Math 222B Notes

Connor Duncan

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Contents

1 Sobolev Spaces

The reference for this section is Evans Chapter 5., and Sung-Jin Oh's 222A lecture notes, section 11.

1.1 Introduction to Sobolev Spaces

We begin with an introduction to the basics of Sobolev Spaces.

Definition 1. Let U be an open subset of $\mathbb{R}^d$, and $u \in \mathcal{D}'(U)$. The k^{th} order L^p-based Sobolev Norm of u is defined as:

$$||u||_{W^{k,p}U} = \left[\sum_{\alpha:|\alpha|<k} ||D^\alpha u||^p_{L^p(U)} \right]^{1/p}$$

Here, D^α is the weak derivative, and $D^\alpha u \in L^p(U)$.[1]

Remark. The sum over $||D^\alpha u||_{L^p}$ is motivated by its appearance in energy-method solutions to PDE's found in 222A.

Definition 2. The L^p-Sobolev space of order k on U is defined as

$$W^{k,p}(U) = \{u \in \mathcal{D}'(U) : ||u||_{W^{k,p}(U)} < \infty\}$$

Similarly, we call the subspace of $W^{k,p}(U)$ which vanish to appropriate order on the boundary

$$W^{k,p}_0(U) = \overline{C^\infty_c(U)}^{||\cdot||_{W^{k,p}(U)}} \subset W^{k,p}(U)$$

[1] This is equivalent to the other definition given in lecture: $||u||_{W^{k,p}(U)} = \sum_{\alpha:|\alpha|<k} ||D^\alpha u||_{L^p(U)}$.

When $p = 2$, we have many extra analytical tools, since the Fourier transform is an L^2 isometry. This justifies special notation for this case.

Notation. We denote by $H^k(U) = W^{k,2}(U)$, and $H^k_0(U) = W^{k,2}_0(U)$.

We also have a special notation for inequalities that hold up to a multiplicative constant.

Notation. If, for some $c > 0$, $A \leq cB$, then $A \lesssim B$. If $A \lesssim B$ and $B \lesssim A$, then $A \simeq B$.

Prop 1.1. *Some basic facts about $W^{k,p}(U)$ and $H^k(U)$.*

i. For all $k \in \mathbb{Z}_{\geq 0}$ and $1 \leq p \leq \infty$, $(W^{k,p}(U), ||\cdot||_{W^{k,p}(U)})$ and $(W^{k,p}_0(U), ||\cdot||_{W^{k,p}})$ are Banach spaces.

ii. For all $k \in \mathbb{Z}_{\geq 0}$, and denoting $\langle u, v\rangle_{H^k(U)} = \sum_{\alpha:|\alpha|\leq k}\langle D^\alpha u, D^\alpha v\rangle_{L^2(U)}$, both $(H^k(U), \langle\cdot,\cdot\rangle_{H^k(U)})$ and $(H^k_0(U), \langle\cdot,\cdot\rangle_{H^k(U)})$ are Hilbert Spaces.

iii. (Fourier Analytic Characterization of H^k). If $u \in H^k(U)$, then $||u||_{H^k} \simeq ||\hat{u}||_{L^2} + |||\xi|^k\hat{u}||_{L^2} \simeq ||(1+|\xi|^2)^{k/2}\hat{u}||_{L^2}$.

Proof. For (i), we first check that $||\cdot||_{W^{k,p}}$ is a norm. The triangle inequality may be verified by the elementary calculation, where $u, v \in W^{k,p}$:

$$
\begin{aligned}
||u+v||_{W^{k,p}} &= \left[\sum_{\alpha:|\alpha|\leq k} ||D^\alpha(u+v)||^p_{L^p}\right]^{1/p} \\
&\leq \left[\sum_{\alpha:|\alpha|\leq k} (||D^\alpha u||_{L^p} + ||D^\alpha v||_{L^p})^p\right]^{1/p} \\
&\leq \left[\sum_{\alpha:|\alpha|\leq k} ||D^\alpha u||^p_{L^p}\right]^{1/p} + \left[\sum_{\alpha:|\alpha|\leq k} ||D^\alpha v||^p_{L^p}\right]^{1/p} \\
&= ||u||_{W^{k,p}} + ||v||_{W^{k,p}}
\end{aligned}
$$

The first inequality follows from the fact that $||\cdot||_{L^p}$ is a norm. It is obvious that $||\lambda u||_{W^{k,p}} = |\lambda|||u||_{W^{k,p}}$.

It remains to check that $W^{k,p}$ is complete. Let $\{f_n\}_1^\infty$ be a Cauchy sequence in $W^{k,p}$. By definition, every $D^\alpha f_i \in L^p$, and since L^p is complete every $D^\alpha f_i$ converges to a function $f_\alpha \in L^p$. So, the claim is that when $\alpha = (0,\ldots,0)$, we have convergence in L^p: $f_m \to f_{(0,\ldots,0)} := f \in W^{k,p}$. To see that $f \in W^{k,p}$, fix a test function $\phi \in C^\infty_0$, and integrate

$$
\begin{aligned}
\int f D^\alpha \phi dx &= \lim_{n\to\infty} \int f_n D^\alpha \phi dx \\
&= \lim_{n\to\infty} (-1)^{|\alpha|} \int (D^\alpha f_n)\phi dx \\
&= (-1)^{|\alpha|} \int f_\alpha \phi dx
\end{aligned}
$$

This shows that every Cauchy sequence of functions and all derivatives of index $|\alpha| < k$ converge in L^p, which proves convergence in $W^{k,p}$. For $W^{k,p}_0$, we need only check that f_α is compactly supported for all α. This is easily accomplished by replacing ϕ with $\varphi \in C^\infty$, and repeating the calculaton.

For (ii), we first check that $\langle\cdot,\cdot\rangle_{H^k}$ is an inner product. Letting $a, b \in \mathbb{C}$, and $f, g, h \in H^k$, we have

$$\begin{aligned}\langle af+bg, h\rangle_{H^k} &= \sum_{\alpha:|\alpha|\le k}\langle D^\alpha(af+bg), D^\alpha h\rangle_{L^2} \\ &= \sum_{\alpha:|\alpha|\le k} a\langle D^\alpha f, D^\alpha h\rangle_{L^2} + b\langle D^\alpha g, D^\alpha h\rangle_{L^2} \\ &= a\langle f,h\rangle_{H^k} + b\langle g,h\rangle_{H^k}\end{aligned}$$

That $\langle y,x\rangle_{H^k} = \overline{\langle x,y\rangle_{H^k}}$ follows from the same fact for the L^2 inner product, as does positivity. Completeness follows from (i), which shows that $(H^k, \langle\cdot,\cdot\rangle_{H^k})$ is a Hilbert space.

