Math 222 B (Partial Differential Equations)

To be covered:

- Sobolev Spaces!
- Linear elliptic (laplace), hyperbolic (wave) PDEs
- nonlinear pdes arising from action principle (i.e. calculus of variations)

Sobolev Spaces

[ref: evans chapter 5., 222A lecture notes sec 11.]

<u>dfn</u> $u \in \mathcal{D}'(U)$ (open $\subseteq \mathbb{R}^d$). The k-th order, L^p-based Sobolev norm of u is defined

$$\|u\|_{W^{k,p}(U)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p}$$

weak derivative.

$D^\alpha u \in L^p$.

(<u>rmk</u> $\|D^\alpha u\|_{L^p}$ arose from energy method in 222A.)

<u>dfn</u> $W^{k,p}(U) = \{u \in \mathcal{D}'(U) : \|u\|_{W^{k,p}(U)} < +\infty\}$

"L^p-Sobolev space of order k on U"

note that $C_c^\infty(U) \subseteq W^{k,p}(U)$

We can also define

$$W_0^{k,p}(U) = \overline{C_c^\infty(U)}^{\|\cdot\|_{W^{k,p}}} \subseteq W^{k,p}(U).$$

set of functions in sobolev class which vanish to appropriate order on the boundary of U.

When $p=2$, we have extra tools since $\hat{\ }$ is an L^2 isometry.

Dfn $H^k(U) = W^{k,2}(U)$, $H_0^k(U) = W_0^{k,2}(U)$.

Prop i) $\forall k \in \mathbb{Z}_{\geq 0}$, $1 \leq p \leq \infty$

$(W^{k,p}(U), \|\cdot\|_{W^{k,p}})$ is a Banach space.

$(W_0^{k,p}(U), \|\cdot\|_{W^{k,p}})$ "

ii) $\forall k \in \mathbb{Z}_{\geq 0}$, $\langle u, v\rangle_{H^k} = \sum_{\alpha : |\alpha| \leq k} \langle D^\alpha u, D^\alpha v\rangle_{L^2}$

$(H^k(U), \langle\cdot,\cdot\rangle_{H^k})$ is a Hilbert space

$(H_0^k(U), \langle\cdot,\cdot\rangle_{H^k})$ " "

iii) (Fourier-Analytic char. of H^k)

$u \in H^k(U)$

$$\|u\|_{H^k} \simeq \|\hat{u}\|_{L^2} + \||\xi|^k \hat{u}\|_{L^2}$$
$$\simeq \|(1+|\xi|^2)^{k/2} \hat{u}\|_{L^2}$$

Notation

$A \lesssim B \iff$ for some $c>0$, $A \leq cB$

$A \simeq B \iff A \lesssim B$ and $B \lesssim A$

TODO: prove these

- Duality & Negative order Sobolev spaces.

Prop $k \in \mathbb{Z}_{\geq 0}$ $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{p'} = 1$.

$$(W_0^{k,p}(U))^* \simeq W^{-k,p'}(U)$$

Dfn (Neg. order Sobolev Space). $k \in \mathbb{Z}_{\geq 0}$, $1 < p < +\infty$
$U \underset{\text{open}}{\subset} \mathbb{R}^d$

$$\|u\|_{W^{-k,p}(U)} = \inf\left\{ \sum_{|\alpha| \leq k} \|g_\alpha\|_{L^p} : u = \sum_{|\alpha|\leq k} D^\alpha g_\alpha \right\}$$

$$W^{-k,p}(U) = \left\{ u \in \mathcal{D}'(U) : u = \sum_{|\alpha| \leq k} D^\alpha g_\alpha,\ g_\alpha \in L^p(U) \right\}$$

Rmk: if $g \in L^p$, then $D_{x^i} g \in W^{-1,p}$.
if $g \in W^{k,p}(U)$, then $D_{x^j} g \in W^{k-1,p}$

Basically space of dist. derivatives of L^p functions up to k-times.

Proof of previous proposition

Prop $k \in \mathbb{Z}_{\geq 0}$ $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{p'} = 1$.

$$(W_0^{k,p}(U))^* \simeq W^{-k,p'}(U)$$

Pf $(W_0^{k,p})^* \supseteq W^{-k,p'}(U)$

$v \in W^{-k,p'}(U) \Rightarrow v = \sum_{|\alpha| \leq k} D^\alpha g_\alpha$

$u \in W_c^{k,p}(U)$

$$\langle v, u \rangle = \int v u \, dx = \sum_{|\alpha| \leq k} \int D^\alpha g_\alpha u \, dx$$

$$= \sum_{|\alpha| \leq k} \int (-1)^{|\alpha|} g_\alpha D^\alpha u \, dx$$

(since $u \in L^p$, $g_\alpha \in L^{p'}$) ← or By distr. derivative. Compact spt gets ridd of bdry terms.

$$\leq \sum_{|\alpha| \leq k} \|g_\alpha\|_{L^{p'}} \|D^\alpha u\|_{L^p} \leq c \|v\|_{W^{-k,p'}} \|u\|_{W^{k,p}}$$

$(W_0^{k,p})^* \subseteq W^{-k,p'}$. General idea is to use the Hahn-Banach theorem.

recall for X a norm space, $Y \subset X$, and a linear functional

$$\ell : Y \to \mathbb{R} \quad \text{st} \quad |\ell(u)| \leq C\|u\|$$

then $\exists$ $\tilde{\ell}(u) \leq C\|u\|$ & $\tilde{\ell}/_Y = \ell$.

let $\ell : W_0^{k,p} \to \mathbb{R}$, ℓ bdd, $u \in C_0^\infty(U)$.

(our objective is $\ell(u) = \langle v, u\rangle = \sum_{|\alpha| \leq k} (-1)^{|\alpha|}\langle g_\alpha, D^\alpha u\rangle$)

to this end, define

(total # of multi-indices of order up to k)

$$T : C_0^\infty(U) \to L^p(U)^{\times K(k)}$$

$$u \mapsto (u, D_{x_1}u, \dots D_{x_d}u, \dots D^\alpha u)$$

$\|T(u)\| \leq C\|u\|_{W^{k,p}}$

T is injective, and an isomorphism onto it's image

what is this norm?

$$(C_0^\infty(U), \|\cdot\|_{W^{k,p}}) \sim (T(C_0^\infty(U)), \|\cdot\|).$$

we map $\ell \to \tilde{\ell} : T(C_0^\infty(U)) \to \mathbb{R}$

$\tilde{\ell}(Tu) = \ell(u) \to$ Bdd.

$\therefore$ By Hahn-Banach theorem, $\tilde{\ell}$ extends to

$$\underbrace{\tilde{\tilde{\ell}} : (L^p(U))^{\oplus K} \to \mathbb{R}}_{\in (L^p(U)^{\oplus K})^* = \{ f = \sum_\alpha \tilde{g}_\alpha \,:\, \tilde{g}_\alpha \in L^{p'}(U)\}} \quad \text{bdd}$$

Then,

$$\tilde{\hat{\ell}}(\tilde{V}) = \sum_\alpha \langle \tilde{g}_\alpha, \tilde{u}_\alpha \rangle \quad \text{for some } \tilde{g}_\alpha \in L^{p'}(U).$$

$\therefore$ for some $\tilde{u} \in T(C_c^\infty(U))$ (i.e. $\tilde{u} = Tu$)
$\tilde{u}_\alpha = D^\alpha u$

so, $\ell(u) = \tilde{\hat{\ell}}(T(u)) = \tilde{\hat{\ell}}(Tu) = \sum_{\alpha \le k} \langle \tilde{g}_\alpha, (Tu)_\alpha \rangle$
$= \sum_{\alpha \le k} \langle \tilde{g}_\alpha, D^\alpha u \rangle.$

setting $g_\alpha = \tilde{g}_\alpha (-1)^{|\alpha|}$ completes the proof.

■ A digression into functional analysis
Let X, Y be Banach spaces, $P : X \to Y$.

2 questions:

- Existence? $f \in Y$, $\exists\, u \in X$ st $Pu = f$?
- Uniqueness? $u, u' \in X$, does $Pu = Pu' \Rightarrow u = u'$?

rmk in our setting, P would be a PDO.

in energy methods, e.g. Laplace, you are usually proving apriori estimates.

$$-\Delta u = f \quad \text{in } U$$
$$u = 0 \quad \text{in } \partial U$$

gives us an a-priori estimate meaning we assumed existence of a solution.

$$\int_U -\Delta u \cdot u = \int_U f \cdot u \to \|Du\|_{L^2} \lesssim \left|\int f u \, dx\right|$$

$$\int_U |Du|^2$$

The claim is that apriori estimates can also ~~prove the existence~~ of a solution.

This is because existence & uniqueness are related by Duality.

in linear algebra, $A \in \mathbb{R}^{m \times n}$

$$Au = f, \quad (Au = 0 \Rightarrow u = 0) \quad \text{unique}$$

$$\Updownarrow$$

$$A^* v = g; \quad (\forall g, \exists v \text{ st } A^* v = g) \quad \text{exist.}$$

Lecture 1/20

Why study Sobolev spaces as Banach spaces?

want to study PDEs as problems in infinite-dimensional linear algebra.

- Digression to functional analysis

X, Y Banach,

$P: X \to Y$ linear & bdd.

Two questions:

- existence
- uniqueness

(Concrete thing to keep in mind.
often, we prove apriori estimates to PDEs.
If we can find apriori estimates for the dual problem, we get existence for the original problem)

Prop Let X, Y be Banach spaces, $P: X \to Y$ bdd, linear.

Let $P^*: X^* \to Y^*$ be the adjoint of P.

Suppose $\exists\, C > 0$ s.t. $\|u\|_X \leq C\|Pu\|_Y \ \forall u \in X$.

Then the following hold:

① (uniqueness for $Pu = f$)

If $u \in X$, $Pu = 0 \Rightarrow u = 0$.

② (existence for $P^* v = g$)

$\forall g \in X^*$, $\exists v \in Y^*$ s.t. $P^* v = g$

and

$\|v\|_{Y^*} \leq C\|g\|_{X^*}$

proof ① is obvious

② apply the Hahn-Banach theorem.

w.t. find $v \in Y^*$ s.t. $P^* v = g \iff \underset{\substack{\| \\ \langle v, Pu \rangle}}{\langle P^* v, u\rangle} = \langle g, u\rangle \quad \forall u \in X$.

Define $\ell: P(X) \to \mathbb{R}$, $\ell(Pu) = \langle g, u\rangle$ | note, since P is injective by ①, ℓ is well-defined.

By def. for $\underbrace{\|Pu\|_Y \leq 1}$, $|\ell(Pu)| = |\langle g, u\rangle| \leq \|g\|_{X^*}\|u\|_X$
$\leq C\|g\|_{X^*}\|Pu\|_Y$
$\leq C\|g\|_{X^*}$

$\therefore$ By Hahn-Banach, $\exists v \in Y^*$ s.t.

$\langle v, Pu\rangle = \ell(Pu) = \langle g, u\rangle \ \forall u \in X$ and $\|v\|_{Y^*} \leq C\|g\|_{X^*}$ ∎

What about existence for $Pu=f$?

we take the easy way out & assume X is reflexive, i.e.

$$\begin{pmatrix} X \to (X^*)^* \\ u \mapsto (v \mapsto \langle v,u\rangle) \end{pmatrix} \text{ is an isomorphism.}$$

to go the other way, switch duality @ the beginning.

<u>prop</u> X, Y Banach, $P: X \to Y$ linear, Bdd, X reflexive

Suppose $\|v\|_{Y^*} \le c \|P^* v\|_{X^*}$

then:

① uq. for $P^* v = g$

$v \in Y^*,\ P^* v = 0 \Rightarrow v = 0$

② existence for $Pu=f$

$\forall f \in Y,\ \exists u \in X$ st $Pu=f$

and

$\|u\|_X \le c \|f\|_Y$

<u>pf</u> same as before, using linear fctnl on X^*

<u>rmk</u> all Sobolev spaces $W_0^{k,p}(U)$, $1<p<\infty$ are reflexive.

will be a homework prob, using fact a closed subspace of a reflexive space is reflexive.

Rmk $\operatorname{range}(P)^{\perp} = \ker(P^*)$, $\ker(P) = {}^{\perp}(\operatorname{range} P^*)$

notation: $U \subseteq Y$, $U^{\perp} = \{v \in Y^* : \langle v, f \rangle = 0 \ \forall f \in U\}$,
$V \subseteq X^*$ ${}^{\perp}V = \{u \in X : \langle g, u \rangle = 0 \ \forall g \in V\}$.

as a consequence: if $\ker P^* = \{0\}$, then
$\operatorname{range}(P)^{\perp} = \{0\} \iff \overline{\operatorname{range}(P)} = Y$.

(1)
In finite dimensions, $\overline{\operatorname{range} P} = \operatorname{range} P$.
Therefore we get the fact regarding solveability for nonsquare matrices from last lecture.

However, in ∞-dim case, we don't necessarily have $\overline{\operatorname{range} P} = \operatorname{range} P$. That's what the estimate is.

(2) (qualitative vs. quantitative).
There is no loss of generality for deriving existence for P from the quantitative bound

$$\|v\|_{Y^*} \leq C \|P^* v\|_{X^*} \qquad (1)$$

Prop X, Y Banach, $P : X \to Y$ bdd, linear.
If $P(X) = Y$, then $\exists C > 0$ st (1) holds.

proof $\|P^* v\|_{X^*} = \sup_{u : \|u\|_X \leq 1} |\langle P^* v, u \rangle| = \sup_{u : \|u\|_X \leq 1} |\langle v, Pu \rangle|$

P is an open map. By the open mapping theorem, $P(B_X)$ is open and contains 0.
unit ball in X

Thus $\exists c>0$ st $P(B_X) \supseteq cB_Y$

$\therefore \|P^* v\|_{X^*} = \sup_{u:\|u\|_X \le 1} |\langle P^* v, u\rangle| = \sup_{\overline{B_X}} |\langle v, Pu\rangle|$

$\geq \sup_{f\in cB_Y} |\langle v, f\rangle|$

$= c\|v\|_{Y^*}$ ∎

e.g. $-u'' = f$ in $(0,1)$, with bdry
$u=0$ at $x=0, x=1$.

Q. solveability in $H_0^1((0,1))$?

$\hookrightarrow \overline{C_c^\infty((0,1))}^{\|\cdot\|_{H^1}}$, $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2$

likewise, recall $(H_0^1((0,1)))^* = H^{-1}(0,1)$.

using $X = H_0^1((0,1))$, $Y = H^{-1}((0,1))$.

$P = -\partial_x^2$

Claim: if $Pu = f$ for $u \in X$, then we have

$\|u\|_X \le c\|f\|_Y$ (2)

$\iff -u''=f$, $u \in H_0^1$, then $\|u\|_{H^1} \le c\|f\|_{H^{-1}}$

proof. use the energy method.

$$\int -u'' u\,dx = \int f u\,dx = \int (u')^2 dx = \|u'\|_{L^2}^2.$$

To prove (2), it suffices to consider $u \in C_c^\infty((0,1))$.
It gets rid of Bdry term in IBP

how about $\|u\|_{L^2}^2$? Use $u\big|_{\partial(0,1)}=0$

Use $u(x) = \int_0^x u'(y)\,dy$.

So $|u(x)| \le \int_0^1 |u'(y)|\,dy \overset{\text{Cauchy-Schwartz}}{\le} \|u'\|_{L^2}$

$\Rightarrow \int_0^1 |u|^2 dy \le \sup_{[0,1)} |u|^2 \le \|u'\|_{L^2}^2$

$\Rightarrow \|\cancel{u}\|_{H^1} \le c|\langle f, u\rangle| \le c\|f\|_{H^{-1}}\|\cancel{u}\|_{H^1}$

$\Rightarrow \|u\|_{H^1} \le c\|f\|_{H^1}$ $\boxed{\equiv}$

from this, (2) & Prop 1 $\Rightarrow$ $-u''=0$ & $u\in H_0^1((0,1)) \Rightarrow u=0$.

(2) + Prop 2 $\Rightarrow$ need to compute P^*

By reflexivity of H_0^1, $(H^{-1})^* = H_0^1$.

So, we compute $\langle v, Pu\rangle = \int_0^1 v(-u'')\,dx = \int_0^1 v'u'\,dx$ $(v\in H_0^1)$

$P^* = -\partial_x^2$ $\longleftarrow$ $(u\in H_0^1) = \int_0^1 -v''u\,dx = \langle P^*v, u\rangle$

$\therefore$ $Y^* = H_0^1$ & the problem is entirely self-adjoint.

$\Rightarrow$ we get $\forall f\in H^{-1}$, $\exists u\in H_0^1$ st $Pu=f$.

From our energy method, we hinted at the Poincaré inequality.

§ Approximation (Density Arguments) for Sobolev spaces.

- convolution & mollifiers.

<u>Lemma:</u> let $\varphi \in C_0^\infty$, $\int \varphi \, dx = 1$.

$u \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$.

<u>mollifiers:</u> $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d} \varphi\left(\frac{x}{\varepsilon}\right)$ (NB: $\int \varphi_\varepsilon = 1$).

$$\| \varphi_\varepsilon * u - u \|_{L^p} \to 0 \text{ as } \varepsilon \to 0$$

$$\varphi_\varepsilon * u = \int \varphi_\varepsilon(x-y) u(y) dy$$

<u>proof</u> key ingredient is continuity of translation on L^p.

$z \in \mathbb{R}^d$, $u \in L^p$, $\tau_z u(x) = u(x-z)$.

$\lim_{|z| \to 0} \| \tau_z u - u \|_{L^p} = 0$. (Uses Dom convergence to prove).

to prove take

$$\varphi_\varepsilon * u - u = \int u(x-y) \varphi_\varepsilon(y) dy - u(x)$$

$$= \int (u(x-y) - u(x)) \varphi_\varepsilon(y) dy$$

$$\| \text{''} \text{''} \|_{L^p} = \left\| \int (u(x-y) - u(x)) \varphi_\varepsilon(y) dy \right\|_{L^p}$$

$$\le \int \|u(\cdot - y) - u(\cdot)\|_{L^p} |\varphi_\varepsilon(y)|\, dy$$

Minkowski inequality

$\Rightarrow$ spt $\varphi_\varepsilon \to \{0\}$ by cpt spt.

$\Rightarrow |y| \to 0$ as $|\varepsilon| \to 0$ where integrand $\neq 0$.

By translation, the whole integrand $\to 0$.

The remainder follows by the dominated convergence, whole integral $\to 0$.

- Smooth Partitions of unity.

<u>Lemma</u> Let $\{U_\alpha\}_{\alpha \in A}$ an open covering of $\mathcal{U}$ in $\mathbb{R}^d$. $\exists$ a smooth partition of unity subordinate to $\{U_\alpha\}_{\alpha \in A}$. i.e.

Partition of unity:

- $\sum_\alpha \chi_\alpha(x) = 1$ on $\mathcal{U}$
- $\forall x \in \mathcal{U}$, $\exists$ only finitely many χ_α.

subordinate:

- spt $\chi_\alpha \subseteq U_\alpha$
- χ_α is smooth.

If start from a continuous partition of unity & apply mollificato to get a smooth object.

1/25/22 Today: - approximation of things in Sobolev space by nicer objects

- prove useful techniques, e.g. extension & trace Theorems on bdd domains.

Approximation Theorems

Our goal is to take something $u \in W^{k,p}(U)$ & approximate by some smooth, cpt supported objects.

Main tools are convolution & mollification.

$$f * g = \int f(x-y) g(y) dy$$

key props: • $\partial_{x_j} f * g = f * \partial_{x_j} g$

• approx of identity by $\varphi_\varepsilon = \frac{1}{\varepsilon^d} \varphi\left(\frac{\cdot}{\varepsilon}\right)$

"$\varphi_\varepsilon * f \to f$" as $\varepsilon \to 0$, convergence is dependent on norm.

