar{x}}$ is the MLE of θ under Ω.

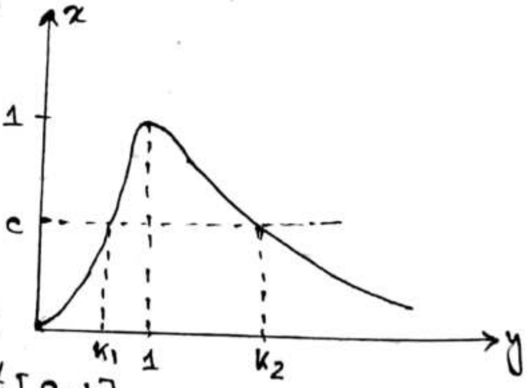
Here, $\lambda = (\theta_0 \bar{x})^n \cdot e^{-n(\theta_0 \bar{x} - 1)}$

$= y^n \cdot e^{-n(y-1)}$, where $y = \theta_0 \bar{x}$

Now, $\frac{d\lambda}{dy} = y^n \cdot e^{-n(y-1)}(-n) + ny^{n-1} \cdot e^{-n(y-1)}$

$= ny^{n-1} \cdot e^{-n(y-1)} \{1 - y\}$

$= \begin{cases} > 0 & \text{if } y < 1 \\ < 0 & \text{if } y > 1 \end{cases}$



From graph, $\lambda < c$

$\Rightarrow y < k_1$ or $y > k_2$

$\Rightarrow 2\theta_0 \sum_{i=1}^n x_i < a$ or $2\theta_0 \sum_{i=1}^n x_i > b$

where, $2nk_1 = a$, $2nk_2 = b$.

Here, the size α LRT is given by:

Reject H_0 iff $\lambda > c$ iff $2\theta_0 \sum_{i=1}^n x_i \notin [a, b]$

, where 'a', 'b' are such that

$\left[\text{As } X_i \sim \text{Exp}(\theta) \therefore 2\theta \sum_{i=1}^n X_i \sim \chi^2_{2n}\right]$

$\alpha = P_{H_0}\left[2\theta_0 \sum_{i=1}^n x_i \notin [a, b]\right]$

$= 1 - P_{H_0}\left[a \leq 2\theta_0 \sum x_i \leq b\right]$

$= 1 - P\left[a \leq \chi^2_{2n} \leq b\right]$

$= 1 - F_{\chi^2_{2n}}(b) + F_{\chi^2_{2n}}(a)$ and $\lambda(k_1) = \lambda(k_2) \Rightarrow k_1^n e^{-n(k_1 - 1)} = k_2^n \cdot e^{-n(k_2 - 1)}$.

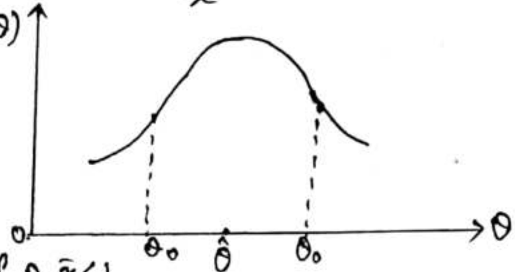
(ii) To test $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$: —

Here $\Omega_0 = \{\theta_0\}$ and $\Omega = \{\theta \geq \theta_0\}$

the likelihood ratio is

$$\lambda = \frac{\sup_{\theta \in \Omega_0} L(\underset{\sim}{x};\theta)}{\sup_{\theta \in \Omega} L(\underset{\sim}{x};\theta)} = \frac{\theta_0^n \cdot e^{-\theta_0 \sum_{i=1}^n x_i}}{\sup_{\theta \geq \theta_0} \{L(\underset{\sim}{x},\theta)\}}$$

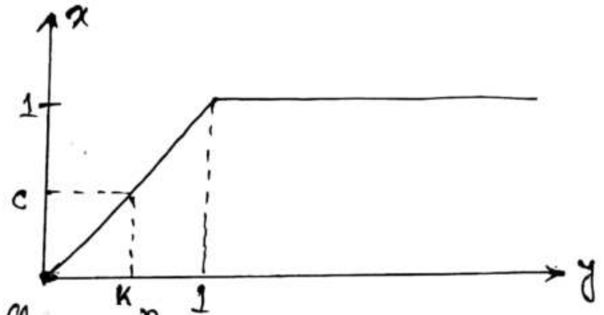
For $\theta > 0$, $L(\underset{\sim}{x},\theta)$ is maximum at $\theta = \frac{1}{\bar{x}} = \hat{\theta}$



$\therefore \sup_{\theta \geq \theta_0} L(\underset{\sim}{x},\theta)$

$$= \begin{cases} (\hat{\theta})^n e^{-\hat{\theta}\sum x_i}, & \text{if } \theta_0 < \hat{\theta} \\ \theta_0^n \cdot e^{-\theta_0 \sum x_i}, & \text{if } \theta_0 > \hat{\theta} \end{cases}$$

Now, $\lambda = \begin{cases} (\theta_0 \bar{x})^n \cdot e^{-n(\theta_0 \bar{x} - 1)}, & \text{if } \theta_0 \bar{x} < 1 \\ 1, & \text{if } \theta_0 \bar{x} \geq 1 \end{cases}$



From graph, $\lambda < c \ (<1)$

$\Rightarrow y < K$

$\Rightarrow 2\theta_0 \sum_{i=1}^n x_i < a$, say

The size α LRT is given by: Reject H_0 iff $\lambda < c$, iff $2\theta_0 \sum_{i=1}^n x_i < a$

where 'a' is such that $\alpha = P_{H_0}\left[2\theta_0 \sum_{i=1}^n x_i < a\right]$

$\therefore \alpha = P\left[\chi^2_{2n} < a\right]$

$\Rightarrow a = \chi^2_{1-\alpha;2n}$.

Therefore the size α LRT is given by:

Reject H_0 iff $\sum_{i=1}^n x_i < \frac{\chi^2_{1-\alpha,2n}}{2\theta_0}$.

(iii) Hint:- $\lambda = \frac{\sup_{\theta \geq \theta_0} L(\underset{\sim}{x},\theta)}{\sup_{\theta \in \Omega} L(\underset{\sim}{x},\theta)}$

The size α LRT is given by: Reject H_0 iff $\sum_{i=1}^n x_i > \frac{\chi^2_{\alpha,2n}}{2\theta_0}$.

Remark:— In fact, the LRT defined above is UMP level α test of $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$. The situation in the example is not merely a coincidence for an OPEF, it can be shown that an LRT of $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$ is UMP of its size.

Ex.(2):- Let $X_1, X_2, \ldots, X_n$ be a r.s. from $N(\theta, \sigma^2)$, σ known. Derive size α LRT for testing

(i) $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$

(ii) $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$.

Show that the LRT's obtained are unbiased.

Solution:— The likelihood function is

$$L(\underset{\sim}{x};\theta) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2} \text{ ; where } \theta \in \mathbb{R}$$

(i) To test $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$:—

Here $\Omega_0 = \{\theta_0\}$ and $\Omega = \{\theta_0 : \theta \in \mathbb{R}\}$

The likelihood ratio is

$$\lambda = \frac{\sup_{\theta \in \Omega_0} L(\underset{\sim}{x};\theta)}{\sup_{\theta \in \Omega} L(\underset{\sim}{x},\theta)}$$

$$= \frac{L(\underset{\sim}{x},\theta_0)}{\sup_{\theta \in \mathbb{R}} L(\underset{\sim}{x},\theta)}$$

$$= \frac{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta_0)^2}}{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

$$= e^{-\frac{1}{2\sigma^2}\left\{\sum(x_i-\theta_0)^2 - \sum(x_i-\bar{x})^2\right\}}$$

$$= e^{-\frac{1}{2\sigma^2} \cdot n(\bar{x}-\theta_0)^2}$$

Note that $\lambda < c$

$$\Rightarrow e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta_0)^2} < c$$

$$\Rightarrow \frac{n(\bar{x}-\theta_0)^2}{\sigma^2} > c_1$$

$$\Rightarrow \left|\frac{\sqrt{n}(\bar{x}-\theta_0)}{\sigma}\right| > k, \text{ say.}$$

The size α LRT is given by:

Reject H_0 iff $\lambda > c$ iff $\left|\frac{\sqrt{n}(\bar{x}-\theta_0)}{\sigma}\right| > k$, where k is such that

$$\alpha = P_{H_0}\left[\left|\frac{\sqrt{n}(\bar{X}-\theta_0)}{\sigma}\right| > k\right] = P\left[|Z| > k\right], \; Z \sim N(0,1)$$

$$\Rightarrow k = \tau_{\alpha/2}.$$

The size α LRT is given by : Reject H_0 iff

$$\left|\frac{\sqrt{n}(\bar{x}-\theta_0)}{\sigma}\right| > \tau_{\alpha/2}.$$

(ii) To test $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$:-

Here $\Omega_0 = \{\theta_0\}$ and $\Omega = \{\theta_0 : \theta \geq \theta_0\}$

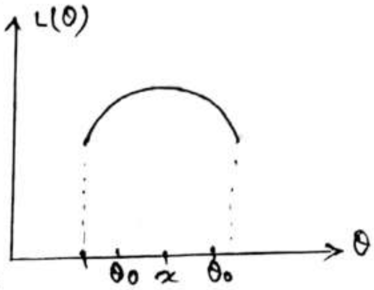
The Likelihood ratio is

$$\lambda = \frac{\sup_{\theta \in \Omega_0} L(\underset{\sim}{x};\theta)}{\sup_{\theta \in \Omega} L(\underset{\sim}{x},\theta)}$$

$$= \frac{L(\underset{\sim}{x}, \theta_0)}{\sup_{\theta \geq \theta_0} L(\underset{\sim}{x}, \theta)}.$$

Here $L(\underset{\sim}{x},\theta)$ is maximum at $\theta = \bar{x} = \hat{\theta}$.

Now, $\sup_{\theta \geq \theta_0} L(\underset{\sim}{x},\theta) = \begin{cases} L(\underset{\sim}{x},\hat{\theta}) & , \text{if } \bar{x} > \theta_0 \\ L(\underset{\sim}{x},\theta_0) & \text{if } \theta_0 \geq \bar{x} \end{cases}$



Here, $\lambda = \begin{cases} e^{-\frac{1}{2\sigma^2} \cdot n(\bar{x}-\theta_0)^2} & , \text{if } \theta_0 < \bar{x} \\ 1 & , \text{if } \theta_0 \geq \bar{x} \end{cases}$

Note that, $\lambda < c \, (<1)$

$\Rightarrow e^{-\frac{1}{2\sigma^2} \cdot n(\bar{x}-\theta_0)^2} < c$, where $\theta_0 < \bar{x}$

$\Rightarrow \frac{n(\bar{x}-\theta_0)}{\sigma} > c_1$, where $\bar{x} > \theta_0$

$\Rightarrow \frac{\sqrt{n}(\bar{x}-\theta_0)}{\sigma} > K$, as $\bar{x} - \theta_0 > 0$

The size α LRT is given by: Reject H_0 iff $\lambda < c$

iff $\frac{\sqrt{n}(\bar{x}-\theta_0)}{\sigma} > K$, where K is such that

$$\alpha = P_{H_0}\left[\frac{\sqrt{n}(\bar{x}-\theta_0)}{\sigma} > K\right]$$

$$= P[Z > K], \; Z \sim N(0,1)$$

$\Rightarrow K = \tau_\alpha$.