For (iii), we have to use some properties of the Fourier transform proved in 222A. Let $u \in H^k$. Since $u \in L^2$, we may write that $\widehat{(D^\alpha u)} = ((i\xi)^\alpha \hat{u})$. Clearly, we have that $||u||_{H^k} \ge C||\hat{u}||_{L^2} + |||\xi|^k\hat{u}||_{L^2}$, since the latter quantity contains only some of the terms present in the H^k norm. That $||\hat{u}||_{L^2} + |||\xi|^k\hat{u}||_{L^2} \ge C||(1+|\xi|^2)^{k/2}\hat{u}||_{L^2}$ follows from the Cauchy-Schwarz inequality and because $|\xi| > 0$, we have $(1+|\xi|^2)^{k/2} \le C(1+|\xi|^2)^{k/2}$. From this, the chain of inequalities (choosing appropriate C so that all constant-dependent inequalities still hold) reads as

$$||u||_{H^k} \ge C\left(||\hat{u}||_{L^2}^2 + |||\xi|^k\hat{u}||_{L^2}^2\right)^{1/2} \ge C||(1+|\xi|^k)\hat{u}||_{L^2} \ge ||(1+|\xi|^2)^{k/2}\hat{u}||_{L^2}$$

All that remains is to show that $||(1+|\xi|^2)^{k/2}\hat{u}||_{L^2} \ge C||u||_{H^k}$. To see this, note that $(1+|\xi|^2)^{k/2} = \sum_{j=0}^{k/2} c_j|\xi|^{2j}$, pick the smallest $C = c_j$, and doing this once more for the sum that appears,

$$||(1+|\xi|^2)^{k/2}\hat{u}||_{L^2}^2 \ge C||\sum_{j=0}^{k}|\xi|^j\hat{u}||_{L^2}^2 \ge C'\sum_{\alpha:|\alpha|\le k}||D^\alpha u||_{L^2}^2$$

Taking a square root completes the proof. □

Naturally, for a vector space like $W^{k,p}(U)$, we ask what the dual of this space is. By an appropriate definition, we can characterize the dual as being a Sobolev space of negative order.

Definition 3. For $k \in \mathbb{Z}_{\ge 0}$, $1 < p < \infty$, and U and open subset of $\mathbb{R}^d$, we define

$$||u||_{W^{-k,p}(U)} = \inf\left\{\sum_{\alpha:|\alpha|<k}||g_\alpha||_{L^p(U)} : u = \sum_{\alpha:|\alpha|<k} D^\alpha g_\alpha\right\}$$

and

$$W^{-k,p}(U) = \left\{u \in \mathcal{D}'(U) : u = \sum_{\alpha:|\alpha|<k} D^\alpha g_\alpha, g_\alpha \in L^p(U)\right\}$$

Remark. If $g \in L^p(U)$, then $D_{x^i}g \in W^{-1,p}(U)$. If $g \in W^{k,p}(U)$, then $D_{x^i}g \in W^{k-1,p}(U)$. In essence, we can characterize $W^{-k,p}$ as the space of L^p functions weakly differentiated up to k times.

With this in mind, we are able to prove the following proposition.

Prop 1.2. *For $k \in \mathbb{Z}_{\ge 0}$, $1 \le p \le \infty$, and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$,*

$$(W_0^{k,p}(U))^* \simeq W^{-k,p'}(U)$$

Proof. We show first that $(W_0^{k,p})^* \supseteq W^{-k,p'}(U)$. Let $v \in W^{-k,p'}(U)$, and $u \in W_0^{k,p}(U)$. By definition, we may write $v = \sum_{\alpha:|\alpha|<k} D^\alpha g_\alpha$. Testing v against u, we find that

$$\begin{aligned}\langle v, u\rangle = \int_U v u dx = \sum_{\alpha:|\alpha|<k} \int_U D^\alpha g_\alpha u dx &= \sum_{\alpha:|\alpha|<k} \int_U (-1)^\alpha g_\alpha D^\alpha u dx \\ &\leq \sum_{\alpha:|\alpha|<k} ||g_\alpha||_{L^{p'}} ||D^\alpha u||_{L^p} \\ &\leq c||v||_{W^{-k,p'}} ||u||_{W^{k,p}}\end{aligned}$$

Here, the third equality follows by integrating by parts, using the fact that $\mathrm{spt}(u)$ is compact in U to disregard boundary terms. The first inequality is a direct application of Hölder's inequality (A.4). The second is the definition of the Sobolev norm, aggregating the constants from each term into c. Thus, we have shown that every v in $W^{-k,p'}(U)$ is a bounded linear functional on $W_0^{k,p}$, i.e. an element of the dual space.

To show that $(W_0^{k,p})^* \subset W^{-k,p'}$, we apply the Hahn-Banach theorem (A.2). First, define the bounded linear functional $\ell : W_0^{k,p} \to \mathbb{R}$, and let $u \in C_0^\infty(U)$, with the end goal that $\ell(u) = \langle v, u\rangle = \sum_{\alpha:|\alpha|<k} (-1)^\alpha \langle g_\alpha, D^\alpha u\rangle$. To that end, we define (where $K(k)$ is the total number of multi-indices up to order k):

$$\begin{aligned} T : \quad & C_0^\infty(U) \to L^p(U)^{K(k)} \\ & u \mapsto (u, D_{x^1} u, \ldots, D_{x^d} u, \ldots D^\alpha u)\end{aligned}$$

We have that $||T(u)|| \leq c||u||_{W^{k,p}}$. Furthermore, T is injective, and an isomorphism onto its image, i.e., $(C_0^\infty(U), ||\cdot||_{W^{k,p}}) \sim (T(C_0^\infty(U)), ||\cdot||)$. So, we may send ℓ to $\tilde{\ell} : T(C_0^\infty(U)) \to \mathbb{R}$ by composing with this isomorphism. In particular, $\tilde{\ell}(Tu) = \ell(u)$ tells us that $\tilde{\ell}$ is similarly bounded. Now, by the Hahn-Banach theorem, $\tilde{\ell}$ extends to the bounded linear functional $\tilde{\tilde{\ell}} : (L^p(U))^{\otimes K} \to \mathbb{R}$. By definition, $\tilde{\tilde{\ell}} \in \left((L^p(U))^{\oplus K}\right)^* = \{\tilde{v} = \sum_{\alpha:|\alpha|<k} \tilde{g}_\alpha : \tilde{g}_\alpha \in L^{p'}(U)\}$. So, for some $\tilde{u} \in (L^p(U))^{\oplus K}$, the pairing with $\tilde{v}$ is exactly $\langle \tilde{v}, \tilde{u}\rangle = \sum_{\alpha:|\alpha|<k} \langle g_\alpha, u_\alpha\rangle$, where $\tilde{u}_\alpha = D^\alpha u$. So,

$$\ell(u) = \tilde{\ell}(Tu) = \tilde{\tilde{\ell}}(Tu) = \sum_{\alpha:|\alpha|<k} \langle \tilde{g}_\alpha, (Tu)\alpha\rangle = \sum_{\alpha:|\alpha|<k} \langle \tilde{g}_\alpha, D^\alpha u\rangle$$

If we choose $g_\alpha = (-1)^{|\alpha|}\tilde{g}_\alpha$, we have shown the remainder of the proof. □

1.2 Existence and Uniqueness Problems

The concrete objective of this section is to explore the duality relationship between the existence and uniqueness of solutions to linear equations on Banach spaces. In particular, apriori estimates of the dual problem prove the existence of solutions to the direct problem, and vice-versa (under certain conditions).

