Intro. Metric Differential Geometry

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Contents

1 Curves in $\mathbb{R}^3$

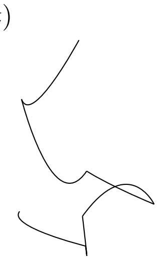
1.1 Quick and Dirty Linear Algebra

We have our definition of $\mathbb{R}^3$ in the usual way. We define $(1,0,0) = \vec{e}_1, (0,1,0) = \vec{e}_2, (0,0,1) = \vec{e}_3$. They satisfy the right-hand rule.
There's an inner product. $\langle \vec{u}, \vec{v} \rangle$. Defines the norm as $|\vec{u}| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$.
Basis is orthonormal iff $\langle \vec{u}_i, \vec{u}_j \rangle = \delta_{ij}$.

1.2 Parametric Functions

Parametric functions exist, we can have some path

$$\vec{\alpha} = (x, y, z)(t)$$



Parametric curves trace out geometric curves, which are the "images" of parametric curves.

Definition 1.1 (Geometric Curve)**.** A **Geometric Curve** is defined as the image of a parametric curve

Parametric curves uniquely determine geometric curves, but not the other way around (i.e. there may be many parameterizations of geometric curves).

Definition 1.2 (C^k regular)**.** A parameterized curve $\vec{\alpha} : (a, b) \to \mathbb{R}^3$ is called C^k-regular ($k \geq 1$) if $\vec{\alpha}$ is of class C^k and $\frac{\partial \vec{\alpha}}{\partial t} \neq 0$ at any $t \in (a, b)$.

e.g. 1.2.1. Consider $\alpha(t) = (t, 0, 0)$. Then, since $\frac{\partial \alpha}{\partial t} = (1, 0, 0) \neq \vec{0}$ for any t, we have α is C^∞ regular.
A different parameterization is not C^∞-regular. Consider $\beta(t) = (t^3, 0, 0)$. This is not a regular parameterization.

Definition 1.3 (C^k-regular curve)**.** A geometric curve with a C^k-regular parameterization is called a **C^k-regular curve**.

Definition 1.4 (C^k reparameterization)**.** Consider $\vec{\alpha} : (a, b) \to \mathbb{R}^3$ a parameterized curve. Consider $g : (c, d) \to (a, b)$ a bijective map, where both g, g^{-1} are of class C^k.

Lemma 1. If $\alpha : (a, b) \to \mathbb{R}^3$ is a C^k-regular curve, and $g :\to (c, d) \to (a, b)$ is a C^k reparameterization, then we claim $B = \alpha \circ g$ is a C^k-regular parameterization.

Proof.
1. $\vec{\beta} : (c, d) \to \mathbb{R}^3$ is of class C^k which implies $\frac{\partial \vec{\beta}(s)}{\partial s} = \frac{\partial (\vec{\alpha} \circ g(s))}{\partial s} = \frac{\partial \vec{\alpha}(g(s))}{\partial t} \cdot \frac{\partial g(s)}{\partial s}$. These are all of the same C^k-class, so we have shown that β is as well.
2. $\vec{\beta}$ is regular, since $\frac{\partial \vec{\beta}}{\partial s} = \frac{\partial g(s)}{\partial s} \cdot \frac{\partial \vec{\alpha}(g(s))}{\partial t}$. We need to show that $\frac{\partial g}{\partial s}$ is never zero for any s. It has to be true, since g^{-1} is C^k class, so it must be at least C^1 class, thus never zero.

$\square$

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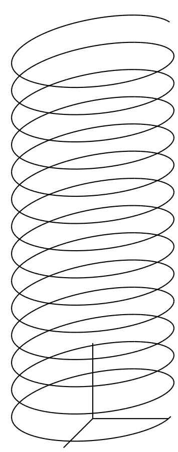
In order to define a C^k regular parameterized curve, we should consider the set of C^k regular parameterizations, and define an equivalence class over the set, with $\vec{\alpha} \sim \vec{\beta}$ if $\exists g$ some C^k regular reparameterization. From here, take $\{C^k\text{-regular paramaterizations}\}/\sim$. If we mod out by positive representaitons $S/\sim_+$, we have equivalence classes of oriented curves.