The size α LRT is given by: Reject H_0 iff $\bar{x} > \theta_0 + \frac{\sigma}{\sqrt{n}}\tau_\alpha$,

which is the UMP test for testing $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$ and is unbiased.

The power function of the LRT of $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$ is $\beta(\theta) = P_{H_0}\left[\left|\frac{\sqrt{n}(\bar{x}-\theta_0)}{\sigma}\right| > \tau_{\alpha/2}\right]$

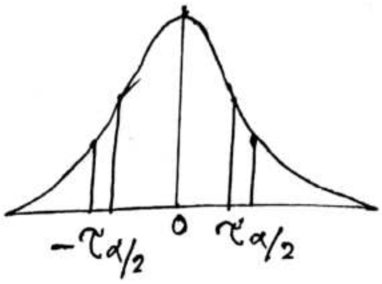
$$= 1 - \Phi\left(\frac{\sqrt{n}(\theta_0-\theta)}{\sigma} + \tau_{\alpha/2}\right) + \Phi\left(\frac{\sqrt{n}(\theta_0-\theta)}{\sigma} - \tau_{\alpha/2}\right)$$

Note that,

$$\beta'(\theta) = \phi\left(\frac{\sqrt{n}(\theta_0-\theta)}{\sigma} + \tau_{\alpha/2}\right)\left(\frac{\sqrt{n}}{\sigma}\right) - \phi\left(\frac{\sqrt{n}(\theta_0-\theta)}{\sigma} - \tau_{\alpha/2}\right) \times \left(\frac{\sqrt{n}}{\sigma}\right);$$

if $\theta > \theta_0$

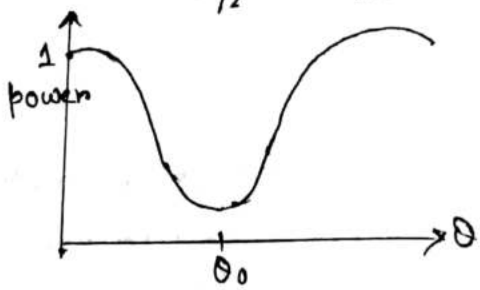
Now, $\beta'(\theta) > 0$ if $\theta > \theta_0$
< 0 if $\theta < \theta_0$



Clearly, $\beta(\theta) > \beta(\theta_0)\ \forall\ \theta \neq \theta_0$

$\Rightarrow$ Power $>$ size

i.e. the LRT is unbiased.



Ex. (3):– Let $X_1, \ldots, X_n$ be an r.s. from $B(1,p)$ popl^n. Derive a LRT of its size of $H_0: p = p_0$ against $H_1: p \neq p_0$.

Solution:– Here $\Omega_0 = \{p_0\}$ and $\Omega = \{p ; 0 < p < 1\}$

The Likelihood function is

$$L(\underset{\sim}{x}, p) = \begin{cases} p^{\sum x_i}(1-p)^{n-\sum x_i} & , \text{if } x_i = 0,1 \\ 0 & , \text{ow} \end{cases}$$

The LR is

$$\lambda = \frac{\sup\limits_{p \in \Omega_0} L(\underset{\sim}{x}, p)}{\sup\limits_{p \in \Omega} L(\underset{\sim}{x}, p)} = \frac{L(\underset{\sim}{x}, p_0)}{L(\underset{\sim}{x}, \hat{p})}, \text{ where}$$

$\hat{p} = \frac{\sum x_i}{n} = \frac{t}{n}$ is the MLE of p under Ω.

$$\lambda = \frac{p_0^{\,t}(1-p_0)^{n-t}}{\hat{p}^{\,t}(1-\hat{p})^{n-t}}, \text{ if } t = \sum x_i$$

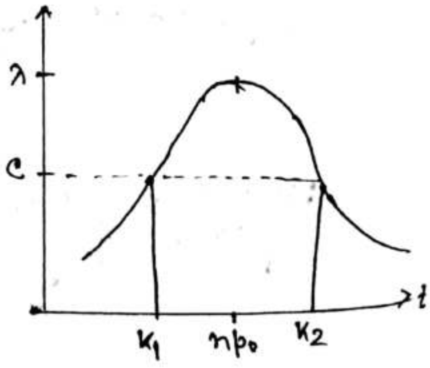
$$= \left(\frac{np_0}{t}\right)^t \left(\frac{n(1-p_0)}{n-t}\right)^{n-t}$$

Now, $\ln\lambda = t\ln\left(\frac{np_0}{t}\right) + (n-t)\ln\left\{\frac{n(1-p_0)}{n-t}\right\}$

and $\frac{d}{dt}\ln\lambda = \ln\left(\frac{np_0}{t}\right) - \ln\left\{\frac{n(1-p_0)}{n-t}\right\}$

$$= \ln\left(\frac{np_0}{t} \cdot \frac{n-t}{n(1-p_0)}\right)$$

$$= \begin{cases} > 0, & \text{if } t < np_0 \\ < 0, & \text{if } t > np_0 \end{cases}$$



From graph, $\lambda < c \Rightarrow t > K_2$ or $t < K_1$

the LRT of its size is given by;

Reject H_0 iff $\lambda < c$ iff $\sum_{i=1}^{n} x_i < K_1$ or $\sum_{i=1}^{n} x_i > K_2 \ni \lambda(K_1) = \lambda(K_2)$

Remark:- The LR test is specifically useful when $\underset{\sim}{\theta}$ is multiparameter and we wish to test hypothesis concerning one of the parameters, the remaining parameter is as nuisence parameter.

EX.(4):- Let $X_1, \ldots, X_n$ be a r.s. from $N(\mu, \sigma^2)$ when μ and σ are unknown. Derive the size α LRT of

(i) $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$

(ii) $H_0: \mu = \mu_0$ against $H_1: \mu_1 > \mu_0$

Solution:- The likelihood function is

$$L(\underset{\sim}{x}, \underset{\sim}{\theta}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2};$$

where, $\mu \in \mathbb{R}$, $\sigma > 0$ and $\underset{\sim}{\theta} = (\mu, \sigma)$.

(i) To test $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$:-

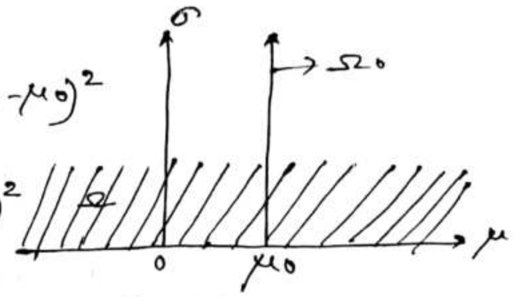
Here $\Omega_0 = \{(\mu_0, \sigma): \sigma > 0\}$ and

$\Omega = \{(\mu, \sigma): \mu \in \mathbb{R}, \sigma > 0\}$

Note that $\sup_{\underset{\sim}{\theta} \in \Omega_0} L(\underset{\sim}{x}, \underset{\sim}{\theta})$

$$= \sup_{\sigma > 0}\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2}\sum_i(x_i-\mu_0)^2}$$

$$= \left(\frac{1}{\hat{\sigma}_0\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\hat{\sigma}_0^2}\sum_i(x_i-\mu_0)^2}$$



where, $\hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2$ is the MLE of σ^2 when $\mu = \mu_0$.

and $\sup_{\underset{\sim}{\theta} \in \Omega} L(\underset{\sim}{x}, \underset{\sim}{\theta})$

$$= \frac{1}{(\hat{\sigma}\sqrt{2\pi})^n} e^{-\frac{\sum(x_i-\hat{\mu})^2}{2\hat{\sigma}^2}};$$

where, $\hat{\mu} = \bar{x}$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$ are the MLE of μ and σ^2 under Ω.

The LR is $\lambda = \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{n/2} = \left\{\frac{\sum(x_i-\bar{x})^2}{\sum(x_i-\mu_0)^2}\right\}^{n/2}$

$$= \left\{\frac{\sum(x_i-\bar{x})^2}{\sum(x_i-\bar{x})^2 + n(\bar{x}-\mu_0)^2}\right\}^{n/2}$$

$$= \left\{\frac{1}{1+\frac{n(\bar{x}-\mu_0)^2}{(n-1)s^2}}\right\}^{n/2}, \text{ where, } s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

Now, $\lambda < c$

$$\Rightarrow \frac{n(\bar{x}-\mu_0)^2}{s^2} > k^2$$

$$\Rightarrow \left|\frac{\sqrt{n}(\bar{x}-\mu_0)}{s}\right| > k, \text{ say}$$

The size α LRT is given by:

Reject H_0 iff $\left|\frac{\sqrt{n}(\bar{x}-\mu_0)}{s}\right| > t_{\alpha/2, n-1}$, which is 'Student's t-test'.

