Assignment 5: Smooth Trajectories and Control

Robot Kinematics and Dynamics

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1 Overview

This assignment reinforces the following topics:

- Improved Trajectories for Robot Control
- Dynamics for Point Masses

2 Background

2.1 Trajectories

Recall that when controlling a large motion of the joints of a robot arm, you can ensure your commands are more physically realizable (and therefore the robot will have a smoother resulting motion) if you subdivide the motion into small steps.

In the last assignment, we used path segments where the joints moved with a constant speed. This simple approach is a step in the right direction, but resulted in theoretically infinite accelerations at the step changes in velocity at the beginning and end of the motion.

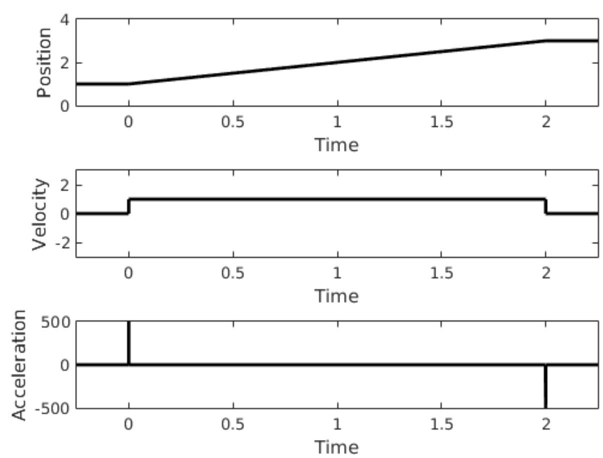
Here we look at the case of moving a single joint between two angles θ_0 and θ_F over the time interval $[t_0, t_F] = [0, 2]$. (When moving multiple joints, this is done independently for each joint.)

The equations for position and velocity of the joint during the interval $[t_0, t_F]$ are:

$$\begin{aligned} q(t) &= v_c(t - t_0) + q_0 \\ \dot{q}(t) &= v_c \end{aligned}$$

The expression for $q(t)$ can be found by inspection (using the formula for a line), or by integrating $\dot{q}(t)$ with appropriate boundary conditions. Also note that given q_0 and q_f, you can easily solve for v_c.

Below we visualize the joint position, velocity, and acceleration for this *constant velocity* trajectory.

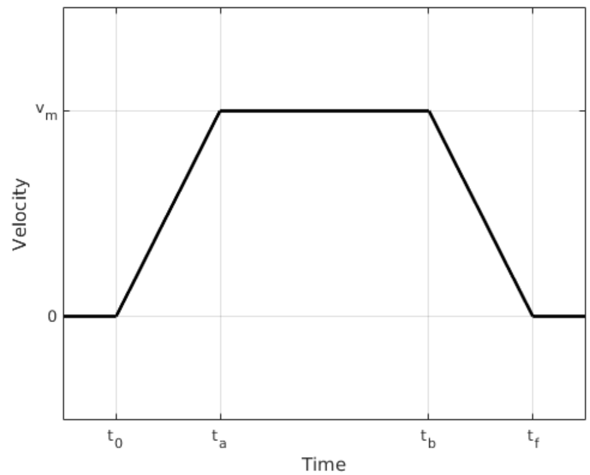


These large accelerations are undesirable because if the arm has any mass then we are commanding large torques that are usually outside the capabilities of the robot. Again, the goal of using trajectories is to always command things the arm can physically do.

2.1.1 Trapezoidal Velocity

A simple solution is to *ramp up* the velocity to the maximum velocity, so the velocity signal is continuous and therefore the acceleration is bounded. You can define the velocity as a piecewise signal, given a ramp time t_r and maximum velocity v_m. (For simplicity, define $t_a = t_0 + t_r$ and $t_b = t_f - t_r$.)

$$\dot{q}(t) = \begin{cases} v_m(t - t_0)/t_r & t_0 \le t \le t_a \\ v_m & t_a < t \le t_b \\ v_m(t_f - t)/t_r & t_b < t \le t_f \end{cases}$$



Note that if q_0, q_f, t_0, and t_f are specified, then t_r fully determines v_m (or v_m determines t_r, depending on the problem constraints). For example, you may have a joint velocity limit that cannot be exceeded. In other cases, you may define t_f, the length of the motion, given a particular maximum acceleration and desired ramp time t_r. Setting these values is application-specific, but we can determine a relationship among them by solving for $q(t)$.

