Assignment 1: Forward Kinematics

Robot Kinematics and Dynamics

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Contents

1 Overview

Welcome to the very first hands-on assignment for Robot Kinematics and Dynamics! In the previous assignment, we ensured that you have all the preliminary material to start diving into the course material from a good base. This assignment will give you a solid foundation on the absolute fundamentals of kinematics:

- degrees of freedom
- homogeneous transformations
- forward kinematics

You will also use the robots for the first time. We will make sure you can properly communicate with the arms, which will prepare you to start taking the reins in the coming assignments.

1.1 Accessing the Robots (Specific to CMU)

The two robots will be in the REL, available immediately. However, to ensure that as many people as possible get an opportunity to work with them, we've instituted a time slot policy. On Piazza, there will be a link to a spreadsheet that you must fill out to get access to the robot. Time slots will fall into two categories: Remote Access and REL. Remote Access means controlling the robot via accessing a robot in the REL from your own computer. REL means coming to the lab in-person. The availability and length of the time slots will vary depending on the lab and TA availability. For this first assignment, all slots will be via Remote Access and will be numerous enough that everyone will have the opportunity to control the robot from home. Future labs, which are more complicated, may have fewer slots. For those labs, there will be an option to complete extra work in simulation as an alternative to spending time working on the actual robots. It is highly, highly recommended that you complete the written section before your time slot. Detailed directions for Remote Access will be posted on Piazza.

1.2 About this assignment

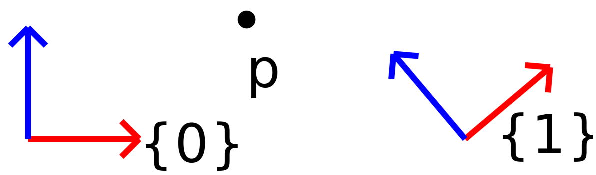
We recognize that, in this assignment, you will be typesetting many matrices. Not only is being able to typeset matrices is a necessary skill for explaining your work in this course, it also is a critical skill for much of the robotics development you may be involved with in the future. Once you have the matrix typesetting templates created in this assignment, you will be well prepared to typeset matrices in the future.

2 Background

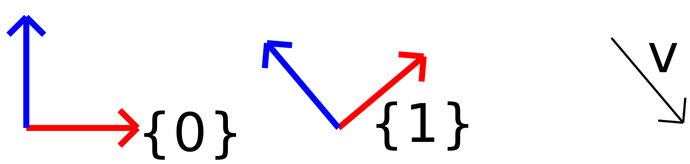
2.1 Preliminary Terms

Coordinate Frame: A coordinate frame consists of a special point called the *origin* and some number of orthonormal[1] axes. Frames provide a reference for measurements of positions and rotations of a body or point. Frame $\mathbf{i}$ is denoted $\{\mathbf{i}\}$. For the purposes of this class, all frames must be right-handed.

Point: A point is a fixed position in space. Its coordinates depend on the frame describing it. Point $\mathbf{p}$ relative to $\{\mathbf{0}\}$ is denoted $\mathbf{p^0}$. For example, in the figure below $\mathbf{p^0} = \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$ and $\mathbf{p^1} = \left[\begin{smallmatrix}-1\\2\end{smallmatrix}\right]$.



Vector: A direction and magnitude that is free in space. Among other things, vectors can describe offsets from a point, forces, or velocities. Just like a point, the direction of the vector depends upon in which frame it is referenced, but the magnitude remains constant in all frames. For example, in the figure below $\mathbf{v^0} = \left[\begin{smallmatrix}.7071\\-.7071\end{smallmatrix}\right]$ and $\mathbf{v^1} = \left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right]$ both have a magnitude of 1.



Rigid Body: A collection of points where the distance between two points and the handedness of the points remains constant while the collection undergoes a displacement.

Rigid Motion: A translation, rotation, or combination of the two that can be applied to a body without changing the distance between any two points in the body or changing the

[1]Orthonormal axes are:
- unit vectors (length 1)
- mutually orthogonal (perpendicular to each other)

handedness of the points.

$\mathbb{R}^2$: Two-dimensional Cartesian space; can be parameterized by x and y.

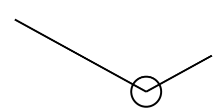
SE(2): It is both 1) the space corresponding to position and orientation of a rigid body in two dimensions and 2) the set of rigid motions in two dimensions. It is usually parameterized by x, y, and θ.

Workspace: The set of all points $\mathbf{p}$ such that there exists some joint configuration which places a robot's end effector at $\mathbf{p}$.

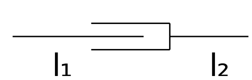
2.2 Robots and Robot Diagrams

For the purpose of this class, a **robot** or a **linkage** is a combination of *links* (which are rigid bodies) and *joints*, which connect two links with a certain constraint on relative motion. Links are generally represented by straight lines in robot diagrams. The two categories of joints we will use in this class are *rotational* and *prismatic* joints.

Revolute Joint: Creates an angular offset between two adjacent links. This offset is typically notated as θ, with a subscript that matches the joint index or name. Positive rotations go from the x axis towards the y axis (i.e., they follow the right hand rule). They are drawn as follows in robot diagrams:

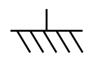


Prismatic Joint: Creates a translational offset between two adjacent links along a single axis. The notation for this offset varies. We will refer to this as d with a subscript to match the joint index or name, but in a robot with both rotational and prismatic joints sometimes $\theta or q$ is used for any joint. They are drawn as follows in robot diagrams. Note that this figure represents two separate links with one prismatic joint in between with a total length is $l_1 + l_2 + d$; however, depending on the system, l_1 and l_2 may be omitted and just combined into d. You can use either convention, but be sure to label any diagrams clearly.

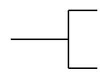


Kinematic Chain: An n-joint kinematic chain is a robot consisting of $n + 1$ links and n joints, connected in series. If the robot is attached to the world, the first link is called the base link. Often, the first link has zero length and is omitted. Many of our examples will have a zero length base link or "no base link." Joint i connects links $i - 1$ and i and joint i moves link i. It is also possible to have a free floating robot of $n+1$ links connected by n joints.

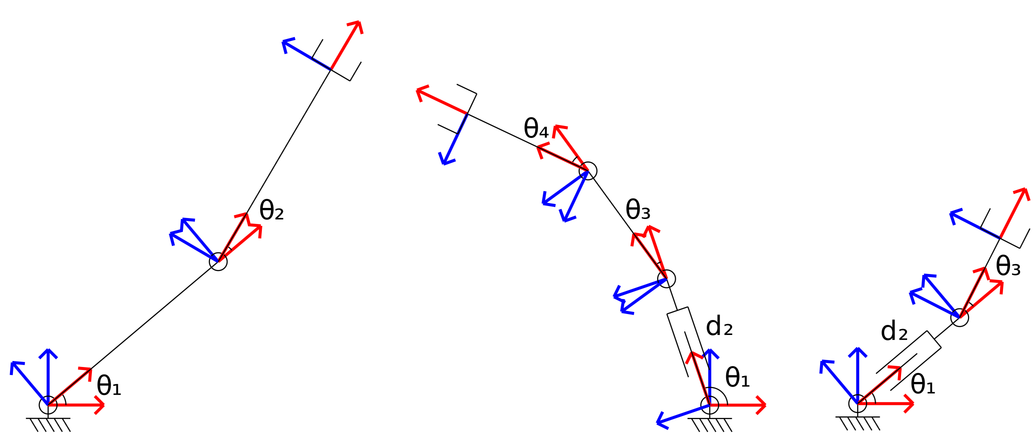
Fixed Base: Drawn as follows, this represents a rigid point of attachment to the world.



End Effector: Drawn as follows, this represents a gripper or other end effector attached to the end of a robot link (or directly to the output of a joint). We generally do not consider the end effector to add a degree of freedom to the robot, as it does not take part in the overall kinematics.



Frame Conventions: We define a *base* frame as a coordinate frame at the point where the robot is fixed to the world, labeled $\{\mathbf{0}\}$ (NOTE: in the OLI material, this frame is omitted). We also define a coordinate frame at the base of each link (the *proximal* frame for that link). Optionally, we can also define a frame at the *end* of each link, where the next link is attached by a joint; this is the *distal* frame. Note that this is also omitted in the OLI material, but is used in this assignment to help define the order of all intermediate transformations. The convention in this class will be that the x axes of these frames points from the base to the end of the link. End effectors of planar robots will have a frame defined at the attachment point to the link, and the x axes will point *out* of the gripper (i.e., this frame will usually match the distal frame of the connecting link). Positive angles of rotation will be counter-clockwise (from the x axis towards the y axis). Below are several simple robots demonstrating these conventions.



