Assignment 6: Rigid Body Dynamics

Robot Kinematics and Dynamics

Prof. Howie Choset

Contents

1 Overview

This assignment reinforces the following topics:

- Rigid Body Dynamics

2 Background

2.1 Standard Form

As a reminder, the standard form for writing equations of motion is as follows:

$$\boxed{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau}$$

Input torques

$n \times 1$ **gravitational forces vector**

$n \times n$ **Coriolis/centrifugal matrix** linear in $\dot{q}$

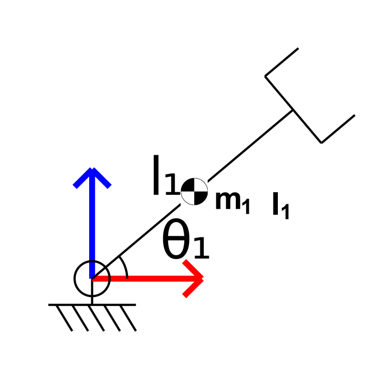
$n \times n$ symmetric positive-definite **mass matrix**

3 In Class Question

The following question will be done in class, as a part of a group. Your group's answer will still need to be turned in with the rest of your assignment, however unlike the rest of the work this is allowed to be done in groups.

1) Rigid Body Dynamics

Please use the diagram of the arm below for the following questions:



The arm has a link length of l_1 where the center of mass is located in the center of the link at $\frac{l_1}{2}$. The link's mass is m_1 and it has moment of inertia I_1.

(1) [5 points] Determine the Kinetic Energy for the R arm shown above.

$$T = \frac{1}{2}m_1\frac{l_1^2}{4}\dot{\theta}_1^{\,2} + \frac{1}{2}I_1\dot{\theta}_1^{\,2}$$

(2) [5 points] Determine the Potential Energy for the R arm shown above.

$V = \frac{1}{2}m_1 l_1 g \sin\theta_1$

(3) [5 points] Write the Lagrangian for the R arm shown above.

$L = \frac{1}{2}m_1 \frac{l_1^2}{4}\dot{\theta}_1^{\,2} + \frac{1}{2}I_1\dot{\theta}_1^{\,2} - \frac{1}{2}m_1 l_1 g \sin\theta_1$

(4) [5 points] Determine the Equations of Motion for the R arm shown above using the Lagrangian.

$\tau_1 = \frac{1}{4} m_1 l_1^2 \ddot{\theta}_1 + I_1 \ddot{\theta}_1^{\,2} + \frac{1}{2} m_1 l_1 g \cos\theta_1$

(5) [5 points] Rewrite the equations of motion into standard form.

$$\tau_1 = \left[\frac{1}{4}m_1 l_1^2 + I_1\right]\ddot{\theta} + \left[0\right]\dot{\theta}\left[\frac{1}{2}m_1 l_1 g\cos\theta_1\right]$$

4 Written Questions

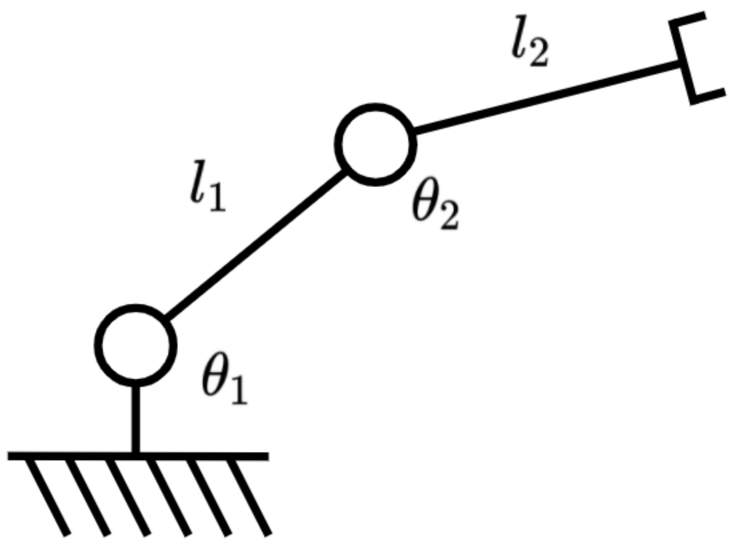
For the following problems, fully evaluate all answers unless otherwise specified.