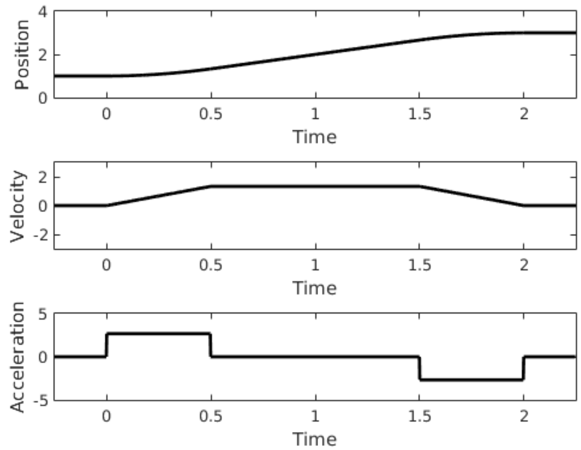
First, consider $q(t)$ for $t_0 \le t \le t_a$:

$$\begin{aligned} q(t) &= q_0 + \int_{t_0}^{t} \dot{q} \\ q(t) &= q_0 + \int_{u=t_0}^{t} \frac{v_m}{t_r}(u - t_0) \\ q(t) &= q_0 + \left(\frac{v_m}{t_r} \left(\frac{u^2}{2} - t_0 u \right) \Bigg|_{u=t_0}^{t} \right) \\ q(t) &= q_0 + \frac{v_m}{2t_r}(t - t_0)^2 \end{aligned}$$

Similarly, we can integrate the other two piecewise linear segments of the velocity curve to get

$$q(t) = \begin{cases} q_0 + \frac{v_m}{2t_r}(t - t_0)^2 & t_0 \le t \le t_a \\ q(t_a) + v_m(t - t_a) & t_a < t \le t_b \\ q(t_b) - \frac{v_m}{2t_r}(t_f^2 - 2t_f t + t^2 - t_r^2) & t_b < t \le t_f \end{cases}$$

This results in the following joint position, velocity, and acceleration curves – note the bounded value for the acceleration.



Note that one can solve for $q(t_f)$, and substitute the expressions for t_a and t_b:

$$
\begin{aligned}
q_f &= q(t_f) = q_0 + v_m \left(\frac{(t_a - t_0)^2 + (t_f - t_b)^2}{2t_r} + t_b - t_a \right) \\
q_f &= q_0 + v_m \left(t_f - t_0 - t_r\right))
\end{aligned}
$$

And finally, rearrange and solve for v_m in terms of t_f, t_r, q_0, and q_f.

$$
v_m = (q_f - q_0)/(t_f - t_0 - t_r)
$$

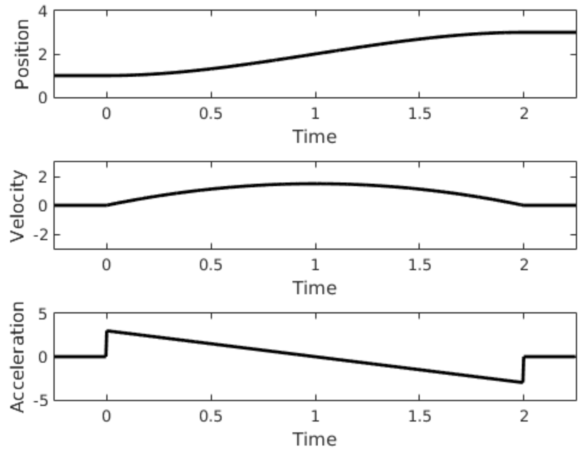
These trajectories are sometimes called S-curves, since the position follows an S-shape with a parabolic beginning / end and linear middle section. This is what a lot of industrial systems and robots do. Overall, it is a good balance between respecting the limits physical actuators without being too complex.

2.1.2 Spline Trajectories

Another method to achieve smooth motions through waypoints is to use a *spline* interpolation. Given a set of waypoints, a spline is a smooth curve that passes through these points, often with some boundary conditions at the start and end. Note that this interpolation gives a curve in joint space, not joint velocity space.

Another benefit to splines over trapezoidal velocity profiles is that for more than two waypoints, the velocity in a spline does not need to be zero at the "middle" (non end) waypoints.

