Physics 105: Analytic Mechanics **Spring 2019**

Lecture 14: π Day

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14.1 Rigid Body Motion

We can write the kinetic energy of a rotating rigid body by the sum of its CM kinetic energy and that due to rotation,

$$T = \frac{1}{2}MV_C^2 + \frac{1}{2}\sum_{\alpha,\beta} I_{\alpha\beta}\omega_\alpha\omega_\beta = \frac{1}{2}MV_C^2 + \frac{1}{2}\vec{\omega}\hat{I}\vec{\omega} \tag{14.1}$$

where the components of the inertia tensor are given by

$$I_{\alpha,\beta} = \iiint \rho(\vec{r})(r^2\delta_{\alpha\beta} - x_\alpha x_\beta)d^3\vec{r} \tag{14.2}$$

We can write the Lagrangian for this system,

$$\mathcal{L} = T - V(\vec{r}_{CM}, \theta, \varphi, \psi) \tag{14.3}$$

We can use conservation of angular momentum to gain more information about this system.

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i \implies \dot{\vec{L}} = \frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i = \vec{N} \tag{14.4}$$

We can write $\vec{F}_i = m_i \times (\vec{\omega} \times \vec{r}_i)$, so the angular momentum is

$$\vec{L} = \sum_i m_i \left(\vec{\omega}\vec{r}_i^2 - \vec{r}_i(\vec{\omega}\vec{r}_i)\right) \tag{14.5}$$

Comparing this to the definition $L_\alpha = I_{\alpha\beta}\omega_\beta$, we see that the above is the definition of the components of the inertia tensor.

14.2 Calculating the Inertia Tensor and Rotational Dynamics

To translate the inertia tensor from being centered around a point O to being centered around a point C, with position vector $\vec{R}$ between them, the formula is

$$I_{ij}^{(O)} = I_{ij}^{(C)} + M(R^2\delta_{ij} - X_i X_j) \tag{14.6}$$

Consider a rectangle in the $x-y$ plane with width a and height b, rotating about its positive-slope diagonal. In this case, we can compute the components of the inertia tensor,

$$I_{xy} = I_{yx} = \int dxdy\sigma_0(r^2\delta_{xy} - xy) = 0 \tag{14.7}$$

$$I_{xz} = I_{zx} = \int dxdz\sigma_0(r^2\delta_{xz} - xz) = 0 \tag{14.8}$$

$$I_{xx} = \int dxdy\sigma_0(r^2 - x^2) = \sigma_0 \int dxdyy^2 = \frac{1}{12}Mb^2 \tag{14.9}$$

$$I_{yy} = \frac{1}{12}Ma^2 \tag{14.10}$$

$$I_{zz} = \int \sigma_0(x^2 + y^2)dxdy = I_{xx} + I_{yy} = \frac{1}{12}M(a^2 + b^2) \tag{14.11}$$

In general, for any rigid body with principal axes along x, y, z, if a body rotates about any of these principal axes (say x) then its inertia tensor is diagonal, so $L_x = I_{xx}\omega_x$ and if there is rotation only along x, then $L_y = L_z = 0$.

Returning to the case of the rectangle, we write

$$L_x = I_{xx}\omega_x = I_{xx}\omega\cos\theta = \frac{1}{12}Mb^2\omega\cos\theta \tag{14.12}$$

$$L_y = I_{yy}\omega_y = I_{yy}\omega\sin\theta = \frac{1}{12}Ma^2\omega\sin\theta \tag{14.13}$$

$$L_z = 0 \tag{14.14}$$

We can use this definition and $\vec{\omega} = (\omega\cos\theta, \omega\sin\theta, 0)$ to write $\vec{N} = \frac{d\vec{L}}{dt}$ as

$$\vec{N} = (I_{xx} - I_{yy})\omega_x\omega_y\hat{z}(t) = \frac{1}{12}M(b^2 - a^2)\omega^2\frac{ab}{a^2 + b^2} \tag{14.15}$$

Consider a cube of side a with the origin at a corner. The components of the inertia tensor can be calculated by shifting relative to the center, $\vec{R} = \left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$. (Get details later)

Cone:

$$\vec{N} = \left(\frac{d\vec{L}}{dt}\right)_L = \left(\frac{d\vec{L}}{dt}\right)_B + \vec{\omega} \times \vec{L} \tag{14.16}$$

From this, we find the components of $\vec{N}$,

$$N_1 = I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\omega_2\omega_3 \tag{14.17}$$

$$N_2 = I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\omega_3\omega_1 \tag{14.18}$$

$$N_3 = I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\omega_1\omega_2 \tag{14.19}$$

These equations describe the rotational dynamics of the system. We can solve these as a coupled system of differential equations to get

$$\vec{\omega} = (A\cos(\Omega t + \varphi), A\sin(\Omega t + \varphi), \omega_3) \tag{14.20}$$