Abstract

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0.1 Guest Discussion while Altman out of town (9/16/19)

0.1.1 System of Two Spin-1/2 Particles

Discussion of this motivated by desire to analyze hyperfine splitting in the hydrogen atom. We begin with discussion of the hamiltonian

$$\hat{H} = \frac{2A}{\hbar^2} S_1 \cdot S_2$$

Hilbert space is given by the tensor product of our two spin-1/2 systems. For two bases, $|\uparrow\rangle, |\downarrow\rangle$, we have our new space as $|\uparrow\downarrow\rangle, |\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle$. We can also write our new hamiltonian as

$$2S_1 \cdot S_2 = S_1^+ S_2^- + S_1^- S_2^+ + 2S_1^z S_2^z$$

where

$$S_{1,2}^{\pm} = S_{1,2}^x \pm iS_{1,2}^y$$

It can be checked explicitly that $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle$ are still eigenstates of the hamiltonian, since. We want to know then what

$$\langle\uparrow\downarrow| S_1^+ S_2^- + S_1^- S_2^+ + 2S_1^z S_2^z |\uparrow\downarrow\rangle$$

By symmetry arguments, a lot of these coefficients go to zero, so we get $\frac{-A}{2}$ for both oriientations on the diagonal, and A on the off. In total, the logic gives the following hamiltonian

$$\hat{H} = \begin{bmatrix} \frac{A}{2} & 0 & 0 & 0 \\ 0 & \frac{-A}{2} & A & 0 \\ 0 & A & \frac{-A}{2} & 0 \\ 0 & 0 & 0 & \frac{A}{2} \end{bmatrix}$$

We can get the eigenvalues, vectors of the central square matrix by taking

Spin Triplet	$\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle + \lvert\downarrow\uparrow\rangle)$	$\frac{A}{2}$
Spin Singlet	$\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle)$	$\frac{-3A}{2}$

Now, we want to see if its possible to invent some symmetry that commutes with our hamiltonian. It seems like it should be symmetric by rotation about $\hat{n}$ by an angle $\delta\theta$. We can write down $R(\delta\theta)$, which is generated by $\vec{S} \cdot \vec{n}$, which gives

$$R(\delta\theta) = e^{-i(\vec{S}\cdot\vec{n})\delta\theta}$$

We can show that $[S \cdot \hat{n}, H] = 0$. What it means to take $S \cdot \hat{n}$ in our tensor product space is

$$S \cdot \hat{n} = S_1\hat{n} \otimes I + I \otimes S_2\hat{n}$$

[1] [2] These all fulfill the basic requirements of the spin algebra,

$$[S_x, S_y] = iS_z$$

and cyclic permutations thereof. Alternately,

$$[S_\alpha, S_\beta] = i\varepsilon_{\alpha\beta\gamma} S_\gamma$$

There's another operator

$$\hat{S}^2 = S_x^2 + S_y^2 + S_z^2 \qquad [\hat{S}^2, H] = 0$$

Theres another way to write this as

$$\begin{aligned} S^2 &= (S_{1x} + S_{2x})^2 + (S_{1y} + S_{2y})^2 + (S_{1z} + S_{2z})^2 \\ &= S_1^2 + S_2^2 + 2S_1 \cdot S_2 \end{aligned}$$

All of those individually commute with the hamiltonian, since $[S_1^2, S_1 \cdot S_2] = 0$, and same for S_2. We call S^2 the *total spin operator*. With $S_{x,y,z}, S^2$ the total spin operators. Now, check the eigenstates of the hamiltonian are eigenstates of the spin operators.

	$\hat{H}$	$\hat{S}^2$	$\hat{S}_z$
$\lvert\uparrow\uparrow\rangle$	$A/2$	$2\hbar^2$	$\hbar$
$\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle + \lvert\downarrow\uparrow\rangle)$	$-A/2$	$2\hbar^2$	0
$\lvert\downarrow\downarrow\rangle$	$A/2$	$2\hbar^2$	$-\hbar$
$\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle)$	$-A/2$	0	0

We can see the first 3 are the spin-1 system, and the final system the spin-0.

0.1.2 Problems 11.16, 11.18 in Townsend

11.16 Considering $H_{\text{Hydrogen}} + \frac{\gamma}{r}$.

[1] Is this basically JCF?
[2] TODO: Write out spin operators for this state explicitly