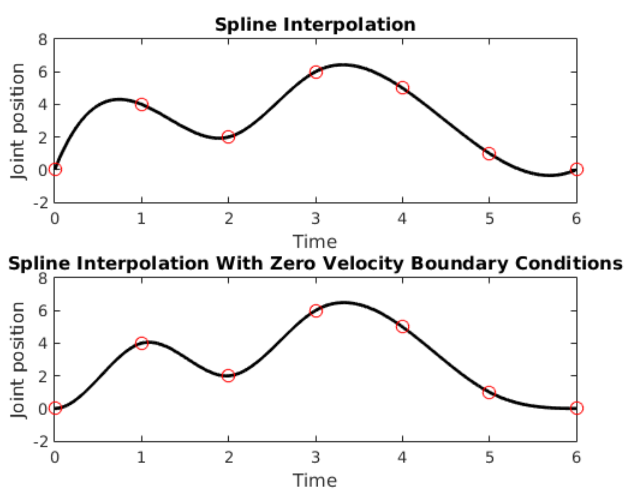
In the figure below, we show the position, velocity, and acceleration for a *cubic spline* between these two waypoints, along with the added boundary condition constraint of zero velocity at the endpoints. (Note – other implementations of spline interpolation allow you to add zero acceleration constraints at the endpoints as well, reducing the jump in acceleration at the ends.



A *cubic* spline in particular is constructed from piecewise third-order polynomials which pass through the waypoints. The equations are straightforward but involved, so we will not expect you to write these yourself. Fortunately, MATLAB's built in 'spline' function can generate a spline interpolation for a given set of waypoints.

As a more complex example, the figure below shows a spline through a number of waypoints. The matlab code to get the y trajectory is below; notice we also show the addition of 'zero velocity' constraints by adding a first and last column of zeros to the waypoints, per the MATLAB 'spline' function documentation.

```
%% Spline through lots of points
waypoints = [0 4 2 6 5 1 0];
waypoint_times = 0:6;
interpolated_times = linspace(0,6,1000);
trajectory = ...
    spline(waypoint_times, waypoints, interpolated_times);
trajectory_zero_vel_ends = ...
    spline(waypoint_times, [0 waypoints 0], interpolated_times);
```



Two final notes: First, splines no longer constrain the motion to straight lines (in joint space) between waypoints, and so the robot can "swing out" as it passes through waypoints that are close together. However, this is often a non-issue.

Also, we won't cover this in the class, but if we keep going down into further derivatives, we see that we still have infinite spikes in the derivative of acceleration which is called jerk. In case you are wondering, the next 3 derivatives are called snap, crackle, and pop. I'm not making this up.

If we want to bound the jerk of a given move, we can specify beginning and end positions, velocities, and accelerations and solve for a 5th-order polynomial that meets these conditions. Such a polynomial will minimize the jerk, and thus command continuous accelerations throughout the trajectory.

We're not going to make you do this in your work, but it's worth knowing about if you want to try to be aware of, or implement, the state-of-the-art in motion control. This approach all started when researchers from MIT in the 1980s did some studies of humans and found that we roughly follow minimum jerk trajectories when we move our limbs. More recently, polynomial trajectories are showing up in other places in robotics. Minimum snap trajectories lie at the heart of a lot of the latest control techniques for quadrotors and small drones, and minimum crackle trajectories are used to gracefully control ballbots.

2.2 Lagrangian

Recall that the Lagrangian is the Kinetic Energy of the system minus the Potential Energy.

$$\mathcal{L}(q, \dot{q}) = T(q, \dot{q}) - V(q, \dot{q})$$

where T stands for the Kinetic Energy and V stands for the Potential Energy.

2.3 Euler-Lagrange Equations

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{q}_j}\right) - \frac{d\mathcal{L}}{dq_j} = \tau_j$$

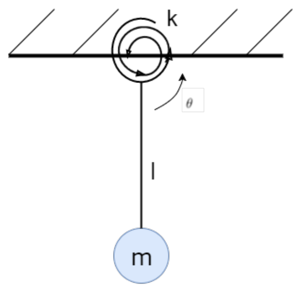
where $\tau_j = 0$ for simplicity.

3 In Class Question

The following question will be done in class, as a part of a group. Your group's answer will still need to be turned in with the rest of your assignment, however unlike the rest of the work this is allowed to be done in groups.

1) Spring Pendulum Lagrangian Dynamics

Given the Spring Pendulum illustrated below.



(1) [5 points] Derive the Kinetic Energy of the System T in terms of θ and $\dot{\theta}$.

