Physics-105-Lecture-Notes-04-18-2019

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April 18, 2019

Contents

Abstract

A single PDF with all lectures in a single document can be downloaded at `https://www.dropbox.com/sh/8sqzvxghvbjifco/AAC9LoSRnsRQDp7pYedgWpQMa?dl=0`. The password is 'analytic.mech.dsp'. This file was automatically generated using a script, so there might be some errors. If there are, you can contact me at `mailto:ctdunc@berkeley.edu`. Lagrangian density for some discrete masses on string, with y_k change in y, and η_k displacement in x, then

$$\mathcal{L} = \frac{1}{2}\rho(x)\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\tau(x)\left(\frac{\partial y}{\partial x}\right)^2 + O(\eta)$$

where $O(\eta)$ higher order terms Recall our equation of motion

$$\frac{\partial}{\partial x}\left(\tau(x)\frac{\partial y}{\partial x}\right) = \rho(x)\frac{\partial^2 y}{\partial t^2}$$

if we require τ constant, then we get $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c(x)}\frac{\partial^2 y}{\partial t^2}$.

1 Solving the Wave Equation

Two methods of solving

- Computer
- Pertubation Theory (only valid for $\lambda \ll [\frac{\mathrm{d}}{\mathrm{d}x}(\ln(l_s''))]^{-1}$ or $\lambda \ll \frac{1}{\frac{1}{c_s}\frac{\mathrm{d}L_s}{\mathrm{d}x}}$. Often called the WKB approximation. Basically λ is less than the second x derivative of a typical length scale. GOOGLE.

Let's take the following new variables,

$$\xi = x - vt$$
$$\eta = x + vt$$

so that $(x,t) \to (\xi, \eta)$. So, we now have

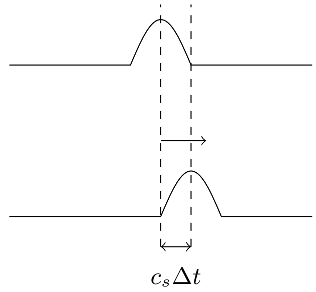
$$\frac{\partial y}{\partial x} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) y$$
$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right) y$$
$$\frac{\partial y}{\partial t} = c_s\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) y$$
$$\frac{\partial^2 y}{\partial t^2} = c_s^2\left(c_s\frac{\partial^2}{\partial \xi^2} - 2\frac{\partial^2}{\partial \eta \partial \xi} + \frac{\partial^2}{\partial \eta^2}\right) y$$

so we can find some solution by setting

$$y(x,t) = f(x - c_s t) + g(x + c_s t)$$

if we set $g \equiv 0$, then

$$y(x,t) = f(x - c_s t) \qquad y(x,0) = f(x)$$



traveling wave solution, it only goes from $x \to x + c_s\Delta t$. In the arbitrary solution, we have

$$y(x,t) = f(x-ct) + g(x+ct)$$

where f, g are determined by initial conditions. (i.e. at $t=0$, we have $f(x)+g(x) = y_0(x)$). taking

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} = \frac{1}{c_s}\dot{y}_0(x)$$

we integrate to see that

$$g(x) - f(x) = \frac{1}{c_s}\int_{x_0}^{x} \dot{y}_0(x')dx'$$

which gives

$$f(x) = y_0(x) = \frac{1}{c}\int_{x_0}^{x} \dot{y}_0(x')dx'$$
$$g(x) = y_0(x) + \frac{1}{c}\int_{x_0}^{x} \dot{y}_0(x')dx'$$

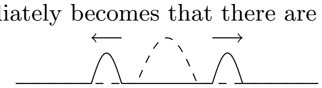
Then, we also have

$$y(0,t) = f(x-ct) + g(x+ct)$$

so we can write the D'lambert solution to the wave equation.

$$y(x,t) = \frac{1}{2}\left[y_0(x-ct) + y_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \dot{y}_0(x')dx'$$

for a small pertubation, the solution of this immediately becomes that there are two pulses



propagating in opposite directions.

1.1 General Solution

The generalized wave equation can be written

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

There are several categories of solutions to consider. First is a standing wave, $\psi(\vec{r},t)$ depends as $A(\vec{r})e^{-i\omega t}$. We get the **Helmholtz Equation** that describes such solutions.

$$\nabla^2 A(r) + \frac{\omega^2}{c^2}A(\vec{r}) = 0$$

The 1-d case (say for a string with fixed boundaries is given as

$$\frac{\partial^2 A}{\partial x^2} + \frac{\omega^2}{c^2}A = 0$$

so given that $A(0) = A(L) = 0$, then the boundary condition for the left gives $A(x) = a\sin\left(\frac{\omega}{c}x\right)$. We also need $\sin\left(\frac{\omega}{c}L\right) = 0$, then possible frequencies given $\Omega = \{\frac{c\pi n}{L} | n \in \mathbb{Z}\}$. So, the general solution is written

$$\psi_n(x,t) = e^{-i\omega_n t}\sin\left(\frac{\pi n}{L}x\right)$$

with $\omega_n = \frac{\pi n}{L}c$. Fun math fact, we can write down the general solution for one dimension as

$$\psi(x,t) = \sum_{n=1}^{\infty}[a_n\cos(\omega_n)t + b_n\sin(\omega_n t)] \times \sin\frac{\pi n}{L}x$$