Definition 1.5 (Velocity (for a regular parameterized curve))**.** If we assume $\vec{\alpha} : (a, b) \to \mathbb{R}^3$, then we have

$$\vec{v} = \frac{\partial \vec{\alpha}}{\partial t} : (a, b) \to \mathbb{R}^3$$

We can also consider helical parameterizations, e.g.



Our velocity depends on our parameterization, but our tangent line does not.
We have that for every t,

$$L(\alpha(t)) = \{\vec{\alpha}(t) + r * \vec{v}(t) | r \in \mathbb{R}\}$$

Lemma 2. Tangent Line is independent of parameterization

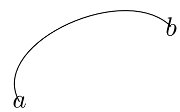
Proof. Assume $\alpha : (a, b) \to \mathbb{R}^3$, $\beta : (c, d) \to \mathbb{R}^3$, with $g : (c, d) \to (a, b)$. Consider $t \in (a, b)$, $s \in (c, d)$ such that $g(s) = t$. Then, we have $L(\alpha(t)) = \{\vec{\alpha}(t) + r\frac{\partial \vec{\alpha}}{\partial t} | r \in \mathbb{R}\}$.
Then, we also have $L(\beta(s)) = \{\vec{\beta}(s) + r\frac{\partial \vec{\beta}}{\partial s} | r \in \mathbb{R}\}$. From the chain rule, we also know that $\frac{\partial \vec{\beta}}{\partial s} = \frac{\partial \vec{\alpha}}{\partial s} \cdot \frac{\partial g}{\partial s}$, so we're free to replace

$$L = \left\{\vec{\alpha}(t) + \left(r \cdot \frac{\partial g}{\partial s}\right) \frac{\partial \vec{\alpha}}{\partial t} | r \in \mathbb{R}\right\}$$

which defines the same set since $\frac{\partial g}{\partial s} \neq 0$. $\square$

Definition 1.6 (Unit Tangent Vector Field)**.** For $\vec{\alpha}(t)$ regular, and $\vec{v}(t)$ is nowhere vanishing, then $\frac{\vec{v}(t)}{|\vec{v}(t)}=\vec{T}(t)$ is the **unit tangent vector**.

Is $\vec{T}$ is defined for any regular curve, but not all reparameterizations. NO. Consider



We could easily go from $a \to b$ and back again. But if we *orient* our curve then it is defined, since we aren't allowed to turn back around.

1.3 Arc Length Parameter

Definition 1.7 (Parameterized Curve Segment)**.** $\vec{\alpha}:(a,b)\to\mathbb{R}^{\nVdash}$, $[c,d]\subseteq(a,b)$. $\vec{\alpha}|_{[c,d]}$ is called a parameterized curve segment.

Letting ℓ :=length, we have

$$\ell(\vec{\alpha}|_{[c,d]})=\int_c^d\left|\frac{\partial\vec{\alpha}}{\partial t}\right|dt$$

Lemma 3. Length is defined for curve segments (equivalence class of parameterized curve segments).

Proof. Take $\alpha:[a,b]\to\mathbb{R}^3$, $\beta:[a,b]\to\mathbb{R}^3$, with $g:[c,d]\to[a,b]$.

$$\ell(\vec{\alpha}|_[a,b])=\int_a^b\left|\frac{\partial\vec{\alpha}}{\partial t}\right|dt \qquad \ell(\vec{\beta}|_[c,d])=\int_c^d\left|\frac{\partial\vec{\beta}}{\partial s}\right|ds$$

By the chain rule, we have

$$\ell(\vec{\beta})=\int_c^d\left|\frac{\partial\vec{\alpha}}{\partial t}\right|\left|\frac{\partial g}{\partial s}\right|ds$$

Either dg positive or negative, so just pick one, and take

$$\int_c^d\left|\frac{\partial\vec{\alpha}}{\partial t}\right|\left|\frac{\partial g}{\partial s}\right|ds=\int_a^b\frac{\partial\vec{\alpha}}{\partial t}dt=\ell(\vec{\alpha})$$

If we reverse orientation, we have

$$\int_c^d\left|\frac{\partial\vec{\alpha}}{\partial t}\right|\left(-\frac{\partial g}{\partial s}\right)ds$$

Since everytihng is parameterized differently, we just reverse the order of integration on our order in α, and we're good, since g has the different mapping. $\square$

We can actually use arc length as a parameter.