$T = \frac{1}{2}ml^2\dot{\theta}^2$

(2) [5 points] Derive the Potential Energy of the System V in terms of θ and $\dot{\theta}$.

$V = mgl(1 - \cos\theta) + \frac{1}{2}k\theta^2$

(3) [2 points] Derive the Lagrangian $\mathcal{L}$ in terms of θ and $\dot{\theta}$.

$\mathcal{L} = T - V$

$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta) - \frac{1}{2}k\theta^2$

(4) [5 points] Derive the Euler-Lagrange Equation using the Lagrangian.

$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{q_j}}\right) - \frac{d\mathcal{L}}{dq_j} = 0$
$ml^2\ddot{\theta} - mgl\sin\theta + k\theta = \tau_\theta$

4 Written Questions

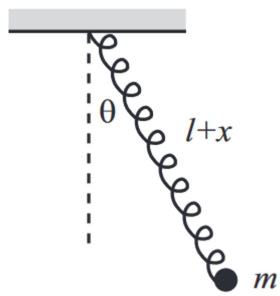
For the following problems, fully evaluate all answers unless otherwise specified.

Answers for written questions must be typed. We recommend LaTeX, Microsoft Word, OpenOffice, or similar. However, diagrams can be hand-drawn and scanned in.

Unless otherwise specified, **all units are in radians, meters, and seconds**, where appropriate.

1) Different Spring Pendulum

We went over the derivations for the mass spring and pendulum systems in the lecture videos. This time we shall combine the two systems with a spring pendulum system. The spring lies in a straight line where it is wrapped around a rigid massless rod. The equilibrium length of the spring is l, but at time t, the length of the spring is $l + x(t)$. The angle of the spring with respect to the vertical is $\theta(t)$.



(1) [5 points] Derive the Kinetic Energy of the System T in terms of $x, \dot{x}, \theta, \dot{\theta}$.

$T = \frac{1}{2}m(l + x)^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2$

(2) [5 points] Derive the Potential Energy of the System V in terms of $x, \dot{x}, \theta, \dot{\theta}$.

$V = \frac{1}{2}kx^2 + mgl(1 - \cos\theta) - mgx\cos\theta$

(3) [2 points] Derive the Lagrangian $\mathcal{L}$ in terms of $x, \dot{x}, \theta, \dot{\theta}$.

$\mathcal{L} = \frac{1}{2}m(l + x)^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - mgl(1 - \cos\theta) + mgx\cos\theta$

(4) [5 points] Derive one Euler-Lagrange Equation by differentiating with respect to x.

$m\ddot{x} - m(l+x)\dot{\theta}^2 - mg\cos\theta + kx = \tau_x$

(5) [5 points] Derive the other Euler-Lagrange Equation by differentiating with respect to θ.

$m(l + x)\ddot{\theta} + 2m\dot{x}(l + x)\dot{\theta} + mgl\sin\theta + mgx\sin\theta = \tau_\theta$

(6) [3 points] Please give a simple explanation for each of the components in both of the Euler-Lagrange equations you derived above.

The x component of the Euler-Lagrange equation represents centripital forces, while the th component of the equation represents Coriolis forces.

5 Feedback

1) Feedback Form *5 points*

We are always looking to improve the class! To that end, we're looking for your feedback on the assignments. When you've completed the assignment, please fill out the feedback form.

6 Code Questions

Copy the Code Handout folder to some location of your choice. Open Matlab and navigate to that location. Whenever you work on the assignment, go into this directory and run `setup.m`.

In the last assignment, we focused on straight-line trajectories, both in workspace and in joint configuration space. We found out that defining paths in workspace is more computationally intensive that defining paths in the joint configuration space. Although carefully controlling workspace paths can be important, it is often enough to define some primary waypoints in workspace, and identify the corresponding joint angles for these waypoints, and then restrict ourselves to joint space to simplify the generation and following of the trajectory.

We also found that when controlling robots using trajectories with piecewise constant velocities, the abrupt changes in velocity can lead to infinite "spikes" in the desired accelerations for the joints, which is physically unrealistic and can lead to unstable behavior.

The focus of this section will be generating trajectories that pass through specified configuration space waypoints. In addition to the basic constant-velocity trajectories from the previous assignment, we will add a function to generate trajectories with trapezoidal velocity profiles, and a function to generate a cubic spline interpolation of waypoints.