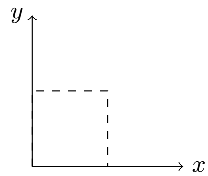
if you do the math out you get thatt

$$a_n = \frac{2}{L}\int_0^L \psi(x,0)\sin\left(\frac{\pi n}{L}x\right)dx$$
$$\omega_n b_n = \frac{2}{L}\int_0^L \frac{\partial\psi}{\partial t}\sin\frac{\pi}{n}Ldx$$

1.2 2-d Helmholz

$$\psi = e^{-i\omega t} A(x, y)$$

Say some membrane, with ψ some oscillation in z over a bounded membrane



Let's take

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2}\right) A = 0$$

under teh assumption that A is separable, i.e. $A = X(x)Y(y)$. Then we can write

$$X''(x)Y(y) + Y''(y)X(x) + \frac{\omega^2}{c^2} XY = 0$$

which, we can divide by XY to get

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{\omega^2}{c^2} = 0$$

so, let's write some stuff down

$$X''(x) = -\lambda X, \qquad \lambda \equiv \text{const}$$
$$Y''(y) = -\mu Y \qquad \mu \equiv \text{const}$$

we have the boundary conditions that $X = a \sin\left(\sqrt{\lambda} x\right)$, also with $\sin\left(\sqrt{\lambda} L\right) = 0$, so we have

$$X_n(x) = a \sin\left(\frac{\pi n}{L} x\right)$$
$$Y_m(y) = b \sin\left(\frac{\pi m}{L} y\right)$$

we also know that

$$\frac{\omega^2}{c^2}(\lambda + \mu) \Rightarrow \frac{\omega_{nm}^2}{c^2} = \frac{\pi^2(n^2 + m^2)}{L^2}$$

which gives some general solution

$$\psi(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{nm} \cos(\omega_{nm} t) + b_{nm} \sin(\omega_{nm} t)) X_n(x) Y_m(y)$$

1.2.1 Circular Boundary

If we have some circular boundary, we still have the same helmholz, and $A = A(r, \theta)$. We rewrite the laplacian in cylindciral coordinates and get

$$\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\partial A}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 A}{\partial \theta^2} + \frac{\omega^2}{c^2} A = 0$$

Apply condition $A(r, 0) = A(r, 2\pi R_0)$, and $A(r, \theta) = R(r) e^{-im\theta}$.

1.2.2 Bessel function

$$y''(x) + \frac{1}{x} y'(x) + \left(\lambda^2 - \frac{n^2}{x^2}\right) y = 0$$

so that the *Bessel Functions* of order n are given as solutions to this bad boy.

$$y(x) = J_n(\lambda x)$$

We have

$$R''(r) + \frac{1}{r} R'(r) + \left(\frac{\omega^2}{c^2} - \frac{m^2}{r^2}\right) R = 0$$

which gives solution

$$R = J_m(\frac{\omega}{c} r)$$

we apply that it must satisfy the boundary condition $J_m(\frac{\omega}{c_s} R_0 = 0$, which give solutions. I don't think it's gonna be super important to know how to solve this, but basically it;s the roots of the bessel function (this is the wave equation for a spherically propagating wave, which is how the double slit experiment works!). so the full on solutions are given as

$$\psi = J_m\left(\frac{\omega_{nm}}{c}\right) \times \begin{Bmatrix} \cos(m\varphi) \\ \sin(m\varphi) \end{Bmatrix} \times \times e^{-i\omega_{nm} t}$$

If we consider the case for oscillaating membranes on a cylinder, we'd write down

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^r A}{\partial \theta^r}+\frac{\partial^2 A}{\partial z^2}+\frac{\omega^2}{c^2}A=0$$

which becomes, with $A = R(r)Z(z)e^{-im\theta}$

$$\frac{1}{Rr}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)-\frac{m^2}{r^2}+\frac{Z''(z)}{Z}+\frac{\omega^2}{c^2}=0$$

we just get anotherr

$$Z''(z)+\lambda Z(z)=0$$

which gives solutions of the bessel equation with differend conditions, we fund

$$R''+\frac{1}{r}R'-\frac{m^2}{r^2}R+\left(\frac{\omega^2}{c^2}-\left(\frac{\pi n}{L}\right)^2\right)R=0$$

which reduces

$$R''+\frac{1}{r}R'+\left(\frac{\omega^2}{c^2}-\left(\frac{\pi n}{L}\right)^2-\frac{m^2}{r^2}\right)R=0$$

$$R(r)=J_k(\sqrt{\frac{\omega^2}{c^2}-\left(\frac{\pi n}{L}\right)}r)$$