Prop 1.3. *Let X, Y be Banach Spaces, and let $P : X \to Y$ be a bounded linear operator. Likewise, let $P^* : Y^* \to X^*$ be the adjoint of P. Suppose that there exists $c > 0$ such that $||u||_X \leq c||Pu||_Y$ for all $u \in X$. Then the following hold:*

i. (Uniqueness for $Pu = f$) If $u \in X$, $Pu = 0 \Rightarrow u = 0$.

ii. (Existence for $P^*v = g$) For all $g \in X^*$, there exists $v \in Y^*$ such that $P^*v = g$, and $||v||_{Y^*} \leq c||g||_{X^*}$.

Proof. The proof of (i) is clear, since $||u||_X \leq 0$, $u = 0$ since X is normed.

As for (ii), we again apply the Hahn-Banach theorem. In particular, our objective is to find $v \in Y^*$ such that for all $u \in X$: $P^*v = g \Leftrightarrow \langle P^*v, u\rangle = \langle g, u\rangle = \langle v, Pu\rangle$. To that end, define $\ell : P(X) \to \mathbb{R}$, where $\ell(Pu) = \langle g, u\rangle$. Since P is injective by (i), ℓ is well-defined. By definition, if $||Pu||_Y \leq 1$, we have that

$$|\ell(Pu)| = |\langle g, u\rangle| \leq ||g||_{X^*}||u||_X \leq c||g||_{X^*}||Pu||_Y \leq c||g||_{X^*}$$

So, by the Hahn-Banach theorem, there exists $v \in Y^*$ such that $\langle v, Pu\rangle = l(Pu) = \langle g, u\rangle$ for all $u \in X$, and $||v||_{Y^*} \leq c||g||_{X^*}$. $\square$

Definition 4. Let X be a normed vector space with member x, and let $\hat{x} : X^* \to \mathbb{C}$ denote $\hat{x}(f) = f(x)$. Let $\hat{X} = \{\hat{x} : x \in X\}$. X is called **reflexive** if $\hat{X} = X^{**}$.

If we want existence for the direct problem, we take the easy way, and assume X is reflexive, which yields Proposition 1.4. In general, $\hat{X} \subseteq X^{**}$.

Prop 1.4. *Let X, Y be Banach Spaces, with X reflexive, and Let $P : X \to Y$ be a bounded linear operator. Likewise, let $P^* : X^* \to Y^*$ be the adjoint of P. Suppose that there exists $c > 0$ such tht $||v||_{Y^*} \leq c||P^*v||_{X^*}$. Then the following hold:*

i. (Uniqueness for $P^*v = g$) If $v \in Y^*$, $P^*v = 0 \Rightarrow v = 0$.

ii. (Existence for $Pu = f$) For all $f \in Y$, there exists $u \in X$ such that $Pu = f$, and $||u||_X \leq c||f||_Y$.

Proof. Exercise. $\square$

Remark. All Sobolev Spaces $W_0^{k,p}$ for $1 < p < \infty$ are reflexive. This will be a homework problem.

Notation. Let $P : X \to Y$ be a linear operator, and P^* its associated adjoint. With $U \subset Y$, and $V \subset X^*$, we define the following sets:

$$U^\perp = \{v \in Y^* : \langle v, f\rangle = 0 \, \forall f \in U\}$$
$$^\perp V = \{u \in X : \langle g, u\rangle = 0 \, \forall g \in V\}$$

Remark. $\text{range}(P)^\perp = \ker(P^*)$, and $\ker(P) = ^\perp \text{range}(P^*)$. As a consequence of this fact, if $\ker P^* = \{0\}$, then $\text{range}(P)^\perp = \{0\} \Leftrightarrow \overline{\text{range}(P)} = Y$.

It's worth noting that in finite dimensions, $\overline{\text{range}(P)} = \text{range}\, P$, which provides the simpler duality relation between uniqueness and existence of solutions for linear operators. In infinite dimensions, this does not always hold, which is what our boundedness estimate provides. There is no loss of generality for deriving existence for P from the qualitative bound

$$||v||_{Y^*} \leq c||P^*v||_{X^*} \tag{1}$$

Notation. We denote by $B_X = \{x \in X : ||x||_X < 1\}$.

Prop 1.5. *Let X, Y be Banach Spaces, and $P : X \to Y$ a bounded linear operator. If $P(X) = Y$, then there exists $c > 0$ such that (1) holds.*

Proof. $||P^*v||_{X^*} = \sup_{\overline{B_X}} |\langle P^*v, u\rangle| = \sup_{\overline{B_X}} |\langle v, Pu\rangle|$. T is a surjective linear map between Banach spaces, and is therefore open by the Open mapping theorem (A.3). Thus, $P(B_X)$ is open and contains 0. Thus, there exists $c > 0$ such that $P(B_X) \supseteq cB_Y$, which implies

$$||P^*v||_{X^*} = \sup_{\overline{B_X}} |\langle P^*v, u\rangle| = \sup_{\overline{B}_X} |\langle v, Pu\rangle| \geq \sup_{f \in cB_Y} |\langle v, f\rangle| = c||v||_{Y^*}$$

which completes the proof. $\square$

Example. We now examine the solvability of the equation $-u'' = f$ in $H_0^1((0,1))$. Note that $||u||_{H^1}^2 = ||u||_{L^2}^2 + ||u'||_{L^2}^2$, and that $(H_0^1)^* = H^{-1}$. Using $X = H_0^1, Y = H^{-1}$, we consider $P = -\partial_x^2$, and claim that if $-u'' = f$ for $u \in H_0^1$, then

$$||u||_{H^1} \leq c||f||_{H^{-1}}.$$

The proof is an application of the energy method. A simple integration by parts yields that

$$\int -u'' u dx = \int f u dx = \int (u')^2 dx = ||u'||_{L^2}^2$$

To obtain the previous inequality from what we have just derived, we use the fact that u is zero on the boundary, and so $u(x) = \int_0^x u'(y)dy$. Using the Cauchy-Schwarz inequality,

$$|u(x)| \leq \int_0^1 |u'(y)|dy \leq ||u'||_{L^2}$$

which implies that

$$\int_0^1 |u|^2 dy \leq \sup_{[0,1)} |u|^2 \leq ||u'||_{L^2}^2,$$

in turn implying that

$$||u||_{H^1}^2 \leq c|\langle f, u\rangle| \leq c||f||_{H^{-1}} + ||u||_{H^1}$$

which completes the proof after dividing out a term $||u||_{H^1}$.