Each of these functions will take as input a series of matrix of joint (configuration) space waypoints. The first column of this matrix is the starting waypoint, the last column is the ending waypoint, and any intermediate columns are intermediate points the robot must pass through.

The other input for these functions is a vector of times at which to hit each point (the starting time is assumed to be zero) and a control frequency (how many columns of joint angles per second should be in the resulting trajectory matrix).

1) Piecewise Constant Velocity Trajectories *15 points*

   In the folder `ex_01`, you will fill in `trajectory_const_vel.m`. This should create a series of straight-line, constant velocity trajectories between the waypoints that are passed in. (Feel free to use the code from your previous assignment to help on this problem).

   To validate your work, run `validate_trajectory_const_vel.m`. Your results should overlay the examples for each test case.

2) Trapezoidal Velocity Trajectories *30 points*

   In the folder `ex_02`, you will fill in `trajectory_trap_vel.m`. This should create a series of trapezoidal velocity trajectories between the waypoints that are passed in

   This function has an additional argument, the "ramp duty cycle". This is used to define the percent of each segment used to ramp up (and also the percent used to ramp down) the velocity. This argument must be between 0 (constant velocity trajectory) and 0.5 (triangular velocity profile), and can be used to find the ramp time for each segment.

   To validate your work, run `validate_trajectory_trap_vel.m`. Your results should overlay the examples for each test case.

3) Cubic Spline Interpolation *15 points*

   In the folder `ex_03`, you will fill in `trajectory_spline.m`. This should create a single spline (for each joint) that passes through the given waypoints.

At the first and last waypoint, the velocity should be zero; see the example in the background material for instructions on how to enforce this constraint with MATLAB's `spline` function.

To validate your work, run `validate_trajectory_spline.m`. Your results should overlay the examples for each test case.

7 Hands-On Questions

For this lab, there are two different robots (one with 2DOF and one with 3DOF).
You will use your code from the previous section for this lab.

1) Playing Through Trajectories *40 points*

In the folder `ex_04`, you will fill in `play_trajectory.m`. This function should, given a trajectory, move the robot smoothly to the start point and then execute the trajectory on the robot.

You will use this function to compare the performance of the trajectories produced by the methods that you created in the previous section. You will do this by testing each on the robot (you may use either robot) with the following waypoints:

- **2-DOF Robot:** $\begin{bmatrix} 0.62 & 0.06 & 0.43 & 0.71 & 0.66 \\ 0.87 & 0.70 & -0.71 & 0.54 & 1.84 \end{bmatrix}$
- **3-DOF Robot:** $\begin{bmatrix} 0.62 & 0.06 & 0.43 & 0.71 & 0.66 \\ 0.87 & 0.70 & -0.71 & 0.54 & 1.84 \\ 1.38 & 0.92 & -0.48 & -1.26 & -1.83 \end{bmatrix}$

Use these waypoints and the following timesteps to create each of:

- Constant Velocity Trajectory
- Trapezoidal Velocity Trajectory with Ramp Duty Cycle of 0.2
- Spline Trajectory

Timesteps: $0, 1, 2, 3, 4$

When generating the trajectory and running this on the robot, use a 100Hz control frequency.

Note that a second argument to `play_trajectory` indicates whether to use velocity commands as well as position. Try each of these trajectories with and without using velocity commands in addition to the position commands.