2.3 Degrees of Freedom

In robot kinematics, a system's *degrees of freedom*, or *DOF*, is the number of parameters needed to completely define a particular configuaration. For example:

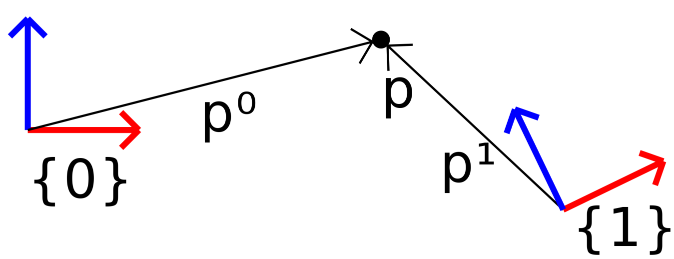
- A point in two dimensions (i.e., $\mathbb{R}^2$) has 2 DOF, since $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ completely defines the point's configuration.
- A rigid body in the plane has 3 DOF, since $\left[\begin{smallmatrix} x \\ y \\ \theta \end{smallmatrix}\right]$ completely defines its position and orientation.
- An fixed-base n-joint kinematic chain in SE(2) has n DOF, since n parameters for the joints completely define the robot's configuration. If the robot isn't attached to the base frame, the base link can move freely (in x, y, and θ), and the robot has $n+3$ DOF.

2.4 Rigid Motions / Transforms in Two Dimensions

There are two basic types of motions/transformation we are concerned with: rotations and translations. Rotations are represented with matrices, and translations are represented by vectors. These can be combined into compound motions/transformations.

Rotations and translations are used for two different concepts. The first is the motion of a point or rigid body within a coordinate frame:

The second concept is a *transform* between two coordinate frames. In other words, if you have the coordinates of a point or vector in $\{\mathbf{0}\}$, what are the coordinates of this same point or vector in $\{\mathbf{1}\}$:

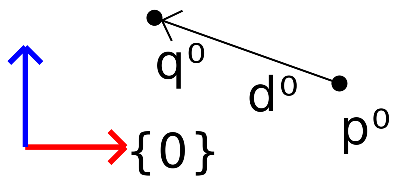


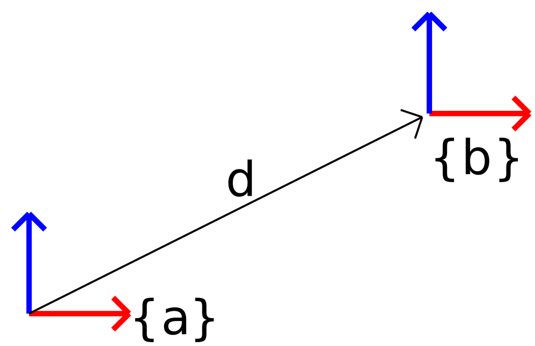
Notationally, transformations follow the convention of a superscript and a subscript, to describe which frames are transformed between. A transform $\mathbf{T}^{\mathbf{a}}_{\mathbf{b}}$ between two frames has several interpretations (notice the order of the superscripts in each case).

- The motion that moves $\{\mathbf{a}\}$ to $\{\mathbf{b}\}$.
- $\{\mathbf{b}\}$ as represented in $\{\mathbf{a}\}$.
- The transform that changes the representation of a point or vector from $\{\mathbf{b}\}$ to $\{\mathbf{a}\}$.

Note that the motion between two frames is the opposite of the transform that changes point representation between these frames. This is something which can often lead to errors! A good rule of thumb is that the top number always represents what frame this motion/transform is represented in. Here, the transform is in the frame of reference $\{\mathbf{a}\}$. Below, we explicitly consider translations, rotations, and combined motions.

2.4.1 Translation



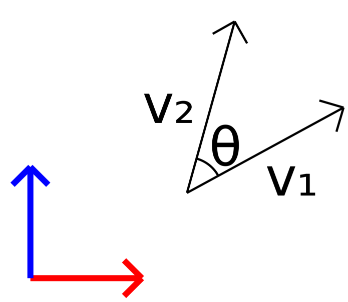


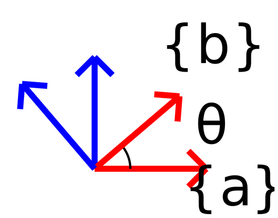
A translation $\mathbf{d} = \left[\begin{smallmatrix}\Delta x \\ \Delta y\end{smallmatrix}\right]$ expresses a change in position of a point or rigid body, or a position offset between coordinate frames. In terms of rigid motions, translating a point is the geometric addition of two positions; above left, $\mathbf{q^0} = \mathbf{p^0} + \mathbf{d^0}$. This is the motion that moves $\mathbf{p}$ to $\mathbf{q}$ in $\{\mathbf{0}\}$.

For transformations, $\mathbf{d_b^a}$ is the offset from $\{\mathbf{a}\}$ to $\{\mathbf{b}\}$ (above right, $\mathbf{d} = \mathbf{d_b^a} = \left[\begin{smallmatrix}4 \\ 2\end{smallmatrix}\right]$). In other words, $\mathbf{d_b^a}$ is:

- The motion that moves (the origin) of $\{\mathbf{a}\}$ to the origin of $\{\mathbf{b}\}$.
- The representation of $\{\mathbf{b}\}$ in $\{\mathbf{a}\}$ (e.g., the coordinates of the origin of $\{\mathbf{b}\}$ in $\{\mathbf{a}\}$).
- The transform that changes the representation of a point p from $\{\mathbf{b}\}$ to $\{\mathbf{a}\}$: $\mathbf{p^a} = \mathbf{d_b^a} + \mathbf{p^b}$.

2.4.2 Rotation





A *rotation matrix* $\mathbf{R}$ represents rigid body rotations or rotational offsets between coordinate frames. In 2 dimensions, the rotation matrix corresponding to a rotation of θ is

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

For rigid motions, this is the rotation of a body or vector by θ; above left, the vector $v_2 = \mathbf{R}(\theta)v_1$.

For transformations, $\mathbf{R}^{\mathbf{a}}_{\mathbf{b}}$ is the rotation from $\{\mathbf{a}\}$ to $\{\mathbf{b}\}$ (above right, $\mathbf{R}^{\mathbf{a}}_{\mathbf{b}} = \mathbf{R}(\theta)$). $\mathbf{R}^{\mathbf{a}}_{\mathbf{b}}$ can be described as:

- The motion that rotates $\{\mathbf{a}\}$ to $\{\mathbf{b}\}$.
- The representation of $\{\mathbf{b}\}$ in $\{\mathbf{a}\}$; $\mathbf{R}^{\mathbf{a}}_{\mathbf{b}}$ can be found by expressing the axes of $\{\mathbf{b}\}$ in $\{\mathbf{a}\}$ as column vectors.
- The rotation that changes the representation of a vector v from $\{\mathbf{b}\}$ to $\{\mathbf{a}\}$: $\mathbf{v}^{\mathbf{a}} = \mathbf{R}^{\mathbf{a}}_{\mathbf{b}}\mathbf{v}^{\mathbf{b}}$.

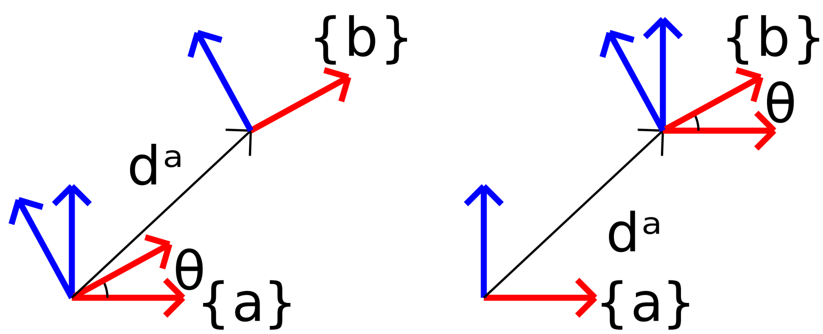
Furthermore, as shown in lecture, for any rotation matrix $\mathbf{R}$, $(\mathbf{R})^{-1} = (\mathbf{R})^{\top}$.

2.4.3 Compound Motions: Homogeneous Transforms

Any rigid motion/transform between two frames can be expressed by combining a single rotation and translation. A *homogeneous transform* is a single matrix which allows us to express this combined rotation and translation:

$$\mathbf{H}^{\mathbf{a}}_{\mathbf{b}} = \begin{bmatrix} \mathbf{R}^{\mathbf{a}}_{\mathbf{b}} & \mathbf{d}^{\mathbf{a}}_{\mathbf{b}} \\ 0\ 0 & 1 \end{bmatrix}$$

As with previous notation, $\mathbf{H}^{\mathbf{a}}_{\mathbf{b}}$ is a motion from $\{\mathbf{a}\}$ to $\{\mathbf{b}\}$, or equivalently the transform that changes the coordinates of points from $\{\mathbf{b}\}$ to $\{\mathbf{a}\}$.



An important point here is that the *order* and *frame of reference* which you apply the rotation and transforms here matter. The following two descriptions are equivalent (and describe $\mathbf{H}^{\mathbf{a}}_{\mathbf{b}}$):

- First apply the rotation relative to $\{\mathbf{a}\}$, and then apply the translation relative to $\{\mathbf{a}\}$ (see above left).