From this, we can deduce from the inequality above, and Proposition 1.3 that $-u'' = 0$ and $u \in H_0^1 \Rightarrow u = 0$. From the inequality and Proposition 1.4, we should compute P^*, and obtain existence for the dual problem. Explicitly, we ue the fact that H_0^1 is reflexive, and compute for $u \in H_0^1$, that

$$\langle v, Pu\rangle = \int_0^1 v(-u'')dx = \int_0^1 v'u'dx = \int_0^1 -v''udx = \langle P^*v, u\rangle$$

So, $P^* = -\partial_x^2$ as well, meaning the problem is entirely self-adjoint, and $Y^* = H_0$ 1. This gives that $\forall f \in H^{-1}$, there exists $u \in H_0^1$ such that $Pu = f$, by applying Proposition 1.4.

This hints at the Poincaré inequality, which we will explore shortly.

1.3 Approximation Theorems

1.3.1 Convolution and Mollifiers

Definition 5. Let $\varphi \in C_0^\infty(U)$, with $\int_U \varphi = 1$. We define the family of functions $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi\left(\frac{x}{\varepsilon}\right)$. Note that $\int_U \varphi_\varepsilon dx = 1$ for every ε.

Lemma 1.6. *Let $\varphi \in C_0^\infty$, with $\int \varphi dx = 1$, $u \in L^p(\mathbb{R}^d$ where $1 \leq p \leq \infty$, and φ_ε as in Definition 5. As $\varepsilon \to 0$, $||\varphi_\varepsilon \star u - u||_{L^p} \to 0$.*

Before proving the lemma, we note that translations are continuous in L^p.

Lemma 1.7. $\lim_{|z|\to 0} ||u(x-z) - u(x)||_{L^p} = 0$ *for $u \in L^p$.*

Proof. Clearly, the conditions of the dominated convergence theorem are satisfied, choosing u_n to be a sequence of functions $u_n(x) = u(x - z_n)$, where $z_n \to 0$. It suffices to dominate u_n by $v(x) = |u(x)| \cdot \mathbf{1}_{B_1(x)}$, under the assumption that z_n is a sequence which is of distance at most 1 from x. □

Proof.(Lemma 1.6) Consider:

$$\begin{aligned}\varphi_\varepsilon \star u - u &= \int_U u(x-y)\varphi_\varepsilon(y)dy - u(x) \\ &= \int_U (u(x-y) - u(x)\varphi_\varepsilon(y)dy\end{aligned}$$

Therefore,

$$\begin{aligned}||\varphi_\varepsilon \star u - u||_{L^p} &= \left\|\int_U (u(x-y) - u(x)\varphi_\varepsilon(y)dy\right\|_{L^p} \\ &\leq \int_U ||u(\cdot - y) - u(\cdot)||_{L^p}|\varphi_\varepsilon(y)|dy\end{aligned}$$

The inequality above is a direct application of the Minkowski inequality. First, we note that $\text{spt}(\varphi_\varepsilon) \to \{0\}$ as $\varepsilon \to 0$. Furthermore, by the L^p-continuity of translations, the entire integrand converges to zero. □

Definition 6. A **partition of unity** on an open set $U \subset \mathbb{R}^d$ is a family of functions $\{\chi_\alpha\}_{\alpha\in A}$ such that the following hold:

1. $\sum_{\alpha\in A} \chi_\alpha(x) = 1$ for all $x \in U$.
2. For every $x \in U$, only finitely many χ_α are nonzero at x.

If $\{U_\alpha\}_{\alpha\in A}$ is an open cover of U, $\{\chi_\alpha\}_{\alpha\in A}$ is called **subordinate to** $\{U_\alpha\}_{\alpha\in A}$ if $\text{spt}\,\chi_\alpha \subseteq U_\alpha$ for all α. If $\chi_\alpha \in C_0^\infty$, then $\{\chi_\alpha\}_{\alpha\in A}$ is called a **smooth partition of unity**.

Lemma 1.8. *Let $\{U_\alpha\}_{\alpha\in A}$ be an open covering of $U \subset \mathbb{R}^d$. Then there exists a smooth partition of unity subordinate to $\{U_\alpha\}_{\alpha\in A}$.*

Proof. Largely omitted. To see this, start from a continuous subordinate partition of unity, and mollify it until you get a smooth partition. □

1.3.2 Density Theorems

In what follows, we prove four density theorems, and an extension theorem. The goal is to provide tools for representing members $u \in W^{k,p}(U)$ by objects with prescribed smoothness and support properties.

Theorem 1.9. *Let $k \in \mathbb{Z}_{\geq 0}$, $1 \leq p < \infty$. Then*

i. $C^\infty(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$.

ii. $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$.

Proof. (i) is a rote application of mollifiers. (ii) will be homework. The main step is to approximate f by $f\chi(1/R)$, with $\chi \in C_c^\infty$, and $\chi(0) = 1$. □

Theorem 1.10. *Let $k \in \mathbb{Z}_{\geq 0}$, $1 \leq p < \infty$, and U be an open set in $\mathbb{R}^d$. Then $C^\infty(U)$ is dense in $W^{k,p}(U)$.*

Proof.
Let $u \in W^{k,p}(U)$, and fix $\epsilon > 0$. We want to find $v \in C^\infty(U)$ such that $||u - v||_{W^{k,p}(U)} \leq c\epsilon$. To that end, consider the family of opens $U_j = \{x \in U : \text{dist}(x, \partial U) < \frac{1}{j}\}$, and define $V_j = U_j \setminus \overline{U_{j+2}}$. Then, since $U \subseteq \bigcup_{j=1}^\infty V_j$, we may choose χ_j to be a partition of unity of U subordinate to V_j, and write

$$u = \sum_{j=1}^\infty u\chi_j := \sum_{j=1}^\infty u_j$$

Note that because $\text{spt}\,\chi_j \subseteq V_j$, $\text{spt}\,u_j \subseteq V_j$, Furthermore, $u_j \in C_0^\infty(\mathbb{R}^d)$, since it is smoothly extended by 0 outside V_j.