(ii) Here $\Omega_0 = \{(\mu_0, \sigma): \sigma > 0\}$ and

$$\Omega = \{(\mu, \sigma^2): \mu \geq \mu_0, \sigma > 0\}$$

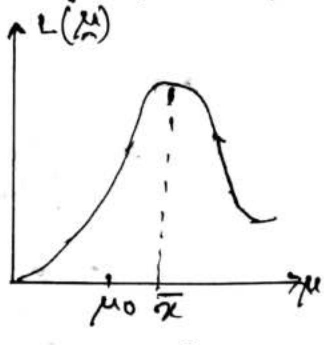
The LR is $\lambda = \dfrac{\sup_{\theta \in \Omega_0} L(\underset{\sim}{x}, \underset{\sim}{\theta})}{\sup_{\theta \in \Omega} L(\underset{\sim}{x}, \underset{\sim}{\theta})}$

Note that, $\sup_{\underset{\sim}{\theta} \in \Omega_0} L(\underset{\sim}{x}, \underset{\sim}{\theta}) = \left(\frac{1}{\hat{\sigma}_0\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\hat{\sigma}_0^2}\sum_{i=1}^{n}(x_i-\mu_0)^2}$

where, $\hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2$

and $\sup_{\underset{\sim}{\theta} \in \Omega} L(\underset{\sim}{x}, \underset{\sim}{\theta}) = \sup_{\mu \geq \mu_0, \sigma > 0} \left\{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}\right\}$ for fixed σ.

$$= \begin{cases} \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma}\sum(x_i-\bar{x})^2}, & \text{if } \mu_0 < \bar{x} \\ \left(\frac{1}{\hat{\sigma}_0\sqrt{2\pi}}\right)^n \cdot e^{-\frac{\sum(x_i-\mu_0)^2}{2\hat{\sigma}_0^2}}, & \text{if } \mu_0 \geq \bar{x} \end{cases}$$



[Here we follow two stage maximization. first we maximize $L(\underset{\sim}{x}, \underset{\sim}{\theta})$ for $\mu \geq \mu_0$ for fixed σ and then maximize w.r.t. σ]

Here the LR is $\lambda = \begin{cases} \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{n/2}, & \text{if } \mu_0 < \bar{x} \\ 1, & \text{if } \mu_0 \geq \bar{x} \end{cases}$

$$= \begin{cases} \left\{\frac{1}{1 + \frac{n(\bar{x}-\mu_0)^2}{(n-1)s^2}}\right\}^{n/2}, & \text{if } \mu_0 < \bar{x} \\ 1, & \text{if } \mu_0 \geq \bar{x} \end{cases}$$

Now, $\lambda < c \ (<1)$

$$\Rightarrow \frac{n(\bar{x}-\mu_0)}{s^2} > K^2, \text{ with } \mu_0 < \bar{x}$$

$$\Rightarrow \frac{\sqrt{n}(\bar{x}-\mu_0)}{s} > K, \text{ as } \bar{x}-\mu_0 > 0$$

The size α LRT is given by:

Reject H_0 iff $\frac{\sqrt{n}(\bar{x}-\mu_0)}{s} > t_{\alpha, n-1}$.

i.e. iff $\bar{x} > \mu_0 + \frac{s}{\sqrt{n}} t_{\alpha; n-1}$. (ANS)

Ex.(5):- Let $X_1, \ldots, X_n$ be a r.s. from $N(\mu, \sigma^2)$, where μ and σ both are unknown. Find the size α LRT of $H_0: \sigma = \sigma_0$ against $H_1: \sigma \neq \sigma_0$. Also, obtain the power function of the LRT.

Solution:- The likelihood function is

$$L(\underset{\sim}{x}, \underset{\sim}{\theta}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}; \text{ where } \mu \in R, \sigma > 0, \ \underset{\sim}{\theta} = (\mu, \sigma).$$

Here, $\Omega_0 = \{(\mu, \sigma_0) : \mu \in \mathbb{R}\}$ and $\Omega = \{(\mu, \sigma) : \mu \in \mathbb{R}, \sigma > 0\}$

then the LR is

$$\lambda = \frac{\sup_{\mu \in R} L(\underset{\sim}{x}; \mu, \sigma_0)}{\sup_{\mu \in R, \sigma > 0} L(\underset{\sim}{x}; \mu, \sigma)}$$

$$= \frac{\left(\frac{1}{\sigma_0\sqrt{2\pi}}\right)^n e^{-\frac{\sum(x_i-\bar{x})^2}{2\sigma_0^2}}}{\left(\frac{1}{\hat{\sigma}(\sqrt{2\pi})}\right)^n \cdot e^{-\frac{\sum(x_i-\bar{x})^2}{2\hat{\sigma}^2}}}$$

$$= \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{n/2} \cdot e^{-\frac{1}{2}\left\{\frac{\sum(x_i-\bar{x})^2}{\sigma_0^2} - n\right\}}$$

$$= \text{constant} \cdot \left(\frac{s^2}{\sigma_0^2}\right)^{n/2} \cdot e^{-s^2/2\sigma_0^2}, \text{ where } s^2 = \frac{1}{n}\sum(x_i-\bar{x})^2 = \hat{\sigma}^2$$

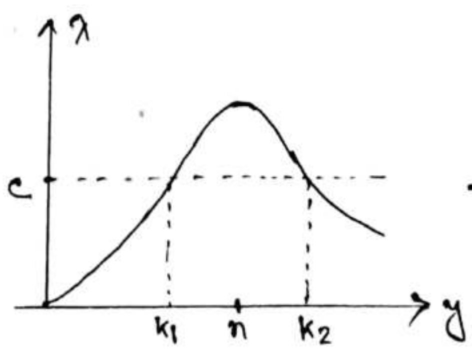
$$= \text{constant} \cdot y^{n/2} \cdot e^{-y/2}, \text{ where } y = \frac{s^2}{\sigma_0^2}.$$

[Let $f(y) = y^{n/2} \cdot e^{-y/2}$

$$f'(y) = y^{n/2} \cdot \left(-\frac{1}{2} e^{-y/2}\right) + e^{-y/2} \cdot \frac{n}{2} \cdot y^{n/2-1}$$

$$= y^{n/2-1} \cdot e^{-y/2}\left(\frac{n-y}{2}\right)$$

$$\begin{cases} > 0, & y < n \\ < 0, & y > n \end{cases}$$



]

Here $\lambda < c$

$\Rightarrow \lambda < k_1$ or $\lambda > k_2$

$\Rightarrow \frac{s^2}{\sigma_0^2} < k_1$ or $> k_2$ with $\lambda(k_1) = \lambda(k_2)$.

The size α LRT is given by:

Reject H_0 iff $\lambda < c$

iff $\frac{s^2}{\sigma_0^2} < k_1$ or $\frac{s^2}{\sigma_0^2} > k_2$, where k_1, k_2 are such that $\lambda(k_1) = \lambda(k_2) \Rightarrow k_1^{n/2} e^{-n/2\, k_1} = k_2^{n/2} e^{-nk_2/2}$

and $\alpha = 1 - P_{H_0}\left[k_1 < \frac{s^2}{\sigma_0^2} < k_2\right]$

$= 1 - P\left[nk_1 < \chi^2_{n-1} < nk_2\right]$

$= 1 - F_{\chi^2_{n-1}}(nk_2) + F_{\chi^2_{n-1}}(nk_1)$

$\therefore$ The Power function is

$$\beta(\sigma) = 1 - P_\sigma\left[k_1 < \frac{s^2}{\sigma_0^2} < k_2\right]$$

$$= 1 - P\left[\frac{\sigma_0^2}{\sigma^2} k_1 < \frac{s^2}{\sigma^2} < \frac{\sigma_0^2}{\sigma^2} k_2\right]$$

__Theorem:__- If for given $\alpha \in (0,1)$, a non-randomized NPMP test and the LRT for a simple null $H_0: \theta = \theta_0$ against simple alternative $H_1: \theta = \theta_1$ exists, then they are equivalent.

__Proof:__- Here $\Omega_0 = \{\theta_0\}$ and $\Omega = \{\theta_0, \theta_1\}$

Note that $\sup_{\theta \in \Omega_0} L(\underset{\sim}{x}, \theta) = L(\underset{\sim}{x}, \theta_0)$

and $\sup_{\theta \in \Omega} L(\underset{\sim}{x}, \theta) = \max\{L(\underset{\sim}{x}, \theta_0), L(\underset{\sim}{x}, \theta_1)\}$

Now, the LR is

$$\lambda(\underset{\sim}{x}) = \frac{\sup_{\theta \in \Omega_0} L(\underset{\sim}{x}, \theta)}{\sup_{\theta \in \Omega} L(\underset{\sim}{x}, \theta)} = \frac{L(\underset{\sim}{x}, \theta_0)}{\max\{L(\underset{\sim}{x}, \theta_0), L(\underset{\sim}{x}, \theta_1)\}}$$

$$= \begin{cases} \frac{L(\underset{\sim}{x}, \theta_0)}{L(\underset{\sim}{x}, \theta_1)}, & \text{if } L(\underset{\sim}{x}, \theta_0) < L(\underset{\sim}{x}, \theta_1) \\ 1, & \text{if } L(\underset{\sim}{x}, \theta_0) \geq L(\underset{\sim}{x}, \theta_1) \end{cases}$$

The size α LRT rejects H_0 iff $\lambda(\underset{\sim}{x}) < c$ such that

$$P_{H_0}[\lambda(\underset{\sim}{x}) < c] = \alpha$$

Note that $\lambda(\underset{\sim}{x}) < c \; (<1)$

$$\Rightarrow \frac{L(\underset{\sim}{x}, \theta_0)}{L(\underset{\sim}{x}, \theta_1)} < c$$

$$\Rightarrow L(\underset{\sim}{x}, \theta_1) > \frac{1}{c} \cdot L(\underset{\sim}{x}, \theta_0)$$

Then the critical function of the LRT is given by the non-randomized test:

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & , \text{if } L(\underset{\sim}{x}, \theta_1) > k \cdot L(\underset{\sim}{x}, \theta_0) \\ 0 & , \text{OW} \end{cases}$$

where, $k \left(= \frac{1}{c}\right)$ is such that $E_{H_0}[\phi(x)] = \alpha$.

By (sufficient part of) NP lemma, the above LRT is an MP level α test of $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$.

<u>Asymptotic Distribution of Likelihood Ratio Test:</u>—

<u>Theorem:</u>— Let $(X_1, X_2, \ldots, X_n)$ be a r.s. from $f(x; \underset{\sim}{\theta})$ where $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k)$ that is assumed to satisfy quite general regularity conditions. Clearly, the particular space is $\Omega \subseteq R^k$.

In testing the hypothesis, $H_0: \theta_1 = \theta_1^0, \ldots, \theta_r = \theta_r^0$,

$$-2\log_e \Lambda_n \overset{a}{\sim} \chi^2_r \text{ as } n \to \infty, \text{ under } H_0.$$

Here Λ_n is a R.V. with an observed value

$$\lambda_n = \frac{\sup_{\theta \in \Omega_0} L(x; \underset{\sim}{\theta})}{\sup_{\theta \in \Omega} L(x; \underset{\sim}{\theta})}, \text{ which is the LR.}$$

The LRT rejects H_0 iff $\lambda_n < c$.

$$\Leftrightarrow -2\log \lambda_n > c' \text{ (say)}$$

The approximate size 'α' LRT is: Reject H_0 iff

$$-2\log \lambda_n > \chi^2_{\alpha; r}$$

Note that, the d.f. r is the no. of parameters that are specified by H_0.

— x —

WORKED OUT PROBLEMS ON TESTING OF HYPOTHESIS [C.U]

1) Distinguish between randomized and non-randomized test. (5) '05

Solution:- The test of a statistical hypothesis H is a rule or procedure for deciding whether to reject H or not.

Let $(X_1, \ldots, X_n)$ be a r.s. from the popln with pdf / pmf $f_\theta(\cdot)$ of size n, where θ being the unknown parameter.