- First apply the translation relative to $\{\mathbf{a}\}$ and then apply to rotation relative to the *new* frame resulting from this translation (see above right).

Note that the other two potential descriptions - rotation then translation in the new frame, or translation then rotation in $\{\mathbf{a}\}$ - are both equivalent but **do not describe $\mathbf{H^a_b}$**.

2.4.4 Points, Vectors, and Homogeneous Transforms

When using Homogeneous Transforms to operate on points and vectors, we must add an extra coordinate to these values. Points are padded with a 1, so $\mathbf{p^i} = \left[\begin{smallmatrix} p_x \\ p_y \end{smallmatrix}\right]$ is now $\left[\begin{smallmatrix} p_x \\ p_y \\ 1 \end{smallmatrix}\right]$, whereas vectors are padded with a 0, so $\mathbf{v^i} = \left[\begin{smallmatrix} v_x \\ v_y \end{smallmatrix}\right]$ is now $\left[\begin{smallmatrix} v_x \\ v_y \\ 0 \end{smallmatrix}\right]$. This allows the matrix dimensions to match when multiplying by such transforms.

If $\mathbf{H^0}$ describes a motion in $\{\mathbf{0}\}$, then the point $\mathbf{q^0}$ resulting from this motion starting at point $\mathbf{p^0}$ is:

$$\begin{aligned}
\mathbf{q^0} &= \mathbf{H^0}\begin{bmatrix} \mathbf{p^0} \\ \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{R^0} & \mathbf{d^0} \\ \\ 0\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{p^0} \\ \\ 1 \end{bmatrix} \\
&= \mathbf{R^0}\mathbf{p^0} + \mathbf{d^0}
\end{aligned}$$

Note that the 1 we added to the bottom of the point results in the addition of $\mathbf{d^0}$; because free vectors have a 0 added, they are only affected by the rotation portion of $\mathbf{H^0}$.

Using $\mathbf{H^a_b}$ as a transformation between coordinate representations of points/vectors, we have

$$\left[\begin{smallmatrix} \mathbf{p^a} \\ 1 \end{smallmatrix}\right] = \mathbf{H^a_b}\left[\begin{smallmatrix} \mathbf{p^b} \\ 1 \end{smallmatrix}\right] \text{ and } \left[\begin{smallmatrix} \mathbf{v^a} \\ 0 \end{smallmatrix}\right] = \mathbf{H^a_b}\left[\begin{smallmatrix} \mathbf{v^b} \\ 0 \end{smallmatrix}\right].$$

2.4.5 Inverting $\mathbf{H^a_b}$

The inverse of $\mathbf{H^a_b}$ is $\mathbf{H^b_a}$, representing the opposite motion, or a reverse transform between the two frames.

To invert $\mathbf{H^a_b}$, we could perform the matrix inversion manually, but there is a faster method using the properties of $\mathbf{H^a_b}$. First, recall that $\mathbf{p^a} = \mathbf{R^a_b}\,\mathbf{p^b} + \mathbf{d^a_b}$. Solving for $\mathbf{p^b}$, we can see that $\mathbf{p^b} = (\mathbf{R^a_b})^\top\mathbf{p^a}$ - $(\mathbf{R^a_b})^\top\mathbf{d^a_b}$. Therefore, given $\mathbf{R^a_b}$ and $\mathbf{d^a_b}$, $\mathbf{R^b_a} = (\mathbf{R^a_b})^\top$ and $\mathbf{d^b_a}$ = -$(\mathbf{R^a_b})^\top\mathbf{d^a_b}$. This finally gives us that

$$(\mathbf{H^a_b})^{-1} = \mathbf{H^b_a} = \left[\begin{smallmatrix} (\mathbf{R^a_b})^\top & (-(\mathbf{R^a_b})^\top\mathbf{d^a_b}) \\ (\mathbf{0})^\top & 1 \end{smallmatrix}\right]$$

Inspection can show that this holds true for $\mathbf{v^a}$ and $\mathbf{v^b}$ as well.

2.4.6 Composing Transforms

Homogeneous transforms are composable. For example:

$$\mathbf{H_2^0} = \mathbf{H_1^0 H_2^1}$$

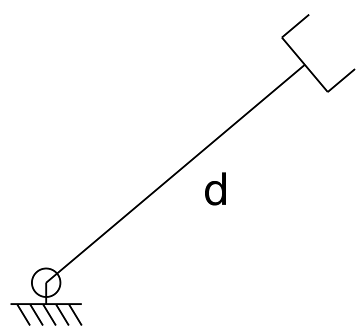
Note that in expressions like $\mathbf{p^1} = \mathbf{H_0^1 p^0}$, the superscripts "cancel" out, which gives you a quick check for your systems of equations. However, note that multiplication of transformations is **not** commutative and this cancellation only works diagonally one way: $\mathbf{H_2^0} \neq \mathbf{H_2^1 H_1^0}$. Using superscripts for the frame of points and vectors will help enforce this consistency.

Furthermore, $\left(\mathbf{H_i^j}\right)^{-1} = \mathbf{H_j^i}$, so if we are given $\mathbf{H_1^0}$, $\mathbf{H_1^2}$, and $\mathbf{H_3^2}$, we can find $\mathbf{H_3^0} = \mathbf{H_1^0}(\mathbf{H_1^2})^{-1}\mathbf{H_3^2}$.

2.5 Forward Kinematics

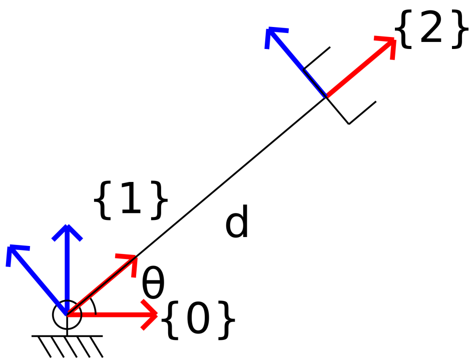
Forward kinematics is the idea of taking a robot's input parameters (joint angles, link lengths, etc.) and finding the position of the end effector. We can do this by writing homogeneous transforms that use the input parameters as variables in order to find the final end effector position. In effect, we want to define frames at each link and joint and describe their differences in terms of the robot parameters.

Let us consider the single link R arm of length d.



We wish to find the end effector position. First, we can find it analytically by noting that $x = d\cos(\theta)$ and $y = d\sin(\theta)$. In vector form, $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} d\cos(\theta) \\ d\sin(\theta) \end{smallmatrix}\right]$. While finding the forward kinematics algebraically will always work, it definitely gets more error prone with larger systems, and when composing multiple arms. The way around this is using homogeneous transforms.

We define one frame at the base, the base frame, usually labeled $\mathbf{0}$. Let $\{\mathbf{1}\}$ be another frame at the base of the link, but rotated by θ. Finally, let $\{\mathbf{2}\}$ be based at the end of the link, such that $\mathbf{p_e^2} = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ in that frame is the end-effector position.



Note that:

- $\mathbf{H_1^0} = \begin{bmatrix} \mathbf{R}_\theta & \mathbf{0} \\ (\mathbf{0})^\top & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $\mathbf{H_2^1} = \begin{bmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Therefore, $\mathbf{H_2^0} = \mathbf{H_1^0}\mathbf{H_2^1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & d\cos(\theta) \\ \sin(\theta) & \cos(\theta) & d\sin(\theta) \\ 0 & 0 & 1 \end{bmatrix}$. Lastly, we can see that $\mathbf{H_2^0}\mathbf{p_e^2} = \mathbf{H_2^0}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} d\cos(\theta) \\ d\sin(\theta) \end{bmatrix}$, which matches our analytical solution.

3 In Class Questions

The following questions will be done in class, as a part of a group. Your group's answer will still need to be turned in with the rest of your assignment, however unlike the rest of the work this is allowed to be done in groups.

1) Matrix Review

Given the Matrix $A = \begin{bmatrix} -s_1 - s_{12} & -s_{12} \\ c_1 + c_{12} & c_{12} \end{bmatrix}$

(1) [1 point] Is A an element of $SO(2)$?

$\det A = c_{12}(-s_1 - s_{12}) + s_{12}(c_1 + c_{12})$
$\det A = s_{12}c_1 - s_1 c_{12}$
$\det A$ does not always equal 1, $A \notin SO(2)$

(2) [1 point] For what values of θ does rank(A) = 1?

rank(A) = 1 when $\det A = 0$
$0 = s_{12}c_1 - s_1 c_{12}$
$0 = c_1(s_1 c_1 + s_2 c_2) - s_1(c_1 c_2 - s_1 s_2)$
$0 = s_2(s_1 s_1 + c_1 c_1)$
$0 = s_2$, so $\theta_2 = 0, \pi, \theta_1$ is unconstrained

(3) [1 point] What is the inverse of A?

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{s_{12}c_1 - s_1 c_{12}} \begin{bmatrix} c_{12} & -c_1 - c_{12} \\ s_{12} & -s_1 - s_{12} \end{bmatrix}$$

(4) [1 point] Using (2) and (3), what values of θ can you not solve Ax = b, where x and b are column vectors?