Definition 1.8. Consider $\alpha:(a,b)\to\mathbb{R}^3$ We have a map

$$\begin{aligned} S:\quad &(a,b)\to\mathbb{R}\\ &t\mapsto\ell\left(\vec{\alpha}|_{[a,t]}\right)\end{aligned}$$

Or, to resolve trouble around endpoints, take $t_0\in(a,b)$, then we have

$$\begin{aligned} S:\quad &(a,b)\to\mathbb{R}\\ &t\mapsto\int_{t_0}^t\left|\frac{\partial\vec{\alpha}(\sigma)}{\partial\sigma}\right|d\sigma\end{aligned}$$

Lemma 4. If α a regular parameterization, its arc length parameterization will also be regular.

Proof. Consider $s'(t)=\left|\frac{\partial\vec{\alpha}}{\partial t}\right|\Rightarrow s$ is C^k. WLOG, let $s'(t)>0; t\in(a,b)\Rightarrow s$ is injective, which gives an inverse map s^{-1} exists. $s\circ s^{-1}=\text{id}$, or $(s^{-1})^1=\frac{1}{s^1}>0$. $\square$

We get a nice property, where for some curve $\vec{\alpha}(t)$, with arc length parameter, $s(t)$, we can let $t = t(s)$, some function of the arc length, which gives a new parameterization $\vec{\beta}(s) = \vec{\alpha}(t(s))$. Calculating the velocity $\vec{v}_\beta$, we have

$$\vec{v}(\vec{\beta}(s)) = \frac{\partial \vec{\beta}}{\partial s} = \frac{\partial \vec{\alpha}}{\partial t} \cdot \frac{\partial t}{\partial s} = \frac{\frac{\partial \vec{\alpha}}{\partial t}}{\frac{\partial s}{\partial t}} = \frac{\frac{\partial \vec{\alpha}}{\partial t}}{|\vec{v}(\vec{\alpha})|} = 1$$

The velocity in terms of our new parameterization is always 1.

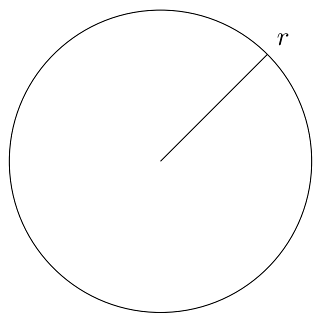
Definition 1.9 (Curvature)**.** Let $\vec{T}(s) = \frac{\partial \vec{\alpha}(s)}{\partial s}$. Then, $\left|\frac{\partial \vec{T}(s)}{\partial s}\right| = \kappa(s)$ is the curvature.

e.g. 1.9.1 (Curvature of a line)**.** Consider some $\vec{\alpha}(t) = \vec{u} + t\vec{v}$, $t \in \mathbb{R}$. First, we compute the arc length parameter,

$$S(t) = \int_0^t \left|\frac{\partial \vec{\alpha}}{\partial \sigma}\right| d\sigma = \int_0^t |\vec{v}| \, d\sigma = |\vec{v}|t$$

This gives $t = \frac{S}{|\vec{v}|}$, which gives $\vec{\beta}(S) = \vec{u} + \frac{S}{|\vec{v}|}\vec{v}$.
Then, we have $\vec{T}(s) = \frac{\vec{v}}{|\vec{v}|}$ independent of s, so when we derive by s, $\kappa = 0$. Straight lines have no curvature! This is good.

e.g. 1.9.2 (Curvature of a plane-curve (constant z))**.** Consider some plane curve, a circle of radius r.



$\vec{\alpha}(t) = (r\cos t, r\sin t, 0$. Arc length parameter is given by

$$S(t) = \int 0^t |v| d\sigma = \int_0^t \sqrt{r^2 \sin^2(\sigma) + r^2 \cos^2(\sigma) + 0^2} d\sigma = \int_0^t r d\sigma = rt$$

or, through an appropriate change of variables, we ave

$$\vec{\beta}(S) = \left(r\cos\frac{s}{r}, r\sin\frac{s}{r}, 0\right)$$

with

$$\vec{T}(s) = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}, 0\right)$$

or $\kappa = 1$.

Definition 1.10. $\vec{N} = \frac{\frac{\partial \vec{T}}{\partial s}}{\kappa(S)}$ is the normal vector

Then, we have

Definition 1.11 (Binormal Vector Field)**.** $\vec{T} \times \vec{N} = \vec{B}$ is the binormal vector field. Note that $\{T, N, B\}$ is an orthonormal basis for $\mathbb{R}^3$.