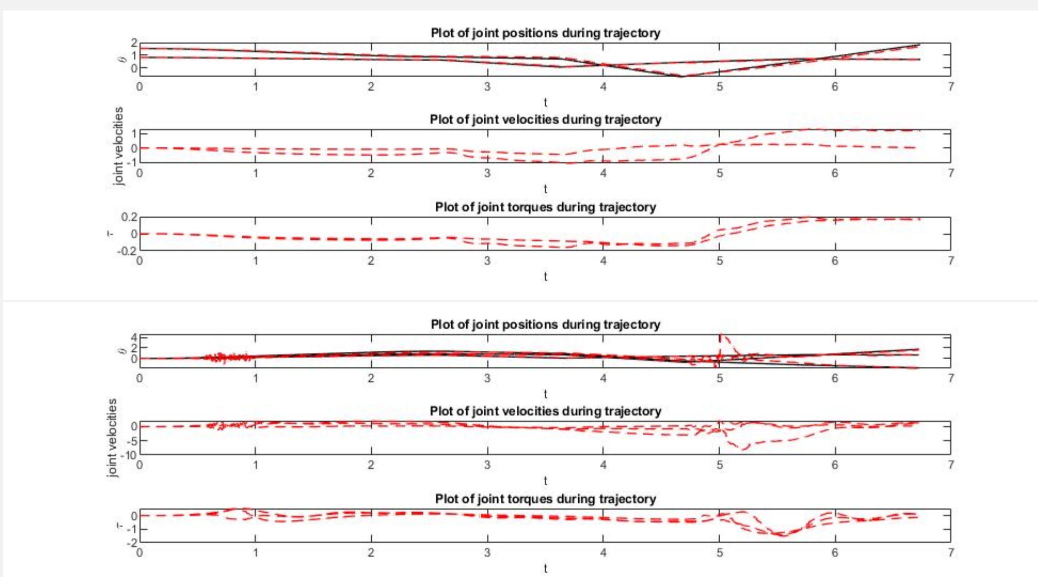
After each test, a figure with the actual/commanded joint angle position along the trajectory will be shown. Save the 6 figures showing the actual/commanded joint angles (3 Trajectories, each with and without additional velocity control).

Insert your figures below as well as a written explanation (no more than a few sentences) of any trends you see in the data.

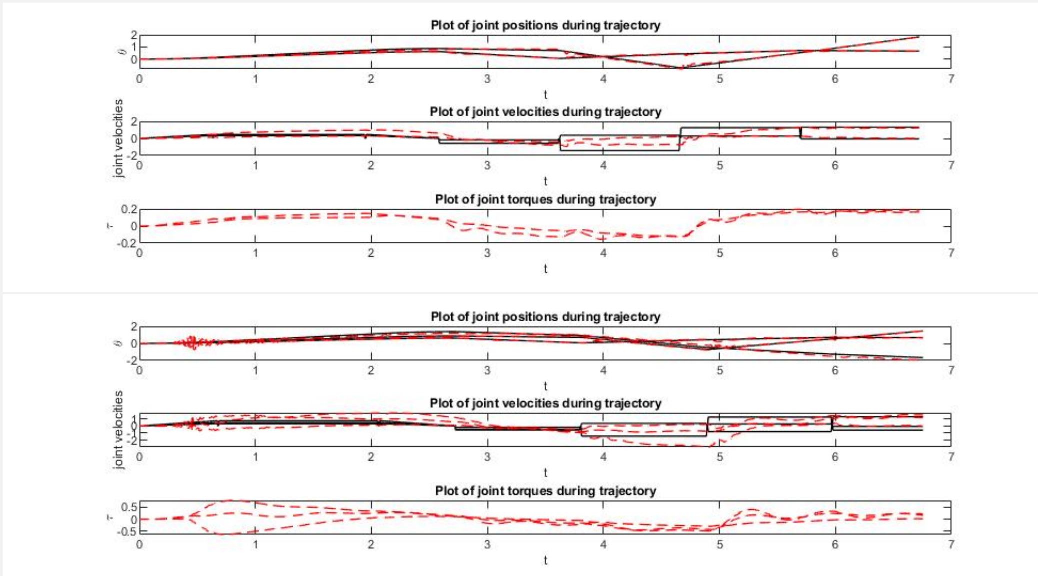
1. Explanation of Trends in the data

Error always tended to be higher near waypoints regardless of trajectory. However, relative to constant velocity, ramp velocity's wp error was smaller, and spline's was almost none. Error was much higher in velocity commands than in position commands, and when velocity changed sharply in a small amount of time, it took the robot a lot of time to reach desired velocity. Torque magnitude tended to be higher near waypoints and especially near sharp velocity changes (these were usually at the same point in time).