Smooth partitions of unity. (usually an open cover)

$\{U_\alpha\}_{\alpha \in A}$ open sets

$\exists \{\chi_\alpha(x)\}_{\alpha \in A}$ smooth which

- smooth
- $\sum_{\alpha \in A} \chi_\alpha$ is locally finite & $= 1$.
- subordinate $\operatorname{spt} \chi_\alpha \subseteq U_\alpha$

Thm Let $k \in \mathbb{Z}_{\geq 0}$, $1 \leq p < \infty$.

i) $C^\infty(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$

ii) $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$

pf

i) solely an application of mollification technique.

ii) (HW) approximate by $f\chi(\frac{\cdot}{R})$, $\chi \in C_c^\infty(\mathbb{R}^d)$

$\chi(0) = 1$, $R \to \infty$.

Thm $k \in \mathbb{Z}_{\geq 0}$ $1 \leq p < \infty$, U is any open set in $\mathbb{R}^d$. $C^\infty(U)$ is dense in $W^{k,p}(U)$.

~~pf~~ $u \in W^{k,p}(U)$, fix $\varepsilon > 0$. We want to find $v \in C^\infty(U)$ st

$\|u - v\|_{W^{k,p}(U)} \leq C\varepsilon$

1. Consider $U_j = \{x \in U : \operatorname{dist}(x, \partial U) > \frac{1}{j}\}$.

$V_j = U_j \setminus \overline{U_{j+2}}$

$U \subseteq \bigcup_{j=1}^\infty V_j \Rightarrow \exists \chi_j$ PCU subordinate to V_j.

$\therefore u = \sum_{j=1}^\infty u\chi_j := \sum_{j=1}^\infty u_j$ | $\operatorname{spt} \chi_j \subseteq V_j \Rightarrow \operatorname{spt} u_j \subseteq V_j$.
u_j is $C_0^\infty(\mathbb{R}^d)$

define $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\int \varphi_\varepsilon = 1$, $\operatorname{spt}\varphi \subseteq B_1(0)$.

now mollify u_j $\qquad\Downarrow$

$\qquad\qquad \operatorname{spt}\varphi_{\varepsilon_j} \subseteq B_{\varepsilon_j}(0)$.

$v_j = \varphi_{\varepsilon_j} * u_j$, where we choose ε_j to achieve:

$\|u_j - v_j\|_{W^{k,p}(U)} \le 2^{-j}\varepsilon$, and $\operatorname{spt} v_j \subseteq \tilde{V}_j = U_{j-1} \setminus \overline{U_{j+2}}$

[__Fact__ $\operatorname{spt}(f*g) \subseteq \operatorname{spt}(f) + \operatorname{spt}(g) = \{x+y : x \in \operatorname{spt} f,\ y \in \operatorname{spt} g\}$]

now, take $v = \sum_{j=1}^{\infty} v_j$ $\to$ well-defined since $\tilde{V}_j$ is locally finite.

$\to \|v-u\|_{W^{k,p}} \le \sum_{j=1}^{\infty} \|v_j - u_j\|_{W^{k,p}} \le \sum_{j=1}^{\infty} 2^{-j}\varepsilon = \varepsilon.$

$\blacksquare$

C^∞ has no control near boundary.

Thm $k \in \mathbb{Z}_{\ge 0}$, $1 \le p < \infty$, U is a bounded open set, with ∂U of class C^1.

$C^\infty(\overline{U})$ is dense in $W^{k,p}(U)$.

__notation__ $C^\infty(\overline{U}) = \{u : U \to \mathbb{R} : u$ is the restriction of a $\tilde{u} \in C^\infty(\tilde{U})$, $\tilde{U} \supseteq \overline{U}\}$.

__recall__ ∂U is of class C^k if $\forall x_0 \in \partial U$, $\exists r = r(x_0) > 0$, s.t. up to relabeling, $B_{r_0}(x_0) \cap U = \{x \in B_r(x_0) : x^d > \gamma(x^1, \dots, x^{d-1})\}$, where the graph γ is C^k on $(B_{r_0} \cap \mathbb{R}^{d-1}_x)$

Basically, it's a C^k manifold w/ bdr.

Pf ① reduce to simpler case.

$u \in W^{k,p}(U)$, $\varepsilon > 0$. By dfn of C^1-regularity of ∂U, ∂U can be covered by balls $\{B_{r_k}(x_k)\}_{k=1}^{K}$, in each of which U can be represented as the region above a C^1 graph.

The # of such balls is finite by compactness of ∂U.

we may add to $U_k = B_{r_k}(x_k)$ an open set $U \setminus \bigcup_{k=1}^{K} U_k$ so that $\{U_0, U_1, \dots U_K\}$ is an open cover of U.

Choose $U_0 \supseteq U \setminus \bigcup_{k=1}^{K} U_k$,

Then $\{U_0, \dots U_K\}$ is an open cover, so, choosing $\{\chi_k\}_{k=0}^{K}$ smooth subordinate POU sub. to $\{U_k\}_{k=0}^{K}$

$$u = \sum_{k=0}^{K} u\chi_k =: u_0 + \sum_{k=1}^{K} u_k$$

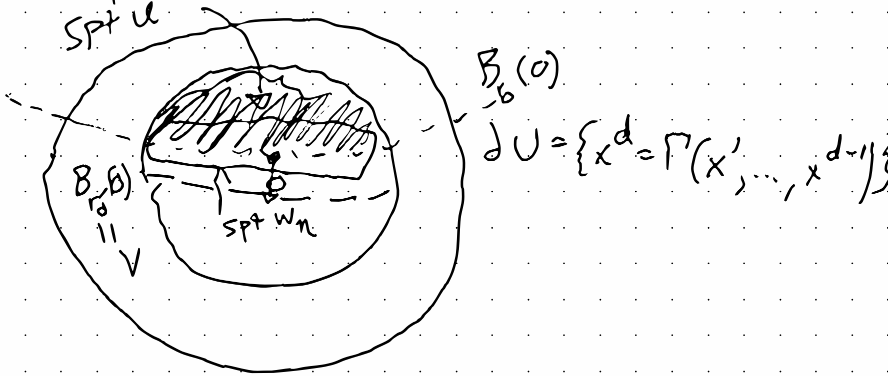
Cpt so we may mollify ✓

suffices to consider $U = B_{r_0}(x_0)$, $\operatorname{spt} u_k \subseteq V \subseteq U$.

$B_{r_0'}(x_0)$.

This gives a very explicit description of the boundary.

Step 2 WLOG assume $x_0 = 0$, $\varepsilon > 0$.



we want to mollify u.

Use a 2 step approximation.

d-th unit vector

(i) let $w_\eta = u(x + \eta e^d)$

as $\eta \to 0$, by continuity prop, w_η approximates u.

$$\|u - w_\eta\|_{W^{k,p}(U \cap B_{r_0}(0))} < \frac{1}{2}\varepsilon.$$

moreover, w_η is defined on $B_{r_0}(0) \cap U - \eta e^d$

(ii) now, choose $v = \varphi_\delta * w_\eta$, and $\delta \ll \eta$, then

v is well-defined on $V \cap \{x^d > \Gamma(x', \dots, x^{d-1})\}$.

& if δ is sufficiently small,

$$\|v - w_\eta\|_{W^{k,p}(U \cap B_{r_0}(0))} < \frac{1}{2}\varepsilon.$$

$$\Rightarrow \|u - v\|_{W^{k,p}(U)} \le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \le \varepsilon.$$

moreover, $v \in C^\infty(\overline{V \cap \{x^d > \Gamma(x^1, \dots, x^{d-1})\}})$ which is enough. $\boxed{\Sigma}$

Can use Lipschitz Bdry & ϕ require bounded.

• Extension Theorems.

tools to deal with $u \in W^{k,p}(U)$ when U is bounded.

<u>Thm</u> $k \in \mathbb{Z}_{\geq 0}$, $1 \leq p < \infty$, U a bounded domain with C^k boundary, V is an open set containing $\overline{U}$, Then, $\exists\, \mathcal{E} : W^{k,p}(U) \to W^{k,p}(\mathbb{R}^d)$ s.t

(i) (extension) $\mathcal{E}u|_U = u$.

(ii) (linear & bdd) $\mathcal{E}$ is linear, and $\|\mathcal{E}u\|_{W^{k,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{k,p}(U)}$

(iii) $\operatorname{spt}(\mathcal{E}u) \subseteq V$.

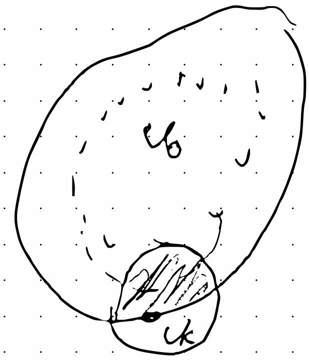
<u>proof</u> observe that by our approximation & bdd. theorem, it suffices to consider $u \in C^\infty(\overline{U})$.

<u>step 1</u> reduction to the half-ball case.

as in step 1 of pf of final approx thm, construct $\{U_0, \dots U_K\}$, $\{\chi_k\}_1^K$ def $u_k = u\chi_K$

• u_0 is already in $W^{k,p}(\mathbb{R}^d)$, since it's extended by 0.

• $u_k \in C^\infty(\bar{U})$, $\operatorname{spt} u_k \subseteq V_k \cap U$



for $\begin{cases} y^j = x^j & 1 \le j < d \\ y^d = x^d - \Gamma(x^1, \dots, x^{d-1}) \end{cases}$

$U_k \cap U \hookrightarrow \{ y \in B_r(0) : y^d > 0 \}$.

$x \mapsto y$ is C^k, U_k smooth,

$\Rightarrow$ $u_k(y) = u_k(x(y))$ satisfies (by chain rule).

$$\| u_k(y) \|_{W_y^{k,p}(\tilde{U})} \le C \| U_k(x) \|_{W_x^{k,p}}$$

<u>step 2</u> Extension in the half ball case.

$U = B_r^+(0)$, $W = B^+_{r/2}(0)$

$\operatorname{spt} u \subseteq W$.

how to extend? higher order reflection method.

$$Eu = \tilde{u} = \begin{cases} u & x^d > 0 \\ \sum_{j=0}^{k} \alpha_j u(x^1, \dots x^{d-1}, -\beta_j x^d) & x^d < 0 \end{cases}$$

$0 < \beta_j < 1$ so that $x^1, \dots x^{d-1}, -\beta_j x^d \in B_r^+(0)$.

want to change α_{j} so that it matches normal derivatives up to order K.

$$u(x^1, \dots x^{d-1}, 0+) = \sum_{j=0}^{k} \alpha_j u(x^1, \dots, x^{d-1}, 0-)$$

$$\partial^j_{x^d} u = (-\beta_j)^j \partial_{x^d}(u)$$

$$\Rightarrow \partial_{x^d} u = \sum \alpha_j (-\beta_j) \partial_{x^d} u$$

$$\vdots$$

$$\partial^k_{x^d} u = \sum \alpha_j (-\beta_j)^k \partial^k_{x^d} u$$

$$\Leftrightarrow 1 = \sum_j \alpha_j \qquad \Leftrightarrow \qquad \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & \cdots & 1 \\ -\beta_1 & \cdots & \beta_1 \\ & \vdots & \\ (-\beta_k)^k & \cdots & (-\beta_k)^k \end{pmatrix}}^{\text{Vandermonde matrix.}} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_k \end{pmatrix}$$

$$1 = \sum_K \alpha_j (-\beta_j)$$

$$1 = \sum_{j=0} \alpha_j (-\beta_j)^k$$

if all β_j's are distinct, this matrix is invertible.

$$\Longleftrightarrow$$

$\exists\, \alpha_1 \dots \alpha_k$ st the eqn holds,

defines $\tilde{U}$ on $B_r(x)$, extending u & matches up derivatives.

Finally, put an appropriate indicator function

$\chi_V = 1$, $\operatorname{spt} \chi_V \subseteq V$, $E u = \chi_V \tilde{U}$.

lecture 1/27:

From next wk, in person lecture. 31 Evans

Today: i) trace & extension (from bdry). this terminology for restricting to bdry

ii) Sobolev inequalities.

- Trace & Extension from bdry.

U: open $\subseteq \mathbb{R}^d$, ∂U is C^1

$1 < p < \infty$, $k \in \mathbb{Z}_{\geq 0}$

recall: $C^\infty(\bar{U}) \subseteq W^{k,p}(U)$ dense.

fix $k = 1$.

$C^\infty(\bar{U}) \subseteq W^{1,p}(U)$ dense.

aim: discuss the restriction of $u \in W^{1,p}(U)$ to ∂U.

(NB, $\mu(\partial U)$ is 0, so if we just deal w/ L^p, it is ? are.).

Take, (to start with) $u \in C^\infty(\bar{U})$

Dfn Let $u \in C^1(\bar{U})$. $tr_{\partial U} u = u|_{\partial U}$

we want to extend tr to all of $W^{1,p}(U)$.

note, $tr_{\partial U}$ is linear.

<u>Thm</u> (trace theorem, nonsharp)

Let U bdd, open in $\mathbb{R}^d$ with ∂U C^1. volume form on the bdry.

$1 < p < \infty$. Then for $u \in C^1(\bar{U})$ $\|\mathrm{tr}_{\partial U} u\|_{L^p(\partial U)} \lesssim \|u\|_{W^{1,p}(U)}$

(i) As a consequence, $\mathrm{tr}_{\partial U}$ is extended uniquely by continuity & density of $C^1(\bar{U}) \subseteq W^{1,p}(U)$ to

$$\mathrm{tr}_{\partial U} : W^{1,p}(U) \to L^p(\partial U).$$

(ii) $u \in W_0^{1,p}(U) \iff \mathrm{tr}_{\partial U} u = 0$

<u>note</u> the extension is <u>not</u> surjective, ie.

$\mathrm{img}(\mathrm{tr}_{\partial U}) \subsetneq L^p(\partial U)$.

Pf in evans. §5.5.

we choose to discuss a sharp version in a restricted setting.

in particular, we will take $p = 2$.

now, we can use fourier transforms, & eventually wind up at fractional sobolev spaces.

We also set $U = \mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^d > 0\}$,

notation $\partial U = \{(x', 0) \in \mathbb{R}^d\} \cong \mathbb{R}^{d-1}$

recall also $\|u\|^2_{H^k} \simeq \|(1+|\xi|^2)^{k/2} \hat{u}\|^2_{L^2}$

we can generalize to $k \in \mathbb{R}$, & use fractional L^2 sobolev spaces.

$$\boxed{\begin{aligned}\hat u(\xi) &= \int u(x) e^{-i\xi x}\,dx \\ u(x) &= \int \hat{u}\, e^{i\xi x}\frac{d\xi}{2\pi}\end{aligned}}$$

<u>Thm</u> (Sharp Trace).

For $u \in C^1(\overline{\mathbb{R}^d_+}) \cap H^1(\mathbb{R}^d_+)$

$$\|tr_{\partial U} u\|_{H^{1/2}(\mathbb{R}^{d-1})} \lesssim \|u\|_{H_1(\mathbb{R}^d_+)}$$

pf Take $u \in C^1(\overline{\mathbb{R}^d_+}) \cap H^1(\mathbb{R}^d_+)$. Using the extension theorem of the previous lecture, we can find

$\tilde u \in C^1(\mathbb{R}^d)$ such that $\|\tilde u\|_{H^1(\mathbb{R}^d)} \le c\|u\|_{H^1(\mathbb{R}^d_+)}$

$$tr_{\partial U} u = u(x',0) = \tilde u(x',0) = \int \tilde{\mathcal{F}}_{x_d} \tilde u(x',\xi_d)\frac{d\xi_d}{2\pi}$$

$$\tilde{\mathcal{F}}_{x'}\, tr_{\partial U} u(\xi') = \int \tilde{\mathcal{F}} \tilde u(\xi',\xi_d)\frac{d\xi_d}{2\pi}$$

we want to get $\|(1+|\xi'|^2+|\xi_d|^2)^{1/2}\tilde{\mathcal{F}}\tilde u\|$

$$\|tr_{\partial U} u\|_{H^s} \simeq \|(1+|\xi'|^2)^{s/2}\tilde{\mathcal{F}}_{x'}\, tr_{\partial U} u(\xi')\|_{L^2_{\xi'}}$$

$$\left\|(1+|\xi'|^2)^{s/2}\int \tilde{\mathcal{F}}\tilde u(\xi',\xi^d)\frac{d\xi_d}{2\pi}\right\|_{L^2_{\xi'}}$$

$$= \left\|\int \frac{(1+|\xi'|^2)^{s/2}}{(1+|\xi'|^2+|\xi^d|^2)^{s/2}}\left((1+|\xi'|^2+|\xi^d|^2)^{s/2}\tilde{\mathcal{F}}\tilde u\right)\frac{d\xi_d}{2\pi}\right\|_{L^2_{\xi'}}$$

$$\le \left\|\left(\int\frac{((1+|\xi'|)^2)^s}{1+|\xi'|^2+\xi_d^2}\,d\xi_d\right)^{1/2}\|(1+|\xi|^2)^{1/2}\tilde{\mathcal{F}}\tilde u\|_{L^2_{\xi_d}}\right\|_{L^2_{\xi'}}$$

$$\leq \sup_{\xi' \in \mathbb{R}^{d-1}} \left[\int \frac{(1+|\xi'|^2)^{\frac{?}{2}}}{1+|\xi'|^2+\xi_d^2} \, d\xi_d \right] \| \cdots\cdots \|$$

$\|$

$\|U\|_{H^1}$

$\|^{<}$

TODO

$\boxtimes$

<u>Thm</u> (extension from ∂U)

$\exists$ Bdd linear map $\mathrm{ext}_{\partial U} : H^{1/2}(\mathbb{R}^{d-1}) \to H^1(\mathbb{R}^d_+)$

s.t. $\mathrm{tr}_{\partial U} \circ \mathrm{ext}_{\partial U} = \mathrm{id}$

~~proof~~ we will use the Poisson semigroup.

$g \in \mathcal{S}(\mathbb{R}^{d-1})$.

Define $u^{=\mathrm{ext}_{\partial U} g}$, $\widetilde{\mathcal{F}}_{x'} u(\xi', x^d) = \eta(x^d) e^{-x^d|\xi'|} \hat{g}(\xi')$

where η is smooth, cpt spt

$\eta(s) = 1 \quad |s| < 1 \qquad \eta = 0 \quad |s| > 2.$

WTS that $u \in H^1(\mathbb{R}^d_+) \iff \begin{cases} i & u, \partial_1, \dots \partial_{d-1} u \in L^2 \\ ii & \partial_d u \in L^2 \end{cases}$

i) $\longrightarrow \|u\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2 + \dots \|\partial_{d-1} u\|_{L^2}^2$

(plancherel)

$$= \|(1+|\xi'|^2)^{1/2} [\widetilde{\mathcal{F}}_{x'} u](\xi', x^d)\|^2_{L^2_\xi L^2_{x^d}}$$

$$= \|(1+|\xi'|^2)^{1/2} \eta(x^d) e^{-x^d|\xi'|} \hat{g}(\xi')\|^2_{L^2_{\xi'}, L^2_{x^d}}$$

$$= \|\underbrace{(1+|\xi'|^2)^{1/4} \|\eta(x^d) e^{-x^d|\xi'|}\|_{L^2_{x^d}}}_{\text{need to show uniform bdd } \forall \xi'} (1+|\xi'|^2)^{1/4} \hat{g}(\xi')\|^2_{L^2_{\xi'}}$$

$$\|\eta(x^d)\, e^{-x^d|\xi'|}\|^2_{L^2_{x^d}} \lesssim 1 \quad \text{(trivial).}$$

$$\Downarrow$$

$$\int \eta^2(x^d)\, e^{-2x^d|\xi'|}\, dx^d \lesssim \frac{1}{|\xi'|} \qquad \text{(substitution}$$

$$\Rightarrow \|\eta(x^d)\, e^{-x^d|\xi'|}\|_{L^2_{x^d}} \lesssim \min\left\{1, \frac{1}{|\xi'|^{1/2}}\right\} \lesssim (1+|\xi'|)^{-1/2}$$

$$\lesssim \|\cancel{(1+|\xi'|^2)^{1/4}}\|\eta(x^d)\, e^{-x^d|\xi'|}\|_{L^2_{x^d}}\, \cancel{(1+|\xi'|^2)^{1/4}}\, \hat{g}(\xi')\|_{L^2_{\xi'}}$$

These cancel, & we have ✓ (i)

(ii) $\partial_{x^d} u = \partial_{x^d}(\eta(x^d) v)$, $\tilde{\mathcal{F}}_{x'} v = e^{-x^d|\xi'|}\hat{g}(\xi)$

$$= \underbrace{\eta'(x^d) v}_{\hookrightarrow\ \le \|v\|_{L^2(x^d \in \operatorname{supp}\eta')}} + \underbrace{\eta \cdot \partial_{x^d} v}_{}$$

$$\longrightarrow \|\eta\, \partial_{x^d} v\|_{L^2_{x'} L^2_{x^d}} \overset{\text{Plancherel}}{=} \|\eta\, \partial_{x^d}(e^{-x^d|\xi'|}\hat{g}(\xi'))\|$$

$$= \|\eta(x^d)\, |\xi'|\, e^{-x^d|\xi'|}\, \hat{g}(\xi')\|$$

$$\overset{(i)}{\le} C\|g\|_{H^{1/2}} \qquad ✓$$

<u>Rmk</u> (i) To generalize to L^2 Sobolev spaces we only need fractional Sobolev spaces on a C^2 boundary.
This just uses C^1-local diffeomorphism to the half space.

