Assignment 7: Denavit-Hartenberg and Inverse Kinematics

Robot Kinematics and Dynamics

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Contents

1 Overview

This assignment reinforces the following topics:

- Parameterizations of 3D Rotations
- Denavit-Hartenberg Parameters
- Inverse Kinematics

2 Background

2.1 Parameterizations of 3D Rotations

Along with the 3x3 rotation matrices, there exists the Euler angles, Roll-Pitch-Yaw, axis-angle, and quaternion conventions for representing 3D rotations. The quaternion convention is used frequently in robotics because it avoids the Euler angles' issues with gimbal lock, uses only 4 numbers as compared to the 9 in rotation matrices, and is overall more numerically stable and more efficient. The conversion between the axis-angle representation and quaternion representation is shown below:

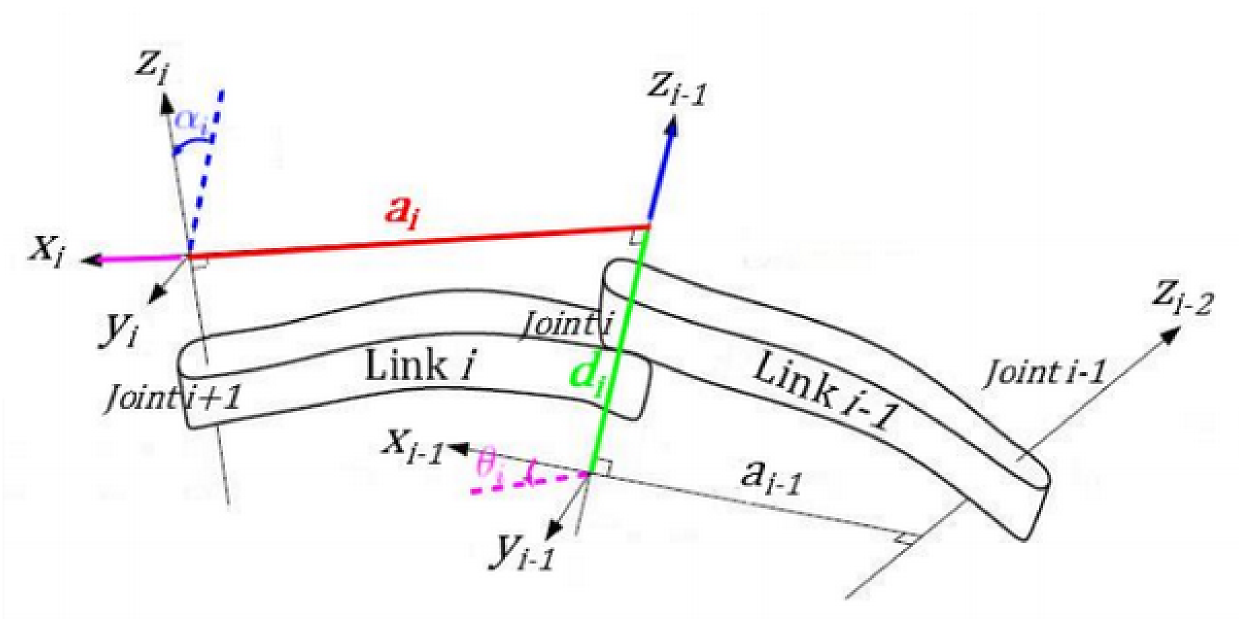
$q_x = a_x * sin(\frac{\theta}{2})$
$q_y = a_y * sin(\frac{\theta}{2})$
$q_z = a_z * sin(\frac{\theta}{2})$
$q_w = cos(\frac{\theta}{2})$

where:

the axis is normalized so: $a_x^2 + a_y^2 + a_z^2 = 1$
the quaternion is also normalized so $q_x^2 + q_y^2 + q_z^2 + q_w^2 = 1$

2.2 Denavit-Hartenberg Parameters

Recall that the Denavit-Hartenberg convention is a systematic and standard way for selecting reference frames for the links of a kinematic chain. There are 4 Denavit-Hartenberg parameters that are used to represent the transformation between the reference frames. We will briefly illustrate the parameters between again below:



a_i	α_i	d_i	θ_i
Link length	Link twist	Link offset	Joint angle

To construct the homogeneous transformation matrix between frames using the 4 D-H parameters, we utilize the following convention:

$$H_i^{i-1} = Rot_{z,\theta_i} Trans_{z,d_i} Trans_{x,a_i} Rot_{x,\alpha_i}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