Now, we define a mollifier $\varphi \in C_0^\infty(\mathbb{R}^d)$, with the usual $\int \varphi dx = 1$, and $\text{spt}\,\varphi \subseteq B_1(0)$. This automatically gives us that $\text{spt}\,\varphi_{\varepsilon_j} \subseteq B_{\epsilon_j}(0)$, for prescribed ε_j. We prescribe ε_j by defining a new $v_j = \varphi_{\varepsilon_j} * u_j$, and choosing each ε_j so that $||u_j - v_j||_{W^{k,p}(U)} \leq 2^{-j}\epsilon$, and $\text{spt}\,v_j \subseteq \tilde{V}_j = U_{j-1} \setminus \overline{U_{j+2}}$. With these prescribed, we take $v = \sum_{j=1}^\infty v_j$. First, v is well-defined since $\tilde{V}_j$ is locally finite. Second, we compute

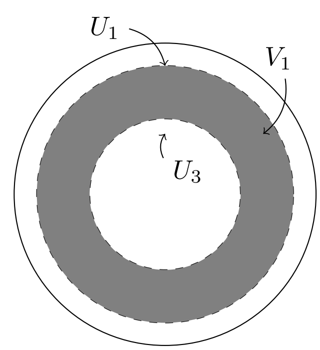
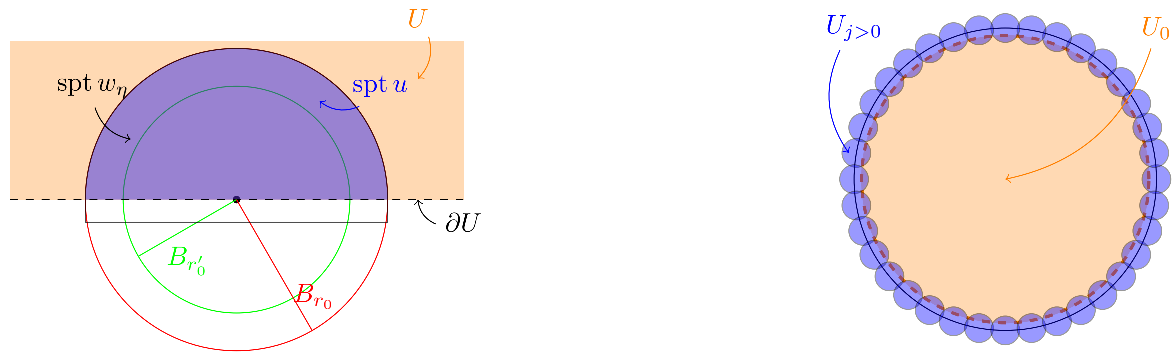


Figure 1: U_j and V_j in the proof of Theorem 1.10
.

$$||v - u||_{W^{k,p}(U)} \leq \sum_{j=1}^\infty ||v_j - u_j||_{W^{k,p}(U)} \leq 2^{-j}\epsilon = c\epsilon$$



(a) Illustration of Step 2.

(b) Open Cover of U

Figure 2: Illustration of the proof of Theorem 1.11

So, v has the desired convergence and smoothness properties, so we are done. $\square$

One issue with Theorem 1.10 is its lack of control over v near the boundary of U. We attempt to resolve this in the following theorem, but first, we define some terms.

Notation. $C^\infty(\overline{U}) = \{u : U \to \mathbb{R} : u \text{ is the restriction of a function } \tilde{u} \in C^\infty(\tilde{U}), \tilde{U} \supseteq \overline{U}\}$.

Definition 7. We say that ∂U is of class C^k if it is locally the graph of a C^k function.

Theorem 1.11. *Let $k \in \mathbb{Z}_{\geq 0}$, $1 \leq p \leq \infty$, and U a bounded open subset in $\mathbb{R}^d$, with ∂U of class C^1.*[2] *Then $C^\infty(\overline{U})$ is dense in $W^{k,p}(U)$.*

Proof. The proof proceeds in two steps. The first is to reduce the problem for a general U to a region where we may consider U to be the graph of a C^1 function. The second is to apply our approximation theorems to these simpler regions, before stitching the function back together.

By the definition of C^1-regularity, and the fact that U is bounded, we may cover ∂U by a finite family of open balls $\{B_{r_j}(x_j)\}_{j=1}^J$, in each of which U may by represented as the region above a C^1 graph. Calling $U_j = B_{r_j}(x_j)$, we choose an open set U_0 such that $U \supseteq U_0 \supseteq U \setminus \bigcup_{j=1}^J U_j$. Then $\{U_j\}_{j=0}^J$ is an open cover of U (which is illustrated in Figure 2b), so we may take $\{\chi_j\}_{j=0}^J$ to be a partition of unity subordinate to U_j, and–as in the proof of Theorem 1.10–write:

$$u = \sum_{j=0}^J u\chi_j := u_0 + \sum_{j=1}^J u_j$$

u_0 already has compact support, so we are free to apply the previous results. For $u_{j>0}$, we need to give a more explicit description of the boundary.

For this portion, we fix $u = u_j$, and make a change of coordinates so that we are centered at the origin, with r_0 and r_0' defined appropriately as in Figure 2a. We note that $\partial U = \{x^d = \Gamma(x^1, \ldots x^{d-1}\}$ possibly after a change of coordinates, where Γ is the C^k graph function. Letting e^d represent the unit coordinate vector in the direction of x^d, we make an approximation in two parts. First, define $w_\eta(x) = u(x + \eta e^d)$. Note that as $\eta \to 0$, by Lemma 1.7, $||u - w_\eta||_{W^{k,p}(U \cap B_{r_0})} < \frac{1}{2}\epsilon$ Moreover, w_η is defined on the set $B_{r_0'} \cap U - \eta e^d$. Second, we choose $v = \varphi_{r_0'} * w_\eta$, where φ is a mollifier. Then, for $r_0' \ll \eta$, v is well-defined on $B_{r_0'} \cap \{x^d > \Gamma(x^1, \ldots, x^{d-1})\}$, and $||v - w_\eta||_{W^{k,p}(U \cap B_{r_0})} < \frac{1}{2}\epsilon$. So, an application of the triangle rule gives that

$$||u - v||_{W^{k,p}(U)} \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon \leq \epsilon$$

[2]This can be relaxed to give U a Lipschitz boundary.

And, since $v \in C^\infty(\overline{V \cap \{x^d > \Gamma(x^1, \dots, x^d)\}})$, we are done. $\square$

1.3.3 Trace and Extension Theorems

Extension theorems can be roughly thought of as tools allowing us to handle $u \in W^{k,p}(U)$ when U is a bounded domain.

Theorem 1.12. *Let $k \in \mathbb{Z}_{\geq 0}$, $1 \leq p < \infty$, U a bounded domain in $\mathbb{R}^d$ with C^k boundary, and V be an open set containing $\overline{U}$. Then there exists $E : W^{k,p}(U) \to W^{k,p}(\mathbb{R}^d)$ such that the following hold:*

i. (Extension property) $Eu_{|U} = u$.

ii. (Linearity and Boundedness) E is linear, and $||Eu||_{W^{k,p}(\mathbb{R}^d)} \leq c||u||_{W^{k,p}(U)}$.

iii. (Support) $\operatorname{spt}(Eu) \subseteq V$.

