Physics-105-Lecture-Notes-03-12-2019

Connor Duncan

March 12, 2019

Contents

0.1 Canonical Transformations (not in book) . . . 1
0.1.1 Type 1 Generator . . . 2
0.1.2 2 Examples . . . 2

1 Rigid Body Motion **3**
1.1 ex: Point mass in a plane . . . 4
1.2 Kinetic Energy . . . 4
1.3 Center of Mass Coordinates . . . 4
1.4 Principal Axes . . . 4
1.4.1 Todo next lecture . . . 4

Abstract

A single PDF with all lectures in a single document can be downloaded at `https://www.dropbox.com/sh/8sqzvxghvbjifco/AAC9LoSRnsRQDp7pYedgWpQMa?dl=0`. The password is 'analytic.mech.dsp'. This file was automatically generated using a script, so there might be some errors. If there are, you can contact me at `mailto:ctdunc@berkeley.edu`.

0.1 Canonical Transformations (not in book)

There's some discussion of this in Hand + Finch. We have some $H(q,p,t)$, where p is the canonical momentum.

$$\dot{p} = -\frac{\partial H}{\partial q}$$
$$\dot{q} = \frac{\partial H}{\partial p}$$

If we have a cyclic coordinate, it simplifies the problem, i.e. if $\frac{\partial H}{\partial q} = 0$, then $\dot{p} = 0 \Rightarrow p$ is a constant. Good example of this principle is central force problems, since $\mathcal{L} = \frac{1}{2}m\dot{r}^2 + r^2\dot{\theta}^2 - V(r)$, potential exclusively depends on r, which means $\frac{\partial L}{\partial \theta} = 0 \rightarrow l = mr^2\dot{\theta} \equiv$ constant. Canonical Transformations are ways of finding convenient coordinates to make the hamiltonian cyclic as well. Lets call the transform $Q = Q(p_i, q_i, t)$, and $P = P(q_i, p_i, t)$. If it makes some $\mathcal{H}$ cyclic in Q, then $\mathcal{H} = \mathcal{H}(P)$, which implies $\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = \omega \equiv$ constant, $Q = \omega t + Q_0$, $\dot{p} = -\frac{\partial \mathcal{H}}{\partial Q} = 0$, $p \equiv$const. (q,p) are called canonically conjugate, that if Hamiltons equations hold for $(q,p) \Leftrightarrow (Q,P)$ under canonical equations. Hamiltons Principle

$$\delta \int L(q,\dot{q},t)dt = 0 = \delta \int \mathcal{L}(Q,\dot{Q},t)dt$$

this means that $\delta(L - \mathcal{L})dt = 0$. Because this is a time integral, the lagrangian and the transformed lagrangian can differ by a total differential and this would still be true.

$$\delta \int_{t_1}^{t_2} \frac{\mathrm{d}F}{\mathrm{d}t} dt = \delta\left(F(t_2) - F(t_1)\right) = 0$$

which implies that

$$L - \mathcal{L} = \frac{\mathrm{d}F}{\mathrm{d}t}$$

with F called the **generating function**. It should have $2n+1$ independent variables. There are four types of generating functions

$$F_1 = F(q_i, Q_i, t)$$
$$F_2 = F(q_i, P_i, t)$$
$$F_3 = F(p_i, Q_i, t)$$
$$F_4 = F(p_i, P_i, t)$$

going back to transformed lagrangian, we take $L = \mathcal{K} + \frac{\mathrm{d}F}{\mathrm{d}t} = \sum p\dot{q} - H$, we can write this as

$$\sum p\dot{q} - H = \sum P\dot{Q} - \mathcal{H} + \frac{\mathrm{d}F}{\mathrm{d}t}$$

0.1.1 Type 1 Generator

This gives that

$$\frac{\mathrm{d}F_1}{\mathrm{d}t} = \frac{\partial F_1}{\partial q}\dot{q} + \frac{\partial F_1}{\partial Q}\dot{Q} + \frac{\partial F_1}{\partial t}$$