Since the binormal vector field is a moving frame of reference, we can transform into the moving frame.
We are also free to differentiate this quantity by S, which we can interpret as a matrix.
So, since $\vec{N}$ is just the unit vector of $\vec{T}'(s)$, we have that $\vec{T}'(s) = \kappa(s)\vec{N}$.
To compute $\vec{N}'(S)$, we can take $\langle N, T\rangle = \langle aT, T\rangle + \langle bN, T\rangle + \langle cB, T\rangle = a$.
So, we have $b = \langle N', N\rangle = 0$, since it is a unit vector.
To compute a, we have $\langle N, T\rangle = 0$, with $\frac{\partial}{\partial s}\langle N, T\rangle = \langle N', T\rangle + \langle N, T'\rangle = \langle N', T\rangle + \kappa(s)$, which gives $a = -\kappa$.
To find c, we want $0 = \frac{\partial}{\partial s}\langle N, B\rangle = \langle N', B\rangle + \langle N, B'\rangle$, which is $\tau(s)$ defined as **torsion**.

Theorem 1.1 (Frenet-Serret)**.** Finally, we have

$$\frac{\partial}{\partial s}\begin{bmatrix}\vec{T}(s)\\ \vec{N}(s)\\ \vec{B}(s)\end{bmatrix} = \begin{bmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{bmatrix}\begin{bmatrix}\vec{T}(s)\\ \vec{N}(s)\\ \vec{B}(s)\end{bmatrix}$$

e.g. 1.11.1. Consider $\vec{\alpha}(s)$ o a sphere with center $\vec{x_0}$ and radius r. We've already shown that if $\tau(s) \neq 0$, then $\vec{\alpha}(s) - \vec{x_0} = -\rho\vec{N} - \rho'\sigma\vec{B}$, where $\rho = \frac{1}{\kappa}, \sigma = \frac{1}{\tau}$.
We have some

Proof. $|\vec{\alpha}(s) - \vec{x}_0|^2 = r^2$, then $\langle\vec{\alpha}(s) - \vec{x}_0, \ldots\rangle$, which, time derived has

$$\frac{\partial}{\partial s}\ldots = 2\langle\vec{\alpha}(s) - \vec{x}_0, \vec{T}\rangle = 0$$

This gives that $\langle T, T\rangle + \kappa\langle\vec{\alpha}(s) - \vec{x}_0, \vec{N}\rangle = 0$, which means that $\kappa \neq 0$ for any s, since otherwise 1=0, which is a contradiction.

□

1.4 Fundamental Theorem for Curves in $\mathbb{R}^3$

Theorem 1.2. For any regular curve with positive curvature, $\kappa > 0$, it is completely determined, up to a starting position, by it's curvature and torsion.
Alternately, for any interval (a, b), with $\bar{\kappa}(s) > 0$ a C^1-function, $\bar{\tau}(s)$ a C^0-function on (a, b), for $\vec{x}_0 \in \mathbb{R}^3$, we pick a frame $\{\vec{D}, \vec{E}, \vec{F}\}$ a right-handed, orthonormal basis at $\vec{x}_0$, then there exists a unique C^3 regular curve $\vec{\alpha}(s)$ defined on (a, b) so that

1. s is the arc length parameter.
2. at $\vec{\alpha}(0) = \vec{x}_0$, $\{\vec{T}(0), \vec{N}(0), \vec{B}(0)\} = \{\vec{D}, \vec{E}, \vec{F}\}$.
3. $\kappa(s) = \bar{\kappa}(s)$, $\tau(s) = \bar{\tau}(s)$.

