# Physics-105-Lecture-Notes-04-11-2019

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## Contents



**Abstract**

A single PDF with all lectures in a single document can be downloaded at `https://www.dropbox.com/sh/8sqzvxghvbjifco/AAC9LoSRnsRQDp7pYedgWpQMa?dl=0`. The password is 'analytic.mech.dsp'. This file was automatically generated using a script, so there might be some errors. If there are, you can contact me at `mailto:ctdunc@berkeley.edu`. Recall we have some $\mathcal{L} = \frac{1}{2}\sum_{i,k} T_{ik}\dot{q}_i\dot{q}_k - V(q)$ is the most general lagrangian, with $\sum_{i,k} T_{ik}\eta_i\eta_k > 0 \forall \eta_i, \eta_1^2 + \ldots + \eta_n^2 \neq 0$. Then, we find stationary point $\frac{\partial V}{\partial q} = 0 \Rightarrow q_i^{(0)}$, $q_2 = q_i^{(0)} + \delta q_i$, which expands to the following form

$$\mathcal{L} = \frac{1}{2}\left(\sum_{i,k} m_{ik}\delta\dot{q}_i\delta\dot{q}_k - \sum_{i,k} V_{ik}\delta q_i \delta q_k\right)$$

with $V_{ik} = \frac{\partial^2 V}{\partial q_i \partial q_k}_{q^{(0)}}$. Guess solution is of the form $q_k = A_k e^{-i\omega t}$, or more generally,

$$\sum_k (V_{ik} - \omega^2 m_{ik})A_k = 0$$

or

$$(\hat{V} - \omega^2\hat{m})\vec{A} = 0$$

### 0.1 Some Hamiltonian

we're using the einstein summation convention.

$$H = p_j\dot{q}_j - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} q_j - \mathcal{L}$$

this expands to

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left[\frac{1}{2}\sum_{i,k} T_{ik}\dot{q}_i\dot{q}_k\right] = \frac{1}{2}\left[\sum_{i,k} T_{ik}\delta_{ij}\dot{q}_k + \sum_{i,k} T_{ik}\dot{q}_i\delta_{kj}\right]$$

which gives generalized momentum (of further simplification

$$p_j = \sum_i T_{ij}\dot{q}_i$$

with

$$H = \sum_{i,j} T_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}T_{ij}\dot{q}_i\dot{q}_j + V(q) = T + V = \frac{1}{2}\sum T_{ik}\dot{q}_i\dot{q}_k + V(q)$$

Recall that $T_{ik}$ a function of $q$. So, we want to expand the hamiltonian around small oscillations or some stationary point, and it becomes

$$H = \frac{1}{2}\left(\sum_{i,k} m_{ik}\dot{q}_i\dot{q}_k + \sum_{i,k} V_{ik}q_iq_k\right)$$

with $V(q^{(0)}) = 0$ by definition.

**Theorem 0.1.** Suppose that $\forall \eta_i | \eta_1^2 + \ldots + \eta_n^2 > 0$ we have $V_{ik}\eta_i\eta_k \gg 0$, then te stationary point $q^{(0)}$ is stable, then $q_i = A_i e^{-i\omega t}$, and $\omega \in \mathbb{R}$.

*Proof.* We have

$$H_0 - \frac{1}{2}\sum_{i,k} V_{ik}q_iq_k = \frac{1}{2}\sum_{i,k} m_{ik}\dot{q}_i\dot{q}_k$$

which, we know the sum term on the right is always positive at a stable point (since it goes up), which gives with $H_0$ constant, that oscillation, with $\omega \in \mathbb{R}$. CONNOR NOTE: I'm pretty sure that $V_{ik}$ is just the hessian matrix for $V$. Let $\omega_s^2, A_k^s$ the eigenvalue, vector correspondent to the root $\omega_s^2$. We have then that

$$(V_{ik} - \omega_s^2 m_{ik}) A_k^s = 0$$

and left multiply by adjoint of $A_k^s$ to get

$$\sum_{ik} = \sum_{ik} A_i^s V_{ik} A_k^s = \omega_s^2 \sum_{i,k} m_{ik} A_i^{*(s)} A_k^{*(s)}$$

where we;re allowed to change multiplication order only in index notation because they're numbers. That gives

$$\omega_s^2 = \frac{\sum_{i,k} V_{ik} A_i^{*(s)} A_k^{(s)}}{\sum_{i,k} m_i k A_i^{*(s)} A_k^{(s)}}$$

want to show that $\forall \eta_i \in \mathbb{C} \sum_{i,k} m_{ik} \eta_i \eta_k > 0$, multiplying the two by complex conjugates makes it poisitive. Let's introduce another $A_i^{(\alpha)}$, and take

$$A_i^{*(\alpha)} A_k^{(s)} V_{ik} = \omega_s^2 m_{ik} A_i^{*(\alpha)} A_k^{(s)}$$
$$A_i^{(s)} A_k^{*(\alpha)} V_{ik} = \omega_\alpha^2 m_{ik} A_k^{*(\alpha)} A_i^{(s)}$$

which subtracting one from the other

$$\omega_s^2 - \omega_\alpha^2 = m_i k A_i^{(\alpha)} A_k^{(s)} \equiv 0$$

for nondegenerate cases, eigenvectors must satisfy

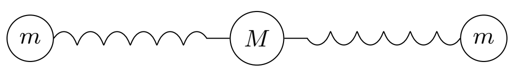
$$\sum_{i,k} m_{ik} A_i^{*(\alpha)} A_k^{(s)} \equiv 0$$

i.e. for nondegeneracy, it's some sort of generalized norm over a space with metric $m_{ik}$. Check out Goldstein for a proof f this. Finally, suppose we have every frequency, eigenvector, how do we write a general solution?

$$q_k = \sum_s C_s A_k^{(s)} e^{-i\omega_s t} \quad C_s \in \mathbb{C}$$

### 0.1.1 Model of a Molecule

consider some




which gives (after writing down the lagrangian)

$$m_{ik} = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}$$
$$V_{ik} = k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

which we diagonalzie to $-k^2(k - \omega^2 m) = 0$, which gives three possible roots

$$\omega = 0$$
$$\omega^2 = \frac{k}{m}$$
$$\omega^2 = \frac{k}{m}(1 + 2\frac{m}{M})$$

we just find the eigenvectors that correspond to these eigenvalues, and we're sitting pretty. For $\omega = 0$, the velocities are symmetric, potential is independent of direction, only depends on displacement of $x_1, x_2, x_3$. for $\omega^2 = \frac{k}{m}$, we get

$$\begin{bmatrix} 0 & -k & 0 \\ -k & 2k - k\frac{M}{m} & -k \\ 0 & -k & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 0$$

which has, so we have $A_2 = 0$, with eigenvector

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

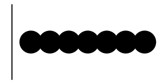
Final eigenfrequency is going to give

$$\begin{bmatrix} -2k\frac{m}{M} & -k & 0 \\ x & x & x \\ 0 & -k & -2k\frac{m}{M} \end{bmatrix}$$

which gives $A_1 = A_3$ and $A_2 = -2\frac{m}{M} A_1$, so eigenvector is

$$A \begin{bmatrix} 1 \\ -2\frac{m}{M} \\ 1 \end{bmatrix}$$

### 0.1.2 Another Example. More Algebra!



basically, it's a bunch of masses on a string that all are oscillating. It's a bit odd. Work this out, there's #TOO #MUCH #ALGEBRA.