MATH 1024 • Spring 2018 • Honors Calculus 2
Problem Set #3 • Section #3.3, #2.7 • Due Date: 09/03/2018, 11:59PM

1. (10 points) Consider the integral $\int_1^3 e^{-x^2}\, dx$.
   (a) Using a spreadsheet program (e.g. Excel), find the approximate values of the above integral using *each* of the rules below with $n = 100$. Either you use LaTeX or scanned submissions, save your spreadsheets as PDFs and upload them together with the main files. The Canvas system does not accept Excel files.
      i. midpoint rule
      ii. trapezoidal rule
      iii. Simpson's rule
   (b) How accurate are these approximations, i.e. accurate up to how many decimal places (at least)? Explain your answers.

2. (10 points) Given a function $f(x)$ which has n-th order derivatives on $[a, b]$. Consider the $(n-1)$-th Taylor's approximation $T_{n-1}(x)$ of $f(x)$ at $c \in (a, b)$, show that for any $x \in (a, b)$ there exists ξ between x and c such that

$$f(x) = T_{n-1}(x) + \frac{f^{(n)}(\xi)}{n!}(x-c)^n.$$

   The last term $\frac{f^{(n)}(\xi)}{n!}(x-c)^n$ is called the *Lagrange's Remainder*, which gives some more concrete idea of what $o((x-c)^{n-1})$ term is like.

   Hint: You can find the key idea from a problem of MATH 1023 Final Exam last semester.

3. (15 points) Consider a function f whose first and second derivatives exist and are bounded on $[a, b]$. Using the results from Problem 2, derive the error bound for the following rules:
   (a) Left-hand rule:

$$\left| \int_a^b f(x)\, dx - L_n \right| \le \frac{(b-a)^2}{2n} \sup_{[a,b]} |f'| \,.$$

   (b) Simpson's rule:

$$\left| \int_a^b f(x)\, dx - S_n \right| \le \frac{(b-a)^5}{Cn^4} \sup_{[a,b]} \left| f^{(4)} \right|$$

   where C is some positive constant.

   Remark: It is not necessary to get $C = 180$ as stated in Theorem 3.3.2. Any positive constant is good enough for this problem. In fact, it is not easy to prove that C can be taken to be 180.

   Hint for (b): Consider the uniform partition $\{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ and denote $\Delta x = x_{i+1} - x_i$. Try to calculate $\int_{x_i}^{x_i+2\Delta x} T_3(x)\, dx$, then estimate the error between this integral and the general term of Simpson's rule, i.e.

$$\int_{x_i}^{x_i+2\Delta x} T_3(x)\, dx - \frac{\Delta x}{3}\big(f(x_i) + 4f(x_i+\Delta x) + f(x_i+2\Delta x)\big).$$

4. (15 points) This problem is about the derivation an error bound for the trapezoidal rule. Using similar approaches as in Problem 3 do not give a bound as good as in Theorem 3.3.1. Follow the outline below instead:

   (a) Consider the uniform partition $\{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ and denote $\Delta x = x_{i+1} - x_i$. Show that there exist numbers A and B, independent of x, such that

$$\Big[(x+A)f(x)\Big]_{x_i}^{x_i+\Delta x} = \text{ area of the } (i+1)\text{-th trapezium in } T_n$$

$$\Big[\big((x+A)^2+B\big)f'(x)\Big]_{x_i}^{x_i+\Delta x} = 0$$

   for any $i = 0, 1, 2, \ldots, n-1$.

   (b) By considering $\dfrac{d}{dx}\left((x+A)f(x) - \frac{1}{2}\big((x+A)^2+B\big)f'(x)\right)$ with the above choice of A and B, show that

$$\int_{x_i}^{x_i+\Delta x} f(x)\,dx = \frac{f(x_i)+f(x_{i+1})}{2}\Delta x + \frac{1}{2}\int_{x_i}^{x_i+\Delta x} \big((x+A)^2+B\big)f''(x)\,dx$$

   for any $i = 0, 1, 2, \ldots, n-1$.

   (c) Finally, show that:

$$\left|\int_a^b f(x)\,dx - T_n\right| \le \frac{(b-a)^3}{12n^2}\sup_{[a,b]}\left|f''\right|.$$

5. (10 points) Suppose $f(x)$ has second-order derivatives on the interval $I = (a, b)$, and that there exist constants $L, M > 0$ such that $f'(x) \ge L$ and $0 \le f''(x) \le M$ for any $x \in I$. Let $c \in (a, b)$ be the unique zero of $f(x)$ (it is unique because $f'(x) > 0$ on I), and we are using Newton's method to find an approximation of this c:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

   (a) Using the results of Problem 2 (wisely), show that for any $n \in \mathbb{N}$, if $x_n \in I$ then there exists $\xi_n \in I$ such that:

$$x_{n+1} - c = \frac{f''(\xi_n)}{2f'(x_n)}(x_n - c)^2.$$

   (b) Show that if x_1 is chosen so that $0 < \dfrac{M}{2L}(x_1 - c) < 1$, then $x_n \in I \cap (c, b)$ for any $n \in \mathbb{N}$, and we have:

$$x_n - c \le \frac{2L}{M}\left(\frac{M}{2L}(x_1 - c)\right)^{2^{n-1}} \qquad \text{for any } n \in \mathbb{N}.$$

   [As a corollary, we have $x_n \to c$ as $n \to \infty$, and the rate of convergence is very fast.]