Abstract

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0.1 Systems of N Particles

Consider a system of N particles. We can arrange this in terms of M single particle basis states. We need to count such systems using our 112 Memes.

- Distinguishable particles: M^N possible arrangements of states.

For fermions and bosons, things behave a bit differently.

Bosons The idea is that for our basis states $n_i = \{a, b, \ldots, \}$, we need to hav $P_{ij}\psi(n_1, \ldots, n_N) = \alpha\psi(n_1, \ldots, n_N)$ where $\alpha = \pm 1$. We need to give our particle state some symmetry, so that

$$|n_1, \ldots, n_N\rangle = \frac{1}{\sqrt{N!}} \sum_{P_N} |P(n_1, \ldots, n_N)\rangle$$

where P_N is the action of taking all possible permutations of group N.

Fermions Again, we have to sum over all possible permutations, but then we need

$$|n_1, \ldots, n_N\rangle = \frac{1}{\sqrt{N!}} \sum_{P_N} \chi(P) |P(n_1, \ldots, n_N)\rangle$$

where $\chi(P)$ is the parity of the permutation, where parity is the number of single particle permutations necessary to achieve a given state. This seems really complicated, but there's actually a trick. Let's try writing in the position basis

$$\begin{aligned} \langle x_1 x_2 x_3 | n_1 n_2 n_3\rangle &= \psi_{n_1 n_2 n_3}(x_1, x_2, x_3) \\ \psi^B_{n_1 n_2 n_3}(x_1, x_2, x_3) &= \frac{1}{\sqrt{3!}} \sum_P \varphi_{P(n_1)}(x_1)\varphi_{P(n_2)}(x_2)\varphi_{P(n_3)}(x_3) \\ \psi^F_{n_1 n_2 n_3}(x_1, x_2, x_3) &= \frac{1}{\sqrt{3!}} \sum_P \chi_P \varphi_{P(n_1)}(x_1)\varphi_{P(n_2)}(x_2)\varphi_{P(n_3)}(x_3) \end{aligned}$$

The trick for fermions is to take the **slater determinant**,

$$= \frac{1}{\sqrt{3!}} = \det \left| \begin{bmatrix} \varphi_{n_1}(x_1) & \varphi_{n_2}(x_1) & \varphi_{n_3}(x_1) \\ \varphi_{n_1}(x_2) & \varphi_{n_2}(x_2) & \varphi_{n_3}(x_2) \\ \varphi_{n_1}(x_3) & \varphi_{n_2}(x_3) & \varphi_{n_3}(x_3) \end{bmatrix} \right|$$

The slater determinant actually spits out exactly the correct permutations and their signs for a fermion, because of the way signs work when taking the determinant. This allows us to return to our original challenge: **counting of Bosonic/Fermionic States**. Consider M orbitals, with N particles. For fermions, this is actually pretty easy. First off, there is no fermionic system with $N > M$. This should come out for $N < M$, there are $\binom{M}{N}$ states. For bosons, we get a stars and bars problem, which gives there should be $\binom{M+N-1}{N-1}$ states.[1]

0.1.1 When does particle statistics matter?

Really only matters when we have a high probability of particle exchange.[2] The example Altman ggives is of a particle in a German lab, and in Haeffners lab. We probably don't need to apply a symmetrization requirement because they're unlikely to exchange. Here's the mathy example. Consider two particles, ψ_E the "earth" hydrogen atom, and ψ_M is the "moon" hydrogen atom. Then, we apply the properly antisymmetric state

$$\psi_A(x_1, x_2) = \frac{1}{\sqrt{2}}(\psi_E(x_1)\psi_M(x_2) - \psi_E(x_2)\psi_M(x_1))$$

and the distinguishable state

$$\psi(x_1, x_2)\frac{1}{\sqrt{2}}\psi_E(x_1)\psi_M(x_2)$$

For the distinguishable case, what we find is

$$P(x) = \int dx_2 |\psi_E(x)|^2 |\psi_M(x_2)|^2 + \int dx_1 |\psi_E(x_1)|^2 |\psi_M(x)|^2 = |\psi_E(x)^2$$

because the integral over x_1 is just obscenely tiny. For the antisymmetric case, we're going to find

$$P(x) = \int dx_2 |\psi_A|(x_1, x_2)|^2 + \int dx_1 |\psi_A(x_1, x_2)|^2 = \int dx_2 |\psi_E(x)\psi_M(x_2) - \psi_E(x_2)\psi_M(x)|^2$$

Basically, the exponentially decaying tails kind of disappear. It can be shown these come out to be the same.

[1] THIS IS A GUESS, but im like 90% sure this is correct. Thank you math 55.
[2] is there a way to calculate the probability of exchange?