θ_2 cannot be 0 or π.

(5) [1 point] When rank(A)=1, what constraints are placed on b?

When rank(A) = 1, it must hold that $b = k\left[\begin{smallmatrix} -s_1 \\ c_1 \end{smallmatrix}\right]$ for some k for $\theta_2 = 0 or$b = a$\left[\begin{smallmatrix} -s_1 \\ c_1 \end{smallmatrix}\right]$ + c$\left[\begin{smallmatrix} -s_1 \\ c_1 \end{smallmatrix}\right] for \theta_2 = \pi$.

2) Basic Transformations

Let $\mathbf{t} = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$, θ =-$\frac{\pi}{4}$

(1) [1 point] Let $\mathbf{H_1}$ be the homogeneous transformation that translates a point by $\mathbf{t}$. What is $\mathbf{H_1}$?

$$\mathbf{H_1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) [1 point] Let $\mathbf{H_2}$ be the homogeneous transformation that rotates a point by θ. What is $\mathbf{H_2}$?

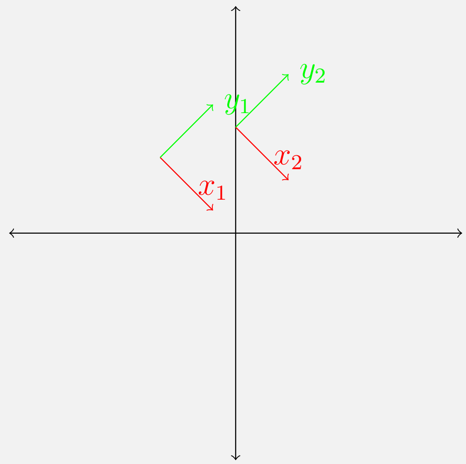
$$\mathbf{H_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3) [1 point] Show algebraically that $\mathbf{H_1H_2}$ is not equal to $\mathbf{H_2H_1}$.

$$\mathbf{H_1H_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -1 \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H_2H_1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

(4) [1 point] Show geometrically that $\mathbf{H_1 H_2}$ is not equal to $\mathbf{H_2 H_1}$.



4 Written Questions

For the following problems, fully evaluate all answers unless otherwise specified.

Answers for written questions must be typed. We recommend LaTeX, Microsoft Word, OpenOffice, or similar. However, diagrams can be hand-drawn and scanned in.

Unless otherwise specified, **all units are in radians, meters, and seconds**, where appropriate.

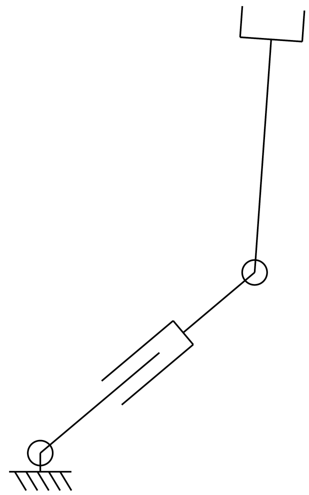
1) Degrees of Freedom

How many degrees of freedom do each of these robots have (assume $\mathbb{R}^2$)?

(1) [1 point]

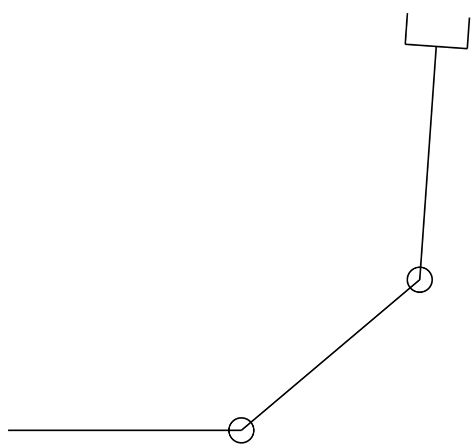
2 DOF

(2) [1 point]



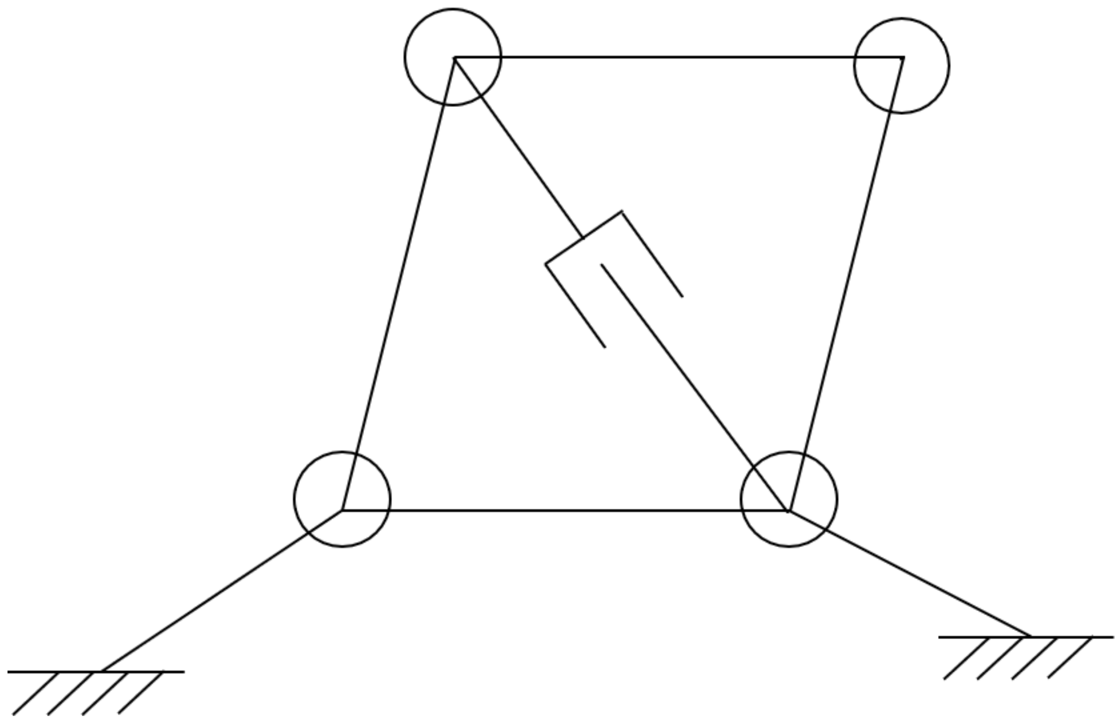
3 DOF

(3) [1 point]



5 DOF

(4) [1 point]



1 DOF

2) Rotation Matrices

Let $\mathbf{p} = \left[\begin{smallmatrix} 10 \\ 0 \end{smallmatrix}\right]$.

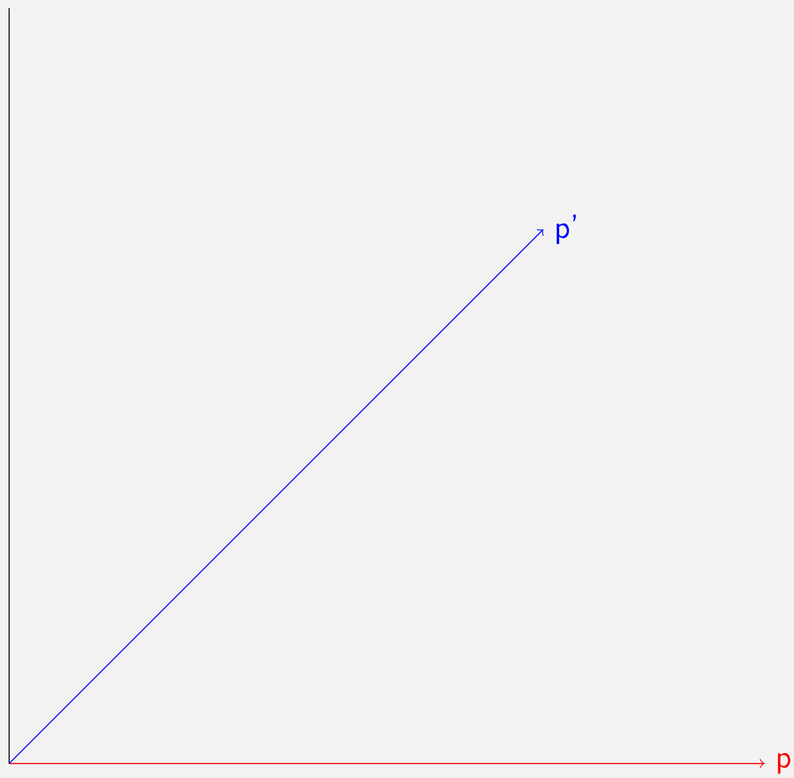
(1) [1 point] Let $\mathbf{R}$ be the rotation matrix representing a rotation by $\frac{\pi}{4}$. Find $\mathbf{R}$.

$$\mathbf{R} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

(2) [1 point] Let $\mathbf{p}'$ be the result of applying $\mathbf{R}$ to $\mathbf{p}$. Find $\mathbf{p}'$.

$$\mathbf{p}' = \begin{bmatrix} 5\sqrt{2} \\ 5\sqrt{2} \end{bmatrix}$$

(3) [1 point] Draw and label a set of axes, $\mathbf{p}$, and $\mathbf{p}'$ in this frame.



