MATH 1024 • Spring 2018 • Honors Calculus 2
Problem Set #2 • Section #3.2 • Due Date: 02/03/2018, 11:59PM

1. (10 points) Let $f(x) = e^x$. Consider the partition P_n of $[0,1]$:

$$P_n : 0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n}{n} = 1.$$

   (a) Find $L(P_n, f)$ and $U(P_n, f)$. Express your answers explicitly without summation signs.

   (b) Hence, show that $f(x)$ is Riemann integrable on $[0,1]$.

2. (10 points) The following classic formula was discovered by Jacob Bernoulli in 1713:

$$1^p + 2^p + \cdots + n^p = \frac{1}{p+1}\sum_{j=0}^{p}(-1)^j C_j^{p+1} B_j n^{p+1-j}, \qquad p \in \mathbb{N}$$

   where B_j's are so-called *Bernoulli's numbers* given by:

$$B_0 = 1, \;\; B_1 = \frac{1}{2}, \;\; B_2 = \frac{1}{6}, \cdots$$

   [You can find the proof of the above formula from standard number theory or complex analysis textbooks.]

   Using this formula without proof, show that x^p (where $p \in \mathbb{N}$) is Riemann integrable on $[0,1]$ and that:

$$\int_0^1 x^p \, dx = \frac{1}{p+1}, \text{ where } p \text{ is a positive integer}$$

   from the definition of integrals.

3. (15 points) First prove the formula:

$$2\sin\frac{x}{2} \cdot (\sin x + \sin 2x + \cdots + \sin nx) = \cos\frac{x}{2} - \cos\left(n + \frac{1}{2}\right)x$$

   for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Hence, show that $\sin x$ is Riemann integrable on $[0, \pi]$, and find the value of

$$\int_0^\pi \sin x \, dx$$

   from the definition of integrals.

4. (15 points) Consider two bounded functions $f(x)$ and $g(x)$ defined on $[a,b]$, where $f(x) = g(x)$ on $[a,b]$ *except* finitely many points $c_1, c_2, \ldots, c_k \in (a,b)$. Show that if f is Riemann integrable on $[a,b]$, then so does g and in this case one has

$$\int_a^b f(x)\, dx = \int_a^b g(x)\, dx.$$

One more problem on the next page! Please turn over...

5. (15 points) Consider the function $f : [0,1] \to \mathbb{R}$ by:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \text{ in the most simplified form } (m, n \in \mathbb{N}) \end{cases}$$

[*Déjà vu* for MATH 1023 students.]

For instance, we have $f(\frac{2}{3}) = \frac{1}{3}$, $f(\frac{8}{14}) = f(\frac{4}{7}) = \frac{1}{7}$.

Show that f is Riemann integrable on $[0,1]$ and $\displaystyle\int_0^1 f(x)\,dx = 0$.

Hint: Given a uniform partition P of $[0,1]$, I believe finding $L(P, f)$ is easy. For $U(P, f)$, it may be difficult to find the exact value, but it suffices to find a sequence of partitions P_n such that $U(P_n, f) \to 0$ as $n \to \infty$. Try to bound $U(P_n, f)$ from above and think of the *worst situation*. You may use the following number theory result without proof:

Let $\varphi(n)$ be the number of positive integers $m \in \{1, \ldots, n-1\}$ which are coprime to n (i.e. $\frac{m}{n}$ is in the simplest form). Then, one has:

$$\sum_{k=1}^{n} \varphi(k) = \frac{3}{\pi^2} n^2 + o(n^2) \quad \text{as } n \to \infty.$$