Consider the following problem of testing

$$H_0: \theta \leq 17 \text{ Vs. } H_1: \theta \geq 17.$$

A test can either be randomized or non-randomized. If for the above testing problem we reject H_0 if $\bar{X} > 17 + \frac{\bar{S}}{\sqrt{n}}$, then the test can be considered as non-randomized. So, we can formalize the definition of a non-randomized test as follows:

A test H is said to be a non-randomized test if the critical region is a subset of the sample space $\mathscr{X}$. If the above example, the critical region is $C = \left\{ \underset{\sim}{x} : \bar{x} > 17 + \frac{\bar{S}}{\sqrt{n}} \right\}$ which is a subset of the sample space $\mathscr{X}$.

Now for the same testing problem, we define the rule to decide the notation of rejection in a random manner. We say that we toss a coin and if head turns up, we reject H_0, i.e. the critical region for a randomized test becomes random. But in a non-randomized test the critical region is deterministic. For Randomized test the critical region is probabilistic.

The actual performance of a non-randomized test is straight forward; one observes a random sample and checks whether the observed sample falls in the critical region or not. On the other hand, to perform a randomized test one first observes the random sample, then evaluates $\psi(x_1, x_2, \ldots, x_n)$, the critical function and finally observes the result of some auxiliary Bernoulli trial has $\psi(x_1, x_2, \ldots, x_n)$ as its probability of success, and if the Bernoulli trial results in a success then the null is rejected. For this reason randomized test is not often used in practice. To attain a preassigned level for a test we opt for randomization.

2) What is uniformly most powerful unbiased test? Why is such test needed? (5) '09

Solution:-

UMPU Test:-

A test $\phi(\underset{\sim}{x})$ is said to be UMPU test for testing a simple null hypothesis against a composite alternative as follows:

$H_0: \theta = \theta_0$ Vs. $H_1: \theta \neq \theta_0$ at level α, if

$$E_{H_0}[\phi(\underset{\sim}{x})] = \alpha \text{ ——— } \langle i \rangle$$

$$E_\theta[\phi(\underset{\sim}{x})] \geq \alpha \quad \forall \theta \neq \theta_0 \text{ ——— } \langle ii \rangle$$

For any other test $\phi^*(\underset{\sim}{x})$ satisfying $E_{\theta_0}[\phi^*(\underset{\sim}{x})] = \alpha$,

$$E_\theta[\phi(\underset{\sim}{x})] \geq E_\theta[\phi^*(\underset{\sim}{x})] \quad \forall \theta \neq \theta_0 \text{ ——— } \langle iii \rangle$$

There three conditions are satisfied.

It is found that in many cases no UMP critical region exists. We then have to bring in some other criterion in addition to level of significance and power to make a choice among available region. In other words, we may now confine our attention first to all regions that are of prescribed size α and have the desired additional property. Next we may require that among all regions of size α have that property, our region should have all the maximum power for all alternatives. A very desirable property is unbiasedness. When our problem is to test a simple hypothesis against a composite alternative in a situation where no UMP region is available, we may take as most satisfactory uniformly most powerful unbiased test.

3) Define a most powerful test. Show that it is necessarily unbiased. (5) '08

Solution:-

Most Powerful test:- Consider the following problem of testing a simple null hypothesis against a simple alternative.

$H_0: X \sim p_0(x)$ Vs. $H_1: X \sim p_1(x)$

A test $\phi(\underset{\sim}{x})$ is said to be the most powerful test of level α if

$\langle i \rangle$ $E_{H_0}[\phi(\underset{\sim}{x})] = \alpha$

$\langle ii \rangle$ For any other test of $\phi^*(\underset{\sim}{x})$ satisfying $\langle i \rangle$

$$E_{H_1}[\phi(\underset{\sim}{x})] \geq E_{H_1}[\phi^*(\underset{\sim}{x})]$$

i.e. the power of the test $\phi(\underset{\sim}{x}) \geq$ power of the test $\phi^*(\underset{\sim}{x})$

where, $\phi^*(\underset{\sim}{x})$ be any other test satisfying $E_{H_1}[\phi^*(\underset{\sim}{x})] = \alpha$.

▨ Let $\phi(\underset{\sim}{x})$ be a most powerful test of level α for testing a simple null against simple alternative as follows:

$$H_0 : X \sim p_0(x) \text{ Vs. } H_1 : X \sim p_1(x)$$

$$\therefore E_{H_0}[\phi(\underset{\sim}{x})] = \alpha$$

Let $\phi^*(\underset{\sim}{x}) \equiv \alpha$ be another test

$$E_{H_0}[\phi^*(\underset{\sim}{x})] = \alpha$$

$$E_{H_1}[\phi^*(\underset{\sim}{x})] = \alpha$$

Since $\phi(\underset{\sim}{x})$ be a most powerful test,

$$E_{H_1}(\phi(\underset{\sim}{x})) \geq E_{H_1}[\phi(\underset{\sim}{x})] = \alpha = E_{H_0}[\phi(\underset{\sim}{x})]$$

$\therefore$ power of $\phi(\underset{\sim}{x}) \geq$ level of $\phi(\underset{\sim}{x})$

$\therefore$ MP test is necessarily unbiased.

4) State the Neyman-Pearson lemma in connection with testing a simple null hypothesis against a simple alternative. Using this lemma obtain the most powerful test for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1 (> \theta_0)$ based on n independent observations from a popl^n. with density

$$f(x|\theta) = \begin{cases} \theta x^{\theta - 1}, & 0 < x < 1 \\ 0, & \text{ow} \end{cases}$$

Solution:—

▨ Neyman Pearson Lemma:— Let us consider the problem of testing of a simple null against a simple alternative as follows: $H_0 : \theta = \theta_0$ Vs. $H_1 : \theta = \theta_1 (> \theta_0)$ where, $X_1, X_2, \ldots, X_n$ be a r.s. drawn from a popl^n. with pdf / pmf $f_\theta(\cdot)$, θ being the unknown parameter. A test $\phi(\underset{\sim}{x})$ is said to be a most powerful test at level α if it has the following form:

$$\phi(\underset{\sim}{x}) = \begin{cases} 1, & \text{if } \dfrac{f_{\theta_1}(x)}{f_{\theta_0}(x)} > K \\ 0, & \text{ow} \end{cases}$$

where, $\phi(\underset{\sim}{x})$ is such that $E_{H_0}[\phi(\underset{\sim}{x})] = \alpha$.

▨ Here a random sample of size n is drawn from the popl^n. having the following pdf

$$f(x|\theta) = \begin{cases} \theta x^{\theta - 1}, & 0 < x < 1 \\ 0, & \text{ow} \end{cases}$$

Here we are to test

$$H_0 : \theta = \theta_0 \text{ Vs. } H_1 : \theta = \theta_1 (> \theta_0).$$

Here, $\lambda(\underset{\sim}{x}) = \dfrac{\prod_{i=1}^{n} f_{\theta_1}(\underset{\sim}{x})}{\prod_{i=1}^{n} f_{\theta_0}(\underset{\sim}{x})} > k$

$$\Rightarrow \frac{\theta_1^n \prod_{i=1}^{n} x_i^{\theta_1 - 1}}{\theta_0^n \prod_{i=1}^{n} x_i^{\theta_0 - 1}} > k$$

$$\Rightarrow \left(\frac{\theta_1}{\theta_0}\right)^n \prod_{i=1}^{n} x_i^{(\theta_1 - \theta_0)} > k$$

$$\Rightarrow \prod_{i=1}^{n} x_i^{(\theta_1 - \theta_0)} > c$$

$$\Rightarrow (\theta_1 - \theta_0) \sum_{i=1}^{n} \ln x_i > \ln c$$

$$\Rightarrow \sum \ln x_i > c' \quad [\because (\theta_1 - \theta_0) > 0]$$

$\therefore$ By Neyman-Pearson lemma, the MP size α test is given by

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \ln x_i > c' \\ 0 & \text{ow} \end{cases}$$

Here c' is such that $E_{H_0}[\phi(\underset{\sim}{x})] = \alpha$

$$\Rightarrow P_{\theta_0}\left[\sum_{i=1}^{n} \ln X_i > c'\right] = \alpha$$

Now, we know that $-2\theta \ln X_i \sim \chi_2^2 \ \forall i$
and $-2\theta \ln X_i$ is independent with each others $\forall$ i.

$$\therefore \sum_{i=1}^{n} -2\theta \ln X_i \sim \chi^2_{2n}$$

Now, $P_{\theta_0}\left[\sum_{i=1}^{n} -2\theta_0 \ln X_i < -2\theta_0 c\right] = \alpha$

$$\Rightarrow P\left[\chi^2_{2n} < -2\theta_0 c\right] = \alpha$$

$$\therefore -2\theta_0 c = \chi^2_{2n, 1-\alpha}$$

$$\Rightarrow c = \frac{\chi^2_{2n, 1-\alpha}}{-2\theta_0}$$

$\therefore$ MP test of size α is given by

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{, if } \sum_{i=1}^{n} \ln X_i > \dfrac{\chi^2_{2n, 1-\alpha}}{-2\theta_0} \\ 0 & \text{, ow} \end{cases}$$

$\therefore$ Hence The Critical Region is $W = \left\{\underset{\sim}{x} \,\middle|\, \prod x_i > \exp[-\chi^2_{1-\alpha, 2n}/2\theta_0]\right\}$

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5) Let $X_1, \dots, X_n$ be a random sample of size n drawn from the normal distribution $N(\mu, 2^2)$. Show that for the likelihood ratio test for testing $H_0: \mu = 0$ Vs $H_1: \mu \neq 0$, the critical region is $|\bar{X}| > c$, where $\bar{X}$ is the sample mean. Find c such that the test is of size α. Find the power function of the test and hence verify whether the test is biased. Compare the powers at $\mu = 1$, $\mu = -1$ and $\mu = 2$ and comment. Show that the test is not UMP for testing H_0 against H_1. (4+2+4+2+3) /10

Solution:- Let $X_1, \dots, X_n$ be a r.s. from $N(\mu, 2^2)$

We are to test

$H_0: \mu = 0$ Vs. $H: \mu \neq 0$

We here adopt likelihood ratio test method to test the above hypothesis,

We define, $$\lambda(x) = \frac{\sup_{\Theta_0} L(\underset{\sim}{x}|\mu)}{\sup_{\Theta} L(\underset{\sim}{x}|\mu)}$$

Here $L(\underset{\sim}{x})$ = Likelihood function of $X_1, \dots, X_n$

$$= \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2/4} \quad ; x_i \in \mathbb{R}$$

MLE of μ is $= \bar{x}$

$$\lambda(x) = \frac{L(0|\underset{\sim}{x})}{L(\bar{x}|\underset{\sim}{x})} = \frac{\frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{8}\sum_{i=1}^{n} x_i^2}}{\frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{8}\sum(x_i-\bar{x})^2}},$$

$$= \exp\left[-\frac{1}{8}\left\{\sum x_i^2 - \sum(x_i-\bar{x})^2\right\}\right]$$

$$= \exp\left[-\frac{1}{8}\left\{\cancel{\sum x_i^2} - \cancel{\sum x_i^2} + n\bar{x}^2\right\}\right]$$

$$= \exp\left[-\frac{\bar{x}^2}{8}\right]$$

We reject H_0 at level α if $\lambda(x) < c'$

$$\Rightarrow \bar{x}^2 > 8c'$$

$$\Rightarrow |\bar{x}| > c$$

Hence the critical region for the test is

$$W = \left\{\underset{\sim}{x} : |\bar{x}| > c\right\}$$

Here c is such that $P_{H_0}[|X| > c] = \alpha$

Under H_0, $\bar{X} \sim N(0, \frac{2^2}{n})$

$\therefore P_{H_0}[|X| > c] = \alpha$

$\Rightarrow P_{H_0}[\bar{X} > c] + P_{H_0}[\bar{X} < -c] = \alpha$

$\Rightarrow 2P_{H_0}[\bar{X} > c] = \alpha$

$\Rightarrow P_{H_0}[\bar{X} > c] = \alpha/2$

$\Rightarrow P\left[\frac{\sqrt{n}\bar{X}}{2} > \frac{\sqrt{n}c}{2}\right] = \frac{\alpha}{2}$

$\therefore \frac{\sqrt{n}c}{2} = \tau_{\alpha/2}$; τ_α being the upper α point of $N(0,1)$.