Independence of the norm from choice of straightening follows from interpolation theory.

See Stein's book, 1970

ii) For $p \neq 2$, $\operatorname{im}\left(\mathrm{tr}_{\partial U} W^{1,p}(U)\right) = B_p^{1-1/p,\,p}(\partial U)$

$L^{1,p}$ Besov space w/ regularity index of order $1-\frac{1}{p}$ of summability index p.

also in Stein.

§ Sobolev inequalities

in a nutshell: quant. generalizations of the FTC.

$$\text{"}\ \underline{u(x)} = \int_{-\infty}^{x} \underline{u'(x')}\,dx'$$

use this to control this

Thm (Gagliardo - Nirenberg - Sobolev inequality)

$d \geq 2$, $u \in C_c^\infty(\mathbb{R}^d)$, we have

$$\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq C_d \|Du\|_{L^1(\mathbb{R}^d)}$$

rmk. The exponent $\frac{d}{d-1}$ need not be remembered because it can be derived from scaling considerations, i.e. dimensional analysis.

$$u \to u_\lambda\left(\frac{x}{\lambda}\right), \quad \lambda > 0.$$

$$\|u_\lambda\|_{L^p} = \Big(\lambda^d \underbrace{\int |u(\tfrac{x}{\lambda})|^p \frac{dx}{\lambda^d}}_{\int |u|^p dx'}\Big)^{1/p} = \lambda^{d/p} \|u\|_{L^p}$$

$Du_\lambda = \frac{1}{\lambda}(Du)_\lambda \Rightarrow \|Du_\lambda\|_{L^p} = \lambda^{\frac{d}{p}-1}\|Du\|_{L^p}$.

so
$$\|Du_\lambda\|_{L^1} = \lambda^{d-1}\|Du\|_{L^1}$$
$$\|u_\lambda\|_{L^p} = \lambda^{d/p}\|u\|_{L^p}$$

$(x^1, \dots, \hat{x^j}, \dots x^d)$ indicates the removal of an index.

$\forall \lambda > 0, \quad \|u_\lambda\|_{L^p} \leq C\|Du_\lambda\|_{L^1}$

$$\lambda^{d/p}\|u\|_{L^p} \overset{(\Rightarrow)}{\leq} C\lambda^{-1+d}\|Du\|_{L^1}. \Rightarrow \frac{d}{p} = d-1$$

$\Rightarrow p = \frac{d}{d-1}$.

// scaling.

$[\|\cdot\|_{L^p}] \sim [x]^{d/p} \qquad [D] \sim \frac{1}{[x]}$.

proof of the Theorem

Key ingredient is another inequality:

Lemma (Loomis - Whitney inequality)

$d \geq 2, \quad j = 1, \dots, d, \qquad f_j = f_j(x^1, \dots, \hat{x^j}, \dots x^d)$

$$\left\|\prod_{j=1}^{d} f_j\right\|_{L^1(\mathbb{R}^d)} \leq \prod_{j=1}^{d}\|f_j\|_{L^{d-1}(\mathbb{R}^{d-1})},$$

Pf integrate along each direction & apply Hölder,

$$\int\left|\prod_{j=1}^{d} f_j\right| dx^1 = |f_1|\int\prod_{j=2}^{d}|f_j|\,dx^1 \leq |f_1|\prod_{j=2}^{d}\|f_j\|_{L^{d-1}_{x^1}}$$

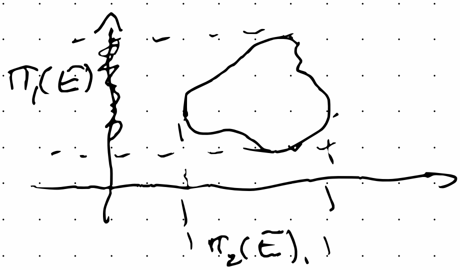
Repeat

$$\iint \Big|\prod_{j=1}^{d} f_j\Big|\, dx^1 dx^2 \le \int |f_1|\, \|f_2\|_{L^{d-1}_{x^1}} \cdots \|f_d\|_{L^{d-1}_{x^1}}\, dx^2$$

$$= \|f_2\|_{L^{d-1}_{x^1}} \|f_1\|_{L^{d-1}_{x^2}} \|f_3\|_{L^{d-1}_{x^1,x^2}} \cdots \|f_d\|_{L^{d-1}_{x^1,x^2}}$$

$$\int \cdots \int \Big|\prod_{j=1}^{d} f_j\Big|\, dx^1 \cdots dx^d \le \prod_{j=1}^{d} \|f_j\|_{L^{d-1}_{x^1,\dots,\widehat{x^j},\dots,x^d}} \qquad \blacksquare$$

<u>Rmk</u> L-W ineq answers a geometric q. $E \subseteq \mathbb{R}^d$, & know $\pi_j(E) = \{x \in \mathbb{R}^d : x^j = 0, \exists x^{j'} \text{ s.t. } x^1, \dots x^{j'}, \dots x^d \in E\}$.



can we bound by product of coord measures?

$$\mu(E) = \int \mathbb{1}_E\, d\mu \le \int \prod_{j=1}^{d} \mathbb{1}_{\pi_j(E)}(x^1, \dots \widehat{x^j}, \dots x^d)\, dx$$

$$\le \prod_{j=1}^{d} \mu(\pi_j(E))^{1/d-1}$$

Lecture 21 OH TuTh 5-6 887 evans

recall, discussing Sobolev inequalities

Thm (Gagliardo–Nirenberg–Sobolev Inequality)

$\mathbb{R}^{d\geq 2}$, $u \in C_0^\infty(\mathbb{R}^d)$ then

$$\|u\|_{L^{d/d-1}} \leq c\|Du\|_{L^1}$$

our approach uses a lemma

Lem [Loomis–Whitney]

$d \geq 2$ $f_1, \dots f_d$: s.t. $f_j = f_j(x_1, \dots \hat{x}_j, \dots x_d)$

$$\left\|\prod_{j=1}^{d} f_j\right\|_{L^1(\mathbb{R}^d)} \leq \prod_{j=1}^{d} \|f_j\|_{L^{d-1}(\mathbb{R}^{d-1})}$$

pf last time, integral var by var & use Hölder.

Pf (of Gagliardo)

Choose $x \in \mathbb{R}^d$.

$$u(x) = \int_{-\infty}^{x^j} \partial_{y_j} u(x_1, \dots, y, \dots, x_d)\, dy.$$

but freedom to choose j coords u.

$$|u(x)| \leq \int_{-\infty}^{x^j} |Du(x_1, \dots y, \dots x^d)|\,dy \leq \underbrace{\int_{-\infty}^{\infty} |Du(\dots y \dots)|\,dy}_{\tilde{f}_j}$$

$$\Rightarrow |u(x)| \leq \left(\prod \tilde{f}_j\right)^{1/d} \Rightarrow |u(x)|^{d/d-1} \leq \prod \tilde{f}^{\frac{1}{d-1}}$$

by Loomis-Whitney

$$\|u\|^{d/d-1}_{L^{d/d-1}} \le \int |u|^{d/d-1} dx \le \int \prod f_j \, dx \le \prod_j \|f_j\|_{L^{d-1}}$$

$$= \prod_{j=1}^{d} \Big(\int \underbrace{|f_j|^{d-1}}_{} \, \widehat{dx^j} \Big)^{1/d-1} =$$

$$\int_{-\infty}^{\infty} |Du(x_1, \dots x^j) .. dx^j| \le \|Du\|^{d/d-1}_{L^1}$$

$\boxtimes$

<u>rmk</u> This is the functional version of the isoperimetric inequality.

If this can be seen by approximating $u = \mathbb{1}$

what about varying p on the RHS of GNS?

<u>Thm</u> (Sobolev inequality for L^p-Based spaces)

$d \ge 2$ $\quad 1 < p < d$ $\quad u \in C_0^\infty(\mathbb{R}^d)$

Jrown value $= q$.

$$\|u\|_{L^{\frac{dp}{d-p}}} = \|u\|_{L^q} \le \|Du\|_{L^p(\mathbb{R}^d)} \quad (*)$$

to find q, we use dim. analysis

$$[x]^{d/q} \sim [x]^{-1+\frac{d}{p}}$$

$$\Rightarrow \frac{d}{q} = \frac{d}{p} - 1 \Rightarrow q = \frac{d}{d/p - 1} = \frac{dp}{d-p}$$

Pf $\qquad \tilde{q} = q\left(\frac{d-1}{d}\right)$

$v = |u|^{\tilde{q}}$ justify using approximation

$|Dv| = \tilde{q}|u|^{\tilde{q}-1}|Du|$ $\leftarrow$ $|x| = \lim_{\varepsilon\to 0^+}(\varepsilon^2+x^2)^{1/2}$, DCT to put in the integral

$$\int |u|^q dx = \int |v|^{d/(d-1)} \overset{GNS}{\leq} \left[\int |Dv|\,dx\right]^{\frac{d}{d-1}} = \left[\int \tilde{q}|u|^{\tilde{q}-1}|Du|\,dx\right]^{\frac{d}{d-1}}$$

now, use Hölder to put $|Du|$ in L^p.

By dim. analysis, it must happen that

$$\left[\int \tilde{q}|u|^{\tilde{q}-1}|Du|\,dx\right]^{d/d} \leq \|u\|_{L^q}^{\frac{d-1}{d}(q-1)} \|Du\|_{L^p}^{\frac{d-1}{d}}.$$

exponentiating & division completes the proof $\boxtimes$

Thm $\quad d \geq 2 \qquad 1 \leq p < d$, U a bounded domain in $\mathbb{R}^d$.

(i) $u \in W^{1,p}(\mathbb{R}^d)$, $(*)$ holds. (by density)

(ii) $u \in W^{1,p}_0(U)$, $(*)$ holds. (completion of $C_c^\infty(U)$)

(iii) ∂U is C^1, $u \in W^{1,p}(U)$, we have

$$\|u\|_{L^q(U)} \leq c\|u\|_{W^{1,p}(U)} \qquad q = \frac{dp}{d-p}.$$

Pf (i, ii) follow from density.

(iii) follows from extension & approximation.

rmk in (iii) we need the full $W^{1,p}$ norm in the extension procedure.

Compare to case ii where no info regarding u itself is needed since it's $\underbrace{0 \text{ on the boundary}}$.

$$u\big|_{\partial U} = u \leftarrow$$

(ii) is sometimes called a poincaré-type inequality.

For the sake of completeness what does control over $\|\cdot\|_{W^{1,p}}$ give when $p \notin [1 \dots d]$? Specifically $p \geq d$.

This disussion will be based on another way to relate u to Du.

let $u \in C^\infty(\mathbb{R}^d)$,

$$|u(x) - u(y)| = \int_0^1 \left|\frac{d}{ds} u(x + s(y-x))\right| ds.$$

$$\frac{1}{|B_r|}\int_{B_r(x)} |u(x) - u(y)|\, dy \leq \frac{c}{|B_r|}\int_{B_r(x)} \int_0^1 \underbrace{\left|\frac{d}{ds} u(x+s(y-x))\right|}\, ds\, dy$$

$$= (y-x) \cdot Du(x+s(y-x))$$

$$\leq c \frac{1}{r^d} \int_{B_r(x)} \int_0^1 |x-y| |Du(x+s(y-x))|\, ds\, dy$$

now, apply polar integration $\rho\omega = y - x$

$\rho = |y-x|$

$$= \frac{c}{r^d} \int_0^r \int_{S^{d-1}} \int_0^1 \rho |Du(x+s\rho\omega)| \, ds \, \rho^{d-1} \, d\omega \, d\rho.$$

now, change vars to $t = s\rho \Rightarrow \frac{dt}{s} = d\rho$

$$\int_0^r \rho^d |Du(x+s\rho\omega)| \, d\rho = \int_0^{sr} \frac{1}{s} \frac{t^d}{s^d} |Du(x+t\omega)| \, dt$$

So,
$$\int_0^1 \int_0^{sr} \frac{1}{s} \frac{t^d}{s^d} |Du(x+t\omega)| \, dt \, ds$$

$$= \int_0^r \int_{t/r}^1 \frac{1}{s^{d+1}} t^d |Du(x+t\omega)| \, ds \, dt$$

$$\leq c \int_0^r r^d |Du(x+t\omega)| \, dt$$

$\Rightarrow$ The whole thing is bounded by

$$\leq c \frac{1}{\cancel{r^d}} \int_{S^{d-1}} \int_0^r \cancel{r^d} |Du(x+t\omega)| \, dt \, d\omega$$

$$= C \int_{S^{d-1}} \int_0^r \frac{1}{t^{d-1}} |Du(x+t\omega)| t^{d-1} \, dt \, d\omega$$

$$= C \int_{B_r(x)} \frac{|Du|}{|x-y|^{d-1}} \, dy.$$

ok! ✓

if $\alpha = 1$, this is Lipschitz.

w/ this in hand we can proceed to the case $p > d$.

$\alpha = 1 - \frac{d}{p} = \frac{p-d}{pd}$

<u>Thm</u> $u \in C^\infty(\mathbb{R}^d)$ $d \geq 2$, then

$$|u(x) - u(y)| \leq C |x-y|^\alpha \|Du\|_{L^p(\mathbb{R}^d)}$$

Pf Use what we just showed

$$⨍_{B_r(x)} |u(x) - u(z)| \, dz \leq C \int_{B_r(x)} \frac{|Du|}{|x-z|^{d-1}} \, dz$$

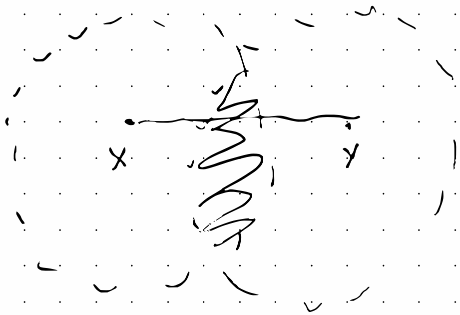
$$|u(x) - u(y)| \leq |u(x) - u(z)| + |u(z) - u(y)|$$

Then we take averages

$$⨍_U |u(x) - u(y)| \, dz \leq ⨍_U |u(x) - u(z)| \, dz + ⨍_U |u(y) - u(z)| \, dz$$

we have to choose U so that

$U \subset B_r(x)$ and $U \subset B_r(y)$



want $|B_r(x)| \simeq |U|$

$$\frac{1}{|U|}\int_U |u(x)-u(z)|\,dz \leq \frac{|B_r(x)|}{|U|}\fint_{B_r(x)} |u(x)-u(z)|\,dz$$

$$\Rightarrow \fint_U |u(x)-u(z)|\,dz \leq \frac{|B_r(x)|}{|U|}\fint_{B_r(x)} |u(x)-u(z)|\,dz$$

$$\leq C\int_{B_r(x)} \frac{|Du|}{|x-z|^{\alpha-1}}\,dz \leq C\|Du\|_{L^p}\left\|\frac{1}{|x-z|^{\alpha-1}}\right\|_{L^{p'}(B_r(x))} \quad p' = p/p-1$$

$$\int_{B_r(0)} \frac{1}{|z|^{(\alpha-1)p'}}\,dz \cong r^{\alpha}$$

& we are done. $\blacksquare$

Hölder spaces here.

what about $p=d$? The Sobolev inequality doesn't hold.

$$\|u\|_{L^\infty(U)} \lesssim \|u\|_{W^{1,d}(U)} \quad \text{FAILS.}$$

e.g. $U = B_1(0) \subseteq \mathbb{R}^2$

$$u(x) = \ln\left(\ln\left(10 + \frac{1}{|x|}\right)\right)$$

There are remedies however!

a popular one is called the BMO:

Bounded Mean Oscillation.

$$\|u\|_{BMO} = \sup_{\substack{x_0 \in \mathbb{R}^d \\ r \in \mathbb{R}}} \left(\fint_{B_r(x_0)} \left| u(x) - \fint_{B_r(x_0)} u(y)\,dy \right| dx \right) < +\infty.$$

Then $\|u\|_{BMO} \leq C \|Du\|_{L^d}$

Lecture 43

recall - Sobolev inequalities for the case $p = d$.

recall also the lemma of the previous lecture.

Lemma $u \in C^\infty(\mathbb{R}^{d \geq 2})$

$$\fint_{B_r(x)} |u(x) - u(z)|\,dz \leq c \int_{B_r(x)} \frac{|Du(z)|}{|z-x|^{d-1}}\,dz$$

Thm $x, y \in B_R$ $\quad \alpha = 1 - \frac{d}{p}$

$$|u(x) - u(y)| \leq C|x-y|^\alpha \|Du\|_{L^p(B_R)}$$

we want to rephrase as an inequality for u, & define a space w/ this regularity prop.

These are Hölder - Cont spaces.

Def (Hölder seminorm) $u \in C(U)$

$$[u]_{C^\alpha(U)} = \sup_{\substack{x,y \in U \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha}$$

$\therefore$ $[u]_{C^\alpha} = 0 \Rightarrow u = \text{constant}.$
$\not\Rightarrow u = 0$

To make $[\ ]$ into a norm, we need to remove under constant invariance

Hölder Norm $\|u\|_{C^\alpha} = [u]_{C^\alpha} + \|u\|_{L^\infty}$

Hölder Space $C^\alpha(U) = \{u \in C(U) : \|u\|_{C^\alpha} < \infty\}.$

Thm (Morrey's Embedding / Inequality).

$d \geq 2$, U bdd $\subseteq \mathbb{R}^d$, ∂U is C^1, $p > d$

- $u \in W^{1,p} \Rightarrow u \in C^\alpha(U)$ with $\alpha = 1 - \frac{p}{d}$,

and

- $\|u\|_{C^\alpha(U)} \leq C\|u\|_{W^{1,p}(U)}$

~~pf~~ suff by extension + density to check the case $u \in C_0^\infty(\mathbb{R}^d)$, spt $u \subset V$ where V is a fixed bounded open which contains $\overline{U}$. (indep of u).