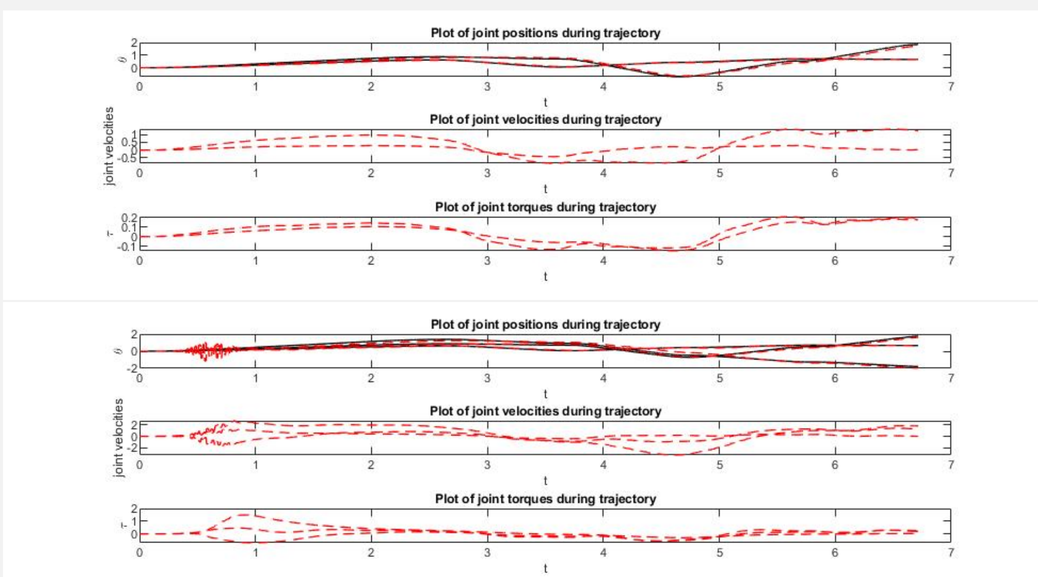
2. 2/3-DOF Robot Constant Velocity Trajectory Without Velocity Commands



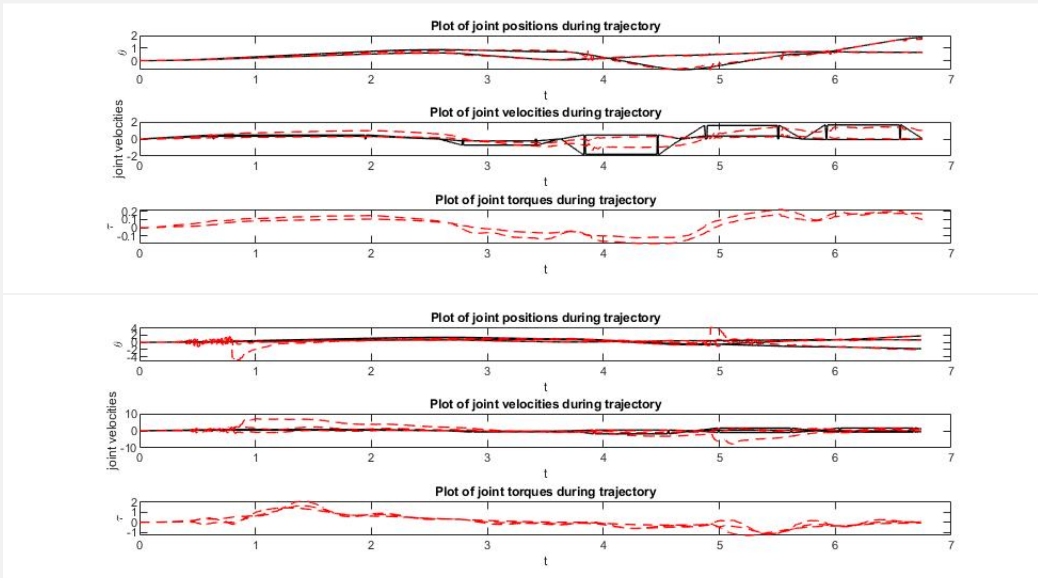
3. 2/3-DOF Robot Robot Constant Velocity Trajectory With Velocity Commands



