Abstract

A single document copy of these notes, as well as a mirror of every note, can be found at `connorduncan.xyz/notes` We first define the **scattering amplitude** $f(\theta, \phi)$, and connect it to the physical quantity $\frac{\partial \sigma}{\partial \Omega}$, the **differential cross section**. We need some spherically symmetric $V(r)$ with what Altman referred to as "finite support", which means it is nonzero in a finite region of space, and decays as $\frac{1}{r^{2+\varepsilon}} \sim V(r)$.

0.0.1 Solving the Schroedinger in Spherical Coordinates

I'm pretty sure we did this in 137A, but basically, these bad boys separate out to

$$\psi_{\ell,m}(\vec{r}) = R_\ell(r) Y_{\ell,m}(\theta, \phi)$$

We then get a radial equation, where $u_\ell = R_\ell(r) r$,

$$E u_\ell = \frac{-\hbar^2}{2m} \frac{\partial^2 u_\ell}{\partial r^2} + \frac{\ell(\ell+1)}{2mr^2} u_\ell$$

In the limit where $r \to \infty$, we get that our solutions are

	Outgoing	Incoming
$R_\ell(r)$	e^{ikr}/r	$e^{-i(kr+Et)}/r$

We also have the current density

$$\frac{j_{sc} d\vec{A}}{j_{inc}} = d\sigma = \frac{d\sigma}{d\Omega} d\Omega = \frac{\text{current scattered into } d\Omega}{\text{incident current per unit area}}$$

we should also note $dA = r^2 d\Omega \hat{r}$.

0.0.2 Probability Current

We can recall the probability current from 137A, where we have $p(r) = |\psi(r)|^2$, and then apply some continuity equation so that

$$\partial_t p(r) = -\nabla j$$

Now, we have

$$\partial_t(\psi^* \psi) = \psi^* \partial_t \psi + \psi \partial_t \psi^*$$

We also have

$$j = \frac{\hbar}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*) = \frac{\hbar}{m} \operatorname{Re}[\psi^* \hat{p} \psi]$$

So, for the incident wave, we have

$$j_{\text{inc}} = \frac{\hbar k}{m}$$

Now, we take j_{sc}.

$$j_{\text{sc}} = \frac{\hbar k}{mr^2} |f(\theta, \phi)|^2 \hat{r}$$

So, we have

$$\frac{\partial \sigma}{\partial \Omega} d\Omega = \frac{j_{\text{sc}} \hat{r} r^2 d\Omega}{j_{\text{inc}}} = |f(\theta)|^2 d\Omega$$

which all yields that

$$\frac{\partial \sigma}{\partial \Omega} = |f(\theta)|^2$$

where we have made the approximation that $r \to \infty$ for the scattered component of the wavefunction.

0.1 Born Approximation

We're going to take $E = \frac{\hbar^2 k^2}{2m}$, with

$$(\nabla^2 + k^2)\psi(r) = \frac{2m}{\hbar^2} V(r) \psi(r)$$

We're going to use Green's functions to solve this, which gives

$$\psi(r) = e^{ikz} + \int d^3r' G_0(k, r, r') \frac{2m}{\hbar^2} V(r') \psi(r')$$

We need $(\nabla^2 + k^2) G(k^2, r, r') = \delta^3(r - r')$, where G is still the greens function. If we apply our operator to the above equation, we get

$$(\nabla^2 + k^2)\psi(r) = (\nabla^2 + k^2) \left(e^{ikz} + \int d^3r' G_0(k, r, r') \frac{2m}{\hbar^2} V(r') \psi(r') \right)$$

As an aside, we can think of greens function as being the inverse of the operator $\nabla^2 + k^2$ so that $\hat{G}_0 = (\nabla^2 + k^2)^{-1}$