Proof. Observe that by Theorem 1.11, and the fact that U is bounded, it suffices to consider $u \in C^\infty(\overline{U})$. The proof proceeds in two steps. First, we reduce to the half-ball case, and second, we prove extension for the half-ball case.

To reduce our problem to the case of the half-ball, it suffices to construct an open cover $\{U_0, \dots, U_J\}$ as in the proof of Theorem 1.11, with similarly constructed partition of unity $\{\chi_j\}_{j=0}^J$, and $u_j := u\chi_j$. Notably, $u_0 \in W^{k,p}(\mathbb{R}^d)$ since it is smoothly extended by 0, and $u_k \in C^\infty(\overline{U})$, and $\operatorname{spt} u_k \subseteq U_k \cap U$. After making a change of coordinates to $y^j = x^j$ for $1 \leq j < d$, and $y^d = x^d - \Gamma(x^1, \dots, x^{d-1})$, we see that $U_k \cap U \mapsto \{y \in B_{\tilde{r}}(0) : y^d > 0\} := \tilde{U}_k$, and $x \mapsto y$ is C^k, with smooth U_j. Therefore, applying the chain rule, we find that $u_j(y) = u_j(x(y))$ satisfies

$$||u_j(y)||_{W_y^{k,p}(\tilde{U}_j)} \leq c||u_k(x)||_{W_x^{k,p}(U_j \cap U)}$$

Thus, it suffices to consider the half-ball case.

The second step is to actually extend u in the case of the half-ball. Here, we define $U = B_r^+(0)$, and $W = B_{r/2}^+(0)$, such that $\operatorname{spt} u \subset W$. In order to extend u, we use the higher order reflection method, defining:

$$Eu = \tilde{u} = \begin{cases} u(x) & x^d > 0 \\ \sum_{j=0}^k \alpha_j u(x^1, \dots, x^{d-1}, -\beta_j x^d) & x^d < 0 \end{cases}$$

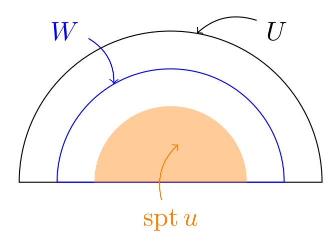


Figure 3: Illustration of the proof of Theorem 1.12.

Our objective here is to match up the normal derivatives of $\tilde{u}$ with u up to order k, i.e. set $u(x^1, \dots, x^{d-1}, 0+) = \sum_{j=0}^k \alpha_j u(x^1, \dots, x^{d-1}, 0-)$, and likewise for all derivatives $\partial_{x^d}^j u = (-\beta_j)^j \partial_{x^d} u$. This sets up the following matrix equation for our coefficients α and β.

$$\begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ -\beta_1 & \cdots & -\beta_1 \\ \vdots & \vdots & \vdots \\ (-\beta_k)^k & \cdots & (-\beta_k)^k \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \vdots \\ \vdots \\ \alpha_k \end{pmatrix}$$

This is the Vandermonde matrix, and if all β_j's are distinct, the matrix is invertible, which implies that the existence of $\alpha_1, \dots, \alpha_k$ such that the equation holds. The existence of such coefficients defines $\tilde{u}$ on $B_r(0)$, extending u and matching up derivatives to order C^k. Finally, to ensure extension to all of $\mathbb{R}^d$, we apply a cutoff function (a lá Urysohn's lemma) χ_V, such that $\chi_V = 1$ on U, and $\operatorname{spt} \chi_V \subset V$. $\square$

Now, we move on to discussing trace theorems, which essentially revolve around the restriction of functions $u \in W^{1,p}(U)$ to ∂U. This is interesting in part because the measure of $\mu(\partial U) = 0$, so using only L^p theory to deal with differentiability on the boundary gives little help, since L^p equivalence is almost everywhere.

Definition 8. Let $u \in C^1(\overline{U})$, and let U be a domain with C^1 boundary. Then the **trace of u on the boundary of U** is defined $\operatorname{tr}_{\partial U}(u) = u_{|\partial U}$.

Our objective is to first extend this definition to all of $W^{1,p}(U)$. We note that $\operatorname{tr}_{\partial U}$ is clearly linear, and will often write tr when ∂U is clear from context. Furthermore, whenever the L^p norm is used on a manifold of dimension less than the ambient space, it is assumed that integration is with respect to the volume fold of the manifold.

Theorem 1.13 (Nonsharp Trace Theorem). *Let U be a bounded, open subset of $\mathbb{R}^d$, with ∂U of class C^1, and $1 < p < \infty$. Then for $u \in C^1(\overline{U})$,*

$$||\operatorname{tr}_{\partial U} u||_{L^p(\partial U)} \lesssim ||u||_{W^{1,p}(U)}$$

As a consequence of this inequality, the following facts hold:

i. $\operatorname{tr}_{\partial U}$ *is extended uniquely by continuity and density of $C^1(\overline{U}) \subseteq W^{1,p}(U)$ to $\operatorname{tr}_{\partial U} : W^{1,p}(U) \to L^p(\partial U)$*

ii. $u \in W_0^{1,p} \Leftrightarrow \operatorname{tr}_{\partial U} u = 0$.

Proof. Evans section 5.5. □

Note that this extension is *not* surjective. $\operatorname{img}(\operatorname{tr}) \subsetneq L^p(\partial U)$.

We direct our attention to a sharp version of Theorem 1.13 in the setting where $p = 2$. This opens up the world of Fourier analysis, and eventually leads to the world of fractional-order Sobolev Spaces. We prove a Sharp Trace theorem for the half-space $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^d > 0\}$, and denote $\partial U = \{(x', 0) \in \mathbb{R}^d\} \simeq \mathbb{R}^{d-1}$.

Notation (Fourier Transform). The convention used for the Fourier and Inverse Fourier transforms is as follows: $\hat{u} = \int u(x)e^{-ix\xi}dx$, and $u(x) = \int \hat{u}e^{i\xi x}\frac{d\xi}{2\pi}$.