$\Rightarrow c = \frac{2}{\sqrt{n}}\tau_{\alpha/2}$.

Power function of the test is given by

$P_\mu[|\bar{X}| > c]$

$= P_\mu[\bar{X} > c] + P_\mu[\bar{X} < -c]$

$= 2P_\mu[\bar{X} > c]$

$= 2P_\mu\left[\frac{\sqrt{n}(\bar{X}-\mu)}{2} > \frac{\sqrt{n}(c-\mu)}{2}\right]$

Now, $Z = \frac{\sqrt{n}(\bar{X}-\mu)}{2} \sim N(0,1)$

Now, if the test is unbiased then power > size

$\therefore 2P_\mu[\bar{X} > c] > \alpha, \quad \mu \neq 0$

$\Rightarrow P_\mu[\bar{X} > c] > \frac{\alpha}{2}$

$\Rightarrow P\left[\frac{\sqrt{n}(\bar{X}-\mu)}{2} > \frac{\sqrt{n}(c-\mu)}{2}\right] > \frac{\alpha}{2}$

$\Rightarrow P\left[Z > \frac{\sqrt{n}(c-\mu)}{2}\right] > \frac{\alpha}{2}$

$\Rightarrow \Phi\left(\frac{\sqrt{n}(c-\mu)}{2}\right) < 1 - \frac{\alpha}{2}$

$\Rightarrow \frac{\sqrt{n}(c-\mu)}{2} < \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$

$\Rightarrow \sqrt{n}\left(\frac{2}{\sqrt{n}}\tau_{\alpha/2} - \mu\right) < 2\Phi^{-1}(\alpha/2)$

$\Rightarrow 2\tau_{\alpha/2} - \sqrt{n}\mu < 2\tau_{\alpha/2}$

$\Rightarrow \mu > 0$ $\therefore$ The test is unbiased if $\mu > 0$.

Again, $P_\mu[|\bar{X}| > c] = 2P_\mu[\bar{X} < -c]$

Proceeding in the same way, we can show that the test is unbiased if $\mu < 0$

i.e. the test is unbiased for $\mu \neq 0$

Thus the power of the test is greater than the size of the test as under the alternative $\mu \neq 0$. Thus $\forall\ \mu \neq 0$, the test is unbiased.

▨ For $\mu = 1$, the power function is given by

$$\text{Power function (at } \mu=1) = 2\left[1 - \Phi\left(\frac{\sqrt{n}\left(\frac{2}{\sqrt{n}}\tau_{\alpha/2} - 1\right)}{2}\right)\right]$$

$$= 2\Phi\left(\frac{\sqrt{n}}{2} - \tau_{\alpha/2}\right)$$

$$\text{Power function (at } \mu=-1) = 2\left[1 - \Phi\left(\frac{\sqrt{n}\left(\frac{2}{\sqrt{n}}\tau_{\alpha/2} + 1\right)}{2}\right)\right]$$

$$= 2\left[1 - \Phi\left(\tau_{\alpha/2} + \frac{\sqrt{n}}{2}\right)\right]$$

$\therefore$ Power function at $\mu = 2$ is $= 2\left[1 - \Phi\left(\tau_{\alpha/2} - \sqrt{n}\right)\right]$

$$= 2\Phi\left(\sqrt{n} - \tau_{\alpha/2}\right)$$

Comparing these three points we can say that the power is maximum at $\mu = 2$ and minimum at $\mu = -1$; i.e. we can say that the power function is a monotone function of μ.

▨ For the given testing problem, we construct the following test function

$$\phi(\underset{\sim}{x}) = \begin{cases} 1, & \frac{f_1(x)}{f_0(x)} > k \\ 0, & \text{ow} \end{cases}$$

where, under H_0, $X \sim f_0(x) \equiv N(0, 2^2)$

under H_1, $X \sim f_1(x) \equiv N(\mu, 2^2)$, $(\mu > 0)$

Now, $\frac{f_1(x)}{f_0(x)} > k$

$$\Rightarrow \exp\left\{-\frac{1}{8}\left[\sum_{i=1}^{n}(x_i-\mu)^2 - \sum x_i^2\right]\right\} > k$$

$$\Rightarrow \mu\sum_{i=1}^{n} x_i > c \quad [\because \mu > 0]$$

$$\Rightarrow \sum x_i > \frac{c}{\mu}.$$

$\therefore$ Critical region for testing $H_0: \mu = 0$ Vs. $H_1: \mu > 0$ is given by $W_1 = \left\{ \underset{\sim}{x} : \sum_{i=1}^{n} x_i > \frac{c}{\mu}, \mu > 0 \right\}$

Similarly, the critical region for testing $H_0: \mu = 0$ Vs. $H_1: \mu < 0$ is given by

$$W_2 = \left\{ \underset{\sim}{x} : \sum_{i=1}^{n} x_i < -\frac{c}{\mu}, \mu > 0 \right\}$$

$\therefore$ For the testing of $H_0: \mu = 0$ Vs. $H_1: \mu \neq 0$, the critical region is given by

$$W = \left\{ \underset{\sim}{x} : \sum_{i=1}^{n} x_i > \frac{c}{\mu} \text{ or } \sum_{i=1}^{n} x_i < -\frac{c}{\mu} \right\}$$

Since the critical region depends on the parameter value of the alternative hypothesis. Hence, we can say that the test is not UMP.

6) Explain the concept of likelihood ratio test for testing a composite null hypothesis against a composite alternative. Discuss its merits and demerits. Derive the likelihood ratio test for testing the equality of the variance of k univariate normal distribution each with mean 7. '09 Give an example where this test can be used. (4+3+5+3).

Solution:- For a r.s. $(X_1, \dots, X_n)$ from a popln. having pmf/pdf $f_\theta(\cdot)$, $\theta \in \Theta$, the parameter space. We seek a test of $H_0: \theta \in \Theta_0$ Vs. $H_1: \theta \in \Theta_1 \subseteq \Theta - \Theta_0$. Here Θ_0 and Θ_1 both are not singleton sets, i.e. here we test a composite null hypothesis against a composite alternative.

To illustrate the concept of likelihood ratio test we at first give the definition of likelihood ratio.

Let $L(\theta|x_1,\dots,x_n)$ be the likelihood function of $x_1,\dots,x_n$. The generalized likelihood ratio is denoted by $\lambda(\underset{\sim}{x})$ and is given by

$$\lambda(\underset{\sim}{x}) = \lambda(x_1,\dots,x_n) = \frac{\sup\limits_{\theta\in \Theta_0} L(\theta|x_1,\dots,x_n)}{\sup\limits_{\theta\in\Theta} L(\theta|x_1,\dots,x_n)}$$

Note, here $\lambda(\cdot)$ is a function of $x_1,\dots,x_n$ and it can be considered as a statistic as it does not depend on θ, the unknown parameters.

Since, λ is the ratio of two non-negative quantity, so $\lambda>0$, and since supremum taken in the denominator is over a larger set of parameter values than that in the numerator, thus the denominator can't be smaller than the numerator. Hence $\lambda \le 1$, i.e. $0<\lambda\le 1$.

If $\lambda=1$, then it means the parameter space Θ_0 and Θ_1 are identical, i.e. the null parameter space coincides with the total parameter space and we accept H_0 trivially.

With the departure from H_0, the null parameter space shrinkers, i.e., the numerator decreases. Thus the likelihood ratio also decreases with the departure from H_0 to H_1. Hence a left tail test based on $\lambda(\underset{\sim}{x})$ will be appropriate where the cut off point depends on the stipulated size of the test, i.e. we reject H_0 at size α.

Merits of LRT :-

(i) Likelihood ratio test is always consistent.

(ii) If for a testing problem, UMP test exists, then it coincides with the LRT for the same testing problem.

(iii) For large sample problem, for the likelihood ratio $\{\lambda(\underset{\sim}{x})\}$ $-2\ln\lambda(\underset{\sim}{x})$ converges in distribution under H_0 in χ^2_δ, where $\delta = $(No. of components of the parameter) $-$ (No. of components to be estimated under null)

Hence it is easy to carry out the test for large sample as the function of Likelihood ratio converges to a standard distribution (chi-square).

(iv) LRT makes a good intuitive sense since $\lambda(\underset{\sim}{x})$ will tend to be small when H_0 is not true.

Demerits of LRT:-

(i) Likelihood may be biased.
(ii) Sometimes it is difficult to obtain Sup $L(\theta|\underset{\sim}{x})$
(iii) In an LRT problem, it can be difficult to find the distn. of λ, which is required to find the power of the test.

▨ Let $X_{11}, X_{12}, \ldots, X_{1i}$ be a r.s. of size n_i from $N(7, \sigma_i^2)$

We are to test: $H_0: \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2$ Vs.

H_1: at least one inequality in H_0.