She sketched a proof of it here, which basically involves taking the frenet apparatus, and recovering T' from it.
There are steps:

1. Solve for $\{\vec{T}, \vec{N}, \vec{B}\}$ from the equation

$$\frac{\partial}{\partial s}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix} = \begin{bmatrix}0 & \bar{\kappa} & 0\\ -\bar{\kappa} & 0 & \bar{\tau}\\ 0 & -\bar{\tau} & 0\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix}$$

2. Construct $\vec{\alpha}(s) = \vec{\alpha}(0) + \int_0^s \vec{T}(\sigma)d\sigma$, your solution.

This is **Cauchy's Problem** in ODE's. Whether a solution exists, and whether it is unique? We can always find a solution, but even with an initial condition, it may not be unique. If we continually apply initial conditions, eventually we will find a unique solution. (e.g. x_0 may not completely specify periodic shapes like an ellipse, but the derivative at $s = 0$ will).
This gives rise to two theorems

- Cauchy's Theorem for continuous $\vec{A}$
- Picards theorem, then $\vec{A}$ is Lipschitz.[1]

Here's the ODE result subject to Lipschitz condition or stronger

Lemma 5 (ODE RESULT)**.** For $|\vec{x} - \vec{F}_0| \leq K$, and $|t - 0| \leq T$, with $\vec{A}$ a C^1 function, then there exists a unique solution $\vec{F}$ of the system $*$ on the time interval $|t| < \min\{T, \frac{K}{M}\}$ where $M = \sup|\vec{A}|$.

[1] wtf does this mean

In our case, the unique solution is the matrix $A_0 =$ our frenet apparatus times $\vec{F}$, $\vec{A}(\vec{F}, t) = A_0(t)\vec{F}$. This will give a unique solution $\{\vec{T}, \vec{N}, \vec{B}\}$.
next time: we'll check that our frenet-serret basis is orthonormal.
A new form for the Frenet-Serret formula is to express it as

$$\frac{\partial}{\partial s}\langle \vec{T}, \vec{N}, \vec{B}\rangle = \langle \vec{T}, \vec{N}, \vec{B}\rangle \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

which is just transposition of our operators.
C^1 implies uniqueness and existence of a solution to the cauchy problem (?).
So, we make the claim

Claim 1.1. For every $t \in (a, b)$, if $\langle \vec{y}_1, \vec{y}_2, \vec{y}_3\rangle(0) = \langle \vec{D}, \vec{E}, \vec{F}\rangle$ where $\langle \vec{D}, \vec{E}, \vec{F}\rangle$ is an orthonormal, right handed basis, then $\langle \vec{y}_1, \vec{y}_2, \vec{y}_3\rangle(t)$ is also an orthonormal basis.
Alternately, we have $\langle \vec{y}_i, \vec{y}_j\rangle = g_{ij}(0) = \delta_{ij}$.

Proof. First, note that

$$\frac{\partial}{\partial t}\vec{y_i} = \sum_{k=1}^{3} \vec{y}_k A_i^k$$

$$\frac{\partial}{\partial t} g_{ij} = \frac{\partial}{\partial \langle} y_i, y_j\rangle$$

This just becomes some nast series of summations over multiple indices. I'll need to do it out later. □

1.5 Non Unit-Speed curves

We can represent everything in our t coordinates if we don't wish to transform to s coordinates. All we really require here is the chain rule. If we have $\beta(t) = \alpha(s)$, then we have $\dot{\beta} = \frac{\partial}{\partial t}\beta$, then $T = \frac{\dot{\beta}}{|\dot{\beta}|}$, which gives $\dot{\beta} = \dot{s}\alpha'$.

2 Plane Curves

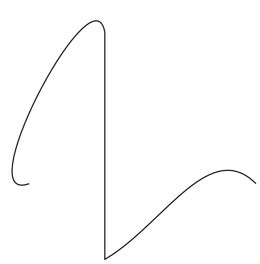
2.1 Introduction

We motivate this from the calculation of angles from middle school. E.g. some triangle has internal angles which sum to π. If we consider some n-gon as a piecewise smooth curve, we can divide it into triangles by taking $n - 1$ triangles, and multiplying by the internal angles of a triangle.
There is another set of angles we can calculate, namely those which are made by extending the side lines of our shapes, we get that for every shape, the angles are the same. This will lead into the Gauss-Benet formula, but for now, we look at plane curves.

2.2 Differential Forms

2.2.1 Forms

Consider some $\vec{\alpha} = (x, y, 0) = (x, y) \subseteq \mathbb{R}^2$ which is piecewise-smooth.



which is defined on $U = [a_1, a_2] \cup [a_1, a_2] \sup [a_2, a_3] \cup \ldots [a_n, b]$. Then, we can define $f, g : U \to \mathbb{R}$, with some **line integral** $\int_C f(x,y)dx + g(x,y)dy$, where C is the geometric curve with $\vec{\alpha}$ as a parameterization. then

$$\int_C f(x,y)dx + g(x,y)dy = \int_a^b \left(f(x(t), y(t))\dot{x}(t) + g(x(t), y(t))\dot{y}(t)\right) dt$$

is defined as our line integral. This integration is independent of our parameterization, but not of orientation.
Consider some region R bounded by a curve C.