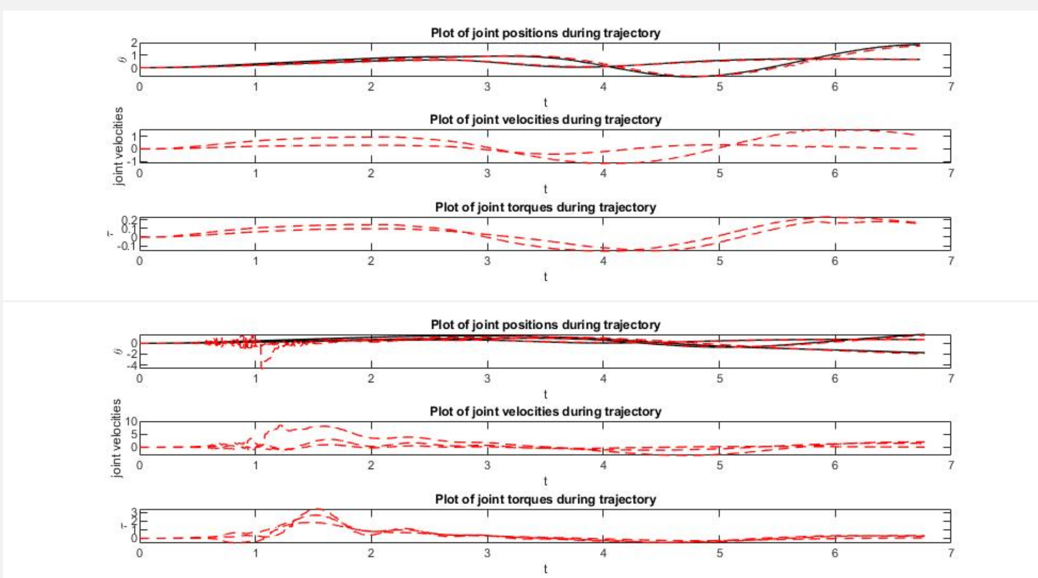
4. 2/3-DOF Robot Robot Trapezoidal Velocity Trajectory with Ramp Duty Cycle of 0.2 Without Velocity Commands



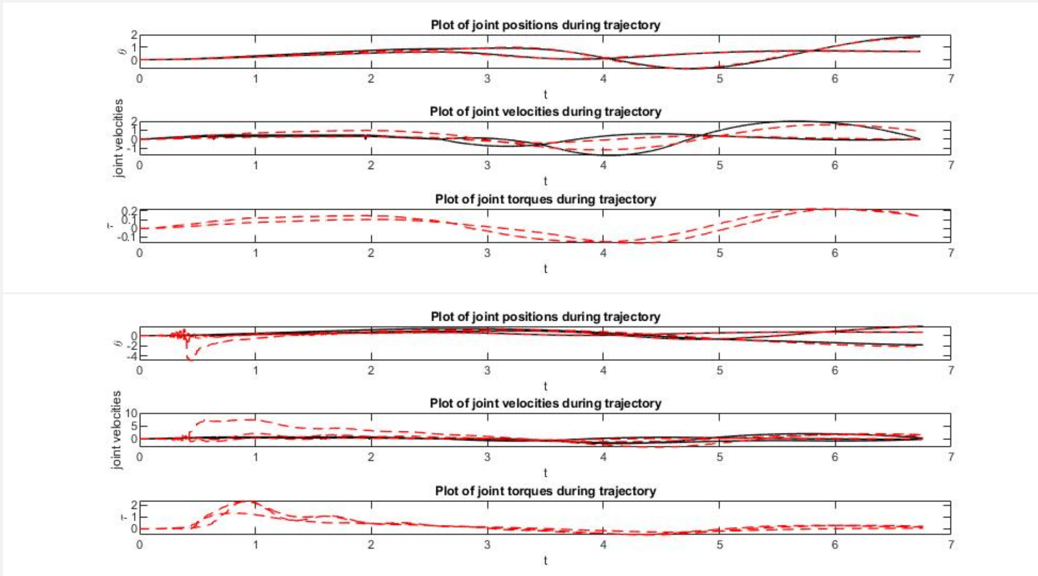
5. 2/3-DOF Robot Trapezoidal Velocity Trajectory with Ramp Duty Cycle of 0.2 With Velocity Commands



6. 2/3-DOF Robot Spline Trajectory Without Velocity Commands



7. 2/3-DOF Robot Spline Trajectory With Velocity Commands



2) Submission

To submit, run `create_submission.m`. It will first check that your submitted files run without error, and perform a small sanity check. Note, this is not going to grade your submission! The function will create a file called `handin-5.tar.gz`. Upload it to Autolab to complete the submission.

8 Submission Checklist

- ☐ Create a PDF of your answers to the written questions with the name `writeup.pdf`.
- ☐ Make sure you have `writeup.pdf` in the same folder as the `create_submission.m` script.
- ☐ Run `create_submission.m` in Matlab.
- ☐ Upload `handin-5.tar.gz` to Canvas and Autolab.
- ☐ Upload `writeup.pdf` to Gradescope.
- ☐ After completing the entire assignment, fill out the feedback form[1].

[1] https://canvas.cmu.edu/courses/11823/quizzes/25762