Physics-105-Lecture-Notes-02-28-2019

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Abstract

A single PDF with all lectures in a single document can be downloaded at `https://www.dropbox.com/sh/8sqzvxghvbjifco/AAC9LoSRnsRQDp7pYedgWpQMa?dl=0`. The password is 'analytic.mech.dsp'. This file was automatically generated using a script, so there might be some errors. If there are, you can contact me at `mailto:ctdunc@berkeley.edu`.

0.1 Central Force Motion, Continued

Recall, we have $m\ddot{r} - \frac{l^2}{mr^3} = f(r)$, with constant energy

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} + V(r) \equiv \text{constant}$$

which allows us to write

$$\begin{aligned}
\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{l}{mr^2} \\
d\theta &= \frac{ldr}{mr^2\sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}} \\
\theta = \theta_0 + \int_{r_0}^{r} &\frac{dr}{mr^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}}} \\
\theta = \theta_0 + \int_{u_0}^{u} &\frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mV(u)}{l^2} - u^2}}
\end{aligned}$$

If we let $f \sim \frac{1}{r^2}$, we get that, with k as a coupling constant (i.e. how much force is scaled by).

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + \sqrt{1 + \frac{2El^2}{mk^2}}\cos(\theta - \theta_0)\right)$$

an example of k for gravity is $V = \frac{GmM}{r}$, leaves $k = GmM$.

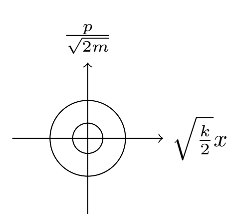
0.1.1 Kepler Orbits/Phase Diagrams

$$\begin{aligned}
\frac{1}{r} &= c\left(1 + \epsilon\cos(\theta - \theta_0)\right) \\
\epsilon = \sqrt{1 + \frac{2El^2}{mk^2}} &\equiv \text{eccentricity of orbit}
\end{aligned}$$

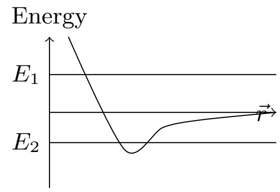
Now, we want to think about phase diagrams. We have

$$(\sqrt{E})^2 = \left(\frac{p}{\sqrt{2m}}\right)^2 + \left(\sqrt{\frac{k}{2}}x\right)^2$$

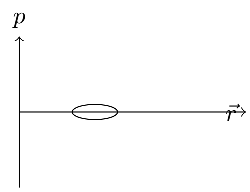
So particles move on circles in this abstract phase space



Recall our diagram of energy from the previous lecture,



It has a corresponding phase diagram,



This is a *suuuuper rough approximation* that you should verify on your own using python or desmos or something. Use $V \sim \frac{1}{r^2} - \frac{s}{r}$ where s is a constant. If curves on the phase diagram are closed, they're *trapped solutions*, i.e. they want to stay within the potential well. For given eccentricity (let $\cos\theta = 1$) we can calculate the minimum r of the orbit in a straightforward manner.

$$r_{min} = \frac{l^2}{mk(1+\epsilon)}$$

and maximum r, we have

$$r_{max} = \frac{l^2}{mk(1-\epsilon)}$$

There are also the unbound orbits, which give you

$$\frac{1}{r} = C(1 + \epsilon\cos(\theta - \theta_0))$$

the right hand side can be zero, which means $r_{max} \to \infty$. Let's examine the orbits. First, let $r = \frac{1}{\alpha}(1 + \epsilon\cos\theta)$, which gives $\alpha = r + \epsilon r\cos\theta = r + \epsilon x$. Then, we get

- $\epsilon = 0 \to \frac{1}{r} = \frac{mk}{l^2}$ which gives constant r, and is thus a circle. Also note it would give $x^2 + y^2 = \alpha^2$ which also describes a circle.
- $0 < \epsilon < 1 \to 0 < 1 + \frac{2El^2}{mk^2} < 1 \to \frac{-mk^2}{2l^2} < E < 0$. This case corresponds to the area of our energy diagram beneath $E = 0$. Completing the square, we alsonote that $\frac{\left(x + \frac{\alpha\epsilon}{1-\epsilon^2}\right)^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a = \frac{\alpha}{1-\epsilon^2}, b = \frac{\alpha}{\sqrt{1-\epsilon^2}}$. a is called the *semimajor axis*, and b the *semiminor axis*. ϵ is a unitless quantity. This centers the ellipse at $x_0 = -\frac{\alpha\epsilon}{1-\epsilon^2}$. The ellipse also has a *focus*, with $c^2 + b^2 = a^2$, c being the focus. You can solve it to be $c = \frac{\alpha\epsilon}{1-\epsilon^2}$, and another focus at the origin.
- $\epsilon = 1$. We get $y^2 = \alpha^2 - 2\alpha x$, since the x^2 terms from our other equation cancel, which gives the parabola $y^2 = -2\alpha\left(x - \frac{\alpha}{2}\right)$.
- $\epsilon > 1$. We find $\frac{\left(x - \frac{\alpha\epsilon}{\epsilon^2-1}\right)^2}{a^2} - \frac{y^2}{b^2} = 1$, which gives a hyperbola!

For a better explanation of what's happening here/pictures, see Taylor fig. 8.11.

0.1.2 Keplers Laws

This lets us derive keplers laws

1. Planets Move in ellipses with one focus at the sun (equiv to condition $0 \le \epsilon < 1$)
2. Radius vector sweeps out equal area at equal time (equiv to conservation of momentum $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2m}$.
3. The square of the period of an orbit (T) is proportional to the cube of the semimajor axis ($T^2 = \frac{4\pi^2 a^3}{Gm_0}$). This can be shown with $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{l}{2m} \to A = \frac{l}{2m}T = \pi ab \equiv$area of ellipse.