Answers for written questions must be typed. We recommend LaTeX, Microsoft Word, OpenOffice, or similar. However, diagrams can be hand-drawn and scanned in.

Unless otherwise specified, **all units are in radians, meters, and seconds**, where appropriate.

1) RR robot

Consider the following robot



Given the above RR arm, each with a mass of m_i located at $l_i/2$ and a moment of intertia of I_i about their com

(1) [5 points] Determine the Kinetic Energy for the RR arm shown above.

$T = \frac{1}{8}m_1 l_1^2 \dot{\theta_1}^2 + \frac{1}{2}I_1\dot{\theta_1}^2 + \frac{1}{2}m_2 l_1^2 \dot{\theta_1}^2 + \frac{1}{8}m_2 l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2}I_2(\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2}m_2 l_1 l_2 \dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2$

(2) [5 points] Determine the Potential Energy for the RR arm shown above.

$\frac{1}{2}m_1 g l_1 \sin\theta_1 + m_2 g(l_1 \sin\theta_1 + \frac{1}{2}l_2 \sin\theta_{12})$

(3) [2 points] Write the Lagrangian for the RR arm shown above.

$L = T - V$

$L = \frac{1}{8}m_1 l_1^2 \dot{\theta_1}^2 + \frac{1}{2}I_1\dot{\theta_1}^2 + \frac{1}{2}m_2 l_1^2\dot{\theta_1}^2 + \frac{1}{8}m_2 l_2^2(\dot{\theta_1} + \dot{\theta_2})^2 + \frac{1}{2}I_2(\dot{\theta_1} + \dot{\theta_2})^2 + \frac{1}{2}m_2 l_1 l_2\dot{\theta}_1(\dot{\theta_1} + \dot{\theta_2})\cos\theta_2 - \frac{1}{2}m_1 g l_1 \sin\theta_1 - m_2 g(l_1 \sin\theta_1 - \frac{1}{2}l_2 \sin\theta_{12})$

(4) [10 points] Determine the Equations of Motion with respect to θ_1 for the RR arm shown above using the Lagrangian.

$\tau_1 = (\frac{1}{4}m_1 l_1^2 + \frac{1}{4}m_2 l_2^2 + I_1 + I_2 + m_2 l_1^2 + m_2 l_1 l_2 \cos\theta_2)\ddot{\theta_1} + (I_2 + \frac{1}{4}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 \cos\theta_2)\ddot{\theta_2} - m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \sin\theta_2 - \frac{1}{2}m_2 l_1 l_2 \dot{\theta_2}^2 \sin\theta_2 + \frac{1}{2}m_1 g l_1 \cos\theta_1 + m_2 g(l_1 \cos\theta_1 + \frac{1}{2}l_2 \cos(\theta_1 + \theta_2))$

(5) [10 points] Determine the Equations of Motion with respect to θ_2 for the RR arm shown above using the Lagrangian.

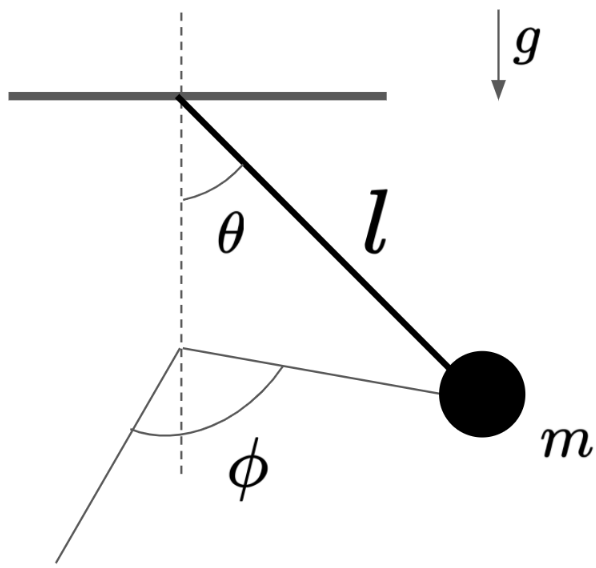
$\tau_2 = (I_2 + \frac{1}{4}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 \cos\theta_2)\ddot{\theta}_1 + (I_2 + \frac{1}{4}m_2 l_2^2)\ddot{\theta}_2 + \frac{1}{2}m_1 l_1 l_2 \sin\theta_2 \dot{\theta}_1^{\,2} + \frac{1}{2}m_2 g l_2 \cos(\theta_1 + \theta_2)$