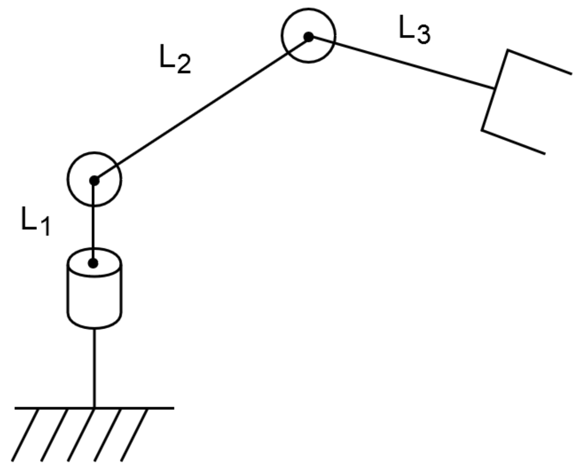
$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3 In Class Question

The following question will be done in class, as a part of a group. Your group's answer will still need to be turned in with the rest of your assignment, however unlike the rest of the work this is allowed to be done in groups.

1) Denavit-Hartenberg Practice

Please use the diagram of the arm below for the following question:



(1) [4 points] Label the X and Z axes for each reference frame using the Denavit-Hartenberg convention.

Convention not completely possible because of R joint before end effector. Opted to make x_3 extend in the reach direction of the end-effector.

(2) [6 points] Determine the Denavit-Hartenberg Parameters for the robot above.

Link	a_i	α_i	d_i	θ_i
1	0	90	l_1	θ_1^*
2	l_2	0	0	θ_2^*
3	l_3	0	0	θ_3^*

4 Written Questions

For the following problems, fully evaluate all answers unless otherwise specified.

Answers for written questions must be typed. We recommend LaTeX, Microsoft Word, OpenOffice, or similar. However, diagrams can be hand-drawn and scanned in.

Unless otherwise specified, **all units are in radians, meters, and seconds**, where appropriate.

1) Parameterizations of Rotations Practice

Please convert between the different parameterizations of rotations below. **Show all of your work.**

(1) [8 points] Rotation Matrix: $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

ZYZ Euler Angles: $\phi = -\frac{\pi}{2}, \frac{\pi}{2}$, $\theta = \frac{\pi}{2}, -\frac{\pi}{2}$, $\psi = \frac{\pi}{2}, -\frac{\pi}{2}$
$\theta = \arctan(\sqrt{1 - r_{33}^2}, r_{33}), \arctan(-\sqrt{1 - r_{33}^2}, r_{33})$
$\theta = \frac{pi}{2}, -\frac{pi}{2}$
$\phi = \arctan(r_{23}, r_{13}), \arctan(-r_{23}, -r_{13})$
$\phi = -\frac{\pi}{2}, \frac{\pi}{2}$
$\psi = \arctan(r_{32}, -r_{31}), \arctan(-r_{32}, r_{31})$
$\psi = \frac{\pi}{2}, -\frac{\pi}{2}$

Roll-Pitch-Yaw Angles: $X = \frac{\pi}{2}$, $Y = 0$, $Z = 0$
$X = \arctan(r_{32}, r_{33})$
$Y = \arcsin(-r_{13})$
$Z = \arctan(r_{21}, r_{11})$

Axis-Angle: Axis = $[1, 0, 0]$ Angle = $\frac{\pi}{2}$
$\theta = \arccos(\frac{r_{11}+r_{22}+r_{33}-1}{2})$
$k = \frac{1}{2\sin(\theta)}[r_{32} - r_{23}, r_{13} - r_{31}, r_{21} - r_{12}]$

Quaternion: $[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0]$
$q = [c_{\theta/2}, s_{\theta/2}\mathbf{k}]$ (θ and k from angle-axis)

(2) [8 points] ZYZ Euler Angles: $\phi = \frac{\pi}{2}$, $\theta = -\frac{\pi}{2}$, $\psi = \pi$

Rotation Matrix: $R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$

$$R = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi s_\theta s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Roll-Pitch-Yaw Angles: $X = 0$, $Y = \frac{\pi}{2}$, $Z = \frac{\pi}{2}$
$X = \arctan(r_{32}, r_{33})$
$Y = \arcsin(-r_{13})$
$Z = \arctan(r_{21}, r_{11})$

