Physics-105-Lecture-Notes-01-31-2019

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Abstract

A single PDF with all lectures in a single document can be downloaded at `https://www.dropbox.com/sh/8sqzvxghvbjifco/AAC9LoSRnsRQDp7pYedgWpQMa?dl=0`. The password is 'analytic.mech.dsp'. This file was automatically generated using a script, so there might be some errors. If there are, you can contact me at `mailto:ctdunc@berkeley.edu`.

1 Calculus of Variations

1.1 Euler-Lagrange Equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}f}{\mathrm{d}x'} - \frac{\mathrm{d}f}{\mathrm{d}x} = 0$$

describes the optimal path along some constraint, using a functional so that

$$dS = \int f dx$$

Recall that if f is not a function of the independent variable, (t in the expression above), then you can take

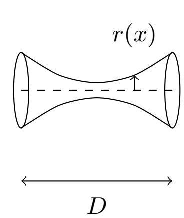
$$H = y'\frac{\mathrm{d}f}{\mathrm{d}y'} - f$$

and discover that

$$\frac{\mathrm{d}H}{\mathrm{d}x} = -\frac{\mathrm{d}f}{\mathrm{d}x}$$

1.1.1 Plateau's Problem, cont.

Take some soap film suspended between two hoops



surface area of a small band given by

$$dS = 2\pi r(x)\sqrt{1 + \left(\frac{\mathrm{d}r}{\mathrm{d}x}\right)^2}dx$$

which implies that the total area is equal to

$$= 2\pi \int_{-D/2}^{D/2} r(x)\sqrt{1 + r'(x)^2}dx$$

we can use the hamiltonian H to say that

$$H = r'\frac{\mathrm{d}f}{\mathrm{d}r} - f$$

with

$$\frac{\mathrm{d}f}{\mathrm{d}r'} = \frac{rr'}{\sqrt{1+r'^2}} \Rightarrow H = \frac{rr'^2}{\sqrt{1+r'^2}} - r\sqrt{1+r'^2}$$

Then, take $\frac{\mathrm{d}H}{\mathrm{d}x} = -\frac{\mathrm{d}f}{\mathrm{d}x}$, so $\frac{\mathrm{d}r}{\mathrm{d}x} = \pm\sqrt{\left(\frac{r}{H}\right)^2 - 1}$, os we integrate

$$\int \frac{dr}{\sqrt{\left(\frac{r}{h}\right)^2 - 1}}$$

we need to use hyperbolic cosines and sines, so we take $\frac{r}{H} = \cosh\psi$, which integrating gives

$$\int \frac{H\sinh\psi dr}{\sinh\psi} = H\psi$$

and thus, taking $\chi = \frac{H}{D}$[1].

$$r(\chi) = H\cosh\frac{\chi}{H}$$

where $\chi = \frac{D}{2}, r = R$. Final result comes out to be that

$$\frac{R}{D} = \chi\cosh\left(\frac{1}{2\chi}\right)$$

This is a number, which has to be equal to the geometry of the system!

1.2 Quantum $\Rightarrow$ Lagrangian Mechanics

Recall we have the transition amplitude, i.e. how probable it is to go from one state to another.

$$q_1(t_1) \to q_2(t_2)$$

would be expressed as

$$\langle q_1(t_1)|q_2(t_2)\rangle$$

which goes to

$$\langle q_2(t+\Delta t)|q_1(t)\rangle = \langle q_2|\, e^{-|\Delta H|}\, |q_1\rangle$$

we take action as

$$S(x(t)) = \int_{t_1}^{t_2} dt\left(\frac{p^2}{2m} - V\right)$$

then, with amplitude functional $A[x(t)] = e^{i\frac{S}{\hbar}}$, we can write an integral across every possible path, with

$$|A|^2 = \int_{\text{all paths}} x(t)e^{i\frac{S(x(t))}{\hbar}}\, dt$$

The path that wins is the one that oscillates the least, i.e. the one that has *stationary phase*, since integrating an oscillator gives zero. This means we want to find a *stationary* form of S, which is called *Hamilton's Principle*, which gives that

$$\delta S(x(t)) = S\int dt\left(\frac{p^2}{2m} - V\right) = 9$$

2 Lagrangian Mechanics

2.1 Defining the Lagrangian

$$L = T - V$$

kinetic minus potential energy. Then, the lagrangian can be put into the Euler-lagrange equation to give that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}q'} - \frac{\mathrm{d}L}{\mathrm{d}q} = 0$$

which contains all of classical mechanics, since we can then write

$$S\int_{t_1}^{t_2} dtL(x, x', t)$$

with mass m, potential V, we have $T = \frac{mv^2}{2}$, so the lagrangian is

$$\frac{mv^2}{2} - V(q)$$

[1] don't worry, I don't totally understand how he did this integral either

now, we write lagrange euler equation as

$$\frac{\mathrm{d}L}{\mathrm{d}q'} = mq' \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}q'} = mq'' \qquad \frac{\mathrm{d}L}{\mathrm{d}q} = \frac{\mathrm{d}V}{\mathrm{d}q}$$