(6) [5 points] Rewrite the equations of motion into standard form.

$$\tau = \begin{bmatrix} \frac{1}{4}m_1 l_1^2 + \frac{1}{4}m_2 l_2^2 + I_1 + I_2 + m_2 l_1^2 + m_2 l_1 l_2 \cos\theta_2 & I_2 + \frac{1}{4}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 \cos\theta_2 \\ I_2 + \frac{1}{4}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 \cos\theta_2 & I_2 + \frac{1}{4}m_2 l_2^2 \end{bmatrix} \ddot{\theta} + \begin{bmatrix} -m_2 l_1 l_2 \dot{\theta}_2 \sin\theta_2 & -\frac{1}{2}m_2 l_1 l_2 \dot{\theta}_2 \sin\theta_2 \\ \frac{1}{2}m_2 l_1 l_2 \dot{\theta}_1 \sin\theta_2 & 0 \end{bmatrix} \dot{\theta} + \begin{bmatrix} \frac{1}{2}m_1 g l_1 \cos\theta_1 + m_2 g(l_1 \cos\theta_1 + \frac{1}{2}l_2 \cos(\theta_1 + \theta_2)) \\ \frac{1}{2}m_2 g l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

2) 3D Pendulum

Consider the following robot



Given a pendulum that can move anywhere in 3D space, with a fixed length l and a mass m, solve for its equations of motion. Hint: use spherical coordinates.

(1) [5 points] Determine the Kinetic Energy for 3D pendulum.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} l\sin\theta\cos\phi \\ l\sin\theta\sin\phi \\ -l\cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} l(\dot{\theta}_1\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi) \\ l(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\cos\phi\sin\theta) \\ l\dot{\theta}\sin\theta \end{bmatrix}$$

$$T = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2)$$

(2) [5 points] Determine the Potential Energy for the 3D pendulum.

$$V = -mgl\cos\theta$$

(3) [2 points] Write the Lagrangian for the 3D pendulum.

$L = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2) + mgl\cos\theta$

(4) [10 points] Determine the Equations of Motion with respect to θ for the 3D pendulum.

$\tau_1 = ml^2\ddot{\theta} - ml^2\sin(\theta)\cos(\theta)\dot{\phi}^2 + mgl\sin\theta$

(5) [10 points] Determine the Equations of Motion with respect to ϕ for the 3D pendulum.

$\tau_2 = ml^2 \sin^2(\theta)\ddot{\phi} + 2ml^2 \sin(\theta)\cos(\theta)\dot{\theta}\dot{\phi}$

(6) [5 points] If ϕ is kept constant, what do the equations above reduce to? What system that we've studied in class has similar EOMs?

It's just a normal pendulum in the plane if ϕ is held constant. The above equations reduce to:
$\tau_1 = ml^2\ddot{\theta} + mgl\sin\theta$
$\tau_2 = 0$

5 Feedback

1) Feedback Form *5 points*

We are always looking to improve the class! To that end, we're looking for your feedback on the assignments. When you've completed the assignment, please fill out the feedback form.

6 Code Questions

There is no coding portion in this homework.

7 Submission Checklist

- ☐ Upload `writeup.pdf` to Gradescope and Canvas.
- ☐ After completing the entire assignment, fill out the feedback form[1].

[1] https://canvas.cmu.edu/courses/11823/quizzes/27788