The likelihood ratio is given by,

$$\lambda(\underset{\sim}{x}) = \frac{\sup\limits_{\theta \in \Theta_0} L(\theta|\underset{\sim}{x})}{\sup\limits_{\theta \in \Theta} L(\theta|\underset{\sim}{x})}, \quad \theta = (\sigma_1, \ldots, \sigma_k)$$

The likelihood function is given by

$$L(\theta|\underset{\sim}{x}) = \prod_{i=1}^{k} \frac{1}{(2\pi\sigma_i^2)^{n_i/2}} \cdot \exp\left[-\frac{1}{2\sigma_i^2}\sum_{j=1}^{n_i}(x_{ij}-7)^2\right]$$

The MLE of σ_i^2 is given by

$$\hat{\sigma}_{i\,MLE}^2 = \frac{1}{n_i}\sum_{j=1}^{n_i}(x_{ij}-7)^2 = s_i^2.$$

Under null the Likelihood function reduces to

$$L_{H_0}(\theta|\underset{\sim}{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-7)^2\right], \text{ where}$$

σ being the common value of $(\sigma_1, \ldots, \sigma_k)$, $n = \sum_{i=1}^{k} n_i$

$$\therefore \ln L_{H_0}(\theta|\underset{\sim}{x}) = -\frac{n}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{i}-7)^2 = L$$

$$\therefore \left.\frac{\partial L}{\partial \sigma}\right|_{\sigma = \hat{\sigma}_{H_0}} = 0$$

$$\Rightarrow -\frac{2n}{2\hat{\sigma}_{H_0}} + \frac{2}{2\hat{\sigma}_{H_0}^3}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-7)^2 = 0$$

$$\Rightarrow \hat{\sigma}_{H_0}^2 = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-7)^2$$

$$\therefore \lambda(x) = \frac{\prod_{i=1}^{k}(s_i^2)^{n_i/2}}{(\sigma^2)^{n/2}} = \frac{\prod_{i=1}^{k} s_i^{n_i}}{\sigma^n}$$

We reject H_0 at size α if $\lambda(x) < c$, where c is such that

$$P_{H_0}[\lambda(x) < c] = \alpha$$

Here, $\lambda(x)$ does not follow any standard distribution.
So, here we adopt large sample procedure.
For a large sample size, $-2\ln\lambda(x) \sim \chi^2_{K-1}$ under H_0.
We reject H_0 at size α if

$$-2\ln\lambda(x) > c'$$, where c is such that

$$P_{H_0}[-2\ln\lambda(x) > c'] = \alpha$$

$\therefore$ $c' = \chi^2_{1-\alpha;K}$, the $(1-\alpha)$th quantile of a chisquare $(K-1)$ distribution.

▨ In the analysis of variance technique we assume that the random errors are homoscedastic (i.e. they have equal variance) normal variates. But to justify the assumption we apply the above test procedure. If the hypothesis of equal variance (s.d.) is accepted, then we proceed with our conventional ANOVA technique. But if the proposed null hypothesis is rejected, then we adopt some other way out.

7) Discuss the relationship between a UMP test and a uniformly most accurate confidence interval. (S) '09

Solution:- Let $A(\theta_0)$ be the acceptance region at level α UMP test for testing $H_0: \theta = \theta_0$ and let for a given $\underset{\sim}{x}$,

$$S(\underset{\sim}{x}) = \{\theta \in \Theta : A(\theta) \ni \underset{\sim}{x}\}$$

Here, we are required to show that $S(\underset{\sim}{x}) \subseteq \Theta$ and it is a UM accurate confidence set at confidence level $(1-\alpha)$.
Let, for testing $H_0: \theta = \theta_0$, $A^*(\theta_0)$ be the acceptance region for another level α test

$$\therefore P_\theta(\underset{\sim}{x} \in A^*(\theta)) \geq 1-\alpha.$$

Let $S^*(\underset{\sim}{x})$ be the confidence set for the above stated acceptance region.

$$\therefore A^*(\theta) = \{\underset{\sim}{x} : S^*(\underset{\sim}{x}) \ni \theta\}$$

Then, $P_\theta\{\underset{\sim}{x} \in A^*(\theta)\} = P_\theta\{S^*(\underset{\sim}{x}) \ni \theta\} \geq 1-\alpha$

Again $A(\theta_0)$ is UMP for $H_0: \theta=\theta_0$ against $H_1: \theta \neq \theta_0$. thus

$$P_\theta\{\underset{\sim}{x} \in A^*(\theta_0)\} \geq P_\theta\{\underset{\sim}{x} \in A(\theta_0)\} \;\forall\; \theta \neq \theta_0.$$

Hence, $P_\theta\{S^*(\underset{\sim}{x}) \ni \theta\} \geq P_\theta\{\underset{\sim}{x} \in A(\theta_0)\} \;\forall\; \theta \neq \theta_0.$

$$= P_\theta\{S(\underset{\sim}{x}) \ni \theta\} \;\forall\; \theta \neq \theta_0$$

Hence, from the definition we can say that $S(\underset{\sim}{x})$ is a uniformly most accurate confidence set.

8) Let X_1 and X_2 be a r.s. of size 2 from $R(0,\theta)$ distn. Define $X_{(2)} = \text{Max}(X_1, X_2)$. Find a $100(1-\alpha)\%$ CI for θ based on $X_{(2)}$. (5) '10

Solution:- X_1 and X_2 be a r.s. of size 2 from $R(0,\theta)$ popln.

$$X_{(2)} = \text{Max}(X_1, X_2)$$

The PDF of $X_{(2)}$ is given by,

$$f_{X_{(2)}}(x) = \frac{n x^{n-1}}{\theta^n}, \quad 0<x<\theta.$$

Let us define, the following ; $T = \frac{X_{(2)}}{\theta}$.

Jacobian of the transformation is $|J| = \theta$ and range of T is $0<T<1$.

$\therefore$ PDF of T is given by $f_T(t) = n t^{n-1}, \; 0<t<1$

Now, $P[\lambda_1 < T < \lambda_2] = 1-\alpha$

$$\Rightarrow \int_{\lambda_1}^{\lambda_2} n t^{n-1}\,dt = 1-\alpha$$

$$\Rightarrow \lambda_2^n - \lambda_1^n = 1-\alpha$$

Let $\lambda_2 = 1$, $\lambda_1 = \alpha^{1/n}$

$$\therefore P[\alpha^{1/n} < T < 1] = 1-\alpha$$

$$\Rightarrow P\left[\alpha^{1/n} < \frac{X_{(2)}}{\theta} < 1\right] = 1-\alpha$$

$$\Rightarrow P\left[X_{(2)} < \theta < X_{(2)} \cdot \alpha^{-1/n}\right] = 1-\alpha$$

$\therefore$ $100(1-\alpha)\%$ CI for θ is given by $\left[X_{(2)}, X_{(2)} \cdot \alpha^{-\frac{1}{n}}\right]$.

9) Let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_n)$ be a r.s. of size n from a univariate normal distn. with mean 0 and unknown standard deviation $\sigma\ (>0)$. Consider the statistic S and T defined by

$$nS^2 = \sum_{i=1}^{n} X_i^2 \ , \quad nT = \sqrt{\frac{\pi}{2}} \sum_{i=1}^{n} |X_i|$$

⟨i⟩ Show that both S and T are consistent estimators of σ, but one of them is not unbiased.

⟨ii⟩ Show that $L(S) \geq L(T)$, where $L(\sigma)$ is the likelihood function.

⟨iii⟩ Let for testing $H_0: \sigma = 1$ against $H_1: \sigma > 1$, W_S and W_T be respectively right tailed size α tests based on S and T. Prove that for any $\sigma \geq 1$, $P_r[\underset{\sim}{x} \in W_S | \sigma] = P_r[\underset{\sim}{X} \in W_T / \sigma]$

[6+4+5] '07

Solution:-

⟨i⟩ Let $X_1, \ldots, X_n$ be a r.s. from $N(0, \sigma^2)$ popln. where σ is known.

Let $|X_i| = Z$

$\therefore$ Pdf of Z is given by $f_Z(z) = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \cdot e^{-\frac{1}{2} \cdot \frac{z^2}{\sigma^2}}, \ z > 0$

$$E(Z) = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \int_0^\infty z \cdot e^{-\frac{1}{2} \cdot \frac{z^2}{\sigma^2}} dz$$

Let $u = \frac{z^2}{\sigma^2}$

$$= \frac{\sigma^2}{2\sigma}\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-u/2} du$$

$\Rightarrow du = \frac{2z\,dz}{\sigma^2}$

$$= \sigma\sqrt{\frac{2}{\pi}}$$

$$\therefore E(|X_i|) = \sigma\sqrt{\frac{2}{\pi}}$$

Here we are given that, $ns^2 = \sum_{i=1}^{n} X_i^2$ and $nT = \sqrt{\frac{\pi}{2}} \sum_{i=1}^{n} |X_i|$

Now, $E(nT) = \sum_{i=1}^{n} \sqrt{\frac{\pi}{2}} E|X_i| = n\sigma \cdot \sqrt{\frac{\cancel{\pi}}{\cancel{2}}} \cdot \sqrt{\frac{\cancel{2}}{\cancel{\pi}}}$

$$\Rightarrow E(T) = \sigma.$$

$\therefore$ T is an unbiased estimator for σ.

Now, $E(nS^2) = \sum_{i=1}^{n} E(X_i^2) = \sum_{i=1}^{n} \{ Var(X_i) + E^2(X_i) \}$

$$= \sigma^2 n.$$

$\therefore$ S^2 is an unbiased estimator for σ^2.

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Now, $Var(S) = E(S^2) - E^2(S)$

$\Rightarrow E^2(S) = Var(S) + \sigma^2$

$\Rightarrow E(S) = \sqrt{Var(S) + \sigma^2}$

Since, $Var(S) > 0$, $E(S) \neq \sigma$, Hence S is unbiased for σ.

Now, for $n \to \infty$, $\lim_{n\to\infty} E(T) = \sigma$

Now, $\lim_{n\to\infty} Var(T) = \lim_{n\to\infty} \frac{1}{n^2}\sqrt{\frac{\pi}{2}}\sum_{i=1}^{n} Var(|X_i|)$

$= 0.$

$\therefore$ T is consistent as well as unbiased for σ.

Here, $S = \frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{n} X_i^2}$

Now, $Var(S) = \frac{1}{n} Var\sqrt{\sum_{i=1}^{n} X_i^2}$

$\therefore \lim_{n\to\infty} Var(S) = 0.$

Again, $\lim_{n\to\infty} E(S) = \lim_{n\to\infty}\sqrt{Var(S) + \sigma^2}$

$= \sqrt{\sigma^2} = \sigma.$

$\therefore$ From the condition of consistency we can say that S is consistent but biased for σ.

(ii) The likelihood function of $X_1, \ldots, X_n$ is given by

$$L(\sigma) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{x_i^2}{\sigma^2}} \; ; \; (x_1, \ldots, x_n) \in \mathbb{R}^n$$

Differentiating $\ln L(\sigma)$ w.r.t. σ and equating with zero, we get

$$\frac{\partial}{\partial\sigma}\ln L(\sigma)\Big|_{\sigma=\hat{\sigma}} = -\frac{1}{2}\Sigma x_i^2\left(-\frac{2}{\hat{\sigma}^3}\right) - \frac{n}{\hat{\sigma}^2} = 0$$

$$\hat{\sigma}^2 = \frac{1}{n}\Sigma x_i^2$$

$\therefore \hat{\sigma}^2 = S^2$ is the MLE of σ^2.

