MATH 1024 • Spring 2018 • Honors Calculus 2
Problem Set #1 • Sections #3.1.1 - #3.2.1 • Due Date: 23/02/2018, 11:59PM

Instructions:

- Only one or two problems (selected by the instructor after the due date) will be graded.
- Only the graded problems will count toward your homework score.
- Every homework will carry equal weight. The score of the lowest homework will be dropped.
- As in the previous semester, a bonus point of 0.1 will be awarded to each homework if you fully type your solution using LaTeX. Just a little extra requirement this semester: Please submit **both** .tex and .pdf files to Canvas.

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1. (15 points) (a) Let m and n be positive integers. Prove that:

$$\int_0^{2\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

Hint: product-to-sum formula

(b) Using (a), or otherwise, find:

$$\int_0^{2\pi} (\cos x + \cos 2x + \cdots + \cos 2018x)^2 \, dx.$$

(c) More generally, let $\{a_1, \ldots, a_N\}$ and $\{b_1, \ldots, b_N\}$ be two finite sequences of real numbers, and let:

$$f(x) := \sum_{n=1}^{N} a_n \cos nx$$
$$g(x) := \sum_{n=1}^{N} b_n \cos nx$$

Evaluate the integral:

$$\int_0^{2\pi} f(x)g(x) \, dx.$$

2. (15 points) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = x - [x] + c$$

where c is a constant, and $[x]$ is the greatest integer not exceeding x.

(a) Sketch the graph of f when $c = 0$.

(b) Define $F(x) : \mathbb{R} \to \mathbb{R}$ by:

$$F(x) = \int_0^x f(t) \, dt.$$

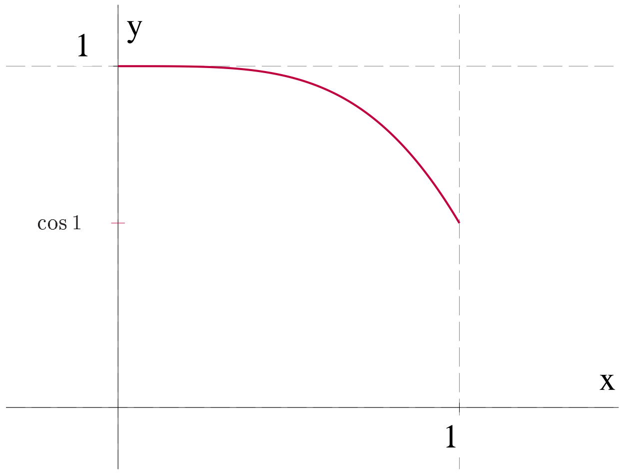
Find the explicit expression of $F(x)$ on the interval $[n, n+1]$ where n is any integer.

(c) Find the value of c so that $F(x)$ is periodic.

3. (15 points) (a) Which of the following integrals is larger? Explain why.

$$\int_0^1 \cos(x^2)\,dx \quad \text{vs} \quad \int_0^1 \cos(x^3)\,dx$$

(b) Define the function $f : [0,1] \to \mathbb{R}$ by $f(x) = \cos(x^2)$. The graph $y = f(x)$ is given in the figure below. Sketch the graph $y = f^{-1}(x)$ in the same figure.



(c) Which of the following integrals is larger? Explain why.

$$\int_0^1 \cos(x^2)\,dx \quad \text{vs} \quad \int_{\cos 1}^1 f^{-1}(x)\,dx.$$

4. (8 points) By considering the graph of the function $f(x) = x^{p-1}$ where $p > 1$, give a *picture proof* of Young's inequality: given $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for any $a, b > 0$. Furthermore, explain when the equality holds based on the picture.

5. (12 points) Consider the triangle in $\mathbb{R}^2$ with base b and height h, and for simplicity let the vertices be $(0,0)$, $(b,0)$, and (c,h) where c can be any real number. Prove the following formula that you learned in primary school (or kindergarten):

$$\text{area of triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$

using two different methods:

(a) from a definite integral

(b) from the definition of area (c.f. Example 3.2.2).