This can be expressed as

$$\sum p\dot{q} - \sum P\dot{Q} - H + \mathcal{H} = \sum \frac{\partial F_1}{\partial q}\dot{q} + \sum \frac{\partial F_1}{\partial Q}\dot{Q} + \frac{\partial F_1}{\partial t}$$

which gives that

$$p_i = \frac{\partial F_1}{\partial q_i}$$
$$P_i = -\frac{\partial F_1}{\partial Q}$$
$$\mathcal{H} = H + \frac{\partial F_1}{\partial t}$$

0.1.2 2 Examples

Coordinate Swap take $F_1(q, Q) = qQ$. Then

$$p_i = \frac{\partial F}{\partial q} = Q$$
$$P_i = -\frac{\partial F}{\partial Q} = -q$$

Simple Harmonic Oscillator Recall $L = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2$, which gives

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \to q = \frac{p}{m}$$
$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$$
$$\omega^2 = \frac{k}{m}$$

Let's try this:

$$p = f(P)\cos(Q)$$
$$q = \frac{f(P)}{m\omega}\sin(Q)$$

if we put these into the old hamiltonian, we get

$$p^2 + m^2\omega^2 q^2 = f(P)^2\cos^2(Q) + f(P)^2\sin^2(Q) = f(P)^2$$

which gives our new hamiltonian as

$$\mathcal{H} = \frac{f(P)^2}{2m}$$

Let's get a new type one generator as defined above, so that

$$p = \frac{\partial F}{\partial q}$$
$$P = -\frac{\partial F}{\partial Q}$$

Carrying on with the hamiltonian that we already have, we get

$$p = m\omega q \cot Q = \frac{\partial F}{\partial q}$$

which gives that

$$F = \int p dq = \frac{1}{2}m\omega q^2 \cot Q$$

it is also easy to see that $p = -\frac{\partial F}{\partial Q} = \frac{1}{2}m\omega^2 q^2 \frac{1}{\sin^2(Q)}$,

$$q = \sqrt{\frac{2P}{m\omega}}\sin(Q)$$
$$p = m\omega\sqrt{\frac{2P}{m\omega}}\cos(Q)$$

putting this all into the hamiltonian, we have

$$H = \frac{1}{2m}\left[2Pm\omega\cos^2 Q + m\omega 2P\sin^2 Q\right]$$

which gives, cancelling out, that

$$\mathcal{H} = \omega P$$

since we knnow it doesn't depend on time, we can write that

$$E \equiv \mathcal{H} = \omega P$$

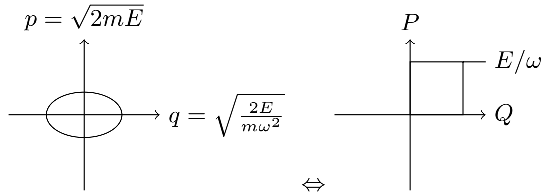
so $P = \frac{E}{\omega}$, or energy per unit angular momentum, so we have that $\dot{Q} = \omega$, which gives

$$Q = \omega t + Q_0$$

Putting these back into the original solution, we find that

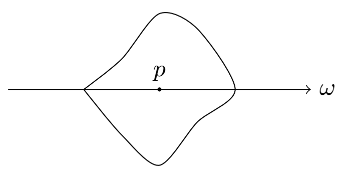
$$p = \sqrt{2mE}\cos(\omega t + Q_0)$$
$$q = \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + Q_0)$$

Let's look at phase space!



1 Rigid Body Motion

Definition 1.1. A rigid body is a body in which the mass elements are fixed with respect to one another.