By prev thm, $[u]_{C^\alpha(V)} \leq C\|u\|_{W^{1,p}(V)}$

it remains to show $\|u\|_{L^\infty(V)} \lesssim \|u\|_{W^{1,p}(V)}$

fix $x \in$ spt u. approximate u by an average.

$$\fint u(x)\,dz - \fint_{B_r(x)} u(z)\,dz \leq \fint_{B_r} |u(x) - u(z)|\,dz \leq \int_{B_r} \frac{|Du|}{|x-z|^{\alpha}}\,dz$$

$$\leq r^\alpha \|Du\|_{L^p(B_r(x))}$$

now, take $r = 1 \Rightarrow |u(x)| \leq C\Big|\underbrace{\int_{B_r(x)} u\,dz}_{\leq \|u\|_{L^1(B^r)} \leq C\|u\|_{L^p},\ C = \|1\|_{L^{p'}}}\Big| + C\|Du\|_{L^p}$

$\leq C(\|u\|_{L^p} + \|Du\|_{L^p})$. ∎

a few words RE $d=p$.

$W^{1,d} \not\hookrightarrow L^\infty$. The standard counterexample is

$U = B_1(0) \subset \mathbb{R}^2$, $u(x) = \ln\left(\ln\left(10 + \frac{1}{|x|}\right)\right)$

a useful substitute for this failure is Bounded Mean Oscillation,

Dfn $u \in L^1_{loc}(U)$. $[u]_{BMO} = \sup \fint_{B_r(x_0)} \left| u(z) - \fint_{B_r(x_0)} u \right| dz$

BMO is a quotient by constants?

Thm $U = \mathbb{R}^d$, $u \in W^{1,d}$, then $[u]_{BMO} < \infty$, and

$$[u]_{BMO} \leq C \|Du\|_{L^d}$$

Excercise: check that $L^\infty \subsetneq BMO$. e.g. $u = \log|x|$ on the unit ball.

proof above also uses completeness of Hölder space.

pf $u \in C^\infty(\mathbb{R}^d)$. Fix $B_r(x)$. WTS c is independent of choice of Ball, & u.

$$\underbrace{\fint_{B_r(x)} \left| u(z) - \fint_{B_r(x)} u(y)\,dy \right| dz}_{} \leq C \|Du\|_{L^d}$$

$$\frac{1}{B_r(x)} \int_{B_r(x)} \left| \fint_{B_r(x)} u(z)\,dy - \int_{B_r(x)} u(y)\,dy \right| dz$$

$$\leq \frac{1}{|B_r(x)|^2} \int_{B_r(x)} \int_{B_r(x)} |u(z) - u(y)| \, dy \, dz$$

Observe $B_r(x) \subset B_{2r}(y)$ by triangle inequality

so

$$\leq \frac{1}{|B_r|^2} \int_{B_r(x)} \int_{B_{2r}(y)} |u(z) - u(y)| \, dy \, dz$$

$$\overset{\text{lemma}}{\leq} \frac{1}{|B_r(x)|^2} \int_{B_r(x)} \underbrace{\int_{B_{2r}(y)} \frac{|Du(z)|}{|z-y|^{d-1}} \, dy}_{F(y).} \, dz$$

$F(y)$ is a convolution, so one might be tempted to use young's inequality.

$$\|f * y\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q} \qquad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

$\to$ fails when we try, since $\frac{1}{|z-x|^{d-1}} \notin L^q$.

instead, we need the Hardy-Littlewood Maximal Theorem.

<u>Thm</u> (Hardy-Littlewood), $\to$ H-L maximal fxn.

$u \in L^1_{loc}(\mathbb{R}^d)$, $\quad Mu(x) = \sup_{r>0} \fint_{B_r(x)} |u| \, dx$.

Thm $\forall 1 < p \le \infty$, $\|Mu\|_{L^p} \le c\|u\|_{L^p}$

We want to use a maximal function to control $F(y)$.

We use <u>dyadic decomposition</u>.

$|y|^\alpha$ has the property, if $2^{k-1} \le |y|, |y'| \le 2^k$, then $|y| \simeq |y'| \simeq 2^{k\alpha}$.

define $A^k = \{2^{k-1} \le |z-y| \le 2^k\}$

$$\int_{B_{2r}(x)} \frac{|Du|}{|z-y|^{d-1}}\, dz \le \sum_{2^k \le 2r} \int_{A^k} \frac{1}{(2^k)^{d-1}} |Du(z)|\, dz$$

$$\le \sum_{2^k \le 2r} \frac{1}{(2^k)^{d-1}} \int_{B_{2^k}(y)} |Du|\, dz \le C \underbrace{\sum_{2^k \le 2r + C}}_{*} 2^k M|Du|(y).$$

suffices to show → we can dispense w/ the sum since it's geometric & of size r.

$\| \sum_{*} 2^k M(Du|(y)) \|_{L^1} \le$

holder

$$\le Cr \|M|Du|(y)\|_{L^d} \|1\|_{L^{d/d-1}(B_r(x))} \le Cr \|M|Du|(x)\|_{L^d} r^{d-1}$$

H-L

$$\le Cr^d \|Du\|_{L^d}.$$

goes to the LHS & we are done b/c it satisfies the averaging condition. ∎

For next time, we have 2 more topics in Sobolev spaces.

1. Compactness property of Sobolev embeddings.
2. Poincaré-Type inequalities.
   ↳ how to get info about u from the Sobolev norm of Du given add'l info for normalizing u.

Compactness

Dfn $T: X \to Y$, linear, X, Y are normed spaces.

T is a compact operator if

— $\overline{T(B_X)}$ is compact in Y.

$\Updownarrow$

— $\forall$ bounded sequences $\{x_n\} \in X$, $\{T(x_n)\}$ has a convergent subsequence.

Dfn suppose $i : X \hookrightarrow Y$ (embedding) & i is linear.

"$X \subseteq Y$" can be naturally identified, & is compact if i is compact.

These are the natural settings for compactness theorems, when examining
"$W^{1,p}(U) \subseteq L^q(U)$".

what's the basic cptness theorem in the function space setting?

Arzelá-Ascoli: K cpt, $\mathcal{A} \subseteq C(K)$, st locally bdd & equicontinuous, then $\mathcal{A}$ is compact.

Lecture 2/8

Compactness of certain embeddings of Sobolev spaces.

recall the dfn X, Y are Banach & $i: X \hookrightarrow Y$ is compact, then X is compact as a subspace of Y.

&, Key theorem is Arzelà-Ascoli

Thm K cpt, $\tilde{\mathcal{F}} \subseteq C(K)$ (equipped w/ uniform topology) is compact if

- (Local Boundedness) $\forall x \in K$ $\exists M_x > 0$ st $|f(x)| \le M_x$ $\forall f \in \tilde{\mathcal{F}}$
- (Equicontinuity) $\forall \varepsilon > 0$, $\exists \delta > 0$ st $\forall f \in \tilde{\mathcal{F}}$, $|f(x) - f(y)| < \varepsilon$ if $|x-y| < \delta$.

Thm (cptness of $C^{0,\alpha}(U) \subseteq C^{0,\alpha'}(U)$)

U is bdd, open in $\mathbb{R}^d$,

$0 < \alpha' < \alpha < 1$. ($\Rightarrow$ clearly $C^{0,\alpha} \subset C^{0,\alpha'}$).

$C^{0,\alpha}(U) \subseteq C^{0,\alpha'}(U)$ is compact.

Pf (sketch),

(i) note that $C^{0,\alpha}(U) \subseteq C(U)$ is compact.

(by a direct app of Arzelà-Ascoli).

(ii) By (i) if $\{u_n\}$ is a sequence in $C^{0,\alpha}$ st bdd, then $\exists$ a subsequence u_{n_j} st it converges in $C(U)$.

(iii) Claim. in fact, $\|u_{n_j} - u_\infty\|_{C^{0,\alpha'}} \to 0$

The idea is to use interpolation.

$$\|v\|_{C^{0\alpha'}} = \underbrace{\|v\|_{L^\infty}}_{\substack{\to 0 \text{ by} \\ \text{convergence} \\ \text{of } v_n \text{ in } C^0}} + [v]_{C^{0\alpha'}}$$

only need to show

$$\star \quad [v]_{C^{0\alpha'}} \le C \|v\|_{L^\infty}^{1-\frac{\alpha'}{\alpha}} [v]_{C^{0,\alpha}}^{\alpha'/\alpha}$$

exponents come from homogeneity.

other is odd

Once we have $\star$, $\Rightarrow [u_{n_j} - u_\infty]_{C^{0\alpha'}} \le \underbrace{\|u_{n_j} - u_\infty\|_{L^\infty}^{1-\frac{\alpha'}{\alpha}}}_{\to 0} \cdots$

Pf of $\star$ is

$$\sup_{\substack{x,y\in U \\ x\neq y}} \left(\frac{|v(x) - v(y)|}{|x-y|^{\alpha'}} \right) \le (|v(x)| + |v(y)|)^{1-\frac{\alpha'}{\alpha}} \left(\cdots \right.$$

$\sim \boxed{\ }$

<u>Thm</u> (Rellich - Kondrachov)

Let $d \geq 2$, U bounded, open, ∂U is C^1.

[recall Sobolev Embedding: $1 \le p < d$, $W^{1,p} \hookrightarrow L^{p^*}(U)$
$p^* = \dfrac{d}{d/p - 1}$]

$1 \le p < d$, and $1 \le q < \dfrac{d}{d/p - 1}$. Then
$W^{1,p} \hookrightarrow L^{q}$ is compact.

The proof uses a lemma regarding mollifiers.

<u>Lemma</u> (Accelerated Convergence of Mollifiers)

[<u>recall</u>: $v \in L^p(\mathbb{R}^d)$ $\varphi \in C_0^\infty(\mathbb{R}^d)$ $\int \varphi = 1$, then $\varphi_\varepsilon * v \xrightarrow[L^p]{} v$.]

Suppose $v \in W^{k,p}$, $1 \le p < \infty$. $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\int \varphi = 1$,

and $\underbrace{\int x^\alpha \varphi \, dx = 0}_{\longleftrightarrow \text{ Moment Condition.}}$ $1 \le |\alpha| < k$.

Then $\|\varphi_\varepsilon * v - v\|_{L^p} \le C \varepsilon^k \|v\|_{W^{k,p}}$

pf
$$\int \varphi_\varepsilon(y) v(x-y) \, dy - v(x)$$

$$= \int \varphi_\varepsilon(y) v(x-y) \, dy - \int \varphi_\varepsilon(y) v(x) \, dy = \int \varphi_\varepsilon(y)(v(x-y) - v(x)) \, dy$$

consider $|y| \lessapprox \varepsilon$ on the spt of the integral.

Now, taylor expand in y, using the integral form of the remainder.

$$\int_0^1 \frac{d}{ds}(v(x-sy)) \, ds \overset{\text{IBP}}{=} -\int \frac{d}{ds}(1-s) \frac{d}{ds} v(x-sy) \, ds$$

$$\overset{\text{IBP}}{=} \underbrace{\frac{d}{ds} v(x-sy)\Big|_{s=0}}_{\substack{0 \text{ By} \\ \text{Moment} \\ \text{condition}}} + \underbrace{\int_0^1 (1-s) \frac{d^2}{ds^2} v(x-sy) \, ds}$$

$k=2$ case.

~~$y \cdot \nabla v(x)$~~

$$+ y^i y^j \int_0^1 (1-s) \partial_i \partial_j v(x-sy) \, ds$$

The whole integral bounded above by

$$|\cdot| \le \int |\varphi_\varepsilon(y)|\,|y|^2 \int_0^1 |\partial^2 v(x-sy)|\,ds\,dy$$

$$\| \ \|_{L^p} \le \|\partial^2 v\|_{L^p} \overbrace{\int |\varphi_\varepsilon|\,|y|^2\,dy}^{\lesssim \varepsilon^2} \lesssim \varepsilon^2 \|\partial^2 v\|_{L^p}$$

generalizing i1 $\Rightarrow$

$$\| \ \|_{L^p} \lesssim \varepsilon^k \|\partial^k v\|_{L^p}$$ $\blacksquare$

<u>Pf of Rellich-Kondrachov</u>

(2) Reduce to cptness of $W^{1,p} \hookrightarrow L^p$

(case 1) $W^{1,p} \hookrightarrow L^q$ $\quad 1 \le q \le p$.
Since U is bdd,

case (i) $\nearrow$ 0 by $\quad \|v\|_{L^q(U)} \le |U|^{1/q - 1/p} \|v\|_{L^p}$ by Hölder.

(case 2) $p < q < p^*$ $\quad$ Bdd by sobolev ineq

$$\|v\|_{L^q} \le \|v\|_{L^p}^{\theta} \|v\|_{L^{p^*}}^{1-\theta}$$ $\leftarrow$ dim analysis gives

$$\frac{d}{q} = \frac{d}{p}\theta + \frac{d}{p^*}(1-\theta)$$

(ii) Suffices to consider $q=p$ & apply the lemma.

in lieu of equicontinuity, we use the lemma.
Given $\{u_n\} \subset W^{1,p}$ & bdd by M.
By extension, we can find $\{\tilde{u}_n\}$ extending u_n, defined on the whole space &

$$\|\tilde{u}_n\|_{W^{1,p}(\mathbb{R}^d)} \leq C\|u_n\|_{W^{1,p}(U)} \leq CM$$

& $\operatorname{spt} \tilde{u}_n \subsetneq V$ bdd open containing U.

Introduce φ_ε mollifier.

$$\tilde{u}_n = \underbrace{\varphi_\varepsilon * \tilde{u}_n}_{v_{n,\varepsilon}} + \underbrace{(\tilde{u}_n - \varphi_\varepsilon * \tilde{u}_n)}_{e_{n,\varepsilon}}.$$ use the lemma.

$v_{n,\varepsilon}$ is smooth, & higher-regularity norms are bounded

By the lemma, $\|e_{n,\varepsilon}\|_{L^p(\mathbb{R}^d)} \leq C\varepsilon M$ ← uniform convergence for the error.

Also, $\|v_{n,\varepsilon}\|_{L^\infty} + \|\nabla v_{n,\varepsilon}\|_{L^\infty} \leq C_\varepsilon M$ (Hölder).

for each l, $\exists$ a subsequence $\tilde{u}_{n_l}$ s.t.

$$\|e_{n_l\varepsilon}\|_{L^p} \leq 2^{-l}$$

$$\|v_{n_{l'}} - v_{n_{l''}}\|_{L^p} \leq 2^{-l} \quad l', l'' \geq l \longleftarrow \text{Argelà-Ascoli.}$$

now, make a diagonal argument, & extract the subsequence, & we are done $\blacksquare$

Poincaré-Type inequalities

loosely speaking, any inequality controlling a function using information about the derivative, & some condition fixing the ambiguity introduced by constants

<u>Thm</u> (Poincaré Inequality).
Let $1 \le p < \infty$, $U \subseteq \mathbb{R}^d$ Bdd, w/ C^1 boundary.

$$\|u\|_{L^p} \le C_U \|Du\|_{L^p},$$

for $u \in W^{1,p}(U)$ with $\int_U u\,dx = 0$.

~~Pf~~ (using compactness).
Argue by contradiction. Assume such a C_U doesn't exist, i.e. $\forall n \ge 1\ \exists \|u_n\|_{L^p} \ge n\|Du\|_{L^p}$
$\int u_n = 0$

By normalization, we may assume $\|u_n\|_{L^p} = 1$.

Then it follows that $\|Du\|_{L^p} \le 1/n$.

$\Rightarrow \|u\|_{W^{1,p}} \le 2.$

By Rellich-Kondrachov, $\exists$ a convergent subsequence of u_n in L^p.

$\Rightarrow \|u_n\|_{L^p} = \|u_\infty\|_{L^p} = 1.$

Du_n converges weakly to Du

$\Rightarrow \|Du_n\| \to \|Du\| \to 0 \Rightarrow u$ is constant on U.

$\|Du_n\| \downarrow 0$

$\int u_n \to \int u = 0.$ So $u = 0$, a.e.

This contradicts $\|u\|_{L^p} = 1. \Rightarrow\Leftarrow$

---

may also see $\|u - \fint_U u\|_{L^p}$

Other examples:

- Freiderich Inequality, condition is $u|_{\partial U} = 0$.

  Pf — cptness.

  └ just use Sobolev for $W^{1,p}_0$.

- Hardy's inequality

  i) $u \in W^{1,p}(U)$ $\quad u|_{\partial U} = 0$, then

$$\left\|\frac{1}{\mathrm{dist}(\partial U)} u\right\|_{L^p} \le C\|Du\|_{L^p}$$

ii) $u \in W_0^{1,p}$, $p<d$

$$\left\|\frac{1}{|x|}u\right\|_{L^p} \leq c\|Du\|_{L^p}$$

next time, elliptic PDEs.

Lecture 2/10

Clarification: in disc. of poincaré inequality, we relied on weak L^p convergence.

Stated $1<p<\infty$, $\int u\,dx=0$, $\|v\|_p \leq c\|Du\|_p$

pf was by contradiction, finding a sequence bounded $\Rightarrow \|Du\|_{L^p}=0$

$Du \to 0$ in $L^p \Rightarrow Du \to 0$ in $\mathcal{D}'$,

we can use this for $p=1$ too.

Thm

Hardy's Inequality.

$u \in C_0^\infty(\mathbb{R}^{d \geq 2})$.

Then $\left\|\frac{1}{|x|}u\right\|_{L^2} \leq c\|Du\|_{L^2}$

Proof we use polar coords. (r, ω)

WTS $\int \frac{1}{r^2}u^2 r^{d-1}\,dr \leq c\int |\partial_r u|^2 r^{d-1}\,dr$ where we have fixed ω.

complete the square.

Study $\left(\partial_r u + \frac{\alpha}{r}u\right)^2 = |\partial_r u|^2 + 2\frac{\alpha}{r}u\partial_r u + \frac{\alpha^2}{r^2}u^2$

$\Rightarrow \int\left(\partial_r u + \frac{\alpha}{r}u\right)^2 r^{d-1}\,dr = \int (\partial_r u)^2 + \frac{2\alpha}{r}u\partial_r u + \frac{\alpha^2}{r^2}u^2 r^{d-1}\,dr$

$$= \int_0^\infty (\partial_r u)^2 r^{d-1} dr + \alpha^2 \int_0^\infty \frac{u^2}{r^2} r^{d-1} + \alpha \underbrace{\int_0^\infty \partial_r u^2 r^{d-2} dr}_{\text{IBP}}$$

$$\Rightarrow \int_0^\infty \partial_r u^2 r^{d-2} dr = u^2 r^{d-2} \Big|_0^{\overset{\infty = 0 \text{ by cpt spt}}{}} - (d-2)\int_0^\infty u^2 r^{d-2} dr$$

$\underbrace{0}_{\to 0 \text{ since } d \geq 2}$

$$\Rightarrow 0 \leq \int_0^\infty \left(\partial_r u + \frac{\alpha}{r} u\right)^2 r^{d-1} = \int_0^\infty (\partial_r u)^2 r^{d-1} - \left((d-2)\alpha - \alpha^2\right) \int_0^\infty \frac{1}{r^2} u^2 r^{d-1}$$

Choose α so that $(d-2)\alpha - \alpha^2$ is maximized,

$\alpha = \frac{d-2}{2}$

$$= \int_0^\infty (\partial_r u)^2 r^{d-1} dr - \left(\frac{(d-2)^2}{2} - \frac{(d-2)^2}{4}\right) \Rightarrow$$

$$\int_0^\infty \frac{1}{r^2} u^2 r^{d-1} dr \leq \left(\frac{2}{d-2}\right)^2 \int_0^\infty (\partial_r u)^2 r^{d-1} dr$$

<u>NB</u> we have an equality here as well

$$\int \frac{1}{r^2} u^2 r^{d-1} dr = \int (\partial_r u)^2 r^{d-1} dr - \underbrace{\int_0^\infty \left(\partial_r u + \frac{d-2}{2r} u\right)^2 r^{d-1} dr}$$

Not in H^1, which is why no equality

Elliptic PDEs (linear).