Definition 2.1 (n-form)**.** Consider $\mathbb{R}^3$ with some fixed orthonormal basis x_1, x_2, x_3. Then we call a

- **0-Form**: Functions on $\mathbb{R}^3$, $f(x_1, x_2, x_3)$.
- **1-Form**: Expressions of the form $f_1(x_1, x_2, x_3)dx_1 + f_2(x_1, x_2, x_3)dx_2 + f_3(x_1, x_2, x_3)dx_3$, where dx_i is a 0-Form.
- **2-Form**: Of the form $\sum f_{i \vee j \vee k} dx_i dx_j$ where $i \neq j \in \{1, 2, 3\}$, and $dx_i dx_j = -dx_j dx_i$.
- **3-Form**: Of form $f(x_1, x_2, x_3)dx_1 dx_2 dx_3$, where permutation of any indices results in a minus sign (antisymmetric under permutation).

Let's see if we can redescribe Green's theorem in terms of n-forms. We're taking an integration over a 1-form on $\mathbb{R}^2$, and translating it to integration over a 2-form.
We can consider the differential operator as a map from the set of 0-forms to the set of 1-forms.

$$\begin{aligned} d: \quad & \{\text{0-form}\} \to \{\text{1-form}\} \\ & f(x_1, x_2, x_3) \mapsto \textstyle\sum_{i=1}^3 \frac{\partial f}{\partial x_i} dx_i \end{aligned}$$

In order to consider 2 forms, maybe we stipulate that d must fulfill the product rule, and distribute

$$d\left(f_1 dx_1 + f_2 dx_2 + f_3 dx_3\right) = d(f_1 dx_1) + d(f_2 dx_2) + d(f_3 dx_3)$$

We consider as an example $(df_1) \cdot dx_1$. Then,

$$(df_1) \cdot dx_1 = \frac{\partial f_1}{\partial x_1} dx_1 dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 dx_1 + \frac{\partial f_1}{\partial x_3} dx_3 dx_1$$

by antisymmetry, $dx_1 dx_1 = -dx_1 dx_1 = 0$, so we have the ability to write this as

$$(df_1) \cdot dx_1 = \frac{\partial f_1}{\partial x_1} dx_1 dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 dx_1 + \frac{\partial f_1}{\partial x_3} dx_3 dx_1 = -\frac{\partial f_1}{\partial x_2} dx_1 dx_2 - \frac{\partial f_1}{\partial x_3} dx_1 dx_3$$

We also define that $d(dx_i) = 0$, so that we can define the following map from 1 to 2-forms.

$$\begin{aligned} d: \quad & \{\text{1-form}\} \to \{\text{2-form}\} \\ & \textstyle\sum_{i=1}^3 f_i dx_1 \mapsto \sum_{i<j\leq 3} \sum_{i=1}^3 \left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}\right) dx_i dx_j \end{aligned}$$

which is just the curl operator!!
If we want to map from 2-forms to 3-forms, we can continue by taking

$$d(f_x dx_2 dx_3 + f_2 dx_3 dx_1 + f_3 dx_1 dx_2)$$

As an example, we consider

$$d(f_1 dx_2 dx_3) = (df_1) dx_2 dx_3 + 0 + 0 = \left(\frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 + \frac{\partial f_1}{\partial x_3} dx_3\right) = \frac{\partial f_1}{\partial x_1} dx_1 dx_2 dx_3$$

So, the overall operator takes

$$d(f_x dx_2 dx_3 + f_2 dx_3 dx_1 + f_3 dx_1 dx_2) = \left(\sum_{i=0}^3 \frac{\partial f_i}{\partial x_i}\right) dx_1 dx_2 dx_3$$

This is just the divergence!!
So, the map from 0-forms to 1-forms. It's the gradient! If we take a 1-form to a 2-form, it's the curl! If we take a 2-form to a 3-form, it's divergence!