Theorem 1.14 (Sharp Trace Theorem). *When $u \in C^1(\overline{\mathbb{R}^d_+}) \cap H^1(\mathbb{R}^d_+)$, we have:*

$$||\operatorname{tr} u||_{H^{1/2}(\mathbb{R}^{d-1})} \lesssim ||u||_{H^1(\mathbb{R}^d_+)}$$

Proof. Let u be as in the Theorem statement. Using Theorem 1.12, we may extend u to $\tilde{u} \in C^1(\mathbb{R}^d)$ such that $||\tilde{u}||_{H^1(\mathbb{R}^d)} \lesssim ||u||_{\mathbb{R}^d_+}$. Then, we may write

$$\operatorname{tr} u = u(x', 0) = \tilde{u}(x', 0) = \int [\mathcal{F}_{x^d}\tilde{u}](x', \xi^d)\frac{d\xi^d}{2\pi}$$

Furthermore,

$$[\mathcal{F}_{x'} \operatorname{tr} u](\xi') = \int [\mathcal{F}\tilde{u}](\xi', \xi^d)\frac{d\xi^d}{2\pi}$$

Using the Fourier characterization from Proposition 1.1, we may write

$$\begin{aligned}
||\operatorname{tr} u||_{H^s} &\simeq ||(1+|\xi'|^2)^{s/2}[\mathcal{F}_{x'} \operatorname{tr} u](\xi')||_{L^2_{\xi'}} \\
&= \left\|(1+|\xi'|^2)^{s/2}\int [\mathcal{F}\tilde{u}](\xi',\xi^d)\frac{d\xi^d}{2\pi}\right\|_{L^2_{\xi'}} \\
&\simeq \left\|\int [\mathcal{F}\tilde{u}](\xi',\xi^d)(1+|\xi'|^2)^{s/2} d\xi^d\right\|_{L^2_{\xi'}} \\
&= \left\|\left\|[\mathcal{F}\tilde{u}](\xi',\xi^d)(1+|\xi'|^d)^{s/2}\frac{(1+|\xi'|^2+|\xi^d|^2)^{1/2}}{(1+|\xi'|^2+|\xi^d|^2)^{1/2}}\right\|_{L^1_{\xi^d}}\right\|_{L^2_{\xi'}} \\
&\leq \left\|\left\|\frac{(1+|\xi'|^2)^{s/2}}{(1+|\xi|^2)^{1/2}}\right\|_{L^2_{\xi^d}} \left\|(1+|\xi|^2)^{1/2}[\mathcal{F}\tilde{u}]\right\|_{L^2_{\xi^d}}\right\| \\
&= \left\|\left(\int \frac{(1+|\xi'|^2)^s}{1+|\xi'|^2+|\xi^d|^2}d\xi^d\right)^{1/2} ||(1+|\xi|^2)^{1/2}[\mathcal{F}\tilde{u}]||_{L^2_{\xi^d}}\right\|_{L^2_{\xi'}} \\
&\leq \left(\sup_{\xi'\in\mathbb{R}^{d-1}, s\in\mathbb{R}}\left[\int \frac{(1+|\xi'|^2)^s}{1+|\xi'|^2+|\xi^d|^2}d\xi^d\right]\right)||u||_{H^1(\mathbb{R}^d_+)} \\
&\simeq ||u||_{H^1(\mathbb{R}^d_+)}
\end{aligned}$$

□

Theorem 1.15 (Extension from the Boundary)**.** *There exists a bounded linear map* $\operatorname{ext}_{\partial U} : H^{1/2}(\mathbb{R}^{d-1}) \to H^1(\mathbb{R}^d_+)$ *such that* $\operatorname{tr}_{\partial U} \circ \operatorname{ext}_{\partial U} = \operatorname{id}$.

Proof. Here, we use the Poisson Semigroup. In particular, define $g \in \mathcal{S}(\mathbb{R}^{d-1})$, and $u = \operatorname{ext}_{\partial U}(g)$, with $[\mathcal{F}_{x'}u](\xi', x^d) = \eta(x^d)e^{-x^d|\xi'|}\hat{g}(\xi')$. Here, η is a smooth cutoff function with $\eta(|s| < 1) = 1$, and $\eta(|s| > 2) = 0$. Our objective is that show that $u \in H^1(\mathbb{R}^d_+)$ if and only if the following statements hold:

i. $u, \partial_{x^1}u, \ldots, \partial_{x^{d-1}} \in L^2$.
ii. $\partial_{x^d}u \in L^2$.

For (i), assume that $u \in H^1(\mathbb{R}^d_+)$. Then

$$\begin{aligned}
||u||^2_{L^2} + ||\partial_{x^1}u||^2_{L^2} + \ldots + ||\partial_{x^{d-1}}u||^2_{L^2} &\simeq ||(1+|\xi'|)^2)^{1/2}[\mathcal{F}_{x'}u](\xi',x^d)||^2_{L^2_{\xi'}L^2_{x^d}} \\
&= ||(1+|\xi'|^2)^{1/2}\eta(x^d)e^{-x^d|\xi'|}\hat{g}(\xi')||^2_{L^2_{\xi'},L^2_{x^d}} \\
&= \left\|(1+|\xi'|^2)^{1/4}||\eta(x^d)e^{-x^d|\xi'|}||_{L^2_{x^d}}(1+|\xi'|^2)^{1/4}\hat{g}(\xi')\right\|^2_{L^2_{\xi'}}
\end{aligned}$$

We want to put a uniform bound on $(1+|\xi'|^2)^{1/4}||\eta(x^d)e^{-x^d|\xi'|}||_{L^2_{x^d}}$ for every ξ'. By the compact support of η, we have the trival inequality:

$$||\eta(x^d)e^{-x^d}||^2_{L^2_{x^d}} \lesssim 1$$

Furthermore, writing out the L^2-norm, and making a substitution of variables inside the integral, we arrive at the substitution inequality:

$$\int (\eta(x^d))^2 e^{-2x^d|\xi'|}dx^d \lesssim \frac{1}{|\xi'|}$$

From these two inequalities, we deduce that

$$||\eta(x^d)e^{-x^d|\xi'|}||_{L^2_{x^d}} \lesssim \min\{1, |\xi'|^{-1/2}\} \lesssim (1+|\xi'|)^{-1/2}$$

So, terms in our initial equality cancel as follows:

$$\left\| \cancel{(1+|\xi'|^2)^{1/4}} \cancel{||\eta(x^d)e^{-x^d|\xi'|}||_{L^2_{x^d}}} \cancel{(1+|\xi'|^2)^{1/4}} \hat{g}(\xi') \right\|^2_{L^2_{\xi'}} \simeq ||\hat{g}(\xi')||_{L^2_{\xi'}}$$

This proves (i), since we may unravel the chain of definitions in the reverse direction exactly the same way.

To see (ii), write $\partial_{x^d} u = \partial_{x^d}(\eta(x^d)v) = \eta'(x^d)v + \eta v'$, $\mathcal{F}_{x'} v = e^{-x^d|\xi'|}\hat{g}(\xi)$. Each term of u' may be bounded, with $\eta'(x^d)v \leq ||v||_{L^2(x^d \in \operatorname{spt}\eta)}$, and

$$||\eta\partial_{x^d} v||_{L^2_{x'}, L^2_{x^d}} = ||\eta\partial_{x^d}(e^{-x^d|\xi'|}\hat{g}(\xi'))||_{L^2_{x^d} L^2_{\xi'}} = ||\eta(x^d)|\xi'|e^{-x^d|\xi'|}\hat{g}(\xi')|| \lesssim c||g||_{H^{1/2}}$$

The final inequality here follows from (i). $\square$

To generalize the above theorems to L^2-based Sobolev spaces, we need fractional Sobolev spaces on a C^1 boundary, under C^1 straightening diffeomorphism to the half-space. The indepedence of the norm under this diffeomorphism follows from some concepts in interpolation theory, which is in a book of Stein from 1970.