Heuristically speaking, these generalize Laplace's Eqn $-\Delta u = f$.

$$-\sum_j \partial_j \partial_j \xrightarrow[\text{symbol}]{\text{(principal)}} -\sum_j (i\xi_j)^2 = \sum_j \xi_j^2 = |\xi|^2$$

$$\partial_j \mapsto i\xi_j$$

Fourier space

Since $|\xi|^2 \neq 0$, it is invertible in Fourier space.

$$|\xi|^2 \hat{u} = \hat{f} \longrightarrow \hat{u} = \frac{1}{|\xi|^2}\hat{f}$$

Suppose P is a linear PDO, and $u : U \to \mathbb{R}^N$.

(Pu) takes values in $\mathbb{R}^N$.

$$(Pu)^I = \underbrace{\sum_{\substack{J,\alpha \\ |\alpha| = k}} A^I_{J\,\alpha_1,\dots,\alpha_d}\, \partial^\alpha u^J}_{\text{principal part}} + (\dots) \text{ lower-order terms}$$

k is the order of P.

We form the principal symbol

$$\sigma_{\text{princ}}(P) = \sum_{|\alpha| = k} i^k A^I_{J\,\alpha_1,\dots,\alpha_d}(x)\, \xi_1^{\alpha_1} \cdots\cdots \xi_d^{\alpha_d}$$

an operator P is elliptic if the principal symbol is invertible $\forall x$ in the domain, and $\xi \neq 0$.

an important subcase is $N=1$, the scalar case

$$P = \sum_{|\alpha|} a_\alpha(x)(\partial^\alpha u)$$

$$\sigma_{prin}(P) = i^k \sum_{|\alpha|=k} a_\alpha(x)\, \xi^\alpha$$

→ First nontrivial example is $k=2$.

$$Pu = a^{ij}\partial_i\partial_j u + b^j\partial_j + c.$$

in this case Ellipticity $\iff$ a^{ij} is definite.

we will adopt the convention a^{ij} is positive definite.

Furthermore, assume uniform ellipticity

i.e. $\exists \lambda > 0$ s.t. $a_{ij}\xi_i\xi_j \geq \lambda \quad \forall |\xi| = 1$.

$\iff$ all eigenvalues are bounded from below.

we can also take a_{ij} to be symmetric.

Examples of elliptic PDEs.

1. optimization problems in geometry, physics, etc.
   ↳ calculus of variations
2. often arise as part of evolutionary problems.

e.g. in the incompressible Euler equation

$$u : \mathbb{R}^{3+1} \to \mathbb{R}^3$$

velocity field of fluid element

$$\partial_t u + u\cdot\nabla u + \nabla p = 0.$$ ← pressure

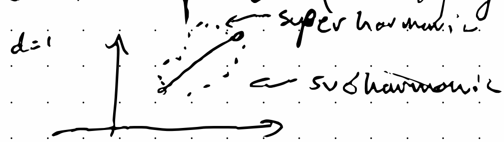
incompressibility $\Rightarrow$ $\nabla\cdot u = 0$ determines p.

$$\nabla(\cancel{\partial_t u} + u\cdot\nabla u + \nabla p) = 0 \Rightarrow -\Delta p = \nabla(u\cdot\nabla u).$$

Boom! Elliptic PDE

What do we want to discuss?

- Boundary-Value Problems, existence + uniqueness.
- Regularity Properties of Solutions. (elliptic regularity)
  - ↳ $Pu = f$, P of order K. when f has regularity k (H^k), u has maximum regularity of order $k+K$
- Maximum Principles. (mostly for the scalar case).



- (if time) Unique Continuation.
  Spectral Theory.

mostly ensure regularity we will deviate a bit (Schauder Theory).

BVPs (elliptic).

$d \geq 2$, $N=1$ (scalar), uniform ellipticity of P.
reg. of a, b, c should be "nice."

$U \subseteq \mathbb{R}^d$, bounded, open, connected.
∂U is "nice".

- Can't prescribe both normal derivative & boundary values.

Dirichlet

$$\begin{cases} Pu = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

Neumann

$$\begin{cases} Pu = f & \text{in } U \\ \frac{\partial}{\partial \nu} u = g & \text{in } \partial U \end{cases}$$

Today, solvability for $u \in H^1(U)$.
$\rightsquigarrow$ we will study Dirichlet BVP, since we need the trace theorem.

Neumann doesn't work, since we need $tr \frac{\partial u}{\partial \nu}$, which would require H^2.

Standard reduction is to understand the case $g=0$.
$\because$ Take any extension of g, $\tilde{g}$ & work w/ $v = u - \tilde{g}$. solving the BdDy problem.

<u>Def</u> (divergence-form operator)
P is divergence-form if $Pu = \partial_i(a^{ij}\partial_j u) + \partial_i(b^i u) + cu$

{ note that for a smooth,
$Pu = a^{ij}\partial_i\partial_j + (\partial_j a^{ij} + b^i)\partial_i u + (\partial_i b^i + c)u$

our discussion of uniqueness for dirichlet depends on apriori estimates.

<u>Thm</u> (a-priori estimate)
Suppose $u \in H^1(U)$ solves Dirichlet, same assumptions on a, b, c.
Then $\exists\, C, \gamma$ s.t.
$$\|u\|_{H^1(U)} \leq C\|Pu\|_{H^{-1}(U)} + \gamma\|u\|_{L^2(U)}$$

Pf. IBP the divergence form

$$\int_U (Pu)u\,dx = \int u(\partial_i(a^{ij}\partial_j u) + \partial_i(b^i u) + cu)\,dx$$

$$= \underbrace{\int_U -a^{ij}\partial_i u\,\partial_j u}_{\lambda\|Du\|^2 \leq \quad a^{ij}\partial_i u\,\partial_j u\,dx} - b^j u\,\partial_j u + cu^2\,dx$$

By uniform ellipticity.

$\int fu\,dx \leq c\|f\|_{H^{-1}}\|u\|_{H^1}$

$$\lambda\|Du\|_{L^2}^2 \leq C\|f\|_{H^{-1}}\|u\|_{H^1} + \int_U |b||\partial u||u|\,dx + \int |c||u|^2$$

$\int |c||u|^2$ if $b, c \in L^\infty$, we can factor them out

$\lesssim A\int |u|^2 \leq \gamma \|u\|_{L^2}^2$

TODO
read Evans here

$A\int |\partial u||u|\,dx \leq A\|\partial u\|\|u\|_{L^2}$

$b \in L^{d+}$ $c \in L^{d/2+}$ is optimal regularity.

I missed lecture for 1/15 – §6.2

Lecture 1/17

Last time: discussed solvability of Dirichlet problem for elliptic PDEs.

Today: elliptic regularity.

e.g. (prototypical)

suppose $\begin{cases} -\Delta u = f, & \text{in } U \end{cases}$, $f \in H^k$ or $\tilde{C}^{k,\alpha}$

note: $C^{k,\alpha} = \{ u \in C^k \mid D^\alpha u \in C^{0,\alpha}(U) \}$ $|\alpha| = k$

$\forall\ V \underbrace{\subseteq\subseteq}_{} U$. $\Rightarrow$ Then u is smoother than

V Bdd, $\overline{V} \subsetneq U$ f by order 2.

called "interior regularity"

$$\|u\|_{H^{k+2}(V)} \leq c\|f\|_{H^{k}(V)} + c\|u\|_{L^2(U)}$$

$\uparrow$ L^2-Based regularity

sp(D3).

{Similarly, $\|u\|_{C^{k+2,\alpha}(V)} \leq c\|f\|_{C^{k,\alpha}(V)} + \underbrace{c\|u\|_{L^\infty(U)}}_{\text{Schauder Theory.}}$ }

c can depend on V.

$0<\alpha<1$.

Schauder Theory will be later (in a few days)

for now, prove L^2-regularity.
The key idea is always IBP & the energy method.

Simplifying assumption: $u \in H^{k+2}(V)$.
we will go back & get rid of this assumption.

for now, $u \in H^{k+2}$ allows us to commute the equation w/ derivatives.

H^1 Bound for $-\Delta w = f$.
we can't just IBP on U, b/c we will have boundary terms.

introduce ζ a smooth cutoff from ζ V to U^c.

$\leftarrow$ no boundary b/c of ζ

$$\int_U -\Delta u \cdot u \cdot \zeta^2 dx = \int_U f u \zeta^2 dx = \sum_j \int_U \partial_j u \cdot \partial_j (u \zeta^2) dx$$

$$= \sum_j \int_U \partial_j u \cdot \partial_j u \, \zeta^2 + 2 \partial_j u \, u \zeta \partial_j \zeta \, dx$$

$$= \sum \int_U (\partial_j u \, \zeta)^2 + 2(\partial_j u)(u \zeta)(\partial_j \zeta) dx$$

rearrange

$$\int_U |Du|^2 \zeta^2 dx \leq \left|\int_U f u \zeta^2 dx\right| + 2\underbrace{\left|\int_U u \zeta Du \cdot D\zeta dx\right|}_{\leq 2\left(\int_U |Du|^2 \zeta^2\right)^{1/2}\left(\int_U u^2 |D\zeta|^2 dx\right)^{1/2}}$$

by $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$
$\quad \frac{1}{\sqrt{\varepsilon}} a \sqrt{\varepsilon} b$

$$\leq \varepsilon\left(\int_U |Du|^2 \zeta^2\right) + \frac{1}{\varepsilon}\int_U u^2 |D\zeta|^2 dx$$

Choose $\varepsilon = \frac{1}{2}$

$$\frac{1}{2}\int |Du|^2 \zeta^2 dx \leq \left|\int f u \zeta^2 dx\right| + 2\int u^2 |D\zeta|^2 dx$$

$$\left|\int f u \zeta^2\right| \le \int_U f^2 dx + \int_U u^2 dx \le \|f^2\|^2_{L^2(U)} + \|u\|^2_{L^2}$$

To get this into H^{-1} we need to introduce a cutoff in H^{-1}.

The whole thing

$$\frac{1}{2}\int |Du|^2 \zeta^2 dx \le \|$$

for Higher regularity, we compute $k+2=2$

$-\Delta u = f \Rightarrow -\Delta \underbrace{\partial_i u}_{\in H^1} = \partial_i f \ldots$ for higher derivatives

$\in H^1$, so apply step 1 & get

$$\int_V |D\partial_j u|^2 dx \le 2\underbrace{\left|\int \partial_j f \, \partial_j u \, \zeta^2\right|} + 2\|\partial_j u\|_{L^2(U)}$$

H^1-regularity justifies IBP,

$$= \left|\int f \partial_j^2 u \, \zeta^2 dx\right| \le \frac{1}{4\varepsilon}\int f^2 \zeta^2 dx + \varepsilon \int |\partial_j^2 u|^2 \zeta^2 dx$$

introduce a new ε

formally on H^1 bound.

$$\int |D\partial_j u|^2 dx \le C\|f\|^2_{L^2(U)} + C\|Du\|^2_{L^2(U)}$$

fix.

replace $V \subset\subset W \subset\subset U$, & repeat the previous bound on a new cutoff for W.

$$\|Du\|_{L^2(W)} \leq \|f\|_{L^2(U)} + \|u\|_{L^2(U)}.$$

combining all inequalities

$$\|D\partial_j u\|_{L^2(V)} \leq c\|f\|_{L^2(U)} + c\|u\|_{L^2(U)}.$$

<u>Setting</u> for the L^2-regularity thm.

$$Pu = -\partial_j(a^{jk}\partial_k u) + b^j\partial_j u + cu.$$

$u: U \to \mathbb{R}$ $\quad$ U open $\subset \mathbb{R}^d$

$\exists\lambda>0$ s.t. $a(x) \geq \lambda I$. $\quad$ $a \in L^\infty(U)$, $b \in L^d$, $c \in L^{d/2}$

is natural for $d \geq 3$.

we assume however, $a, b, c \in L^\infty(U)$.

★ also, in H^2 bound $\underline{\partial a \in L^\infty(U)}$.

<u>Thm</u> (H^2 elliptic regularity)
Let $u \in H^1$ be a weak solution to $Pu = f$ on U, $f \in L^2(U)$.
Then $\forall V \subset\subset U$, u is in $H^2(V)$, and

$$\|u\|_{H^2(V)} \leq c\|f\|_{L^2(U)} + c\|u\|_{L^2(U)}$$

IBP + Ellipticity.

$$\int_U -\partial_j(a^{jk}\partial_k v) v \zeta^2 dx$$

$$= \int_U a^{jk}\partial_k v\, \partial_j v\, \zeta^2 dx + \int_U (a^{jk}\partial_k v)(v\zeta)\partial_j \zeta dx$$

Ellipticity

$$\geq \lambda \int_U |Dv|^2 \zeta^2 dx - \|a\|_{L^\infty} \int_U \underbrace{|Dv|\zeta|v||D\zeta|} dx$$

$$\leq \frac{\lambda}{2}\frac{1}{\|a\|_{L^\infty}} |Dv|^2 + \frac{1}{\lambda}\|a\|_{L^\infty} |v|^2 |D\zeta|^2$$

$$\geq \frac{\lambda}{2}\int |Dv|^2 \zeta^2 dx - \frac{\|a\|^2_{L^\infty}}{\lambda}\int |v|^2 |D\zeta|^2 dx$$

proving that u is $H^2(V)$, we need to modify our computation to commute with difference quotients.

Dfn (diff quotient) $k \in \{1, \ldots, d\}$ $h \in \mathbb{R}$

$$D_k^h v = \frac{v(x + he^k) - v(x)}{h}$$

$$\lim_{h\to 0} D_k^h = \partial_k$$

Pf

Step 0. $u \in H^1(U)$, $Pu = f$ in $U \Leftrightarrow \forall \varphi \in C_0^\infty(U)$

$\langle Pu, \varphi \rangle = \langle f, \varphi \rangle \Leftrightarrow$ By regularity $Pu \in H^{-1}$, $f \in L^2 \subseteq H^{-1}$ we can use the dual of $(H^{-1})^* = H_0^1$

$\Leftrightarrow \forall \varphi \in H_0^1(U)$

$$\langle Pu, \varphi \rangle = \langle f, \varphi \rangle$$

$$\Leftrightarrow \int_U a^{jk} \partial_j u \, \partial_k \varphi + b^j \partial_j u \, \varphi + c u \varphi \, dx = \int_U f \varphi \, dx \quad \forall \varphi \in H_0^1(U).$$

Step 1: commute w/ difference quotient. D_j^h

$$D_j^h(u \cdot v) = D_j^h(u(x)) \, v(x) + \underbrace{u(x+h)}_{\substack{\shortparallel \\ u^h(x)}} D_j^h(v(x))$$

Take. $D_j^h\left((-\partial_j a^{jk} \partial_k u) + b^j \partial_j u + cu\right) = D_j^h f$

$=$

$- \partial_\ell (a^h)^{\ell k} \partial_k D_k^h + (b^h)^\ell \partial_\ell D_j^h u + c^h D_j^h u$

$- \partial_\ell (D^h a)^{\ell k} \partial_k u + (D_j^h b)^\ell \partial_\ell u + D_j^h c u$

Collect all to RHS

$$- \partial_\ell \left((a^h)^{\ell k} \partial_k \underbrace{D_j^h u}_{\text{still } H^1} \right) = \tilde{f}^h,$$

$$\langle -\partial_\ell (a^h)^{\ell k} \partial_k D_\gamma^h u, \varphi \rangle = \langle \tilde{f}_1^h, \varphi \rangle$$

$$\parallel$$

$$\int (a^h)^{\ell k} \partial_k D_\gamma^h u \, \partial_\ell \varphi \, dx$$

<u>Step 2</u> Choose "$\varphi = \partial_\gamma u \, \zeta^2$"

in the general case, $\varphi = D_\gamma^h u \, \zeta^2 \in H_0^1(U)$.

now, By the IBP argument, we see that

$$\frac{\lambda}{2} \int_U |D D_\gamma^h u| \, \zeta^2 dx \leq \underbrace{\cdots \tilde{f}_1^h \cdots D_\gamma^h u}$$

we can treat this as before.

move 2^{nd} derivs of u to LHS using Cauchy-Schwarz.

Treat 1^{st} derivs using the H^1 Bound.

<u>requires</u> a lemma from ch 5 of evans.

$V \Subset U$

i) $\| D_\gamma^h u \|_{L^p(V)} \leq C \| \partial_\gamma u \|$ $\quad |h| \ll 1$

ii) $\forall |h| \ll 1$ $\quad \| D_\gamma^h u \|_{L^p(V)} \leq A \Rightarrow \partial_\gamma u \in L^p$ and $\| \partial_\gamma u \|_{L^p(V)} \leq A$

Lecture 2/22

recall — discussed L^2-based regularity theory.
2nd order, scalar elliptic PDE.
Proved H^2 interior regularity:

U open in $\mathbb{R}^d$, $|Da| + |a| + |b| + |c| \leq \Lambda$ $\forall x \in U$
$u \in H^1(U)$ a weak solution to $Pu = f$ in U, $f \in L^2$
then $\forall V \subset\subset U$, $u \in H^2(V)$, we have

$$\|u\|_{H^2(V)} \leq \underset{\substack{\swarrow \\ \text{dep on } V, U, \lambda, \Lambda}}{C}\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

Basic ideas

- IBP & ellipticity $\xrightarrow{(a(x) \geq \lambda I)}$ ctrl of Du in L^2
- Commute the equation w/ ∂_{x^j}
  (summary; look @ eqn for $\partial_j u$, & apply IBP & ellipticity to control the gradient).
  to deduce $u \in H^2$, we had to use difference quotients.

Thm (Higher-order Elliptic Interior regularity)
Same hypotheses, w/ the following exceptions:

- $f \in H^{k-2}(U)$,

- $|D^\alpha a|$ bdd $\forall |\alpha| \leq k-1$
- $|D^\alpha b| + |D^\alpha c|$ bdd $\forall |\alpha| \leq k-2$.

Then $\forall V \subset\subset U$, $u \in H^k$, and $\exists C$ st

$$\|u\|_{H^k(V)} \leq C\left(\|f\|_{H^{k-2}(U)} + \|u\|_{L^2(U)}\right)$$

Pf (Sketch)
we essentially do the commutation step $k-1$ times.

$$\zeta^2 D^\beta f = \zeta^2 D^\beta P u$$
$$= \zeta^2 D^\beta\left(\partial_j a^{jk} \partial_k u + b^j \partial_j u + cu\right)$$
$$= \underbrace{\zeta^2 \partial_j a^{jk} \partial_k D^\beta u}_{\text{controls } \|DD^\beta \zeta^2 u\|_{L^2}} + \underbrace{D^\beta(b^j \partial_j u)}_{\text{IBP 1 time}} + \underbrace{D^\beta(cu)}_{\text{IBP 1 time}}$$
$$+ \sum_{\substack{\gamma \leq \beta \\ \gamma \neq \beta}} \partial_j (D^{\beta-\gamma} a^{jk} \partial_k D^\gamma u) \times C_\gamma$$

This proof also requires difference quotients & induction.
Details are in Evans.

L^2-Based Boundary Regularity.

Thm (H^2 Boundary Regularity)

Same hypotheses as H^2 interior reg, except:

- $u \in H_0^1(U)$. (Dirichlet, $\iff u|_{\partial U} = 0$).
- ∂U is C^2.