Axis-Angle: Axis = $[0.577, 0.577, -0.577]$ Angle = $\frac{2\pi}{3}$
$\theta = \arccos(\frac{r_{11}+r_{22}+r_{33}-1}{2})$
$k = \frac{1}{2\sin(\theta)}[r_{32} - r_{23}, r_{13} - r_{31}, r_{21} - r_{12}]$

Quaternion: $[0.5, 0.5, 0.5, -0.5]$
$q = [c_{\theta/2}, s_{\theta/2}\mathbf{k}]$ (θ and k from angle-axis)

(3) [8 points] Roll-Pitch-Yaw Angles: $X = 0$, $Y = -\frac{\pi}{2}$, $Z = 0$

Rotation Matrix: $R = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$R = \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta s_\psi \end{bmatrix}$$

ZYZ Euler Angles: $\phi = 0$, $\theta = -\frac{\pi}{2}$, $\psi = 0$
$\theta = \arctan(\sqrt{1 - r_{33}^2}, r_{33}), \arctan(-\sqrt{1 - r_{33}^2}, r_{33})$
$\phi = \arctan(r_{23}, r_{13}), \arctan(-r_{23}, -r_{13})$
$\psi = \arctan(r_{32}, -r_{31}), \arctan(-r_{32}, r_{31})$

Axis-Angle: Axis = $[0, 1, 0]$ Angle = $-\frac{\pi}{2}$
$\theta = \arccos(\frac{r_{11}+r_{22}+r_{33}-1}{2})$
$k = \frac{1}{2\sin(\theta)}[r_{32} - r_{23}, r_{13} - r_{31}, r_{21} - r_{12}]$

Quaternion: $[\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, 0]$
$q = [c_{\theta/2}, s_{\theta/2}\mathbf{k}]$ (θ and k from angle-axis)

(4) [8 points] Axis-Angle: Axis = $[0, 0, 1]$ Angle = π

Rotation Matrix: $R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Axis is the z axis so:
$R = R_z(\theta)$

ZYZ Euler Angles: $\phi = 0$, $\theta = 0$, $\psi = \pi$
$\theta = \arctan(\sqrt{1 - r_{33}^2}, r_{33}), \arctan(-\sqrt{1 - r_{33}^2}, r_{33})$
$\theta = 0$
$\phi + \psi = \arctan(r_{21}, r_{11})$
Gimbal lock present in Euler angles ($\theta = 0$), any solution such that $\phi + \psi = \pi$ satisfies the eqn

Roll-Pitch-Yaw Angles: $X = 0$, $Y = 0$, $Z = \pi$
$X = \arctan(r_{32}, r_{33})$
$Y = \arcsin(-r_{13})$
$Z = \arctan(r_{21}, r_{11})$
Quaternion: $[0, 0, 0, 1]$
$q = [c_{\theta/2}, s_{\theta/2}\mathbf{k}]$ (θ and k from angle-axis)

(5) [8 points] Quaternion: $[0, 0, 0, 1]$

Rotation Matrix: $R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

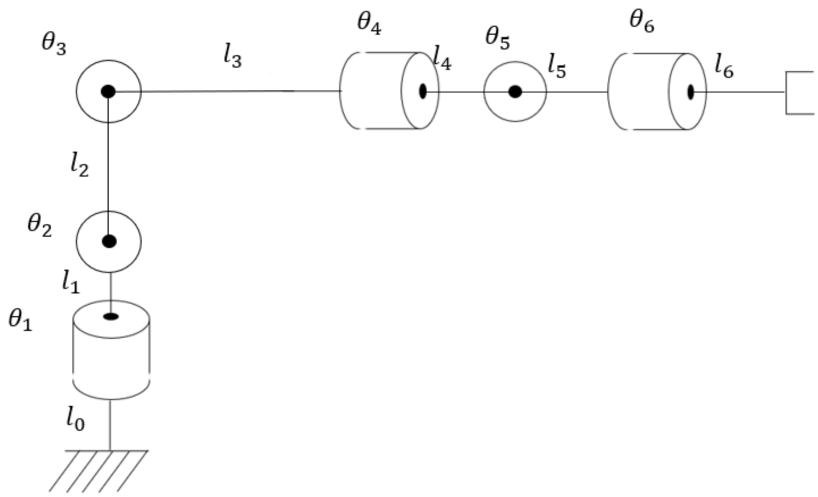
$$R = \begin{bmatrix} 1-2y^2-2z^2 & 2xy-2wz & 2xz+2wy \\ 2xy+2wz & 1-2x^2-2z^2 & 2yz-2wx \\ 2xz-2wy & 2yz+2wx & 1-2x^2-2y^2 \end{bmatrix}$$