3) Inverting Homogeneous Transformations *5 points*

Algebraically show that if $\mathbf{H_i^j} = \begin{bmatrix} \mathbf{R_i^j} & \mathbf{d_i^j} \\ (\mathbf{0})^\top & 1 \end{bmatrix}$, then $\left(\mathbf{H_i^j}\right)^{-1} = \begin{bmatrix} \left(\mathbf{R_i^j}\right)^\top & -\left(\mathbf{R_i^j}\right)^\top \mathbf{d_i^j} \\ (\mathbf{0})^\top & 1 \end{bmatrix}$.

You do not need a formal proof, but you must show your steps clearly.

$$\mathbf{H_i^j} = \mathbf{TR}$$

$$(\mathbf{H_i^j})^{-1} = R^{-1}T^{-1}$$

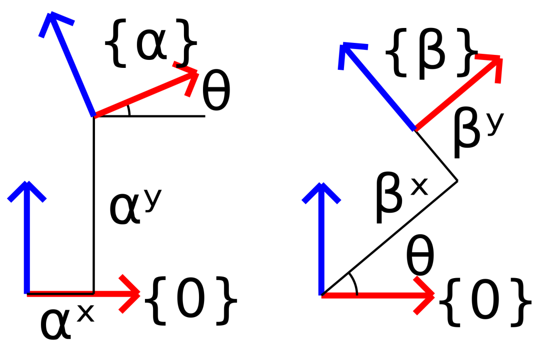
$$(\mathbf{H_i^j})^{-1} = \begin{bmatrix} \mathbf{R_i^j} & \mathbf{0} \\ (\mathbf{0})^\top & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} & \mathbf{d_j^i} \\ (\mathbf{0})^\top & 1 \end{bmatrix}^{-1}$$

$$(\mathbf{H_i^j})^{-1} = \begin{bmatrix} \left(\mathbf{R_i^j}\right)^\top & \mathbf{0} \\ (\mathbf{0})^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{d_j^i} \\ (\mathbf{0})^\top & 1 \end{bmatrix}$$

$$(\mathbf{H_i^j})^{-1} = \begin{bmatrix} \left(\mathbf{R_i^j}\right)^\top & -\left(\mathbf{R_i^j}\right)^\top \mathbf{d_i^j} \\ (\mathbf{0})^\top & 1 \end{bmatrix}$$

4) Homogeneous Transformations

Let $\mathbf{t} = \left[\begin{smallmatrix} 2 \\ 5 \end{smallmatrix}\right]$, $\theta = \frac{\pi}{4}$, and $\{\alpha\}$, $\{\beta\}$ be frames as shown in the following diagrams:



(1) [1 point] Let $\mathbf{T_1}$ be the homogeneous transformation that translates a point by $\mathbf{t}$. What is $\mathbf{T_1}$?

$$\mathbf{T_1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) [1 point] Let $\mathbf{T_2}$ be the homogeneous transformation that rotates a point by θ. What is $\mathbf{T_2}$?

$$\mathbf{T_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3) [5 points] Find $\mathbf{H}_{\alpha}^{\mathbf{0}}$, using the frames shown in the figure. Verify your solution by using this matrix to transform the following points (given in $\{\alpha\}$) to $\{\mathbf{0}\}$

(1) $\mathbf{p}^{\alpha} = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$

$$\mathbf{H}_{\alpha}^{\mathbf{0}} = \begin{bmatrix} \cos\theta & -\sin\theta & \alpha_x \\ \sin\theta & \cos\theta & \alpha_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_{\alpha}^{\mathbf{0}}\mathbf{p}^{\alpha} = \left[\begin{smallmatrix} \alpha_x \\ \alpha_y \end{smallmatrix}\right]$$

(2) $\mathbf{q}^{\alpha} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$

$$\mathbf{q}^{\alpha} = \begin{bmatrix} \alpha_x + \cos\theta \\ \alpha_y + \sin\theta \end{bmatrix}$$

and the following vectors (given in $\{\alpha\}$) to $\{\mathbf{0}\}$.

(3) $\mathbf{v}^\alpha = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$

$$\mathbf{v}^\alpha = \begin{bmatrix} -\sin\theta \\ \\ \cos\theta \end{bmatrix}$$

(4) $\mathbf{u}^\alpha = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$

$$\mathbf{u}^\alpha = \begin{bmatrix} \cos\theta - \sin\theta \\ \\ \sin\theta + \cos\theta \end{bmatrix}$$

Ensure that the coordinates make sense in the new representation.

(4) [5 points] Find $\mathbf{H}^{\mathbf{0}}_{\beta}$, using the frames shown in the figure. Verify your solution by using this matrix to transform the following points (given in $\{\beta\}$) to $\{\mathbf{0}\}$

(1) $\mathbf{p}^{\beta} = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$

$$\mathbf{H}^{\mathbf{0}}_{\beta} = \begin{bmatrix} \cos\theta & -\sin\theta & \beta_x \cos\theta - \beta_y \sin\theta \\ \sin\theta & \cos\theta & \beta_x \sin\theta + \beta_y \cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{p}^{\beta} = \begin{bmatrix} \beta_x \cos\theta - \beta_y \sin\theta \\ \beta_x \sin\theta + \beta_y \cos\theta \end{bmatrix}$$

(2) $\mathbf{q}^{\beta} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$

$$\mathbf{q}^{\beta} = \begin{bmatrix} (\beta_x + 1) \cos\theta - \beta_y \sin\theta \\ (\beta_x + 1) \sin\theta + \beta_y \cos\theta \end{bmatrix}$$

and the following vectors (given in $\{\beta\}$) to $\{\mathbf{0}\}$.

(3) $\mathbf{v}^{\beta} = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$

$$\mathbf{v}^{\beta} = \begin{bmatrix} \cos\theta - (1)\sin\theta \\ \\ \sin\theta + (1)\cos\theta \end{bmatrix}$$

(4) $\mathbf{u}^{\beta} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$

$$\mathbf{u}^{\beta} = \begin{bmatrix} (1)\cos\theta - (1)\sin\theta \\ \\ (1)\sin\theta + (1)\cos\theta \end{bmatrix}$$

Ensure that the coordinates make sense in the new representation.

(5) [1 point] Find $\mathbf{H}_{\beta}^{\alpha}$.

$\mathbf{t}_{\beta}^{\alpha} = \mathbf{t}_{\beta}^{\mathbf{0}} - \mathbf{t}_{\alpha}^{\mathbf{0}}$

$\mathbf{R}_{\beta}^{\alpha} = \begin{bmatrix} \cos(\theta_\beta - \theta_\alpha) & -\sin(\theta_\beta - \theta_\alpha) \\ \sin(\theta_\beta - \theta_\alpha) & \cos(\theta_\beta - \theta_\alpha) \end{bmatrix}$

$$\mathbf{H}_{\beta}^{\alpha} = \begin{bmatrix} \cos(\theta_\beta - \theta_\alpha) & -\sin(\theta_\beta - \theta\alpha) & \beta_x \cos\theta_\beta - \beta_y \sin\theta_\beta - \alpha_x \\ \sin(\theta_\beta - \theta_\alpha) & \cos(\theta_\beta - \theta_\alpha) & \beta_y \cos\theta_\beta + \beta_x \sin\theta_\beta - \alpha_y \\ 0 & 0 & 1 \end{bmatrix}$$

(6) [2 points] Compute $\mathbf{H}_{\alpha}^{\mathbf{0}}$ and $\left(\mathbf{H}_{\alpha}^{\mathbf{0}}\right)^{-1}$. Verify that $\left(\left(\mathbf{H}_{\alpha}^{\mathbf{0}}\right)^{-1}\mathbf{H}_{\alpha}^{\mathbf{0}}\right) = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$.

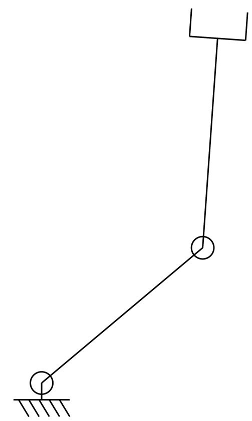
$$\mathbf{H}_{\alpha}^{\mathbf{0}} = \begin{bmatrix} \cos\theta_\alpha & -\sin\theta_\alpha & \alpha_x \\ \sin\theta_\alpha & \cos\theta_\alpha & \alpha_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left(\mathbf{H}_{\alpha}^{\mathbf{0}}\right)^{-1} = \begin{bmatrix} \cos\theta_\alpha & \sin\theta_\alpha & \alpha_x\cos\theta_\alpha + \alpha_y\sin\theta_\alpha \\ -\sin\theta_\alpha & \cos\theta_\alpha & -\alpha_x\sin\theta_\alpha + \alpha_y\cos\theta_\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left(\left(\mathbf{H}_{\alpha}^{\mathbf{0}}\right)^{-1}\mathbf{H}_{\alpha}^{\mathbf{0}}\right) = \mathbf{I}$$

5) Workspace and Frames

Consider the following robot:



Draw and clearly label the following, **using the frame conventions given in the background for this assignment**.

