Abstract

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0.1 Physical Effects of Particle Statistics

0.1.1 Effect on Spectrum

Take some example hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1) + V(x_2)$$

or $H = H_1 + H_2$. Naiveley, we could just solve the schrodinger equation by looking at wavefunctions that are products of eigenstates of H_1, H_2. If we restrict to a square potential of finite magnitude, we have

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a} nx\right)$$

or, we can let

$$E_n = Kn^2; n = \frac{\pi^2\hbar^2}{2ma^2}$$

In the distinguishable case, we get then that

$$E_0^D = E_1 + E_1 = \frac{\pi^2\hbar^2}{ma^2}$$

If we look at E_1^D, there's a double degeneracy. Maybe it's useful to put this in a table

Energy	Degeneracy	WF
$E_0 = 2K$	1	$\psi_1\psi_1$
$E_1 = 5K$	2	$\psi_2\psi_1; \psi_1\psi_2$

If we have two **bosons**, however, we're going to get indistinguishable particles We're going to end up with

Energy	Degeneracy	WF
$E_0 = 2K$	1	$\psi_1\psi_1$
$E_1 = 5K$	1	$\frac{1}{\sqrt{2}}(\psi_1\psi_2 + \psi_2\psi_1)$

Fermions have yet a different spectrum. The lowest allowed energy level for a fermion is $5K$, since it must be antisymmetric. Wild stuff!

0.1.2 "Exchange Forces"

The effect of this statistics manifests itself as an inability of fermions to fill the same state, which causes them to space out more. It's NOT a force.[1] We want to check in on $\langle (x_1 - x_2)^2 \rangle_{(1,2)}$. Basically, we want to know what the expecation value of distinguishable particles, fermions, and bosons are when $E = 5K$.

Distinguishable Particles We can take

$$\langle 1,2| (x_1 - x_2)^2 |1,2\rangle = \langle 1| x_1^2 |1\rangle + \langle 2| x_2^2 |x\rangle - 2 \langle 1| x_1 |1\rangle \langle 2| x_2 |2\rangle$$

Since the final term contains the product of two odd integrals squared, it goes to zero, so we'll get

$$\langle 1,2| (x_1 - x_2)^2 |1,2\rangle = \langle 1| x_1^2 |1\rangle + \langle 2| x_2^2 |x\rangle$$

Bosons, Fermions This changes the calculation to be, collecting terms

$$\frac{1}{2} (\langle 1,2| \pm \langle 2,1|) (x_1 - x_2)^2 (|1,2\rangle \pm |1,2\rangle)$$

$$= \frac{1}{2} \left[\langle 1| x_1^2 |1\rangle + \langle 2| x_2^2 |2\rangle + \langle 2| x_1^2 |2\rangle + \langle 1| x_2^2 |1\rangle \pm \langle 1| x^2 |2\rangle \pm \langle 2| x_2^2 |1\rangle \mp \langle 1| x_1 |2\rangle \langle 2| x_2 |1\rangle \pm \langle 2| x_1^2 |1\rangle \pm\right]$$

Just a whole bunch of shit. Altman is going to write it down after cancelling, and what the heCK this is a lot of algebra. Cancels to (after noting $x_1 - x_2$ is really just $\hat{x} \otimes I - I \otimes \hat{x}$, so that our integrals cancel out)

$$\frac{1}{2} (\langle 1,2| \pm \langle 2,1|) (x_1 - x_2)^2 (|1,2\rangle \pm |1,2\rangle) = I_1 + I_2 \mp |\langle 1| x |2\rangle|^2$$

where the minus corresponds to bosons. ¡++¿

[1] TODO: Anyons are particles that have special statistics under the 2d operator called braiding. This sounds hella cool.