Physics-105-Lecture-Notes-04-23-2019

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Contents

Abstract

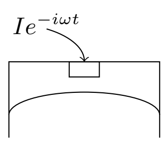
A single PDF with all lectures in a single document can be downloaded at `https://www.dropbox.com/sh/8sqzvxghvbjifco/AAC9LoSRnsRQDp7pYedgWpQMa?dl=0`. The password is 'analytic.mech.dsp'. This file was automatically generated using a script, so there might be some errors. If there are, you can contact me at `mailto:ctdunc@berkeley.edu`.

1 Nonlinear Mechanics + Chaos

You're gonna need to read the chapter and code to solve the majority of these problems. There are only a few of these that we can solve and understand analytically.

1.1 Van der Pol Oscillator

This is a van der pol oscilator.



1.2 Duffing's Oscillator

two springs attached to a mass in a plane with relaxed length l_0. So we have

$$U = \frac{1}{2}k(\sqrt{x^2+l^2}-l_0)^2 2$$

which we expand to be (taylor about 0)

$$\begin{aligned}
\frac{\partial U}{\partial x} = 2k(\sqrt{l^2+x^2}-l_0)\left(\frac{1}{2}(l^2+x^2)^{1/2}2\lambda\right) &= 2k\left(x-\frac{xl_0}{\sqrt{l^2+x^2}}\right)\\
\frac{\partial^2 U}{\partial x^2} &= 2k\left(1-\frac{l_0l^2}{(l^2+x^2)^{3/2}}\right)\\
\frac{\partial^3 U}{\partial x^3} &= 6kl_0l^2(x^2+l^2)^{-5/2}x\\
\frac{\partial^4 U}{\partial x^4} &= \frac{6kl_0}{l^3}
\end{aligned}$$

So, when we put in the right taylor expansion coefficients, the effective potential beomes

$$U(x) \approx U(0) + k\left(1-\frac{l_0}{l}\right)x^2 + \frac{1}{4}\frac{kl_0}{l^3}x^4 + \dots$$

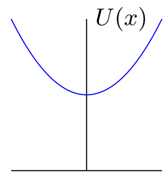
So the force becomes

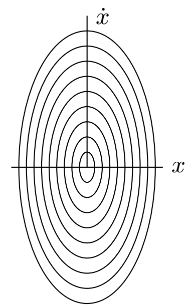
$$F = -\frac{\partial U}{\partial x} = -2k\left(1-\frac{l_0}{l^2}\right)x - \frac{kl_0}{l^3}x^3$$

There's a spring term, and a cubic term. So we want to solve the following differential equation

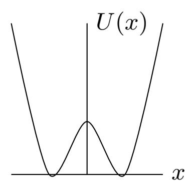
$$m\ddot{x} + 2\beta m\dot{x} + 2k\left(1 - \frac{l_0}{l}\right)x + \frac{kl_0}{l^3}x^3 = f(t)$$

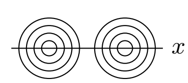
If we check this out in phase space, we look at





if we actially account for the cubic term in potential, however, we're going to get a taylor expansion that looks more like





i.e. there are multiple stable points in phase space. If we go back to solving this bad boi in generalit, we'll take

$$\ddot{x} + \frac{\dot{x}}{Q} + x + \epsilon x^3 = f\cos\omega t$$

Now, let's let $Q \to \infty$ so there's no damping, and fourier expand this,

$$x(t) = \sum_n A_n(\omega)\cos(n\omega t)$$

with the differential equation being now

$$\ddot{x} + x + \epsilon x^3 = f\cos(\omega t)$$

So now, we want to take a look at the harmonics so we have (even terms go away since they correspond to the sin components))

$$\begin{aligned} x(t) &= A_1\cos(\omega t) + A_3\cos(3\omega t) + \ldots \\ \dot{x} &= -A_1\omega\cos\omega t - 3\omega A_3\sin(3\omega t) \\ \ddot{x} &= -\omega^2 A_1\cos\omega t - 9\omega^2 A_3\cos\omega t \end{aligned}$$

Making use of the trig id that

$$\cos^3 x = \frac{3\cos(x) + \cos(3x)}{4}$$

we have our differential equation as

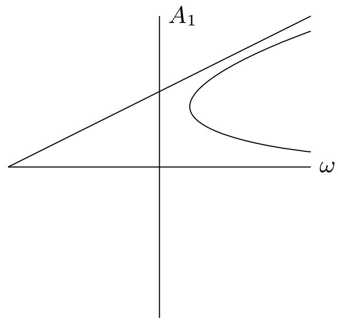
$$\begin{aligned} (1-\omega^2)A_1\cos\omega t + (1-9\omega^2)A_3\cos(3\omega t) + \ldots \\ +\tfrac{\epsilon}{4}(3A_1^3\cos\omega t + A_1^3\cos(3\omega t) + \ldots) \\ = f\cos\omega(t) \end{aligned}$$

if we group our coefficients by n, we get

$$(1-\omega^2)A_1 + \epsilon\frac{3}{4}A_1^3 = f$$

$$(1-9\omega^2)A_4 + \epsilon\frac{1}{4}A_1^3 = 0$$

So, graphically it looks a bit like



where solutions are given as intersections of these curves. If we put damping back in, we have (i.e. Q finite)

$$\begin{aligned} x &= a\cos\omega t + b\sin\omega t \\ \dot{x} &= -a\omega\sin\omega t + \omega b\cos\omega t \\ \ddot{x} &= -a\omega^2\cos\omega t - b\omega^2\sin\omega t = -\omega^2 x \end{aligned}$$

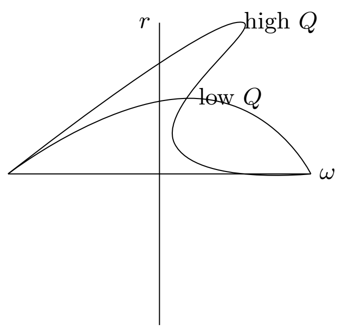
, which gives us an equation from our initial requirements that

$$\begin{aligned} a(1-\omega^2)\cos\omega t + b(1-\omega^2)\sin\omega t + \tfrac{b\omega}{Q}\cos\omega t + \tfrac{3\epsilon a r^2}{4}\cos\omega t \\ + \tfrac{3\epsilon b r^2}{4}\sin\omega t - \tfrac{a\omega}{Q}\sin\omega t \\ = f\cos\omega t \end{aligned}$$

which gives a solution for r^2 as a function of ω

$$r^2 = \frac{f^2}{\left(1-\omega^2+\frac{3\epsilon r^2}{4}\right)^2 + \frac{\omega^2}{Q^2}}$$

which gives you pictures that kind of look like this



this gives rize to a field of study called catastrophe theory, in which there are discontinuous changes in phase and r. This is like magnetic fields and hystereisis. Look this up in free time.