Then $u \in H^2(U)$, and

$$\|u\|_{H^2(U)} \leq c\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

Pf

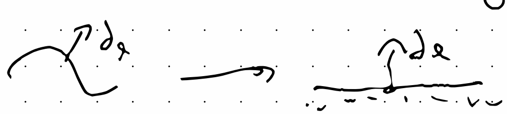
Let's try the pf & see what fails, assuming $u \in H^2(U)$, omit contribution of b and c.

$$f = \partial_j(a^{jk}\partial_k u) + \cdots$$

$$\partial_\ell f = -\partial_\ell \partial_j (a^{jk} \partial_k u)$$

we cant evaluate this when $\partial_\ell \pitchfork \partial U$

w/ an appropriate change of coordinates, only the normal derivative is incapable of being transformed into a coherent object



only $d-1$ tangential directions to U are admissible, so we will deal only w/ those for now.

for simplicity consider the special case
$\mathrm{spt}\, u \subset B_{\frac{1}{2}}(0) \cap \mathbb{R}^d_+$, $V = B_1 \cap \mathbb{R}^d_+$

in this case, the ∂ we can take are superclear, $\partial_\ell = 1, \dots, d-1$.

Then commuting ∂_ℓ yields the exact same proof as before.

The important note is that in the IBP we end up w/ an additional boundary term which goes to 0 by the trace assumption.

now, we control only $\|\partial_{x^d} \partial_{x^d} u\|_{L^2}$.

we can use the eqn to express $D_\gamma D_\gamma u$ in terms of everything else.

$-\partial_j(a^{jk}\partial_k u) = f \qquad a^{ij} \geq \lambda I \iff a^{jk}\xi_j \xi_k \geq \lambda |\xi|^2$

$\Rightarrow a^{dd} \geq \lambda$

$\Rightarrow -\partial_d(a^{dd}\partial_d u) - \partial_{j\neq d}(a^{jk}\partial_{k\neq d} u) = f$

$= -a^{dd}\partial_d^2 u - (\partial_d a^{dd})\partial_d u - \partial_{j\neq} \dots$

now, divide by a^{dd} from the lower bound.

$$\partial_d^2 u = \frac{1}{a^{dd}}\left(f + (\partial_d a^{dd})\partial_d u + \cdots \right)$$

Taking a norm gives the desired control.

Finally a partition of unity extends the argument to the nonflat case, as in the case of extension.

for each $\chi_k u$, when we straighten the bdry, we have to check that it is in fact a straightening of the boundary,

Spt & geometry are easy to check.

we need to verify that the ellipticity constant of the resulting eqn is bounded from below.

also must check Bounded ∂a in the new variables.

The boundedness is from conjugating by the Jacobian.

Q does this mean ellipticity is independent of coordinates?

from the H^2 bound for $\chi_R(y)U(y)$, coming back to $\chi_K(x)u(x)$ requires C^2 since we will have derivatives of the Jacobian.

$\blacksquare$

Overview of Schauder Theory,

L^2-based theory & weak solutions in H^1 are useful in a nonlinear context for deriving existence.

regularity theory has costs & benefits in a nonlinear setting:

- adv - we only need $a \in L^\infty$ to derive the H^1 bound.

This will spit out some bounded a for even low regularity.

$$-\partial_j(a^{jk}(U)\,\partial_k U) = f.$$

Controlling the derivative of $a^{ij}(U)$ is really hard to do here.

we need a regularity theory which can handle these growth derivatives.

We use Hölder continuous functions, & use Schauder estimates

$C^{k,\alpha}$

→ where famous de Georgi-Nash-Moser estimates come in.

Lecture 2/24

announcements:
- next tuesday - no lecture, notes on Schauder Theory forthcoming
- midterm thm following Tuesday march 8.

Schauder Theory - "Hölder-Based Elliptic Regularity".

Two main thm classes: interior & boundary reg.

<u>Thm</u> (Schauder, interior, divergence form).

$$Pu = \underbrace{-\partial_j(a^{jk}\partial_k u)}_{\text{main focus}} + \underbrace{(b^j\partial_j u + cu)}_{\substack{\text{we omit these for simplicity.}\\ \text{regularity is case-by-case}\\ \text{+ easily added.}}}$$

$U \subseteq \mathbb{R}^d$ open,

$u \in C^{k,\alpha}(\overline{U})$, $\leftarrow$ making an a-priori estimate here.

a^{jk} unif. elliptic everywhere and $a \in C^{k-1,\alpha}(\overline{U})$ $0<\alpha<1$

$$Pu = f \quad \text{in } U. \qquad f \in C^{k-2,\alpha}(\overline{U}) \quad k \geq 1$$

$\forall V \Subset U$, $\exists C_V$ s.t.

$$\|u\|_{C^{k,\alpha}(V)} \leq C\left(\|u\|_{C^0(U)} + \|f\|_{C^{k-2,\alpha}(U)}\right)$$

($k-2 \geq 0$ ✓

$C^{-1,\alpha} \Rightarrow f$ is of form

$f_0 + \sum_j \partial_j f^j$ where

$f^0, f^j \in C^{0,\alpha}$

Thm (Schauder, interior, non-divergence).

$Qu = -a^{jk}\partial_j\partial_k u$

U open in $\mathbb{R}^d$, $u \in C^{k,\alpha}(\bar{U})$, a unif elliptic,
$a \in C^{k-2,\alpha}(\bar{U})$, $k \geq 2$, $0 < \alpha < 1$.

$\forall V \subset\subset U$, $\exists c = c_V$ s.t.

$$\|u\|_{C^{k,\alpha}(V)} \leq c_V\left(\|u\|_{C^0(U)} + \|f\|_{C^{k-2,\alpha}(U)}\right)$$

Dfn ($C^{k,\alpha}$ domains)

∂U is $C^{k,\alpha}$ if $\forall x \in \partial U$, $\exists r$ st
∂U is the graph of a $C^{k,\alpha}$ function on B_r.

Thm (Schauder, bdry, divergence).
Same as previous, + $\partial U \in C^{k,\alpha}$, U is bdd.
also need dirichlet bdry: $u|_{\partial U} = 0$.

Then $\exists c$ st the bound holds on the entire domain

$$\|u\|_{C^{k,\alpha}(U)} \leq c\left(\|u\|_{C^0(U)} + \|f\|_{C^{k-2,\alpha}(U)}\right)$$

Thm (Sch, bdry, non div)
exactly what you'd expect.

overall strategy for doing these proofs.

Interior: ① corresponding result in the constant coefficient case,

② method of freezing coefficients — approx. the general case by local const coefficients.

↳ relies on the fact that elliptic regularity is local, & reg. of coefficients allows approximation.

Boundary: ① + ② + Boundary straightening + PoU, solve the half ball case.

we prove the constant coefficient methods today.

2 approaches — (Paley?)

- Littlewood-Paley Theory. (Fourier-Heavy)
- Compactness + Contradiction

II Littlewood-paley.

Thm (const-coeff interior case). focus on $k=2$.

$$Pu = -\partial_j a^{jk} \partial_k u = -a_0^{jk} \partial_j \partial_k u$$

constant on $\mathbb{R}^d$, elliptic.

For $u \in C_0^{k,\alpha}(\mathbb{R}^d)$, $f \in C^{k-2,\alpha}(\mathbb{R}^d)$, s.t. $Pu = f$,

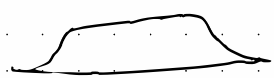
$$\|u\|_{C^{k,\alpha}(\mathbb{R}^d)} \le c\left(\|f\|_{C^{k-2,\alpha}(\mathbb{R}^d)} \right)$$

NB The compact spt of u gets rid of the C_0 term.

Dfn Littlewood-Paley Projections.

$$\chi_{\le 0}(\xi) = \begin{cases} 1 & \text{on } |\xi| \le 1 \\ 0 & \text{on } |\xi| \ge 2 \end{cases}$$ nonnegative.

$$\chi_{\le k}(\xi) = \chi_{\le 0}\left(\frac{\xi}{2^k}\right)$$



$$\chi_k = \chi_{\le k+1} - \chi_{\le k}$$

note $\operatorname{spt} \chi_k = \{\xi \mid 2^{k+2} \ge |\xi| \ge 2^k\}$

also define for $v \in \mathcal{S}'(\mathbb{R}^d)$

$$P_k(v) = \mathcal{F}^{-1}\left[\chi_k \cdot \mathcal{F}[v]\right]$$ ⟵ Projection!

$$v = P_{\le k_0} v + \sum_{k > k_0} P_k(v)$$

for suff regular v, we can prove

$$P_{\le k_0} v \to 0 \text{ as } k_0 \to \infty,$$

our partition sets $\mathrm{spt}\,\chi_k$ st $\forall \xi \in \mathrm{spt}\,\chi_k$, $|\xi| \simeq 2^k$

<u>Lemma</u> (L-P char of $C^{0,\alpha}$)

$v \in C^{0,\alpha}(\mathbb{R}^d)$

$$[v]_{C^{0,\alpha}} \simeq \sup_{k \in \mathbb{Z}} 2^{k\alpha} \|P_k v\|_{L^\infty}$$

(recall $[v]_{C^{0,\alpha}} = \sup_{x \ne y} \left[\frac{|v(x) - v(y)|}{|x-y|^\alpha} \right]$)

<u>Proof</u> Both are scale invariant

suff to consider $k=0$ for $\gtrsim$

WTS $|P_0 v| \lesssim [v]_{C^{0,\alpha}}$

$$P_0 v = \int \check{\chi}_0(x-y) v(y)\, dy$$

N.B. $\int \check{\chi}_0(y)\, dy = 0$ $(\Leftarrow \chi_0(0) = 0)$

$\therefore$ we may take

$$= \int \check{\chi}_0(x-y)(v(y) - v(x))\, dy \le \underbrace{\int \check{\chi}_0(x-y)|x-y|^\alpha dy}_{\text{finite + definite since } \mathcal{S}(\mathbb{R}^d)} [v]_{C^{0,\alpha}}$$

other direction ($\lesssim$)

$$V(x)-V(y) = P_{\le k_0} V(x) - P_{\le k_0} V(y) + \sum_{k>k_0} P_k V(x) - P_k V(y)$$

Choose k_0 s.t. $L^{-1} \simeq 2^{k_0}$

Take
$$\sum_{k\ge k_0} \|P_k V\|_{L^\infty} \le \sum_{k\ge k_0} 2^{-k\alpha}[V]_{C^{0,\alpha}}$$

FTC

$$\simeq L^{\alpha}[V]_{C^{\alpha}}$$

$$|P_{\le k_0} V(x) - P_{\le k_0} V(y)| \le \|\nabla P_{\le k_0} V\|_{L^\infty} L$$

$$\le \sum_{k\le k_0} \|\nabla P_k V\|_{L^\infty} L$$

$$\lesssim C \sum_{k\le k_0} 2^k 2^{-k\alpha}[V]_{C^{0,\alpha}} \simeq C\,\overbrace{L^{-1+\alpha}}^{\Rightarrow L^{\alpha}}[V]_{C^{0,\alpha}}$$

$\boxed{\Sigma}$ Lemma

now, using the lemma,

$$P(P_k U) = P_k f \xrightarrow[\mathcal{F}]{} a_{j\ell}\xi^j\xi^\ell \quad \widehat{(P_k U)} = \widehat{P_k f}$$

$$\widehat{P_k U} = \frac{1}{a^{jk}\xi_j\xi_\ell}\widehat{P_k f}\,\tilde{\chi}_k = \underbrace{\frac{2^{2k}}{a^{j\ell}\xi_j\xi_\ell}}_{\eta_k}\tilde{\chi}_k\widehat{P_k f}\,\frac{1}{2^{2k}},$$

$\tilde{\chi}_k$ $\|$ 1 on spt χ_k

supported on a slightly larger annulus

$P_k u$ ends up as $\tilde{\eta}_k * P_k f \cdot 2^{-2k}$

$\|P_k u\|_{L^\infty} \le C 2^{-2k} \|P_k f\|_{L^\infty} \lesssim C 2^{-2k-\alpha k} [f]$

$\boxed{\cancel{S}}$?

#2 Compactness + contradiction:

(1) assume it fails,

$\Rightarrow \exists\, a_n^{\sigma k}, u_n, f_n$ after renormalization,

$P_n u_n = f_n$

$[u_n]_{C^{2,\alpha}} = 1$, $[f_n]_{C^{0,\alpha}} \le \frac{1}{n}$

after translation

$|D^2 u_n(\eta_n) - D^2 u_n(0)| \ge c > 0$

some pt in $\mathbb{R}^d$ $\quad |\eta_n| \le 1$

Taylor.

(2) $v_n = u_n(x) - x D u_n(x) - \frac{1}{2} x^2 D^2 u_n(x)$

$P_n v_n = \tilde{f}_n$, satisfies

③ take the limit

$P_\infty v_\infty = 0$ w/ $[D^2 v] \leq 1$

Then somewhere, $D^2 v(\eta) \neq 0$ By compactness,

now apply Liouville to P_∞ & v is a constant.

That contradicts $D^2 v(\eta) \neq 0$.

Lecture 2/3

Announcements – Schauder Theory notes next wk.

Today: Maximum Principles

→ notably, max principles require scalar behavior.
Can't be generalized to elliptic systems as previous sections could.

For this lecture, we use non-divergence form of P for convenience.

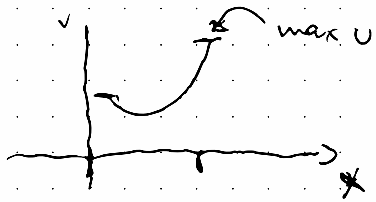
$$P = -a^{jk}\partial_j\partial_k u + b^j \partial_j u + cu$$
$\in L^\infty$

a is elliptic, a, b, c are bounded, (often, starting w/ simpler case $c=0$).

Max princ is the generalization of convex functions on $\mathbb{R}$.

In 1-D, the priciple for convex 1D fxns is

$u: I \to \mathbb{R}$ convex, then $\max_I u = \max_{\partial I} u$



generalizes in 2 ways → Hessian positive definite, (too restrictive)
→ as subsolutions to elliptic PDEs.

<u>Dfn</u> $u \in C^2(U)$ is a (classical) subsolution to
$Pu = 0$
if $Pu \leq 0$ on U.

<u>Thm</u> (weak max) $u \in C^2(U) \cap C(\bar{U})$, $U \subseteq \mathbb{R}^d$ is open, bdd + connected,
and $Pu \leq 0$, with $c = 0$.

Then $\max_U u = \max_{\partial U} u$

~~Pf~~ <u>step 1</u> consider strict subsolutions; $Pu < 0$.

WTS No interior maximum is possible.

suppose for the purpose of contradiction $x_0 \in U^0$ is a local maximum

Then $Du(x_0) = 0$

And $D^2 u(x_0)$ has no positive eigenvalues by 2nd der test.

by local critical ~~min~~ 0

$Pu(x_0) < 0$, and $Pu(x_0) = -a^{jk}\partial_j\partial_k u(x_0) + \cancel{b^j\partial_j u(x_0)}$

$$= -a^{jk}\partial_j\partial_k u(x_0).$$

$$= -\underbrace{\mathrm{tr}(\overset{>0}{a}\,\overset{\leq 0}{D^2 u})}_{\Longrightarrow} \geq 0.$$

$\Longrightarrow \Longleftarrow$ using a diagonalization of a makes this clear.

<u>Step 2</u> consider subsolution $Pu \leq 0$

Can we approx subsoln's by strict subsolutions?

introduce $u_\varepsilon = u + \varepsilon v$, where v is a strict subsolution, $Pv < 0$, $v \in C^2(U) \cap C(\bar{U})$.

Then $u_\varepsilon \to u$ on U, so apply step 1 to each u_ε.

How to construct v? Choose it to be

$V(\vec{x}) = \exp(x^1)$

$-a^{jk}\partial_j\partial_k e^{x^1} = -a^{11}e^{x^1} < 0$.

(1^{st} derivative might mess it up.

instead

$v(\vec{x}) = \exp(Nx^1)$. $\quad -a^{11}e^{Nx^1}N^2 < 0$

$b^1 N e^{x^1} < 0$

Then $-a''\mu^2 e^{x'} < -\lambda \mu^2 e^{x'}$

$|b^{\partial} \partial_{\partial} e^{\mu x'}| = |b' \mu e^{x'\mu}| \leq \sup(|b|\mu e^{\mu x'})$

so take $\mu >> 0$ & $Pv < c$. ✓

If we flip all the signs to sering supersolutions, we get the weak minimum principle by the same proof.

Clearly u is a solution iff it is both a sub + super solution.

So $Pu = 0 \Rightarrow \max_{U} |u| = \max_{\partial U} |u|$.

<u>Corr</u> (WMP for $c \geq 0$). Same conditions otherwise.

Then

$$\begin{cases} Pu \leq 0 \Rightarrow \max_{U} u \leq \max_{\partial U} u^+ \\ Pu \geq 0 \Rightarrow \min_{U} u \leq \min_{\partial U}(-u^-) \end{cases}$$

where $u^+ = \begin{cases} u(x) & \forall x \mid u \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$u^- = \begin{cases} 0 & \text{otherwise} \\ -u(x) & \forall x \mid u(x) \leq 0 \end{cases}$

<u>Pf</u> $V \subseteq U$ on which u is positive.

Cut V into components & apply max principle to $Qu = Pu - cu$ on V.

for Q on V, $\max_{\bar{V}} u \leq \max_{\partial V} u$

This implies our statement, $\square$

Thm Comparison Principle

U open, bdd, connected in $\mathbb{R}^d$

P satisfies usual hypotheses, $c \geq 0$.

$u, v \in C^2(U) \cap C(\bar{U})$

$\underbrace{Pu \leq 0, \quad Pv \geq 0}_{\text{on } U}$, $\quad u \leq v$ on ∂U.

Then $u \leq v$ on all of U.

Pf apply WMP to $u - v$, which is a subsolution. $\blacksquare$

Thm (Strong max Principle)

U open, bdd, connected $\subseteq \mathbb{R}^d$.

$c = 0$, P as usual.

$u \in C^2(U) \cap C(\bar{U})$. $\quad Pu \leq 0$.

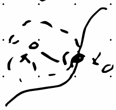
If at any $x_0 \in U^\circ$, $u(x_0) = \max_{\bar{U}} u$, then u is constant.

our proof is based on Hopf's Lemma.

<u>Thm</u> (Hopf's Lemma)

U open bdd conn $\subseteq \mathbb{R}^d$

for some $x_0 \in U$,



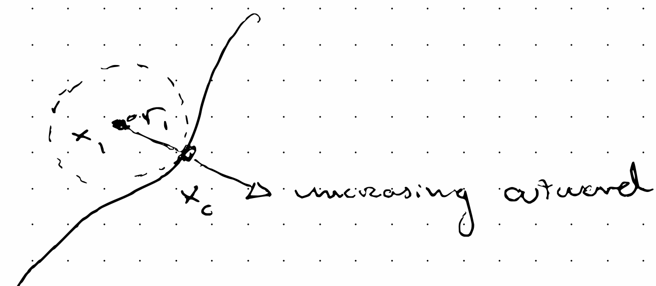
i) $\exists\, x_1 \in U,\ r_1 > 0$ st $B_{r_1}(x_1) \subseteq U$ and $B_r(x_1) \cap \partial U = \{x_0\}$

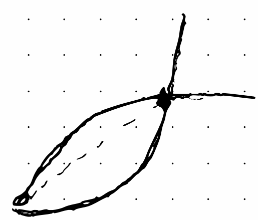
ii) $u(x_0) \geq u(x)\ \forall x \in \overline{B_{r_1}(x_1)}$ and $u(x_0) > u(x)$ $\forall x \in (B_{r_1}(x_1))^o$

wrt the ball.

Then $\left.\dfrac{d}{dr}\right|_{x=x_0} u > 0$.

<u>rmk</u> The surprising bit is strict positivity.





Pf (of Hopf).

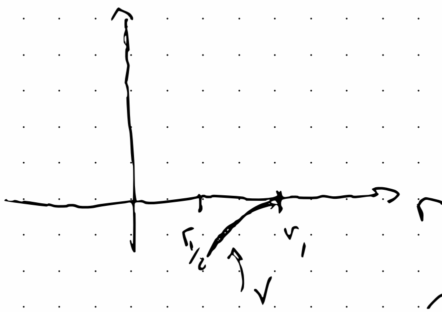
Goal is to compare a supersoln to a subsoln a la picture.