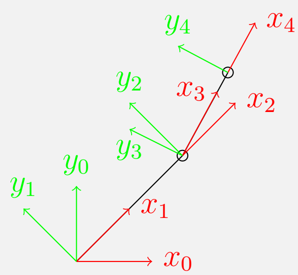
(1) [1 point] The base frame of the entire robot: $\{\mathbf{0}\}$.

(2) [1 point] The starting frame for the first link: $\{\mathbf{1}\}$

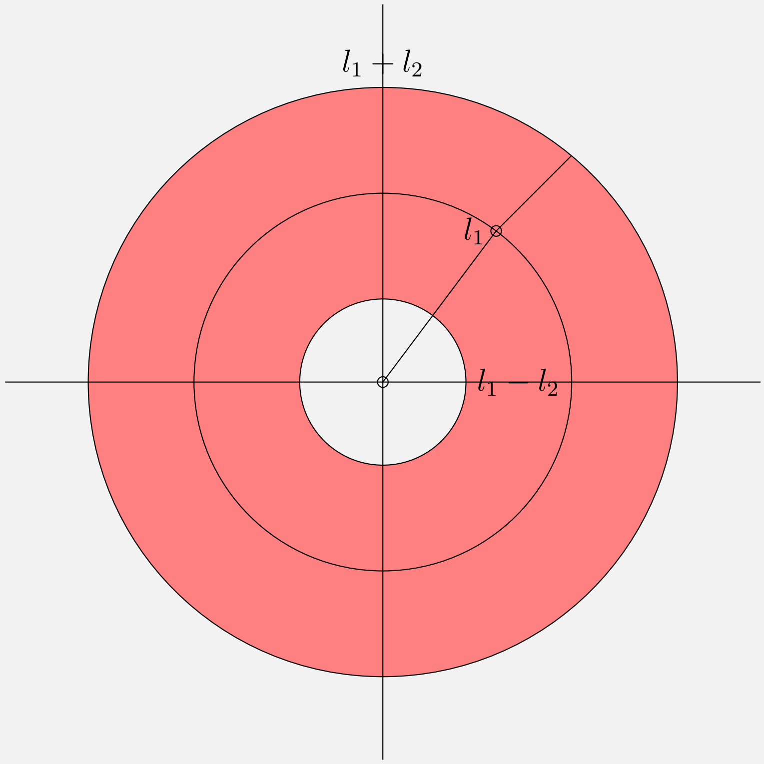
(3) [1 point] The frame at the end of the first link: $\{\mathbf{2}\}$

(4) [1 point] The starting frame for the second link: $\{\mathbf{3}\}$

(5) [1 point] The end effector frame: $\{\mathbf{4}\}$


y_4
x_4
y_2
x_3
x_2
y_3
y_0
y_1
x_1
x_0

(6) [3 points] Assume $l_1 > l_2$. On a separate plot, shade the workspace (in $\mathbb{R}^2$), ignoring self-collision. Label the dimensions of this plot.



6) Forward Kinematics of an RR Arm

For the robot in the previous question, compute the following in terms of l_1, l_2, θ_1, and θ_2:

(1) [1 point] $\mathbf{H_1^0}$

$$\mathbf{H_1^0} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) [1 point] $\mathbf{H_2^1}$

$$\mathbf{H_2^1} = \begin{bmatrix} 1 & 0 & l_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3) [1 point] $\mathbf{H_3^2}$

$$\mathbf{H_3^2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(4) [1 point] $\mathbf{H_4^3}$

$$\mathbf{H_4^3} = \begin{bmatrix} 1 & 0 & l_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(5) [1 point] $\mathbf{H_4^0}$, the transform from the end effector to the base frame (in terms of the previous transforms).

$$\mathbf{H_4^0} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & l_1\sin\theta_1 + l2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 \end{bmatrix}$$

(6) The position of the origin of the end effector frame $\{\mathbf{4}\}$, represented in the base frame $\{\mathbf{0}\}$, for the following values of $\begin{bmatrix}\theta_1\\\theta_2\end{bmatrix}$. Solving for both "nice" numbers and intermediate numbers will build intuition of RR arm forward kinematics.

(1) [1 point] $\begin{bmatrix}0\\0\end{bmatrix}$

$$\mathbf{H_4^0}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2)\\ l_1\sin\theta_1 + l2\sin(\theta_1+\theta_2)\\ 1\end{bmatrix}$$

$$\mathbf{H_4^0}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}l_1 + l_2\\ 0\\ 1\end{bmatrix}$$

(2) [1 point] $\begin{bmatrix}0\\\frac{\pi}{2}\end{bmatrix}$

$$\mathbf{H_4^0}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}l_1\\ l_2\\ 1\end{bmatrix}$$

(3) [1 point] $\begin{bmatrix} \frac{\pi}{2} \\ -\frac{\pi}{2} \end{bmatrix}$

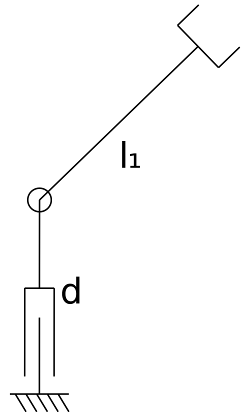
$$\mathbf{H_4^0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} l_2 \\ l_1 \\ 1 \end{bmatrix}$$

(4) [1 point] $\begin{bmatrix} \frac{\pi}{3} \\ \frac{\pi}{2} \end{bmatrix}$

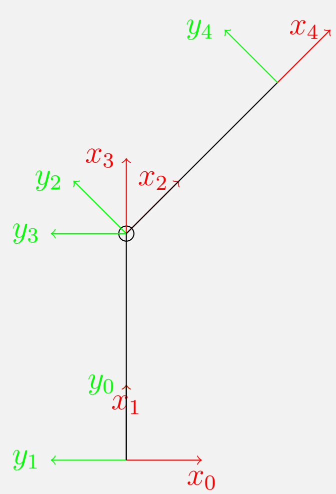
$$\mathbf{H_4^0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}l_1 - \frac{\sqrt{3}}{2}l_2 \\ \frac{\sqrt{3}}{2}l_1 + \frac{1}{2}l_2 \\ 1 \end{bmatrix}$$

7) Workspace and Frames of a PR Arm *8 points*

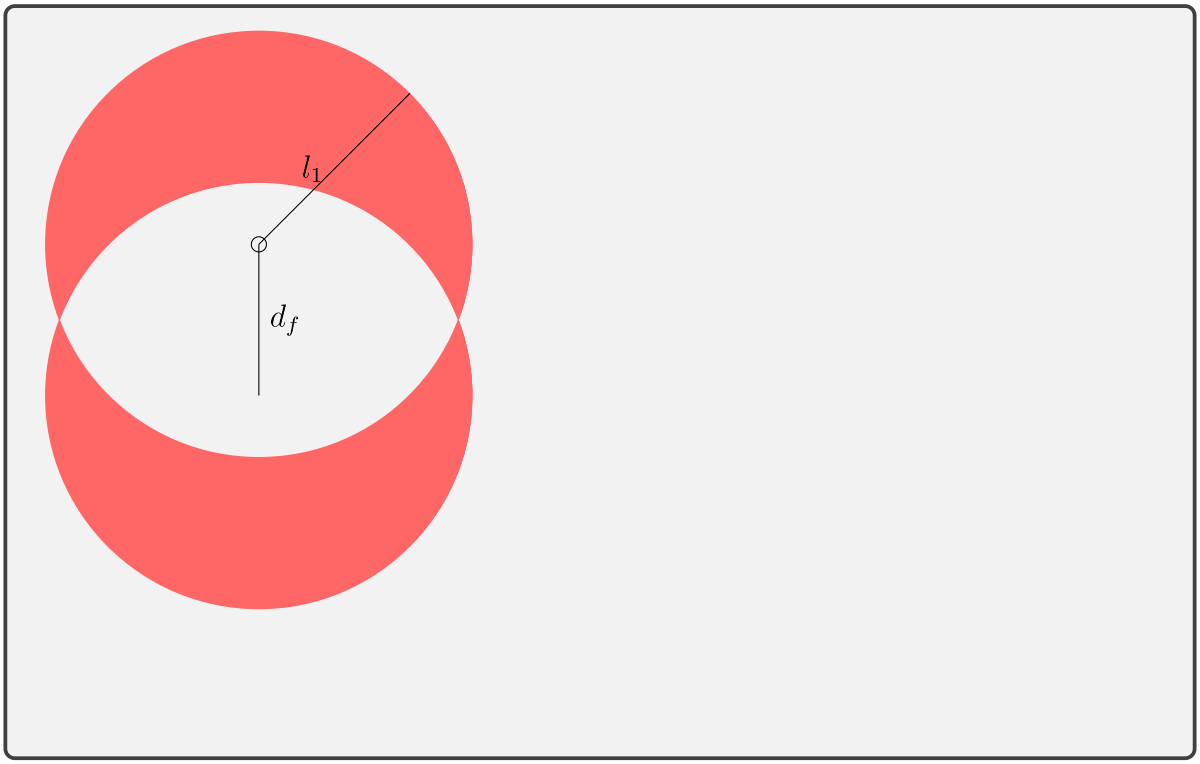
Repeat problem 5 for the following arm:



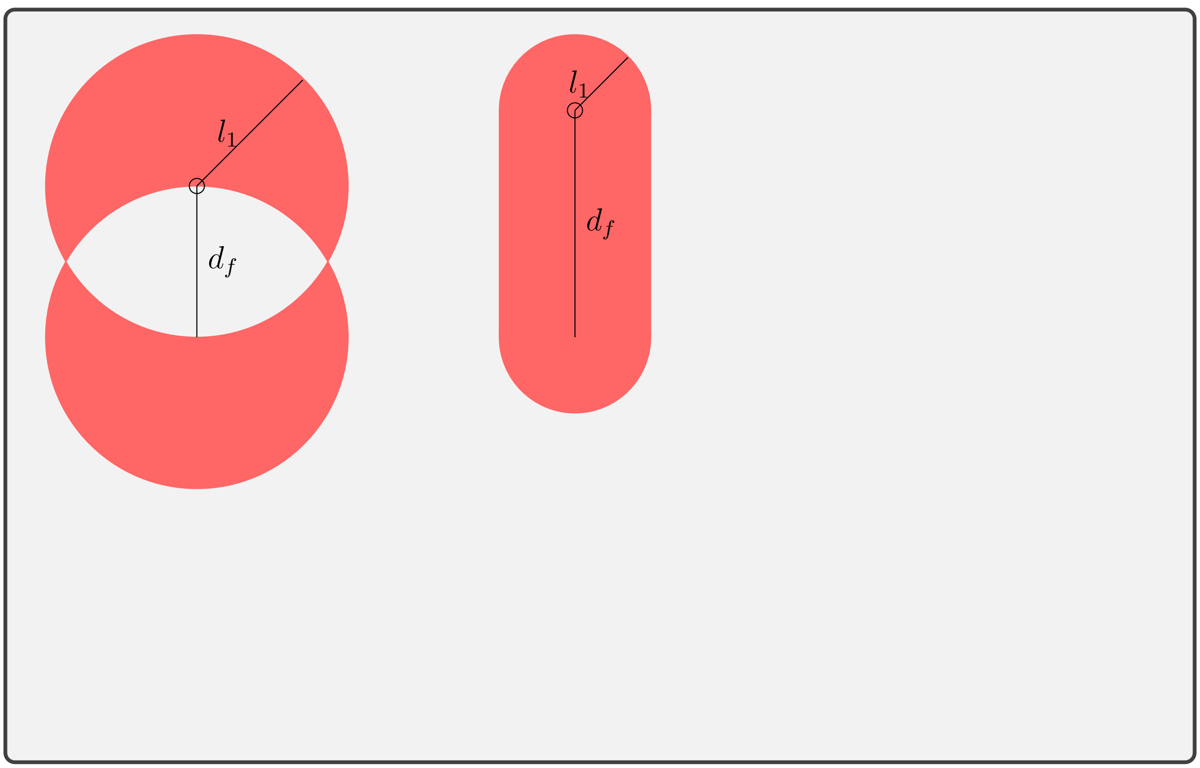
Remember to use the frame conventions given in the background!



For the workspace, assume the prismatic joint can go from $d = 0$ to $d = d_f$.



Draw a separate image for $d_f = l_1$ and $d_f > 2l_1$.



8) Forward Kinematics of a PR Arm *9 points*

Repeat problem 6 with the arm in the previous problem.

(1) [1 point] $\mathbf{H_1^0}$

$$\mathbf{H_1^0} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H_1^0} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) [1 point] $\mathbf{H_2^1}$

$$\mathbf{H_2^1} = \begin{bmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3) [1 point] $\mathbf{H_3^2}$

$$\mathbf{H_3^2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(4) [1 point] $\mathbf{H_4^3}$

$$\mathbf{H_4^3} = \begin{bmatrix} 1 & 0 & l_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(5) [1 point] $\mathbf{H_4^0}$, the transform from the end effector to the base frame (in terms of the previous transforms).

$$\mathbf{H_4^0} = \begin{bmatrix} -\sin\theta_2 & -\cos\theta_2 & -l_1 \sin\theta_2 \\ \cos\theta_2 & -\sin\theta_2 & l_2\cos\theta_2 + d \\ 0 & 0 & 1 \end{bmatrix}$$

Substitute in the following values for the joint positions $\begin{bmatrix} d \\ \theta \end{bmatrix}$ for question 6 part f. Solving for both "nice" numbers and intermediate numbers will build intuition of PR arm forward kinematics.

(1) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ l_1 \end{bmatrix}$$

(2) $\begin{bmatrix} 3 \\ \frac{\pi}{2} \end{bmatrix}$

$$\begin{bmatrix} -l_1 \\ 3 \end{bmatrix}$$

(3) $\begin{bmatrix} 1 \\ -\frac{\pi}{2} \end{bmatrix}$

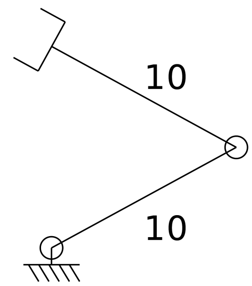
$$\begin{bmatrix} l_1 \\ 1 \end{bmatrix}$$

(4) $\begin{bmatrix} 3 \\ \frac{\pi}{4} \end{bmatrix}$

$$\begin{bmatrix} -l_1\frac{\sqrt{2}}{2} \\ 3 + l_1\frac{\sqrt{2}}{2} \end{bmatrix}$$

9) Singularities

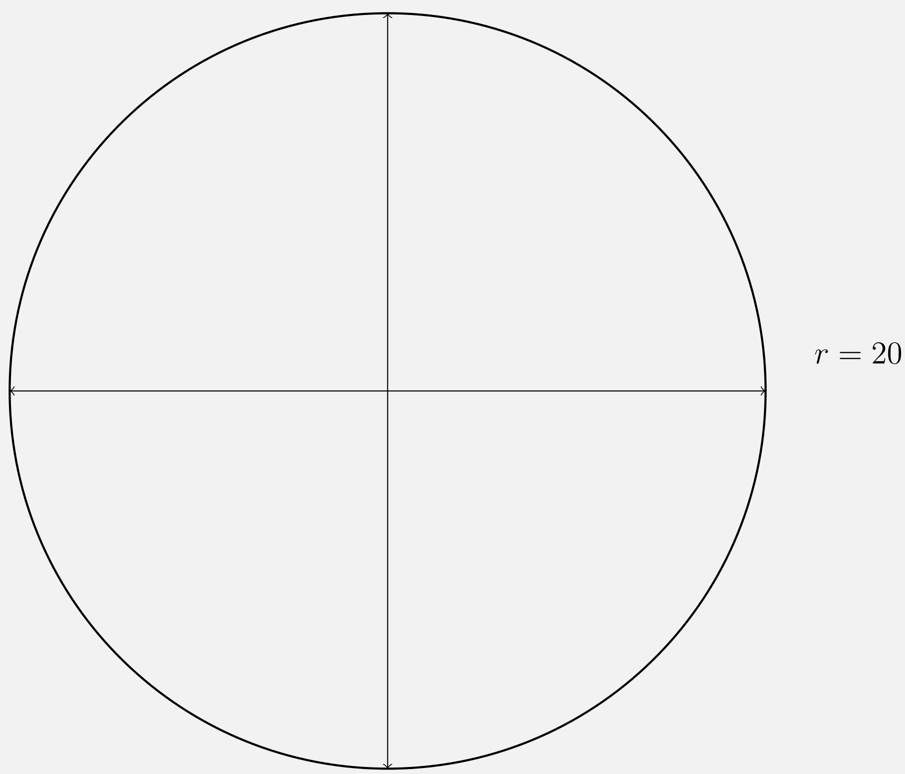
Consider the following RR arm:



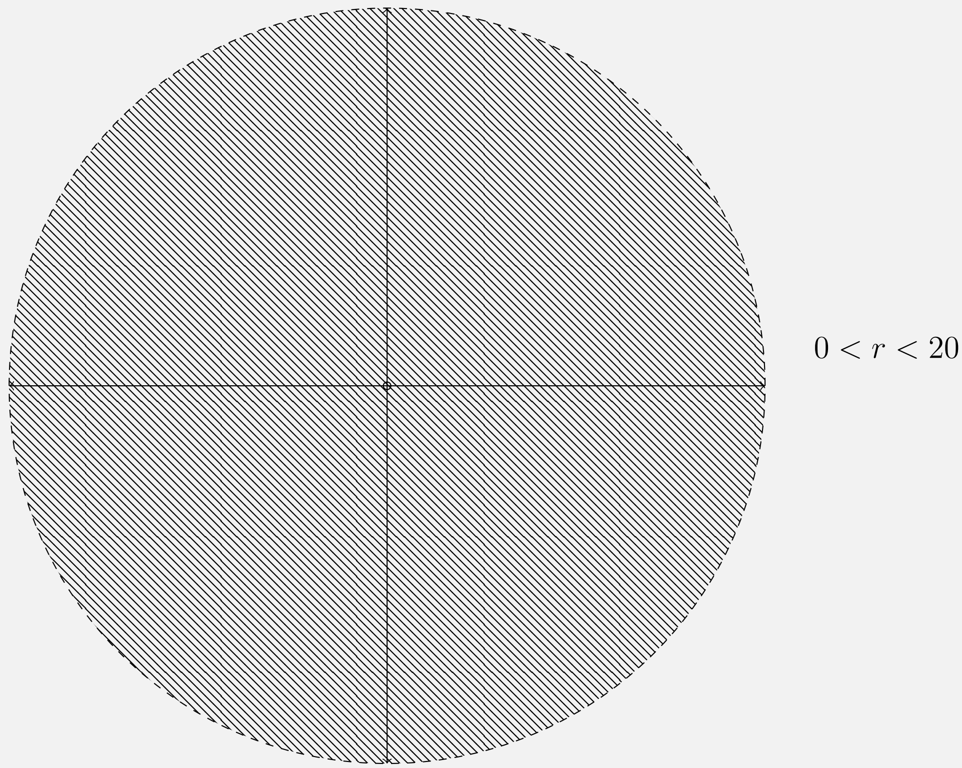
Draw labeled plots. Please draw each part separately.

For which $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ in the workspace can $\theta = \left[\begin{smallmatrix} \theta_1 \\ \theta_2 \end{smallmatrix}\right]$ have:

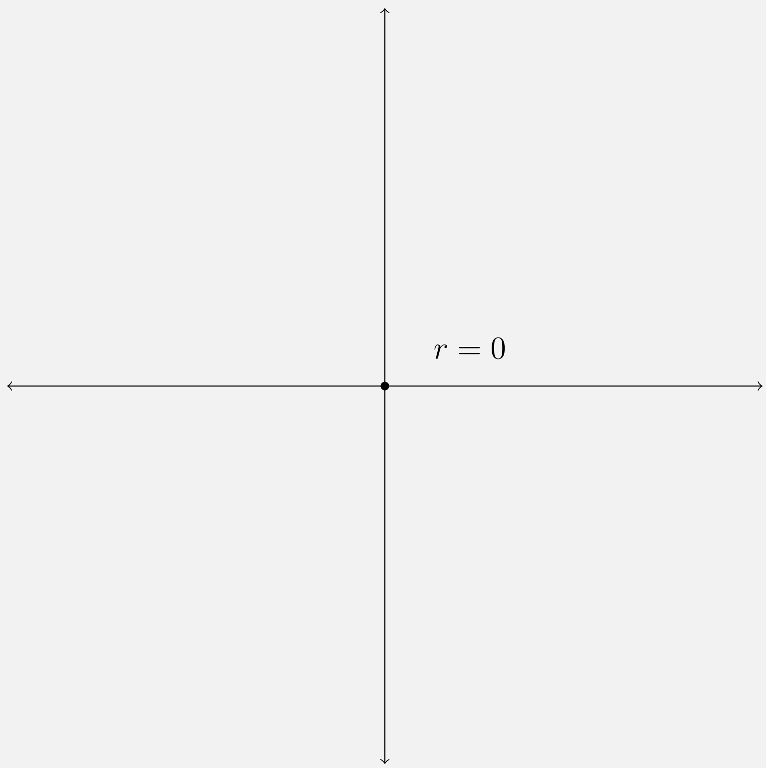
(1) [2 points] Only 1 value?



(2) [2 points] Exactly 2 values?



(3) [2 points] Infinite values?



5 Feedback

1) Feedback Form *5 points*

We are always looking to improve the class! To that end, we're looking for your feedback on the assignments. When you've completed the assignment, please fill out the feedback form.

6 Code Questions

In these problems, we'll analytically test some of the work from the previous section.

Copy the Code Handout folder to some location of your choice. Open Matlab and navigate to that location. Whenever you work on the assignment, go into this directory and run `setup.m`.

1) Forward Kinematics *10 points*

Begin by opening the `ex_01` directory in Matlab.

In this exercise, you must fill out the `forward_kinematics_RR.m` file. When completed correctly, this function should define the forward kinematics for an RR arm with angles `theta_1` and `theta_2`, and link lengths l_1 and l_2 (these values are given in the `forward_kinematics_RR.m` file).

You will define four matrices in this file: $\mathbf{H_1^0}$, $\mathbf{H_2^0}$, $\mathbf{H_3^0}$, and $\mathbf{H_4^0}$. Use the following conventions for the frames (hint: these should match the frames for problem 6 in the written section).

- The base frame of the entire robot is $\{\mathbf{0}\}$.
- The starting frame for the first link is $\{\mathbf{1}\}$.
- The frame at the end of the first link is $\{\mathbf{2}\}$.
- The starting frame for the second link is $\{\mathbf{3}\}$.
- The end effector frame is $\{\mathbf{4}\}$.
- In the code, transform `H_i_j` refers to the transform $\mathbf{H_i^j}$.

Confirm that you have the correct forward kinematics before moving to the next problem. First, we suggest testing a couple of simple cases – for example, angles of 0 or $\pi/2$ – and ensuring the resulting transformations properly transform points in the respective frames to the base frame.

We have also provided a validation script to test your kinematics:

- Run the script `sample_path`. This loads a log of the robot moving, and plots the end effector trajectory generated by your kinematics against ground truth end effector positions. Verify that these paths match.

Before continuing, ensure that the kinematics are correct using the above script!

2) Workspace Analysis *10 points*

Begin by opening the `ex_02` directory in Matlab.

In this exercise, you will create a workspace analysis for a PR arm.

First, run the `workspace_analysis_RR.m` file. This is a function which takes one argument that is a vector of link lengths l_1 and l_2. Executing this should generate a figure showing the end effector positions for a complete sweep of joint angles for an RR arm. Compare this to the results from problem 6.

Your task is to create a `workspace_analysis_PR.m` file, which generates the workspace analysis for the PR arm you studied in problem 7 in the written section. Use the RR analysis code as a starting point for this PR analysis. As before, the prismatic joint can go from $d = 0$ to $d = d_f$. Note that the input for this function should be the link length l_1.

The deliverables for this problem are two figures, saved as MATLAB .fig files (click file->save from the figure menu):

- For the first, let $l_1 = 1$, and $d_f = 1$. Save this as `workspace_pr_1.fig`
- For the second, let $l_1 = 0.5$, and $d_f = 2$. Save this as `workspace_pr_2.fig`

Check that these match the results that you gave in problem 7!

7 Hands-On Questions

1) Tracing Objects with an Arm *10 points*

   Begin by opening the `ex_03` directory in Matlab.

   In this section, we will verify that you can indeed connect to the robot arm, and that your forward kinematics are correct.

   First, follow the directions on Piazza to begin Remote Access. Then, once you have MATLAB open on the remote desktop, type `HebiLookup` into the MATLAB command line, and ensure there are two modules (joint1 and joint2) listed in the table of available modules. If not, notify the TA to plug in wall ethernet cord to the desktop first and the robot ethernet cord second.

   With the forward kinematics you've written, you should be able to run `trace_object`, which takes control of the robot arm, and puts it in a passive mode. This should also bring up a live plot of the robot arm, using your forward kinematics to properly show the robot's position and configuration. Ask the TA to move the arm around. Does your code work properly?

   When you're ready, hit spacebar to begin recording the robot's motion. Ask the TA to use the arm to trace the star shape taped to the table. When you're done, hit space bar once again. If the plot that comes up is shaped like you expect, you've correctly completed the lab! If not, double check your forward kinematics.

   Whenever you use the spacebar to load a plot, you create a file called `star_trace.hebilog`. This is your deliverable for this question. MAKE SURE TO PUSH EVERYTHING TO YOUR PRIVATE GITHUB REPOSITORY!

2) Submission

   After you pull everything back onto your laptop, check that you have everythin to submit. To submit, run `create_submission.m`. It will first check that your submitted files run without error, and perform a small sanity check. Note, this is not going to grade your submission! The function will create a file called `handin-1.tar.gz`.

8 Submission Checklist

- ☐ Create a PDF of your answers to the written questions with the name `writeup.pdf`.
- ☐ Make sure you have `writeup.pdf` in the same folder as the `create_submission.m` script.
- ☐ Run `create_submission.m` in Matlab.
- ☐ Upload `handin-1.tar.gz` to Canvas
- ☐ Upload your `writeup.pdf` to Gradescope.
- ☐ After completing the entire assignment, fill out the feedback form[2].

[2] https://canvas.cmu.edu/courses/18336/quizzes/39693