which gives an equation of motion

$$mq'' = -\frac{\mathrm{d}V}{\mathrm{d}q} \Leftrightarrow F = ma$$

2.2 Ex: spherical pendulum

$$\begin{aligned}
\vec{r} &= l\cos\varphi\sin\theta\hat{x} + l\sin\varphi\sin\theta\hat{y} + l\cos\theta\hat{z} \\
\dot{\vec{r}} &= (-l\dot{\varphi}\sin\varphi\sin\theta + l\cos\varphi\dot{\theta}\cos\theta)\hat{x} \\
&+(l\dot{\varphi}\cos\varphi\sin\theta + l\sin\varphi\dot{\theta}\cos\theta)\hat{y} \\
&-l\dot{\theta}\sin\theta\hat{z} \\
\dot{\vec{r}}\dot{\vec{r}} &= l^2\dot{\varphi}^2\sin^2\theta + l^2\dot{\theta}^2
\end{aligned}$$

Then, we have

$$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}\left(ml^2\dot{\theta}^2 + ml^2\dot{\varphi}^2\sin^2\theta\right)$$

Which gives a lagrangian

$$L = T - V = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}\left(ml^2\dot{\theta}^2 + ml^2\dot{\varphi}^2\sin^2\theta\right) - mgl\cos\theta$$

Now, we want to apply the ELE, which gives two constraints

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}\dot{\theta}} - \frac{\mathrm{d}L}{\mathrm{d}\theta} &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}\dot{\varphi}} - \frac{\mathrm{d}L}{\mathrm{d}\varphi} &= 0
\end{aligned}$$

There's no φ dependence, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}L}{\mathrm{d}\dot{\varphi}}\right) = 0$$

which makes it a constant of motion, so we say

$$\left(\frac{\mathrm{d}L}{\mathrm{d}\dot{\varphi}}\right) = p_\varphi = ml^2\dot{\varphi}\sin^2\theta$$

We also have

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}\theta} &= mgl\sin\theta + ml^2\dot{\varphi}^2\sin\theta\cos\theta \\
\frac{\mathrm{d}L}{\mathrm{d}\dot{\theta}} &= ml^2\dot{\theta} \to \frac{\mathrm{d}}{\mathrm{d}t}\{dvL\dot{\theta} = ml^2\ddot{\theta}
\end{aligned}$$

Then, we can find an equaiton of motion for θ, which gives

$$\ddot{\theta} = \frac{g}{l}\sin\theta + \varphi^2\sin\theta\cos\theta$$

with another term for $\dot{\varphi}$,

$$\begin{aligned}
\dot{\varphi}^2 &= \frac{p_\varphi^2}{m^2l^2\sin^4\theta} \\
\ddot{\theta} &= \frac{g}{l}\sin\theta + \frac{p_\varphi^2}{m^2l^2}\frac{\cos\theta}{\sin^2\theta}
\end{aligned}$$

Integrating this is just rude. So let's analyze some cases.

2.2.1 case $\dot{\varphi} = 0$

implies $p_\varphi = 0$, which is then

$$\ddot{\theta} = \frac{g}{l}\sin\theta$$

which is just a regular harmonic oscillator

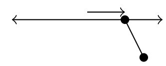
2.2.2 case $\dot{\theta}$ =constant

Implies that $\ddot{\theta} = 0$, which gives then that

$$\begin{aligned}
\frac{g}{l}\sin\theta + \dot{\varphi}^2\sin\theta_0\cos\theta_0 &= 0 \\
\left(\frac{g}{l} + \dot{\varphi}^2\cos\theta_0\right)\sin\theta_0 &= 0
\end{aligned}$$

if $\theta_0 = 0, \pi$ etc, then the pendulum is just balanced at the top, not moving. if $\cos\theta_0 < 0$, we have $\theta_0 > \pi/2$, which gives that $\dot{\varphi}^2 > \frac{g}{l} = \omega_0$ We could also integrate this and get complex motion, but these are the stable forms.

2.3 Driven Pendulum



Point on axis is being pushed, what is the motion of the point at the bottom of the pendulum?

$$
\begin{aligned}
x &= a\cos\omega t \\
r &= x\hat{x} + l(\sin\theta\hat{x} - \cos\theta\hat{y}) \\
\dot{\vec{r}} &= (\dot{x} + l\dot{\theta}\cos\theta)\hat{x} + l\dot{\theta}\sin\theta\hat{y} \\
\dot{r}^2 &= \dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}l\cos\theta
\end{aligned}
$$

we set up the lagrangian

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}^2\dot{\theta}^2 l\cos\theta) + mgl\cos\theta$$

and try to find simple solutions.