2.2.2 notation

To denote forms of different n, we use the set of n-forms on set S as $\Omega^n(S)$. E.g., the set of zero forms on $\mathbb{R}^3$ is denoted by $\Omega^0(\mathbb{R}^3)$.
Every n form has addition, multiplication, and cyclic permutation. We typically call multiplication a **wedge**. The wedge product is antisymmetric, and always keeps lower indices low, where permutatation is by low indices.
If we want the set of all forms, while preserving linear structure, we express

$$\Omega^*(\mathbb{R}^3) = \bigoplus_{n=0}^{3} \Omega^n(\mathbb{R}^n)$$

We also have the **differential operator**:

$$\begin{aligned} d: \quad & \Omega^*(\mathbb{R}^3) \to \Omega^*(\mathbb{R}^3) \\ & \Omega^k(\mathbb{R}^3) \mapsto \Omega^{k+1}(\mathbb{R}^3) \end{aligned}$$

The differential operator forms what is called a deRham chain complex, which defines a Cohomology from degree H^0. Basically a chain from

$$0 \to \Omega^0(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3) \to$$

2.2.3 General Greens/Stokes Theorem

Let's consider Green's theorem in light of these new definitions. Take

$$d(fdx + gdy) = dfdx + dgdy = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

Then, if we let R represent some region, and ∂R the boundary of R, greens theorem can be restated as

$$\int_{\partial R} \text{(1-form)} = \int_R d\text{(1-form)}$$

or,

$$\int_R d\omega = \int_{\partial R} \omega$$

This is generalizable. Consider some V a solid geometric object (e.g. a ball), with ∂V a 2-dimensional surface (the sphere surrounding the ball).
Then for some $\omega \in \Omega^2(\mathbb{R}^3)$ and $d\omega \in \Omega^3(\mathbb{R}^3)$, then we have

$$\int_V d\omega = \int_{\partial V} \omega$$

2.3 Rotation Index

Recall, since we're in a plane curve, Torsion is constant zero. So, we stay in the plane spanned by $\{\vec{T}, \vec{N}\}$, which gives $\vec{\alpha}(s) = (x(s), y(s)) \in \mathbb{R}^2$.

Lemma 6. Reversing orientation changes the sign of the plane curvature

Proof. Left as excercise. □

Note that we can define th eplane curvature $\vec{n}$ with $\vec{T}' = 0$, since it's canonically defined by rotating $\vec{T}'$ by $90°$.

Definition 2.2. Consider $\vec{\beta} : \mathbb{R} \to \mathbb{R}^3$. We say that $\vec{\beta}$ is **closed** if $\exists a > 0$ such that for any $t \in \mathbb{R}$, $\vec{\beta}(t) = \vec{\beta}(t + a)$. I.e. $\vec{\beta}$ is periodic. Where $\min(a)$ is the period of β.

If we take some closed, smooth, regular curve β, and consider $\vec{T}(t)$, we note that $\vec{T}$ is also periodic.

Lemma 7. Let $\vec{\beta}(t)$ be a smooth parameterized curve with a as it's period. Then, we let $\vec{\alpha}(s) = \vec{\beta}(t(s))$, where s is the arc length, the period of α is different from a, but can be computed by

$$L = \int_0^a \left| \frac{\partial \vec{\beta}}{\partial t} \right| dt$$

Proof. We want to show that $\vec{\alpha}(s) = \vec{\alpha}(s+L)$. ☐

Definition 2.3 (Simple Curve)**.** Assume $\vec{\beta}(t)$ is a regular parameterization. It is simple if either

- $\vec{\beta} : \mathbb{R} \to \mathbb{R}^3$ is injective, or
- $\vec{\beta}$ is closed.

I missed a lecture because of the Stanford debate tournament, but in the last lecture we proved the rotation index theorem.

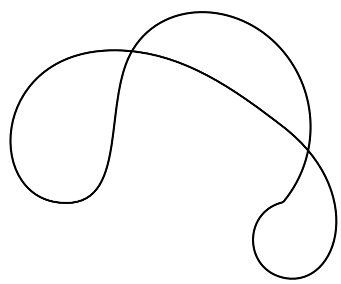
2.4 Convex Curves

We are going to work in $\mathbb{R}^2$ as the target space. Consider some line ℓ through $\mathbb{R}^2$. It will divide $\mathbb{R}^2$ into halves. There are two, H_1, H_2, with $H_1 \cap H_2 = \ell$ and $H_1 \cup H_2 = \mathbb{R}^2$. If we have some $\vec{\alpha} \subseteq H_1$ or H_2, then it lives in the half.