WLOG set $x' = (0, \dots, 0)$.

$V(x) = e^{-\mu r_1^2} - e^{-\mu|x|^2}$ so $V\big|_{\partial B_{r_1}(0)} = 0$.

$Pv \geq 0$ on $B_{r_1} \setminus B_{\frac{r_1}{2}}$

Consider $w = \varepsilon v + u(x_0)$



for $\mu >> 0$

$Pw = P\varepsilon v + Pu(x_0) = P\varepsilon v \geq 0$ in $V = B_{r_1} \setminus B_{r_1/2}$

on ∂B_{r_1} $w = u(x_0) \geq u$

on ∂B_{r_1} $w = \varepsilon v + u(x_0)$ and $u(x_0) > u(x)$,

so $u(x_0) > u(x) + \varepsilon(-v)$ for sufficiently small ε.

$\therefore$ $w \geq u$. By comparison princ

$w > u$ on V

$\Rightarrow \left.\dfrac{\partial u}{\partial r}\right|_{x = x_0} \geq \left.\dfrac{\partial w}{\partial r}\right|_{x = x_0} > 0$ check. ✓ ∎

~~pf of smp~~ consider $V = \{x \in U \mid u(x) < \sup_U u\}$

$x_0 \in U$, $u(x_0) = m \Rightarrow V \not\subseteq U$,

assume for cont. $V \neq \phi$

find x_1 w/ largest r_1 st hopf.
$B_{r_1}(x_1) \subseteq V$ $x_0 \in B_{r_1}(x_1) \cap \partial V)$

Thm sg hopf, $\left.\frac{\partial u}{\partial r}\right|_{x_0} > 0$ is a contradiction of maximum of x_0.

wrt ball

Missed last Thursday

Lecture 3/15

Unique Continuation for elliptic PDEs.
- didn't cover this in lecture.

Thm (Aronszajn)
Let P be an elliptic operator
$-\partial_j(a^{jk}\partial_k u) + b^j \partial_j u + cu$

If $Pu = 0$ in U and $u = 0$ on an open subset $W \subseteq U$, then $u \equiv 0$ on U.

pf (Carleman (?) estimate)

Take $\psi \in C_0^\infty(\mathbb{R}^d)$, $\nabla \psi \neq 0$.

ref for proof in Lerner's <u>Carleman's Ests</u>

Linear Hyperbolic PDEs.

what is meant by Hyperbolicity? one can argue it only matters / is precise in the const-coeff case

- Hyperbolic PDEs are <u>evolutionary</u>

- # of t-derivs = # of ∇-derivs
  - ↳ e.g. classical wave eqn.
  - ↳ e.g. any transport eqn $(\partial_t + X^j \partial_j)\phi = 0$
  - ↳ <u>not</u> e.g. $(-\partial_t + \Delta)\phi = 0$.
  - ↳ <u>not</u> e.g. $(i\partial_t + \Delta)\phi = 0$.

ruled out laplace $(\partial_t^2 + \Delta)\phi = 0$

(local)

- Well-posedness of the IVP - i.e.

$$\begin{cases} P\phi = 0 \\ (\phi, \partial_t\phi, \dots \partial_t^{N-1}\phi) = (g_0, \dots g_{N-1}) \end{cases}$$

is well posed,

where N is the order of ∂_t.

we will try to find algebraic conditions guaranteeing that WP condition.

It's also closely related to the existence of energy estimates.

eg linear, const-coeff systems

WLOG, assume PDE is of form

$$\underline{\Phi} = \begin{pmatrix} \phi^{(1)} \\ \vdots \\ \phi^{(n)} \end{pmatrix}$$

$$\partial_t \underline{\Phi} + A^{\partial} \partial_{x^{\partial}} \underline{\Phi} = F \quad \text{where } A^{\partial} \text{ is } M_{n \times n}(\mathbb{R})$$

Q, how to guarantee uniqueness? ← To IVP

Try an energy estimate to find conditions on A.

$$\int_{\mathbb{R}^d} \phi^{(k)} \partial_t \phi_{(k)} + \phi^{(k)} A^{\partial} \partial_{\partial} \phi_{(k)} \, dx = \int_{\mathbb{R}^d} \phi^{(k)} F_{(k)} \, dx$$

← IBP half of the integral.

$$= \frac{1}{2} \int_{\mathbb{R}^d} |\phi|^2 dx + \frac{1}{2} \int (A^{\partial})^{(k)}_{(\ell)} \phi^{(k)} \partial_{\partial} \phi^{(\ell)} dx = \int_{\mathbb{R}^d} \phi^{(k)} F_{(k)} \, dx$$

$$- \frac{1}{2} \int (A^{\partial})^{(k)}_{(\ell)} \phi^{(k)} \phi^{(\ell)} dx$$

$$= \frac{1}{2} \int |\phi|^2 dx + \frac{1}{2} \int \left((A^{\partial})^{(k)}_{(\ell)} - (A^{\partial})^{(\ell)}_{(k)} \right) \phi^{(k)} \partial_{\partial} \phi^{(\ell)} dx =$$

symmetry kills the problematic derivative.
of A^{∂}

Thm A 1^{st}-order const-coeff
$$\partial_t \phi + A^\gamma \partial_\gamma \phi = F \quad \text{is hyperbolic}$$
(i.e. IVP is WP in L^2) iff A^γ is symmetric.

we will detail later, but it follows from using energy estimate to show uniqueness

finding a counterexample is an excercise in Fourier analysis, basically non-symmetry produces plane-wave solutions..

e.g. (1^{st} order $\Box\phi = f$).

make this 1^{st} order in time by converting to a system. $\psi = \partial_t \phi$

$$\partial_t \psi = \Delta \phi - f$$

$$\partial_t \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} - \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$\widehat{\partial_t \begin{pmatrix} \phi \\ \psi \end{pmatrix}} = \begin{pmatrix} 0 & 1 \\ -|\xi|^2 & 0 \end{pmatrix} \begin{pmatrix} \hat\phi \\ \hat\psi \end{pmatrix} - \widehat{\begin{pmatrix} 0 \\ f \end{pmatrix}}$$

anti-Hermitian.

diagonalize

$$= \begin{pmatrix} +i|\xi| & 0 \\ 0 & -i|\xi| \end{pmatrix} \widehat{\begin{pmatrix} \phi \\ \psi \end{pmatrix}} - \widehat{\begin{pmatrix} 0 \\ f \end{pmatrix}}$$

how energy est holds in the diag'd variables

what are our goals for the section?

① (local) well-posedness of the IVP for variable coefficient wave equations.

$$P\phi = \partial_\mu(g^{\mu\nu}\partial_\nu \phi) + b^\mu \partial_\mu \phi + c\phi$$

where g is a nondegenerate symmetric matrix w/ signature $(-,+,+,+)$

Ie g is a Lorentzian metric.

② long-time behavior of solutions

LWP of the IVP for var coeff wave eqn's

- references: - Ringström: The cauchy problem in GR. [Chap. 6, 7]

Goal - derive energy estimates for

$$\begin{cases} P\phi = f & \mathbb{R}_t \times \mathbb{R}^d \\ (\phi, \partial_t\phi) = (g, h) & \{t=0\} \times \mathbb{R}^d \end{cases}$$

Prelim Tool:

Gronwall's inequality

suppose $E(t) \in C_t([0,T])$, $E \geq 0$

$r(t) \in L^1_t([0,T])$, $r \geq 0$

$$E \leq E(0) + \int_0^t r(t')E(t')\,dt'$$

Then $E(t) \leq E(0) \exp\left(\int_0^t r(t')\,dt'\right)$

pf (using bootstrap)

$$E(t) \leq E_0 + \int_0^t r(t')E(t')\,dt$$

Take the RHS & plug in $E(t') \leq E_0 \exp \cdots$

$$E(t) \le E_0 + \int_0^t r(t')E(t')\,dt' \le E_0 + E_0\int_0^t r(t') \exp\Big(\underbrace{\int_0^{t'} r(\tau)\,d\tau}_{R(\tau)}\Big)\,dt',$$

$$dR(t') = r(t')\,dt' \longrightarrow \quad = E_0 + E_0\left(\exp\left(\int_0^t r(t')\,dt'\right) - 1\right)$$

$$= E_0 \exp\left(\int_0^t r(t')\,dt'\right)$$

now,

$$E(t) \le E_0(1+\delta)\exp\left(\int_0^t r(t')\,dt'\right) \quad \text{on } [0,T)$$

$$\Rightarrow$$

$$E(t) \le E_0 + E_0(1+\delta)\int_0^t \dots\, dt$$

$$= E_0 + E_0(1+\delta)\left(\exp\left(\int_0^t r(t')\,dt'\right) - 1\right)$$

$$= E_0 + E_0 \exp$$

Today: want to achieve the following result about variable coeff. wave equations

<u>Dfn</u>

$$P\phi = \partial_\mu(g^{\mu\nu}\partial_\nu\phi) + b^\mu\partial_\mu\phi + c\phi$$

where g is lorentzian: i.e. symmetric & w/ signature $\begin{pmatrix} - & & & \\ & + & & \\ & & + & \\ & & & \ddots \end{pmatrix}$

heuristically P is the generalized D'alembertian

study the IVP

$$(\text{IVP}) \begin{cases} P\phi = f & \text{in } (0,\infty)_t \times \mathbb{R}^d \\ (\phi, \partial_t\phi) = (g, h) & \text{in } \{t=0\} \times \mathbb{R}^d \end{cases}$$

Further assumptions: • $g^{\mu\nu}, b^\mu, c$ smooth & have bdd derivatives of all order,

• g is of restricted form: $g^{tt} = -1$ $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ \vdots & & \bar{g} & \\ 0 & & & \end{bmatrix}$ ∂_α^k

• no shift $g^{t\nu} = 0$, • $\bar{g}$ is unif elliptic

<u>Thm</u> IVP is well-posed in $H^k \times H^{k-1}$ $\forall k \in \mathbb{Z}$.

i.e. the following hold

i) (existence) given $(g, h) \in H^k \times H^{k-1}$ $\exists$ soln ϕ in the class $C_t \mathcal{H}^k$

ii) Uniqueness

iii) (continuous dependence) $\sup_t \|(\phi, \partial_t\phi)\|_{\mathcal{H}^k} \le C_k \|(g, h)\|_{\mathcal{H}^k}$

notation $t = x^0$, $x^1 \dots x^d$ are space.

greek indices run over $[0..d]$

latin indices run over $[1..d]$

$R_{t_0}^{t_1} = (t_0, t_1) \times \mathbb{R}^d$ $\qquad \tilde{\Phi} = (\phi, \partial_t \phi) \in \mathcal{H}^k$

$\Sigma_{\tilde{\tau}} = \{t = \tilde{\tau}\} \times \mathbb{R}^d$

---

rmk IVP is time reversible.

refer to Ringström chap. 6 & 7.

Our basic ingredient will be an energy inequality.
idea is to multiply IVP by $\partial_0 \phi$ & IBP.

for simplicity we examine 2^{nd} order terms.

$$\partial_\mu (g^{\mu\nu} \partial_\nu \phi) \partial_0 \phi = (-\partial_0^2 \phi) \partial_0 \phi + \partial_j (\bar{g}^{jk} \partial_k \phi) \partial_0 \phi$$

$$= \partial_t \left(-\tfrac{1}{2} (\partial_t \phi)^2 \right) + \partial_j \left(\bar{g}^{jk} \partial_k \phi \partial_t \phi \right) - \bar{g}^{jk} \phi \partial_j \partial_t \phi$$

$$= \partial_t \left(-\tfrac{1}{2} (\partial_t \phi)^2 - \tfrac{1}{2} \bar{g}^{jk} \partial_j \phi \partial_k \phi \right) + \partial_j \left(\bar{g}^{jk} \partial_k \phi \partial_t \phi \right) + \tfrac{1}{2} \partial_t \bar{g}^{jk} \partial_j \phi \partial_k \phi$$

here we've put max # of ∂_ν into divergence form

$$\partial_t\left(-\frac{1}{2}(\partial_t\phi)^2-\frac{1}{2}\bar{g}^{jk}\partial_j\phi\,\partial_k\phi\right)+\partial_j\left(\bar{g}^{jk}\partial_k\phi\,\partial_t\phi\right)+\frac{1}{2}\partial_t\bar{g}^{jk}\partial_j\phi\,\partial_k\phi$$

now, integrate over $(t_0,t_1)\times\mathbb{R}^d$ assuming a vanishing boundary term:

$(t_0,t_1)\times\mathbb{R}^d = R_{t_0}^{t_1}$

$$\int_{R_{t_0}^{t_1}}\partial_\mu(g^{\mu\nu}\partial_\nu\phi)\partial_t\phi\,dx$$

$$=-\int_{\Sigma_{t_1}}\frac{1}{2}\left((\partial_t\phi)^2+\bar{g}^{jk}\partial_j\phi\,\partial_k\phi\right)+\int_{\Sigma_{t_0}}\frac{1}{2}\left((\partial_t\phi)^2+\bar{g}^{jk}\partial_j\phi\,\partial_k\phi\right)$$

$$\underbrace{+\lim_{r\to\infty}\int_0^t\int_{\partial B_r}\nu_j\left(\bar{g}^{jk}\partial_k\phi\,\partial_t\phi\right)dA\,dt}_{\text{assumed}\to 0,}$$

Lemma For $\phi \in C_t \mathcal{H}^1$

$$\sup_{t\in[0,T]} \|\vec{\phi}\|_{\mathcal{H}^1} \leq C_T\left(\|\phi(t=0)\|_{\mathcal{H}^1} + \int_0^t \|P\phi\|_{L^2}\, dt\right)$$

proof WLOG assume $\phi \in C^\infty(\overline{R_0^T})$ & $\operatorname{spt}(\phi(t))$ is cpt $\forall t \in [0,T]$.

By our big integral computation, define

$$E[\phi](t) = \frac{1}{2}\int_{\Sigma_t} (\partial_t\phi)^2 + \bar{g}^{jk}\partial_j\phi\,\partial_k\phi\, dx$$

Then

$$E[\phi](t_1) = E[\phi](0) - \iint_{R_0^{t_1}} \partial_\mu(g^{\mu\nu}\partial_\nu\phi)\,\partial_t\phi$$

$$+\frac{1}{2}\iint_{R_0^{t_1}} \partial_t\bar{g}^{jk}\,\partial_j\phi\,\partial_k\phi$$

$$\Rightarrow E[\phi](t_1) = E[\phi](0) + \iint_{R_0^{t_1}} (P\phi)\,\partial_t\phi\, dx\,dt + \underbrace{\iint_{R_0^{t_1}} \partial_t\phi\,(b^\mu\partial_\mu\phi + c\phi + \partial_t\bar{g}^{jk}\partial_j\phi\,\partial_k\phi)}_{\text{call this the error term}}$$

So

$$\mathcal{E} = \iint_{R_0^{t_1}} |\partial_t\phi\,(b^\mu\partial_\mu\phi + c\phi + \partial_t\bar{g}^{jk}\partial_j\phi\,\partial_k\phi)|\, dx\,dt$$

then:

$\mathcal{H}$ $\mathcal{H}$ $\mathcal{H}$

$$\sup_{t\in[0,T]} E[\phi](t) \lesssim E[\phi](0) + \sup_{t\in[0,T]} \left| \iint_{R_0^t} (P\phi)\partial_t\phi \, dx\,dt \right| + \mathcal{E}_0^T$$

note that

$$\int |\phi|^2(t)\,dx = \iint_0^t (\partial_t\phi)\,\phi\,dx\,dt \leq 2\int |E(t')|^{1/2}\int |\phi|^2(t')\,dy\,dt' + \int |\phi|^2(0)\,dx$$

$$\Rightarrow \sup_{t\in[0,T]} \int |\phi|^2(t)\,dt \leq \int |\phi|^2(0)\,dx + CT \sup_{t\in[0,T]} E(t)$$

Combining these controls, we have

$$\sup_{t\in[0,T]} \|\vec{\phi}\|_{\mathcal{H}^1} \leq C_T\left(\|\vec{\phi}\|_{\mathcal{H}^1} + \sup_{t\in[0,T]} \left| \iint_{R_0^t} (P\phi)\partial_t\phi\,dx\,dt \right| + \mathcal{E}_0^T \right)$$

$$\sup_{t\in[0,T]} \left| \iint_{R_0^t} P\phi\,\partial_t\phi\,dx\,dt \right| \overset{\text{cauchy}}{\leq} \int_0^T \|P\phi\|_{L^2} \|\partial_t\phi\|_{L^2}\,dt$$

$$\leq C\int_0^T \|P\phi\|_{L^2}\sqrt{E[\phi](t)}\,dt$$

$$\leq C\sup_t E[\phi]^{1/2} \int_0^T \|P\phi\|_{L^2}$$

C-S & absorb- $0 \le t_1 \le T$

$$\underbrace{\sup_{t\in[0,t_1]} \|\vec{\Phi}\|^2_{\mathcal{H}^1}}_{\mathcal{D}(t_1)} \le C_T\Big(\underbrace{\|\vec{\Phi}(0)\|^2_{H^1} + \int_0^T \|P\phi\| dt}_{\mathcal{D}_0} + \int_0^T \|\vec{\Phi}\|^2_{\mathcal{H}^1} dt\Big)$$

$$\mathcal{D}(t_1) \le \mathcal{D}_0 + \int_0^{t_1} \mathcal{D}(t)\, dt$$

by Grönwall, then

$$\mathcal{D}(t) \le \mathcal{D}_0 \exp\Big(\int_0^t dt'\Big) \le \mathcal{D}_0 \exp(T) \qquad \blacksquare$$

This was an A-priori estimate for $\mathcal{H}^1$.
doing this for $\mathcal{H}^k$ gives uniqueness.
we should examine the adjoint problem
for existence.

$$\text{"} P: C_t \mathcal{H}^k \to L^1_t \mathcal{H}^{k-1} \text{"}$$

$$\text{"} P^*: C_t \mathcal{H}^{-(k-1)} \to L^1_t \mathcal{H}^{-k} \text{"}$$

Dual $\Rightarrow$ we need negative order Sobolev spaces.

Lemma $\forall k \in \mathbb{Z}$, the following holds:

$\phi \in C_t H^{k+1} \cap C^\infty_{t,x}$

$$\sup_{t\in[0,T]} \|\vec{\phi}(t)\|_{\mathcal{H}^{k+1}} \leq C_{T,k}\left(\|\vec{\phi}(0)\|_{\mathcal{H}^{k+1}} + \int_0^T \|P\phi\|_{H^k}\, dt\right)$$

Pf for $k>0$, commute w/ derivs up to order k, apply the first lemma, & Grönwall.

(cf. higher elliptic reg bounds), w/out diff quotients.

for $k<0$, give $\Phi = (1-\Delta)^{-|k|}\phi$.