We have $\vec{L} = \vec{r} \times \vec{p}$, and $p = m\omega$, with point p rotating at an angle θ away from ω, and momentum p at $r' = r\sin\theta$ along that vector Some mass element δm, we get $\delta\vec{L} = \vec{r} \times \delta\vec{p} = \vec{r} \times \vec{v}\delta m$, which gives

$$\vec{L} = \int \delta m(\vec{r} \times (\vec{\omega} \times \vec{r})$$

or for discrete mass elements, we have

$$\vec{L} = \sum_i m_i \vec{r}_i \times (\omega \times \vec{r}_i)$$

Then, we actually do the cross product out, we get

$$\vec{\omega} \times \vec{r} = (\omega_2 z - \omega_3 y)\hat{x} + (\omega_3 x - \omega_1 z)\hat{y} + (\omega_1 y - \omega_2 x)\hat{z}$$

, so the whole thing comes out to be, after crossing with r again,

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} \int (y^2 + z^2) dm & -\int xydm & -\int zxdm \\ -\int xydm & \int (z^2 + x^2) dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int (x^2 + y^2) dm \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_4 \end{bmatrix}$$

with that big ol matrix defined as the *Inertia Tensor*, $\vec{I}$. Inertia Tensor has a couple of properties

- Symmetric and Positive Definite.
- Depends only on the *shape* of the system, not ω.
- Can only be calculated after choosing an origin and coordinate system.
- Is diagnoalizable.

We could also write it in the following way,

$$I_{ij} = \int_{\text{all } V} \rho(\vec{r}) \left(\delta_{ij} \sum_k (x_k)^2 - x_i x_j \right) dV$$

1.1 ex: Point mass in a plane

Some mass orbiting $\hat{z}$ in the $x-y$ plane, m, wiht $\omega = (0, 0, \omega_3)$ and $x^2 + y^2 = r^2$, we have

$$\vec{L} = \begin{bmatrix} \int y^2 & -\int xy & 0 \\ -\int xy & \int x^2 & 0 \\ 0 & 0 & \int (x^2+y^2)dm \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_3 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ \omega_3 \int (x^2+y^2)dm \end{bmatrix}$$

which just reduces to $\vec{L} = mr^2\omega_3\hat{z} = mvr\hat{z}$, which is what we expected anyways. There's also the parallel axis theorem, which we will discuss later.

1.2 Kinetic Energy

We can also examine the kinetic energy, which is given by

$$dT = m\frac{v^2}{2} = \frac{dm|\vec{\omega} \times r|^2}{2}$$
$$T = \frac{1}{2}\int \left((\omega_2 z - \omega_3 y)^2 + (\omega_3 x - \omega_1 z)^2 + (\omega_1 y + \omega_3 x)^2\right) dm$$
$$= \frac{1}{2}\vec{\omega} \cdot (\vec{I} \cdot \vec{\omega}) = \frac{1}{2}\vec{\omega} \cdot \vec{L}$$

1.3 Center of Mass Coordinates

Say $r = \vec{R} + \vec{r}'$, where $\vec{R}$ goes from origin to center of mass. we have

$$\vec{L} = \int dm(\vec{r} \times \vec{v}) = \int \left((\vec{R} + \vec{r}') \times (\vec{r}' + (\vec{\omega} \times \vec{r}'))\right)$$
$$(\vec{R} + \vec{r}') \times (\vec{V} + (\vec{\omega} \times \vec{r}') = \vec{r'} \times \vec{v}' + \vec{r}' \times (\vec{\omega} \times \vec{r}')$$

which gives that

$$\vec{L} = m\vec{R} \times \vec{V} + \vec{L}_{cm}$$

We can also do this for KE, which would give us that

$$T = \frac{1}{2}\int dmV^2 = \frac{1}{2}\int dm|\vec{V} + (\vec{\omega}' \times \vec{r}')|^2$$
$$= \frac{1}{2}MV^2 + \frac{1}{2}\vec{\omega}' \cdot \vec{L}_{cm}$$

1.4 Principal Axes

Goal is to diagonalize the inertial tensor.

$$\vec{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

where I_1, I_2, I_3 are defined as the principal moments. This kind of thing should be somewhat familiar from freshman mechanics classes, since we have something like a plate in $\mathbb{R}^3$, ortthogonal to the z axis, the cross terms like $xydm$ cancel in the inertia tensor, but you get and eigenvalue problem. We want $\vec{L} = \vec{I} \cdot \vec{\omega_1} = I_1\omega_1$, or ω_1 lies along a principal axis. Want

$$\det \left| \begin{bmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} - I & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} - I \end{bmatrix} \right| = 0$$

1.4.1 Todo next lecture