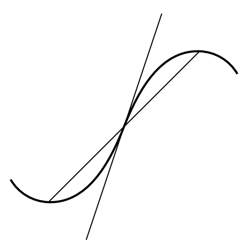
Definition 2.4. Let $\vec{\alpha}$ a regular curve. Then $\vec{\alpha}$ is **convex** if $\vec{\alpha}$ lives on one side of every tangent line.

Theorem 2.1. If $\vec{\alpha}$ is simple, closed and regular, then $\vec{\alpha}$ is convex if and only if $\kappa(s)$ has constant sign.

We just give examples of this. Consider some curve which is not simple.



The proof is roughly done by computing $\kappa(s) = \frac{\partial \theta(s)}{\partial s}$, proving that $\kappa(s)$ has constant sign is to say that $\theta(s)$ is monotonic (not strictly though, we may have zero curvature). $\theta(s)$ is the angle between $\vec{T}$ and some arbitrarily chosen line.



Corollary 2.1.1. Let $\vec{\alpha}$ simple, closed and convex. If $S_1 < S_2$ so that $\theta(S_1) = \theta(S_2)$, then on $[S_1, S_2]$, $\vec{\alpha}$ is a straight line.

Corollary 2.1.2. If we assume $\vec{\alpha}$ is convex, and there exists some line ℓ which intersects $\vec{\alpha}$ at three points, then the whole line segment must stay in $\vec{\alpha}$.

2.4.1 The Isoperimetric Inequality

For some convex, closed curve with fixed perimeter (total length), then we have a reasonable question: what is the maximum area the curve can bound?
Consider the case of a rectangle with $2(a+b) = L$, and $A = ab$. What is $\max(ab)$? Intuition tells us it's $\frac{L^2}{16}$. But we can impose different requirements.
If it's a regular curve, the largest area will be a circle. The conjecture that we make is that

Theorem 2.2 (Isoperimetric Inequality)**.** For some simple, closed curve with fixed perimeter L, the area bounded by the curve will be at most

$$A \leq \frac{L^2}{4\pi}$$

Moreover, A achieves this maximum if and only if, $\vec{\alpha}$ is a circle.

Proof. Let R denote the region bounded by the curve with positive orientation. Let $\vec{\alpha}(s) = (x(s), y(s))$, with $s \in \mathbb{R}$, with both x, y periodic by L. We give a way to calculate the area,

$$A = \iint_R dxdy$$

Since we've introduced differential forms, let's consider $d(xdy) = dxdy$, and $d(ydx) = -dxdy$. We can write $dxdy = \frac{1}{2}d(xdy - ydx)$. Then, we can rewrite as an integral of the differential of a 1-form over the region

$$A = \iint_R dxdy - \iint_R d\left(\frac{dxy - ydx}{2}\right)$$

We may rewrite this as the integration over the boundary of the one-form by

$$A = \iint_R dxdy - \iint_R d\left(\frac{dxy - ydx}{2}\right) = \int_{\partial R} \frac{1}{2}(xdy - ydx)$$

Otherwise, the line integration $\frac{1}{2}\int_{\vec{\alpha}} xdy - ydx$. We have imposed the requirement that x, y are of period L. If we rescale to some $\hat{x}(t), \hat{y}(t)$, such that they are 2π-periodic, we have $\hat{x}(t) = x\left(\frac{Lt}{2\pi}\right)$, then we have $s = \frac{L}{2\pi}t$, whicch gives a periodic function in 2π. Then, we can fourier expand these functions.

$$\hat{x} \sim \sum a_n e^{int} \qquad \hat{y} \sim \sum b_n e^{int}$$

We can think about the derivative $\hat{x}' = \sum_{n\in\mathbb{Z}} a_n(in)e^{int}$, and similary for $\hat{y}$. With these expressions then, we can rewrite our integral as[2]

$$A = \frac{1}{2}\int_0^{2\pi}(xy' - yx')dt = \pi\left|\sum_{n\in\mathbb{Z}}(a_n\overline{inb_n} - b_n\overline{ina_n})\right| = \pi\sum_{n\in\mathbb{Z}} n(|a_n|^2 + |b_n|^2)$$

$\square$

[2]we got kjind of lost here at the end, but we can check for general change of variables.