ZYZ Euler Angles: $\phi = 0$, $\theta = 0$, $\psi = \pi$
$\theta = \arctan(\sqrt{1-r_{33}^2}, r_{33}), \arctan(-\sqrt{1-r_{33}^2}, r_{33})$
$\theta = 0$
$\phi + \psi = \arctan(r_{21}, r_{11})$
Gimbal lock present in Euler angles ($\theta = 0$), any solution such that $\phi + \psi = \pi$ satisfies the eqn

Roll-Pitch-Yaw Angles: $X = 0$, $Y = 0$, $Z = \pi$
$X = \arctan(r_{32}, r_{33})$
$Y = \arcsin(-r_{13})$
$Z = \arctan(r_{21}, r_{11})$

Axis-Angle: Axis = $[0, 0, 1]$ Angle = π
$\theta = \arccos(\frac{r_{11}+r_{22}+r_{33}-1}{2})$
$k = \frac{1}{2\sin(\theta)}[r_{32} - r_{23}, r_{13} - r_{31}, r_{21} - r_{12}]$

2) Denavit-Hartenberg Parameters Practice



Side View of 6DOF

(1) [10 points] Label the X and Z axes for each reference frame using the Denavit-Hartenberg convention.

(2) [20 points] Determine the Denavit-Hartenberg Parameters for the robot.

Link	a_i	α_i	d_i	θ_i
1	0	90	l_1	${\theta_1}^*$
2	l_2	0	0	90 + ${\theta_2}^*$
3	0	90	0	90 + ${\theta_3}^*$
4	0	-90	l_3 + l_4	${\theta_4}^*$
5	0	90	0	${\theta_5}^*$
6	0	0	l_5 + l_6	${\theta_6}^* - 90$

(3) [12 points] Write down the Transformation Matrices between each link using the Denavit-Hartenberg Parameters you determined above.

$$H_1^0 = \begin{bmatrix} \cos(\theta_1^*) & 0 & \sin(\theta_1^*) & 0 \\ \sin(\theta_1^*) & 0 & -\cos(\theta_1^*) & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} \cos(90+\theta_2^*) & -\sin(90+\theta_2^*) & 0 & l_2\cos(90+\theta_2^*) \\ \sin(90+\theta_2^*) & \cos(90+\theta_2^*) & 0 & l_2\sin(90+\theta_2^*) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^2 = \begin{bmatrix} \cos(90+\theta_3^*) & 0 & \sin(90+\theta_3^*) & 0 \\ \sin(90+\theta_3^*) & 0 & -\cos(90+\theta_3^*) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_4^3 = \begin{bmatrix} \cos(\theta_4^*) & 0 & -\sin(\theta_4^*) & 0 \\ \sin(\theta_4^*) & 0 & \cos(\theta_4^*) & 0 \\ 0 & -1 & 0 & l_3+l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_5^4 = \begin{bmatrix} \cos(\theta_5^*) & 0 & \sin(\theta_5^*) & 0 \\ \sin(\theta_5^*) & 0 & -\cos(\theta_5^*) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_6^5 = \begin{bmatrix} \cos(\theta_6^*{=}90) & -\sin(\theta_6^*-90) & 0 & 0 \\ \sin(\theta_6^*-90) & \cos(\theta_6^*-90) & 0 & 0 \\ 0 & 0 & 1 & l_5+l_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5 Feedback

1) Feedback Form *5 points*

We are always looking to improve the class! To that end, we're looking for your feedback on the assignments. When you've completed the assignment, please fill out the feedback form.

6 Submission Checklist

- ☐ Upload `writeup.pdf` to Gradescope and Canvas.
- ☐ After completing the entire assignment, fill out the feedback form[1].

[1] https://canvas.cmu.edu/courses/18336/quizzes/39686