The idea is to use duality in a minimal way. Using the fact

$$(1-\Delta)^{-\ell} : H^s \to H^{s+2\ell}$$

w/ the lemma $s \in \mathbb{R}$,

$$\|v\|^2_{H^s} = \|(1+|\xi|^2)^{s/2}\hat{v}\|^2_{L^2_\xi} = \|(1-\Delta)^{s/2} v\|_{L^2}$$

Using these, we wind up w/

$$\|P\Phi\|^2_{H^{|k|}} = \|(1+\xi^2)^{\frac{|k|}{2}}\widehat{P\Phi}\|_{L^2} = \langle (1+|\xi|^2)^{|k|}\widehat{P\Phi}, \widehat{P\Phi}\rangle$$

$$= \langle (1-\Delta)^{|k|} P\Phi, P\Phi\rangle$$

so,

$$(1-\Delta)^{|k|} P \Phi = P((1-\Delta)^{|k|}\Phi) + \overbrace{[(1-\Delta)^{|k|}, P]}^{\text{commutator}} \Phi$$

$$= P\phi + \underbrace{[(1-\Delta)^{|k|}, P]}_{\text{order } 2|k|+2-1} \Phi$$

now, the lemma gives

$$\|\vec{\Phi}\|_{\mathcal{H}^{1+|k|}} = \|\vec{\phi}(t)\|_{\mathcal{H}^{1+|k|-2|k|}} = \|\vec{\phi}\|_{\mathcal{H}^{1+k}}$$

Lecture 3/29

Linear, Hyperbolic PDEs.

last time, we examined a more concrete IVP w/ variable coefficient wave equations,

$$P\phi = \partial_\mu(g^{\mu\nu}\partial_\nu\phi) + b^\mu\partial_\mu\phi + c\phi$$

$$(\text{IVP})\begin{cases} P\phi = f \\ (\phi, \partial_t\phi)|_{t=0} = (g, h) \end{cases}$$

assumptions:

- $g^{\mu\nu}$ is lorentzian w/ sig. $(-,+,\dots,+)$

reductions we can make:

- $g^{0j} = 0$
- $g^{00} = -1$

[- g^{ij} is uniformly elliptic

• $g^{\mu\nu}, b^{\mu}, c$ are uniformly bdd, w/ unif. bdd derivatives of all order.

<u>Thm</u> (LWP of (IVP))
Let $s \in \mathbb{Z}_+$. Given $(g, h) \in H^{s+1} \times H^s$ on $\mathbb{R}^d$, $f \in L^1_t([0,T], H^s(\mathbb{R}^d))$, there exists unique soln ϕ with

$$\bullet \quad \phi \in C_t([0,T], H^{s+1})$$
$$\partial_t \phi \in C_t([0,T], H^s).$$

$$\bullet \quad \|\phi\|_{H^{s+1}} + \|\partial_t \phi\|_{H^s} \lesssim \|(g,h)\|_{H^{s+1} \times H^s} + \|f\|_{L^1_t([0,T], H^s)}$$

$$\|f\|_{L^1([0,T], H^s)} = \Big\| \, \|f(t)\|_{H^s(\mathbb{R}^d)} \Big\|_{L^1([0,T])}$$

<u>Rmk</u> - LWP entails continuous dependence on initial data.

Idea of proof is to use an apriori estimate & a functional analytic lemma.

<u>Prop</u> (a-priori est) Let $s \in \mathbb{Z}$ (either $\geq$ or <0).
Let $\phi \in C_t([0,T]; H^{s+1})$ & $\partial_t \phi \in C_t([0,T], H^s)$.

$$\|\phi\|_{C_t([0,T]; H^{s+1})} + \|\partial_t \phi\|_{C_t([0,T]; H^s)}$$
$$\lesssim \|(\phi, \partial_t \phi)|_{t=0}\|_{H^{s+1} \times H^s} + \|P\phi\|_{L^1_t([0,T], H^s)}$$

Pf (a-priori, $s \geq 0$).

Use energy method. - a natural strategy is to commute $P\phi$ with D^α $|\alpha| \leq s$, and then apply energy est ("$\times \partial_t \phi$" + "IBP").

instead, vary the multiplier in the energy method:

$$(P\phi) \cdot (\partial_t \phi)$$

$$\int_{\mathbb{R}^d} (P\phi) \cdot (1-\Delta)^s \partial_t \phi \, dx = \langle P\phi, (1-\Delta)^s \partial_t \phi \rangle$$

note $\left| \int_0^T \langle P\phi, (1-\Delta)^s \partial_t \phi \rangle \right| \overset{\text{duality}}{\lesssim} \|P\phi\|_{L^1_t([0,T]; H^s)} \|\partial_t \phi\|_{L^\infty_t([0,T]; H^s)}$

we IBP s times & then Cauchy-Schwarz

another view is to use duality

$$|\langle f, g \rangle| \lesssim \|f\|_{H^s} \|g\|_{H^{-s}}$$

$$\|Q g\|_{H^{+s}} \lesssim \|g\|_{H^{r+s}} \quad (s \in \mathbb{R}).$$

order r op w/ unif bdd coeffs & derivs of all order

follows from commuting derivatives onto g.

This uses the fact

$$\|(1-\Delta)^s g\|_{L^2} \cong \|g\|_{H^{2s}}$$

$$|\langle (1-\Delta)^s g, g\rangle| \cong \|g\|^2_{H^s}$$ ← uses char of H^s norm from fourier analysis.

now, key computation:

$$P\phi = \partial_\mu(g^{\mu\nu}\partial_\nu \phi) + b^\mu \partial_\mu \phi + c\phi$$

$$= -\partial_t^2 \phi + \partial_j(g^{jk}\partial_k \phi) + b^\mu \partial_\mu \phi + c\phi$$

focus on the $\partial_t^2\phi$ term

$$\langle -\partial_t^2 \phi, (1-\Delta)^s \partial_t \phi\rangle = -\partial_t \langle \partial_t \phi, (1-\Delta)^s \partial_t \phi\rangle$$
$$+ \langle \partial_t \phi, (1-\Delta)^s \partial_t^2 \phi\rangle$$
$$= -\frac{1}{2}\partial_t \langle \partial_t \phi, (1-\Delta)^s \partial_t \phi\rangle$$

nca,

$$\langle \partial_j(g^{jk}\partial_k \phi), (1-\Delta)^s \partial_t \phi\rangle \overset{IBP}{=} -\langle g^{jk}\partial_k \phi, (1-\Delta)^s \partial_t \partial_j \phi\rangle$$

Pull out ∂_t
$$= -\partial_t \langle g^{jk}\partial_k \phi, (1-\Delta)^s \partial_j \phi\rangle$$
$$+ \langle (\partial_t g^{jk}) \partial_k \phi, (1-\Delta)^s \partial_j \phi\rangle$$
$$+ \langle (\partial_k \partial_t \phi) g^{jk}, (1-\Delta)^s \partial_j \phi\rangle$$

$$-\langle g^{jk}\partial_k\partial_t\phi, (1-\Delta)^s\partial_j\phi\rangle = -\langle \partial_t\phi, \partial_k([g^{jk},(1-\Delta)^s]\partial_j\phi)\rangle$$

$$= -\langle \partial_t\phi, \partial_k(1-\Delta)^s(g^{jk}\partial_j\phi)\rangle$$

$$= -\langle (1-\Delta)^s\partial_t\phi, \partial_k(g^{jk}\partial_j\phi)\rangle$$

$\therefore$

The whole thing $\Rightarrow$

$$= \frac{1}{2}\Big(-\partial_t\langle g^{jk}\partial_k\phi, (1-\Delta)^s\partial_j\phi\rangle + \langle \partial_t g^{jk}\partial_k\phi, (1-\Delta)^s\partial_j\phi\rangle$$

$$-\langle \partial_t\phi, \partial_k([g^{jk},(1-\Delta)^s]\partial_j\phi)\rangle\Big)$$

using $[Q,P]$ of order $r+n-1$

$(PQ-QP)g$

$\therefore$

$$\langle P\phi, (1-\Delta)^s\partial_t\phi\rangle = -\frac{1}{2}\partial_t\overbrace{\left(\langle \partial_t\phi, (1-\Delta)^s\partial_t\phi\rangle + \langle g^{jk}\partial_k\phi, (1-\Delta)^s\partial_j\phi\rangle\right)}^{E_s[\phi](t)}$$

$$+\mathcal{O}(\langle g_1\partial\phi, \partial^{2s}\partial\phi\rangle) + \mathcal{O}(\langle g_2\partial\phi, \partial^{2s-1}\partial\phi\rangle + \dots$$

∂g b $\qquad\qquad$ $\partial^2 g$ ∂bc

a bunch of lower-order operators

now

$$-\int_0^t \langle P\phi, (1-\Delta)^s \partial_{t'}\phi \rangle \, dt' \geq -E_s[\phi](0) + E_s[\phi](t)$$
$$\underbrace{- C\left(\int_0^T \|\phi\|^2_{H^{s+1}} + \|\partial_t \phi\|^2_{H^s} \, dt' \right)}_{\text{estimate for the remainder.}}$$

$\Rightarrow$

$$E_s[\phi](t) \lesssim E_s[\phi](0) + \|P\phi\|_{L^1_t([0,T], H^{s+1})} \|\partial_t \phi\|_{C_t([0,T], H^s)}$$
$$+ \int_0^t \|\phi\|^2_{H^{s+1}} + \|\partial_t \phi\|^2_{H^s} \, dt'$$

NB: $E_s[\phi](t) \simeq \|\phi\|^2_{H^{s+1}} + \|\partial_t \phi\|^2_{H^s}$

once proven, we apply Grönwall to get the bound.

Take a $\sup_{t \in [0,T]}$ & use $\frac{a^2+b^2}{2} \geq ab$.

now, <u>let $s<0$.</u>

$$\Phi = (1-\Delta)^{-|s|}\partial_t \phi$$

$$\|\Phi\|_{H^{|s|+1}} \cong \|\phi\|_{H^{-|s|+1}} \cong \|\phi\|_{H^{s+1}}$$

we basically apply the previous proof w/ no changes.

$(\phi \to \Phi$

$s \to |s|$

need to compute

$$|\langle P\Phi, (1-\Delta)^{|s|}\partial_t \Phi\rangle| = |\langle (1-\Delta)^{|s|} P\Phi, \partial_t \Phi\rangle|$$

$$= |\underbrace{\langle P\phi, \partial_t \Phi\rangle}_{\lesssim \|P\phi\|_{H^{-|s|}} \|\partial_t \Phi\|_{H^{|s|}}} + \underbrace{\langle \underbrace{[(1-\Delta)^{|s|}, P]}_{2|s|+2-1}\Phi, \partial_t\Phi\rangle}_{\lesssim \|\phi\|_{H^{|s|+1}} \|\partial_t \Phi\|_{H^{|s|}}}|$$

$$\underbrace{\lesssim \|\phi\|_{H^{|s|+1}} \|\partial_t \Phi\|_{H^{|s|}}}$$

Grönwall This.

Pf ($\exists!$ for IVP)

note that uniqueness & a-priori estimate follow from the estimate

only need existence. - use duality.

① $\left(P: L^\infty_t([0,T], H^{s+1} \times H^s) \to L^1_t([0,T]; H^s)\right)$

maybe?

①

want to reduce to the case $g = h = 0$. (via extension & modifying f)

② $\phi \in L^\infty_t([0,T]; H^{s+1}) = \left(L^1_t([0,T]; H^{-s-1})\right)^*$

duality only for finite measure spaces.

want this equality.

$$\left(\int_0^T \langle P\phi, \psi\rangle dt = \int_0^T \langle f, \psi\rangle dt\right) \int_0^T \langle \phi, P^*\psi\rangle dt$$

want

$$\ell(P^*\psi) = \int_0^T \langle f, \psi\rangle dt$$

using apriori estimate.

$$\Rightarrow \|\ell\| \lesssim \|f\|_{L^1 H^s} \|P^*\psi\|_{L^1 H^{-s-1}}$$

Hahn Banach extension on $(L^1 H^{-s-1})^*$

③ upgrading $L^\infty H^{s+1} \to \phi \in C_t H^{s+1}$ $\partial_t \phi \in C_t H^s$
via Ringström.

idea is approximation by smoother objects.

Lecture 3/31 — Today: survey of hyperbolicity more broadly + geometric interp of hyperbolicity

next time: dispersion

recall "working dfn" of hyperbolicity:

- order of ∂_t = order of ∇ — (satisfies finite speed of propagation $\leftrightarrow$ Lorentzian geometry.)
- (local) well-posedness of IVP.

algebraic defn of Hyperbolicity - not always perfect. Many caveats!

recall $P\phi = f = -\partial_t^2 \phi + \partial_j(a^{jk}\partial_k \phi) + b^j \partial_j \phi + c\phi$
$\rightsquigarrow$ LWP.

This was a special case of hyperbolicity for 2nd order linear PDEs.

We turn to 1^{st}-order systems.

$$\Phi : \mathbb{R}_t \times \mathbb{R}^d \to \mathbb{R}^n$$

$$\partial_t \Phi^J + (B^j)^J_K \partial_j \Phi^K = F^J \qquad (\star)$$

(in matrix notation $\partial_t \vec{\Phi} + B^j \partial_j \vec{\Phi} = F$) I.C. $\Phi|_{t=0} = \Phi_0$

<u>Dfn</u> (1-D hyperbolicity).
Take the symbol of B

$$\sigma(t,x;\xi) = \xi_j B^j(t,x). \qquad \xi \in \mathbb{R}^d$$

$(\star)$ is hyperbolic if $\sigma(t,x;\xi)$ has n real eigenvalues for each t, x, ξ.

<u>Thm</u> (const coeff).
Assume B^j is independent of t, x $\forall j$.
Then hyperbolicity $\Rightarrow$ (L)WP of IVP.

Pf (Fourier analysis, can be found in Evans §7.3)

- If B is const coeff, & σ has a nonreal complex eigenvalue, then $\exists$ a plane wave solution of the form $\Phi = \tilde{A} e^{ix\cdot\xi + i\omega t}$

& ω has a nonzero imaginary part,

If $\operatorname{im}\omega < 0$, we have ill-posedness
$e^{i(-i\,\operatorname{im}\omega t)} \to \infty$

In the variable-coeff case, we don't know.
We have to add stronger conditions to get LWP.

<u>Dfn</u> (symmetric hyperbolicity).

($\star$) is symmetric-hyperbolic if each B^j is symmetric $\forall (t,x)$.
If $\exists$ a similarity transformation $\Phi \to \tilde{\Phi} = P\Phi$ s.t. the new equation is symmetric hyperbolic, then ($\star$) is symmetrizable hyperbolic.

<u>Thm</u> ($\star$) symmetrizable hyperbolic & boundedness ~~of coeffs~~ $\Rightarrow$ LWP.

Pf (in symmetric-hyperbolic case, use the energy method) (§7.3 evans) (§7 in ringström)

Dfn (Strict Hyperbolicity)

A hyperbolic system (*) is said to be strictly hyperbolic if all n real eigenvalues are distinct at every point.

Usually expressed $\lambda_1(t,x) < \cdots < \lambda_n(t,x)$

This definition is useful when $d=1$, since it allows us to use the method of characteristics

"Hyperbolic Conservation Laws" by Dafermos.

2nd order case – (going to Lorentzian Geometry)

we are considering scalar equations.

$$P\phi = \partial_\mu(g^{\mu\nu}\partial_\nu\phi) + b^\mu\partial_\mu\phi + c\phi$$

want to ask what generalizes $\Box\phi = 0$

Lorentzian (Inverse) Metric ← Technically on the cotangent bundle.

$g^{\mu\nu}(t,x)$ is symmetric w/ signature $(-,+,\dots,+)$

only the signature is invariant under similarity transforms.

$$g = g_{\mu\nu}\ ,\quad g^{-1} = g^{\mu\nu}$$

(M, g) is a Lorentzian manifold when M is a smooth manifold and g is a symm, covariant 2-tensor w/ signature $(-,+,\dots,+)$

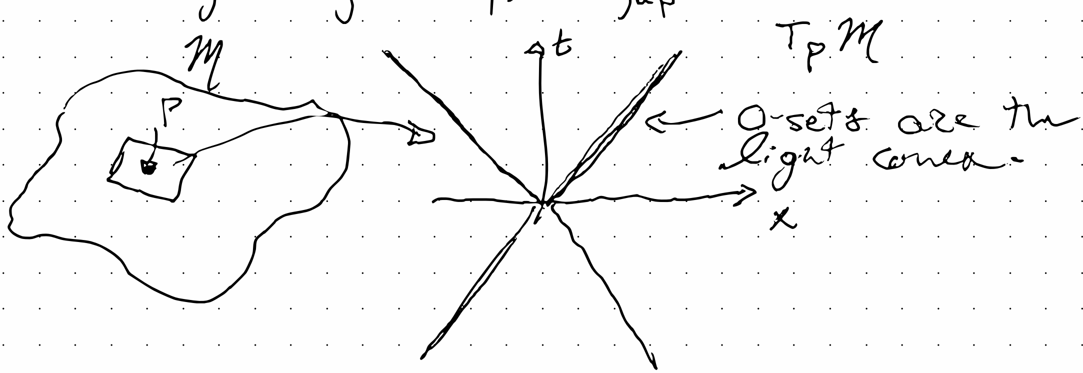
<u>Lemma</u> (a key diff from riemannian geometry)
Consider $Q(\xi)$ a quadratic form $g^{\alpha\beta}\xi_\alpha\xi_\beta$

If g is nondegenerate and has one negative & one positive eigenvalue, then the 0-set of Q determines g up to a real factor.

pf exercise.

<u>Rmk</u> in riemannian geometry, the 0-set of the metric is unhelpful.
here, $\{g^{\alpha\beta}\xi_\alpha\xi_\beta = 0\}$ determines $\pm g^{\alpha\beta}$

In Lorentzian geometry, $\{g_{\alpha\beta}V^\alpha V^\beta = 0\}$



Lemma

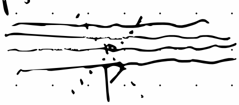
g be a Lorentzian metric, $p \in M$.

There exists a nbhd $p \in U \subset M$, w/ local coordinates $(x^0, \dots, x^d)$ s.t. $g^{0j} = 0 \ \forall j$, $g^{00} < 0$, & ellipticity of the spatial part.

I.E. Locally, we can always choose coords to turn

$$P\phi = -\partial_\mu (g^{\mu\nu} \partial_\nu \phi) \longrightarrow -\partial_t^2 \phi + \partial_j (g^{jk} \partial_k \phi)$$

~~Pf~~



x^0 pick x^0 s.t. dx^0 is timelike, i.e.

$$g^{\mu\nu}(dx^0)_\mu (dx^0)_\nu < 0$$

normalize $x^0 = 0$

take x^j local coordinate near p

now, transport x^j onto other level surfaces of x^0 so that $g^{0j} = 0$

$$(dx^0)_\mu \longrightarrow g^{\mu\nu}(dx^0)_\mu = \nabla x^0 = (g^{0\mu}\partial_\mu) x^0$$

1-form

vector field

algebraically, we want $(g^{0\mu}\partial_\mu) x^j = 0$

finally, finite speed of propagation

Geometric formulation of WP of IVP.

Dfn Global Hyperbolicity.

need new concepts: • Domain of Dependence

• Cauchy Hypersurface

we let (M, g) be manifold,

∇ the levi-civita connection

Dfn γ is a geodesic if parallel transported

$\partial_t(g(\dot\gamma, \dot\gamma)) = 0$

geodesic.

$g(\dot\gamma, \dot\gamma) = 0$ ← null

$g(\dot\gamma, \dot\gamma) < 0$ ← timelike

causal

$g(\dot\gamma, \dot\gamma) > 0$ ← spacelike

Dfn

Let $V \subseteq M$, Then $J^{+}(V) = \{$all $q \in M$ s.t. $\exists$ a causal future curve from $p \in V$ to $q\}$

Causal future

Dfn Time orientable $\exists$ nonvanishing vector field timelike everywhere, Cauchy Hypersurface i.e. all tangent vectors are spacelike.

Σ: Spacelike Hypersurface for which
$\mathcal{D}^+(\Sigma) \cup \Sigma \cup \mathcal{D}^-(\Sigma) = \mathcal{M}$.

$\mathcal{D}^+(V) = \{ q : \text{all causal past-pt curves met } V \}$

Global hyperbol $\Leftrightarrow$ Time orientable, $\exists$ cauchy hypersurface.

Laplace Beltrami

Thm $\begin{cases} \square_g \phi + B\phi + c\phi = f \\ (\phi, n_\Sigma \phi)|_\Sigma = (g, h) \end{cases}$

(M, g) Glob Hyp, & Σ is cauchy hypersurface
then IVP is well posed.

The converse is also true.

Cf Ringström Ch 10-12.