Since MLE maximizes, the likelihood function,

hence, $L(S^2) \geq L(T)$.

iii) Consider the following problems of testing

$H_0: \sigma = 1$ Vs. $H_1: \sigma > 1$.

Here we apply LRT method to find out critical region. The generalised likelihood ratio is given by

$$\lambda(x) = \frac{\sup_{\sigma \in \Theta_0} L(\sigma)}{\sup_{\sigma \in \Theta} L(\sigma)}$$

$$= \frac{L(1)}{L(\hat{\sigma})} \quad \left[\because \hat{\sigma} \text{ be the MLE of } \sigma,\ \hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n} x_i^2\right]$$

$$= \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sum x_i^2}}{\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2} \cdot \frac{\sum x_i^2}{\frac{1}{n}\sum x_i^2}}} = \frac{e^{-\frac{1}{2}\sum x_i^2}}{e^{-n/2}}.$$

We reject H_0 at level α, if

$$\lambda(x) < K$$

$$\Rightarrow e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2} < K'$$

$$\Rightarrow \sum_{i=1}^{n} x_i^2 > c'$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} x_i^2 > c$$

$$\Rightarrow s^2 > c$$

$\therefore$ level α critical region is given by $W = \{\underset{\sim}{x} : s^2 > c\}$

Now if W_S and W_T be two right tailed test based on S and T respectively, then the critical region W coincides with the critical region of the test W_S.

Again, we know that if for the given problem of testing if the UMP test exists for level α, it coincides with the likelihood ratio test.

Hence, $W \equiv W_S$ is the UMP size α test for testing H_0 against H_1.

$\therefore$ Power of $W_S \geq$ Power of W_T.

$$\Rightarrow P_r\left[\underset{\sim}{X} \in W_S \mid \sigma\right] \geq P_r\left[\underset{\sim}{X} \in W_T \mid \sigma\right] \text{ for } \sigma \geq 1.$$

10) Explain the concept of shortest expected length confidence interval. Illustrate with an example. (5) $\underline{\underline{/10}}$

Solution:—

Shortest expected length C.I.: — Uniformly shortest length interval usually do not exist among all $(1-\alpha)$ level confidence interval even for most commonly used distributions. This can't be taken as a measure of precision of a confidence interval. In this light Pratt, 1961 suggested to take expected length of a confidence interval as a measure of its precision.

Now we formalize the definition of a shortest expected length C.I..

If $(\underline{\theta}(\underset{\sim}{X}), \overline{\theta}(\underset{\sim}{X}))$ and $(\underline{\theta}^*(\underset{\sim}{X}), \overline{\theta}^*(\underset{\sim}{X}))$ are two C.I. for a parameter θ with same confidence level $(1-\alpha)$, then one will prefer the former if

$$E_\theta\left[\overline{\theta}(\underset{\sim}{X}) - \underline{\theta}(\underset{\sim}{X})\right] < E_\theta\left[\overline{\theta}^*(\underset{\sim}{X}) - \underline{\theta}^*(\underset{\sim}{X})\right]$$

i.e. if the expected length of the former is smaller than the latter. A confidence interval with minimum expected length is called the shortest expected length C.I.

Illustration: — Let $X \sim N(\mu, \sigma^2)$; σ^2 is unknown. Let us start from the pivotal function $T = \frac{\sqrt{n}(\overline{X}-\mu)}{s} \sim t_{n-1}$.

Here s be the sample s.d. with divison $(n-1)$.

Now,

$$P_\mu\left[\lambda_{1\alpha} < \frac{\sqrt{n}(\overline{x}-\mu)}{s} < \lambda_{2\alpha}\right] = 1-\alpha;$$

$$\Rightarrow P\left[\overline{x} - \frac{\lambda_{2\alpha}}{\sqrt{n}}\cdot s < \mu < \overline{x} + \frac{\lambda_{1\alpha}}{\sqrt{n}}\cdot s\right] = 1-\alpha;$$

So, $\left(\overline{x} - \frac{\lambda_{2\alpha}}{\sqrt{n}}s, \ \overline{x} + \frac{\lambda_{1\alpha}}{\sqrt{n}}s\right)$ is a C.I. of μ with confidence coefficient $(1-\alpha)$.

The expected length of the C.I. is

$$(\lambda_{2\alpha} - \lambda_{1\alpha})\cdot\frac{E(s)}{\sqrt{n}} = E(L) \longrightarrow \text{(i)}$$

Also, $$\int_{\lambda_{1\alpha}}^{\lambda_{2\alpha}} f_t(t)\,dt = 1-\alpha \longrightarrow \text{(ii)}$$

Now, $$\frac{dE(L)}{d\lambda_{2\alpha}} = 0$$

$$\Rightarrow \left(1 - \frac{d\lambda_{1\alpha}}{d\lambda_{2\alpha}}\right) = 0 \longrightarrow \text{(iii)}$$

Again from {ii},

$$f_t(\lambda_{2\alpha}) - f_t(\lambda_{1\alpha}) \frac{d\lambda_{1\alpha}}{d\lambda_{2\alpha}} = 0$$

$$\Rightarrow \frac{d\lambda_{1\alpha}}{d\lambda_{2\alpha}} = \frac{f_t(\lambda_{2\alpha})}{f_t(\lambda_{1\alpha})} \longrightarrow \text{(iv)}$$

$\therefore$ From (iii) and (iv),

$$f_t(\lambda_{2\alpha}) = f_t(\lambda_{1\alpha})$$

$\therefore$ either $\lambda_{1\alpha} = \lambda_{2\alpha}$ or $\lambda_{1\alpha} = -\lambda_{2\alpha}$

If $\lambda_{1\alpha} = \lambda_{2\alpha}$ then $E(L) = 0$

Thus we take $\lambda_{1\alpha} = -\lambda_{2\alpha}$

$$\therefore \int_{-\lambda_{2\alpha}}^{\lambda_{1\alpha}} f_t(t)\,dt = 1-\alpha.$$

$$\Rightarrow 2F(\lambda_{2\alpha}) = 2-\alpha$$

$$\Rightarrow F(\lambda_{2\alpha}) = 1 - \frac{\alpha}{2}$$

$$\therefore \lambda_{2\alpha} = t_{\frac{\alpha}{2}; n-1}$$

$$\therefore \lambda_{1\alpha} = -t_{\alpha/2; n-1}$$

$\therefore$ The shortest C.I. for μ is given by,

$$\left[\bar{X} - t_{\alpha/2; n-1} \cdot \frac{s}{\sqrt{n}}, \bar{X} + t_{\alpha/2; n-1} \cdot \frac{s}{\sqrt{n}} \right]$$

11) Given a r.s. of size n from an exponential distribution with unknown mean λ. Find the likelihood function $L(\lambda)$. Show that $L(\bar{X}) \geq L(1)$, where $\bar{X}$ is the sample mean. Hence derive the likelihood ratio test for $H_0: \lambda = 1$ Vs. $H_1: \lambda \neq 1$. Show that such a test can be performed using a chi-square statistic. Discuss how will you find the cut of point of the test. Also mention the nature of power function of the test. (2+2+3+3+3+2) '08

Solution:- Let us consider that $x_1, \dots, x_n$ be a r.s. of size n from the exponential distribution with unknown mean λ.

Now, pdf of x_i is $f_{x_i}(x) = \theta e^{-\theta x}$, $x > 0$ and θ is such that $E(x_i) = \lambda \; \forall i$.

Now, $E(X_i) = \int_0^\infty \theta x e^{-\theta x} dx$

$$= \theta \cdot \frac{\sqrt{2}}{\theta^2} = \frac{1}{\theta} = \lambda$$

$$\Rightarrow \lambda = \frac{1}{\theta}.$$

$\therefore$ Likelihood function of $x_1, \ldots, x_n$ is given by,

$$L(\lambda) = \left(\frac{1}{\lambda}\right)^n \cdot e^{-\sum_{i=1}^n x_i/\lambda} \quad ; x_i > 0 \ \forall i$$

▨ Now, we find out the maximum likelihood estimator of λ. Differentiating $\ln L(\lambda)$ w.r.t. λ and equating with zero we have

$$\frac{\partial}{\partial \lambda} \ln L(\lambda) \Big|_{\lambda = \hat{\lambda}} = 0$$

$$\Rightarrow -\frac{n}{\hat{\lambda}} + \frac{\sum_{i=1}^n x_i}{\hat{\lambda}^2} = 0$$

$$\Rightarrow \hat{\lambda} = \frac{1}{n} \sum x_i = \bar{x}$$

$\therefore$ MLE of λ is $\hat{\lambda} = \bar{x}$ = sample mean.

Since MLE maximized the likelihood function,

$$L(\lambda) \geq L(1).$$

▨ Here we are to test,

$H_0: \lambda = 1$ vs. $H_1: \lambda \neq 1$

Here we opt for likelihood ratio test, the generalised likelihood ratio is given by,

$$\lambda(\underset{\sim}{x}) = \frac{\sup_{\lambda=1} L(\lambda)}{\sup_{\lambda \neq 1} L(\lambda)} = \frac{L(1)}{L(\bar{x})} \quad [\because \bar{x} \text{ is the MLE of } \lambda]$$

$$= \frac{e^{-\sum_i x_i}}{\frac{1}{\bar{x}} e^{-\sum_i \frac{x_i}{\bar{x}}}} = \frac{\bar{x} \cdot e^{-\sum x_i}}{e^{-n}}.$$

We reject H_0 at level α if $\lambda(x) < k$

$$\Rightarrow \bar{x} \cdot e^{-\sum x_i} < k'$$

$$\Rightarrow \ln \bar{x} - \sum x_i < c$$

$$\Rightarrow \sum x_i > c \quad \text{——— (i)}$$

Here c is such that $E_{H_0}[\sum X_i > c] = \alpha$

$\therefore$ The size α critical region is given by

$$W = \left\{ \underset{\sim}{x} : \sum_{i=1}^n x_i > c \right\}$$

From the LR criterion we have that we reject H_0 if
$\sum_{i=1}^{n} x_i > c$.

