Physics-105-Lecture-Notes-02-19-2019

Connor Duncan

February 19, 2019

Contents

Abstract

A single PDF with all lectures in a single document can be downloaded at `https://www.dropbox.com/sh/8sqzvxghvbjifco/AAC9LoSRnsRQDp7pYedgWpQMa?dl=0`. The password is 'analytic.mech.dsp'. This file was automatically generated using a script, so there might be some errors. If there are, you can contact me at `mailto:ctdunc@berkeley.edu`.

1 Central Force Motion

Recall, newton, we have $F = m\vec{a} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$. So, for some direction $\hat{s}$, if we have $F \cdot \hat{s} = 0$, then $\vec{p} \cdot \vec{s} = 0$, also true for torque. Implies conservation of true, also true for angular momentum.

Definition 1.1. Central force is a force such that $\vec{F}(\vec{r}) = f(r)\hat{r}$, i.e. force depends only on vector between objects. This means that the torque $\vec{r} \times \vec{F} = \vec{r}\hat{r}f(r) = 0$, which implies that $\vec{L} \equiv$ constant.

1.1 Two Body Problem

let r_1, r_2 be vectors pointing to two masses, m_1, m_2 respectively. let $\vec{R}$ be the vector pointing to the center of mass, and $\vec{r}$ be the vector pointing from m_1 to m_2. Let r_n' be the vector pointing from m_n to the cm. We can think about the lagrangian. noting that

$$\vec{r_n} = \vec{R} + \vec{r_n}'$$

Now, writing down the component bits of T, we have

$$T = \frac{1}{2}(m_1\dot{\vec{r}}_1^2 + m_2\dot{\vec{r}}_2^2)$$

which expands (after some substitution) to

$$T = \frac{1}{2}\left(m_1\dot{R}^2 + m_1\dot{r}_1'^2 + 2m_1\dot{R}\dot{r}_1' + m_2\dot{R}^2 +_2 \dot{r}_2'^2 + 2m_2\dot{R}\dot{r}_2'\right)$$

If we define the center of mass as

$$\sum_i m_i r_i = \sum_i m_i \vec{R}$$

then the cross terms from our dot product cancel, and T simplifies to

$$T = \frac{1}{2}(m_1 + m_2)\dot{R}^2 + \frac{1}{2}m_1\dot{r'}_1^2 + \frac{1}{2}m_2\dot{r'}_2^2$$

we also have

$$\vec{r'}_2 = \frac{-m_1}{m_1 + m_2}\vec{r} \qquad \vec{r'}_1 = \frac{m_2}{m_1 + m_2}\vec{r}$$

The lagrangian then becomes, using this simplification for the reduced mass

$$\frac{1}{2}(m_1\dot{r'}_1^2 + m_2\dot{r'}_2^2) = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\dot{r}^2$$

We can call this reduced mass μ, and the total mass M. Now, we get

$$T = \frac{1}{2}(M\dot{R}^2 + \mu\dot{r}^2)$$

Now, the lagrangian

$$\mathcal{L} = T - U = \frac{1}{2}(M\dot{R}^2 + \mu\dot{r}^2) - U(r)$$

Immediately, we can tell that R is cyclic, (i.e. $\frac{\partial L}{\partial R} = 0$, which implies that $m\dot{R} \equiv$constant, which can be edrived from the euler-lagrange equations easily) Since the momentum of the center of mass is conserved, we're just going to drop the $M\dot{R}^2$ term, since we're just changing to a frame that's moving with the center of mass. The problem is now basically a single-body problem. Conservative forces that depend on only r, so we have $F(r) = f(r)\hat{r}$[1] so

$$\vec{F} = -\vec{\nabla}V(r) = f(r)\hat{r}$$
$$V(r) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}')d\vec{r}'$$

We can convert this to a 2-d problem in polar coordinates, with the knowledge that $\frac{\mathrm{d}L}{\mathrm{d}t} = 0$, so, we can write $\mathcal{L}$ as

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

We need to choose a form for the potential, take

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial \mathcal{L}}{\partial \theta} = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t} = 0 \qquad l = mr^2\dot{\theta} \equiv \text{angular momentum}$$

This immediately yields one of keplers laws, since

$$\frac{\mathrm{d}}{\mathrm{d}t} = 0$$

is the areal velocity, we get keplers second law, since that tells you the area swept out by a radius vector per unit time is always the same. This gives the equation of motion

$$m\ddot{r} - mr\dot{\theta}^2 = f(r)$$

with the knowledge that $\dot{\theta} = \frac{l}{mr^2}$, we can write now that

$$m\ddot{r} - \frac{l^2}{mr^3} = f(r)$$

which is a one dimensional equaiton of motion, which we know how to solve. Total energy is

$$E = \frac{1}{2}(m\dot{r}^2 + \frac{l^2}{mr^3} + V(r) \equiv \text{const}$$

Let's integrate the equation of motion

$$\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{l}{mr^2}$$
$$\int d\theta = \int \frac{l}{mr(r)^2}dt$$
$$\Delta\theta = l\int_0^t \frac{dt}{mr^2(t)}$$

We also have

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}$$

which gives

$$t = \int dt = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}}$$

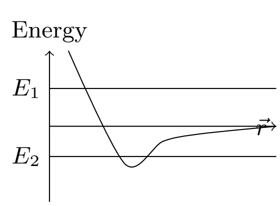
So, what can we qualitatively get out of this problem? WE STILL HAVENT SPECIFIED THE POTENTIAL :eyeroll:. Let's define an *effective potential*. Let $V'(r) = V(r) + \frac{l^2}{+2mr^2}$, then we can rewrite energy as

$$E = \frac{1}{2}m\dot{r}^2 + V'(r)$$

Finally, we should take $V = -\frac{k}{r}$ (like gravity, or a coulomb force). Then,

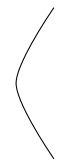
$$V'(r) = \frac{-k}{r} + \frac{l^2}{2mr^2}$$

This looks qualitatively like the following



[1]note that I'm dropping a lot of over-arrows, but these objects are still vectors

Between the certain energy levels, there is different behavior. E_1 corresponds ot a hyperbola, since ther's only one turning point, it will make one turn, which is a path that looks like



at E_2, it will be an ellipse, from r_1 to r_2 (i.e. the places E_2 intersectes energy). At E_4, you should get a parabola (lowest point on the energy graph). We can also solve for $\theta(t), r(t)$ using L, E are constant. In principle, we coould solve and make plots in terms of time etc, but we also want to see plots of $r(\theta, \theta(r)$. We can use conservation of angualr momentum to achieve this goal. $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{l}{mr^2}$, which gives us $ldt = mr^2 d\theta$. We can write this as

$$d\theta = \frac{ldr}{mr^2\sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}}$$

with potential written as inverse r, we get

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{l}{r^2}}}$$

making a u-sub, for $r = \frac{1}{u}$, we get

$$\theta = \theta_0 \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - u^2}}$$

To be continued thursday.