For $p \neq 2$, $\operatorname{img}(\operatorname{tr}_{\partial U} W^{1,p}(U)) = B_p^{1-1/p,p}(\partial U)$, the L^p-Besov space with regularity index of order $1-1/p$, and summability index p. This is also in Stein.

1.4 Sobolev Inequalities

In a nutshell, Sobolev inequalities are quantitative generalizations of the Fundamental theorem of Calculus, allowing us to control the size of a function by the growth of it's derivative.

Theorem 1.16 (Gagliardo-Nirenberg-Sobolev Inequality)**.** *For $d \geq 2$, $u \in C_0^\infty(\mathbb{R}^d)$, we have that for a constant c_d which depends only on the dimension d:*

$$||u||_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq c_d||Du||_{L^1(\mathbb{R}^d)}$$

Remark. The factor of $\frac{d}{d-1}$ can be derived using *dimensional analysis*. It's basically a statement about the fact derivatives have dimensions $[D] \sim \frac{1}{\lambda}$, where λ is the scalaing factor, and that the L^p norm has dimension $[||\cdot||_{L^p}] \sim [\lambda]^{d/p}$. So $[||D\cdot||_{L^p}] \sim [\lambda]^{d-1}$, and the rest follows from equating the dimensions.[3]

The key ingredient in the proof of Theorem 1.16 is actually another inequality.

Lemma 1.17 (Loomis-Whitney Inequality)**.** *For $d \geq 2$, $j = 1, \ldots, d$, and $f_j = f_j(x^1, \ldots, \widehat{x^j}, \ldots, x^d)$, we have:*

$$\|\prod_{j=1}^d f_j\|_{L^1(\mathbb{R}^d)} \leq \prod_{j=1}^d ||f_j||_{L^{d-1}(\mathbb{R}^{d-1})}$$

Proof. The proof of Lemma 1.17 is pretty direct, we just integrate in each direction, applying Hölder's inequality as we go.

$$\int |\prod_{j=1}^d f_j| dx^1 = |f_1| \int \prod_{j\neq 1}^d |f_j dx^1 \leq |f_1| \prod_{j\neq 1}^d ||f_j||_{L^{d-1}_{x^1}}$$

[3] I absolutely did not expect to see dimensional analysis show up anywhere *near* graduate mathematics, and am equal parts happy and confused.

Doing this d times gives

$$\int \cdots \int |\prod_{j=1}^{d} f_j| dx^1 \cdots dx^d \leq \prod_{j=1}^{d} ||f_j||_{L^{d-1}_{x^1,\ldots,\widehat{x_j},\ldots,x^d}}$$

□

Remark.

A Frequently Cited Theorems and Definitions

A.1 Real Analysis

Theorem A.1 (Existence of Smooth Partitions of Unity)**.**

A.2 Functional Analysis

Definition 9. Let X be a real vector space. A map $p : X \to \mathbb{R}$ is called a **sublinear functional** if it satisfies the following for all $x, y \in X$:

1. $p(x+y) \leq p(x) + p(y)$,
2. For all $\lambda \geq 0$, $p(\lambda x) = \lambda p(x)$.

Theorem A.2 (Hahn-Banach)**.** *Let X be a real vector space, p a sublinear functional on X, M a subspace of X, and f a linear functional on M such that $f(x) \leq p(x)$ for all $x \in M$. Then there exists a linear functional F on X such that $F(x) \leq p(x)$ for all $x \in X$, and $F_{|M} = f$.*

Proof. The following proof is due to Folland. We first show that for $x \in X \setminus M$, f may be extended to a linear functional g on $M + \mathbb{R}x$ which satisfies $g(y) \leq p(y)$. For $y_1, y_2 \in M$, we have

$$f(y_1) + f(y_2) = f(y_1 + y_2) \leq p(y_1 + y_2) \leq p(y_1 - x) + p(x + y_2)$$

Rearranging gives

$$f(y_1) - p(y_1 - x) \leq p(x + y_2) - f(y_2)$$

Since this applies to every $y_1, y_2 \in M$, we have

$$\sup_{y \in M} \{f(y) - p(y - x)\} \leq \inf_{y \in M} \{p(x + y) - f(y)\}$$

Let α be any number which satisfies

$$\sup_{y \in M} \{f(y) - p(y - x)\} \leq \alpha \leq \inf_{y \in M} \{p(x + y) - f(y)\}$$

and define $g : M + \mathbb{R}x \to \mathbb{R}$ by $g(y + \lambda x) = f(y) + \lambda\alpha$. g is linear by the linearity of f and multiplication by λ. Furthermore, $g_{|M} = f$, since any input in M has $\lambda = 0$, which gives $g(y) \leq p(y)$ for $y \in M$. Moreover, if $\lambda > 0$, and $y \in M$, we have

$$g(y + \lambda x) = \lambda \left[f\left(\frac{y}{\lambda}\right) + \alpha \right] \leq \lambda \left[f\left(\frac{y}{\lambda}\right) + p\left(x + \frac{y}{\lambda}\right) - f\left(\frac{y}{\lambda}\right) \right] = p(y + \lambda x)$$

If instead, we say $\lambda = -\mu < 0$,

$$g(y + \lambda x) = \mu \left[f\left(\frac{y}{\mu}\right) - \alpha \right] \leq \mu \left[f\left(\frac{y}{\mu}\right) - p\left(-x + \frac{y}{\mu}\right) - f\left(\frac{y}{\mu}\right) \right] = p(y + \lambda x)$$

So, we have $g(z) \leq p(z)$ for all $z \in M + \mathbb{R}x$.

Importantly, the above logic doesn't really depend on the fact $x \in X \setminus M$. If F is any linear extension of f, then $F \leq p$ on it's domain, which shows that the domain of a maximal linear functional satisfying $F \leq p$ must be X. The family $\mathcal{F}$ of linear extensions F of f satisfying $F \leq p$ is partially ordered by inclusion when we consider maps from subspaces of X to $\mathbb{R}$ as subsets of $X \times \mathbb{R}$. Since the union of any increasing family of subspaces of X is also a subspace of X, the union of a linearly ordered subfamily of $\mathcal{F}$ also lies in $\mathcal{F}$. So, we may invoke Zorn's lemma to guarantee the existence of a maximal element $F \in \mathcal{F}$, which completes the proof. □

Theorem A.3 (Open Mapping)**.** *Let X, Y be Banach spaces. If $T \in L(X, Y)$ is surjective, then T maps open sets to open sets.*

Proof. See Folland 5.10. □

A.3 L^p Spaces

Theorem A.4 (Hölder's Inequality).

Theorem A.5 (Minkowski Inequality).

B References