Now, $X_i \overset{iid}{\sim} \text{Exp}(\lambda)$

$Y = \sum_{i=1}^{n} X_i \overset{iid}{\sim} \text{gamma}(\frac{1}{\lambda}, n)$

The PDF of Y is given by

$$f_Y(y) = \frac{y^{n-1} \cdot e^{-y/\lambda}}{(\lambda)^n \, \Gamma n}, \quad y > 0$$

$$\therefore \frac{y^{n-1} \cdot e^{-y/\lambda}}{(\lambda)^n \cdot \Gamma n} > c$$

Let $Z = \frac{2y}{\lambda}$

$$\therefore \left| \frac{dy}{dz} \right| = \frac{\lambda}{2}$$

$\therefore$ PDF of Z is given by :- $f_Z(z) = \dfrac{\lambda^{n-1} \cdot z^{n-1} \cdot e^{-z/2}}{2^n \cdot \lambda^n \cdot \Gamma n}$

$$= \frac{z^{n-1} \cdot e^{-z/2}}{2^n \cdot \Gamma n}$$

$$\therefore Z \sim \chi^2_{2n}$$

$$\therefore \frac{2\sum X_i}{\lambda} > k$$

$$\Rightarrow Z > k$$

$$\Rightarrow \chi^2_{2n} > k$$

$\therefore$ This is a critical region based on k.

Here k is such that, $P_{H_0}[\chi^2_{2n} > k] = \alpha$

$\therefore k = \chi^2_{\alpha; 2n}$, the upper α point of a Chisquare $2n$ distn..

The power function is given by $P_{H_1}[\chi^2_{2n} > \chi^2_{\alpha; 2n}]$

WORKED OUT PROBLEMS ON NON-PARAMETRIC INFERENCE [C.U.]

(1). Describe Wilcoxon signed rank test. Why is it a non-parametric test? Give an example where it can be used. Can you use sign test for the problem mentioned by you? Justify your answers. How is a signed rank test differ from sign test? '07

Solution:—

▨ Wilcoxon signed rank test:— Let $X_1, \ldots, X_n$ be a r.s. of size n drawn from a continuous popln. $F(\cdot)$, with unknown median M. Here we assume that F is symmetric about M.

Consider the problem of testing,

$$H_0: M = M_0 \quad Vs. \quad H_1: M \begin{matrix} > \\ \neq \\ < \end{matrix} M_0$$

Let us define, $D_i = X_i - M_0$, $i=1(1)n$.

Under H_0, the differences D_i's are symmetrically distributed about '0'.

Now, let us define, $Z_i = \begin{cases} 1, & \text{if } D_i > 0 \\ 0, & \text{if } D_i < 0 \end{cases}$ '=' holds with probability zero.

Then $W^+ = \sum_{i=1}^{n} z_i \text{ Rank}(|D_i|)$

and $W^- = \sum_{i=1}^{n} (1 - z_i) \text{ Rank}(|D_i|)$

The Wilcoxon signed rank statistics defined by both W^+ and W^-. Now,

W^+ = Sum of ranks of +ve D_i's.

W^- = " " " -ve " .

WLG, let $|D_1| < |D_2| < \cdots < |D_n|$

Then $\text{Rank}(|D_i|) = i$.

so that $W^+ + W^- = \sum_{i=1}^{n} \text{Rank}(|D_i|)$

$$= \frac{n(n+1)}{2}.$$

Because of this linear constraint the test statistic based on W^+ and W^- only and $(W^+ - W^-)$ are linearly related and therefore equivalent criterion.

If the true popln. median exceeds M_0. Therefore a right tail test is appropriate based on W^+ as most of the larger ranks will correspond to the +ve differences.
Hence we reject H_0 if

$W^+ > W_\alpha$; where W_α is such that

$$P_{H_0}[W^+ > W_\alpha] = \alpha$$

For the alternative $H_1: M < M_0$ a left tail test based on W^- is appropriate.
We reject H_0 in favour of H if $W^- < W_\alpha'$, where W_α' is such that, $P_{H_0}[W^- < W_\alpha'] = \alpha$.
Apart from these two cases a both tailed test is appropriate.

Non-parametric justification:—

Here under H_0, Z_i's are bernoulli $(\frac{1}{2})$ which is independent of the parent popln.. Hence W^+ being a linear function of Z_i's has its distribution independent of the parameter of the parent population under H_0. Hence the test provided by W^+ is exactly distribution free under H_0 and hence non-parametric.

Example of signed rank test :—

Let us draw a random sample of marks in statistics of
20 students of a certain class. Here we are interested about the standard of students in statistics. Here median is quite a good measure. So, here we have to infer about the median of marks in statistics in the class. But we have no prior knowledge about the probability distribution of the marks. So, here we opt for non-parametric method. Here we can apply Wilcoxon signed rank test procedure. First we set the null hypothesis by choosing a tentative value of popln median and then we compute D_i's by the null hypothesis value of the median from the sample value of the marks and we rank the absolute values of D_i and we check the no. of +ve & -ve values. In this way we compute the W^+ and W^- and compare these realized values with the tabulated critical on our desired level of significance and we draw the conclusion for the popln. median.

☒ We can use sign test for the above stated example as for sign test, we are just required to compute the no. of '+' signs and no. of '−' signs and procedure for getting these signs is same as signed rank test.

☒ The ordinary single sample sign test utilizes only the signs of the differences between each observations and the hypothesized median θ_0, the magnitude of these observations relative to θ_0 are ignored. But in signed rank test we consider the signs as well as the magnitude of these differences. this modified test statistic is expected to give better performance.

(2). Describe World-Wolfowitz run test specifying clearly the null and alternative hypothesis for which it is appropriate. Derive the exact distribution of the ~~total~~ total number of runs in the sample under null hypothesis and hence compute its mean and variance. (15) 10

Solution:- World-Wolfowitz run test :—

Definition of run:- A run is a sequence of similar objects or symbols proceded and followed by dissimilar one.

Testing Problem:- Let $X_1, \dots, X_m$ be a r.s. of size m from a popln. with continuous distn. function $F(\cdot)$ and $Y_1, \dots, Y_n$ be a r.s. of size n from popln with d.f. $G(\cdot)$ such that

$$G(x) = F(x-\delta), \delta \in \mathbb{R}$$

the samples are drawn are of independent type.

Here we are to test;

$$H_0: \delta = 0 \text{ vs. } H_1: \begin{cases} \delta > 0 \\ \delta < 0 \\ \delta \neq 0 \end{cases}$$

Test procedure:- Let $\underset{\sim}{Z} = (X_1, \dots, X_m; Y_1, \dots, Y_n)$ be the combined sample.

(i) At first we arrange the combined sample observations in ascending order of magnitude.

(ii) Replace each observation by either X or Y according as the popln. it comes from.

(iii) Count the total number of runs in the sequence obtained. this is our 'run test ~~statistic~~' denoted by r.

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Critical region:— Under each of three kind of alternative, the numbers of runs is expected to be smaller than that under the null hypothesis. So this test has always a left tailed critical region.

At level α, we reject Ho, if $r \le r_\alpha$, where r_α is the largest integer satisfying $P_{H_0}[r \le r_\alpha] \le \alpha$

or, If r_0 is the observed value of r, then reject Ho if $P_{H_0}[r \le r_0] \le \delta$.

Exact distribution of total number of runs:—

If Ho is true, then the no. of distinguishable arrangement's of m X's and n Y's in a line is $\binom{m+n}{n}$ and they are equally likely.

To find $P_{H_0}[r = r_0]$, we need to find the total no. of distinguishable arrangement among these $\binom{m+n}{m}$ which will give us a total of r_0 runs.

Case I:– $r_0 = \text{even} (= 2d)$, say

If $r = 2d$, then if there are d runs of X and d runs of Y. The first run may be either an X or an Y. Now to get d runs of X, we have to partition these m X's in d groups, none of which are non-empty.

This can be done by placing $(d-1)$ bars between the m X's and there are $(m-1)$ places between X's. So this can be done in $\binom{m-1}{d-1}$ distinguishable ways. By a similar argument d runs of Y can be obtained in $\binom{n-1}{d-1}$ ways.

$$\therefore P[r = 2d] = \frac{2\binom{m-1}{d-1}\binom{n-1}{d-1}}{\binom{m+n}{m}}$$

Case II:– $r = \text{odd} (= 2d+1)$

If $r = 2d+1$, then we may have the following two mutually exclusive ways:

i) d runs of X's and $(d+1)$ runs of Y's.

ii) $(d+1)$ runs of X's and d runs of Y's.

Applying similar logic, we have

$$P[r = 2d+1] = \frac{\binom{m-1}{d}\binom{n-1}{d-1} + \binom{n-1}{d}}{\binom{m+n}{m}}$$

—: IMPORTANT QUES :—

(Testing for independence of X and Y)

Let X_i, Y_i, $i=1(1)n$ be a r.s. from $BN(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$. Derive the LRT for testing $H_0: \rho = 0$ vs. $H_1: \rho \neq 0$.

<u>Solution</u>:— Here $\Omega = \{(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) : \mu_i \in \mathbb{R}, i=1,2; \sigma_i > 0, |\rho| < 1\}$,

and $\Omega_0 = \{(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) : \rho = 0, \mu_i \in \mathbb{R}, \sigma_i > 0, i=1,2\}$

The likelihood function is

$$L = \left\{ \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \right\}^{n/2} \cdot e^{-\frac{1}{2(1-\rho^2)} \sum_{i=1}^{n} \left\{ \left(\frac{x_i - \mu_1}{\sigma_1}\right)^2 - 2\rho \left(\frac{x_i - \mu_1}{\sigma_1}\right)\left(\frac{y_j - \mu_2}{\sigma_2}\right) + \left(\frac{y_j - \mu_2}{\sigma_2}\right)^2 \right\}}$$

$$\sup_{\underset{\sim}{\theta} \in \Omega} L = \left(\frac{1}{2\pi \hat{\sigma}_1 \hat{\sigma}_2 \sqrt{1-\hat{\rho}^2}} \right)^n e^{-n/2} \text{ ; where,}$$

$$\hat{\sigma}_1^2 = \frac{1}{n} \sum (x_i - \hat{\mu}_1)^2, \quad \hat{\sigma}_2^2 = \frac{1}{n} \sum (y_j - \hat{\mu}_2)^2, \quad \hat{\rho} = r.$$

and
$$\sup_{\underset{\sim}{\theta} \in \Omega_0} L = \left(\frac{1}{2\pi \hat{\sigma}_1 \hat{\sigma}_2} \right)^n \cdot e^{-n/2}.$$

The LR is
$$\lambda = \frac{\sup_{\underset{\sim}{\theta} \in \Omega_0} L}{\sup_{\underset{\sim}{\theta} \in \Omega} L} = (1 - r^2)^{n/2}$$

Now, $\lambda < c \Rightarrow |r| > k$.

$$\Rightarrow \frac{|r| \sqrt{n-2}}{\sqrt{1-r^2}} > \frac{k\sqrt{n-2}}{\sqrt{1-r^2}} = k'$$

The size α LRT : Reject H_0 iff $\lambda < c$

iff $\left| \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \right| > t_{\alpha/2; n-2}$.

Here, $t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \sim t_{n-2}$, under $H_0: \rho = 0$.

END