### 0.0.1 Optical Theorem (Guest Lecutrer: M. Zaletel)

We have the differential cross section

$$\frac{\partial\sigma}{\partial\Omega} = |f(\theta)|^2 = \frac{1}{k^2}\sum_{\ell,\ell'}(2l+1)(2\ell'+1)f_\ell f^*_{\ell'}P_\ell(\cos\theta)P_{\ell'}(\cos\theta)$$

Legendre polynomials are orthonormal, so we can take

$$\sigma_T = 2\pi\int_{-1}^{1} d\cos\theta|f^2(\theta)|^2 = \frac{4\pi}{k^2}\sum_\ell(2\ell+1)\sin^2(\delta_\ell)$$

There's also a unitarity bound. He erased before I was able to write this down so TODO: GO to the Tong lecture.

### 0.0.2 Hard-Sphere Scattering

Let's consider the following hard-sphere scattering.

$$\psi(r,\theta) = \sum_\ell R_\ell(r)P_\ell(\cos\theta)$$

Then, for $r > a$,

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\ell(\ell+1)}{r^2} + k^2\right](rR_\ell(r))$$

Basically, we write it down in terms of the bessell functions

$$\rho \gg 1 \Rightarrow \begin{cases} j_\ell(\rho) = \frac{1}{\rho}\sin(\rho - \ell\pi/2) \\ n_\ell(\rho) = \frac{1}{\rho}\sin(\rho-\ell\pi/2)\end{cases} \qquad \rho \ll 1 \Rightarrow \begin{cases} j_\ell(\rho) = \frac{\rho^\ell}{(2\ell+1)!!} \\ n_\ell(\rho) = (2\ell-1)!!\rho^{-(2\ell+1)}\end{cases}$$

Taking asymtotics then, we take

$$\lim_{\rho\to\infty} R_\ell(\rho) \propto \left[(-1)^{\ell+1}\frac{e^{-i\rho}}{\rho} + e^{2i\delta_\ell}\frac{e^{i\rho}}{\rho}\right] = \frac{e^{i\delta\ell}e^{i\pi\ell/2}}{\rho}\left[-e^{i(\rho+\delta_\ell-\pi\ell/2)} + e^{i(\rho+\delta_\ell-\pi\ell/2)}\right]$$

We can solve for various $\ell$ by substituting in the asymptotics for $\ell$. For instance, in the limit where $\rho \ll 1$, $\tan(\alpha_\ell = \delta_\ell) = \frac{j_\ell(ka)}{n_\ell(ka)}$, which gives

$$\lim_{\rho\ll 1}\tan(\delta_\ell) = \frac{(ka)^{2\ell+1}}{(2\ell+1)!!(2\ell-1)!!}$$

Now, we can take the total contribution from all sections

$$\sigma_T = \sum_{\ell=0}^{\infty}\sigma_\ell = \frac{4\pi}{k^2}(ka)^2 + \cdots = 4\pi a^2 + \dots$$

We can also always define $a_\ell = \lim_{k\to 0}\frac{\tan(\delta_\ell)}{k}$