**Abstract**

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## 0.1 Method of Partial Waves (Guest Lecturer N. Yao)

Idea is we want to learn about scatter-er by takoing $\varphi(r) = \varphi_{\text{inc}}(r) + \varphi_{\text{sc}}(r)$. Partial waves is distinct from born approximation. We want to calculate phase shifts in low energies relative to the potential. We're going to assume a radial potential $V(r)$, will give us that angular momentum is a good basis. We can break this down into $\ell = 0, 1, 2 \ldots$, and get that for each $\ell$ we have an individual scattering problem.

$$g(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell m}(\theta, \varphi)$$

since $f$ independent of $\varphi$, we really have some $f_k(\theta)$, where

$$f_k(\theta) = \sum_{\ell=0}^{\infty} f_{\ell 0} Y_{\ell 0}(\theta)$$

Now, we're going to change into the *Legendre Basis*, where

$$f_k(\theta) = \sum_{\ell=0}^{\infty} f_{\ell 0} Y_{\ell 0}(\theta) = \sum \frac{2\ell+1}{k} f_\ell P_\ell(\cos\theta)$$

where now, our $f_\ell$ are known as the **partial wave scattering amplitudes** $f_\ell(k)$. So now, if we have

$$\varphi_{\text{inc}}(r, \theta) = e^{ikr\cos\theta} = \sum_\ell (2\ell+1) u_\ell(kr) P_\ell(\cos\theta)$$

The Legendre polynomials have a nice property

$$\int_{-1}^{1} \omega P_\ell(\omega) P_\varpi(\omega) = \frac{2}{2\ell+1}\delta_{\ell\varpi}$$

Now, we're going to solve for $u_\ell(kr)$. Let's multipy $\varphi_{\text{inc}}$ by $P_\ell(\cos\theta)$ and integrate[1]

$$u_\ell(kr) = \frac{1}{2}\int_{-1}^{1} d\omega e^{ikr\omega} P_\ell(\omega)$$

If we integrate this by parts, we have

$$u_\ell(kr) = -\frac{i}{2kr}\left[e^{ikr\omega}P_\ell(\omega)\right]_{-1}^{1} + \frac{1}{2ikr}\int_{-1}^{1} d\omega e^{ikr\omega}\frac{\partial P_\ell}{\partial\omega}$$

where

$$\frac{1}{2ikr}\int_{-1}^{1} d\omega e^{ikr\omega}\frac{\partial P_\ell}{\partial\omega} \sim \frac{1}{r^2}$$

The fina answer ends up being

$$\varphi_{\text{inc}}(r) = \sum_{\ell=0}^{\infty}\frac{2\ell+1}{2ik}\left[(-1)^{\ell+1}\frac{e^{-ikr}}{r} + \frac{e^{ikr}}{r}\right]P_\ell(\cos\theta)$$

If we put this togehter with the scattered component, we get the full wavefunction

$$\sum_\ell \frac{2\ell+1}{k}\left[(-1)^{\ell+1}\frac{e^{-ikr}}{r} + (1 + 2if_\ell(k))\frac{e^{ikr}}{r}\right]P_\ell(\cos\theta)$$

So we want a unitarity property, i.e. conservation of probability density.

$$|(-1)^{\ell+1}| = |1 + 2if_\ell(k)|$$

and

$$1 + 2if_\ell(k) = e^{2i\delta_\ell(k)}$$

where $\delta_\ell(k)$ is known as the **Partial Wave Scattering Phase Shift**. This all gives that

$$f(\theta) = \frac{1}{k}\sum_\ell (2\ell+1)\sin(\delta_\ell(k))e^{i\delta_\ell(k)}P_\ell(\cos\theta)$$

Intuitively, we should have that the net effect of the potential on the wavefunction will be a phase shift. This is because we have reduced the equation in $f(k, \theta)$ in two variables to an infinite series of $f_\ell(k)$, where for small potentials we can consider finitely many $\ell$ to a good approximation. For future reference, we have the scattering length $a_\ell \frac{\delta_\ell}{k}$.

[1] check out david Tong's lectures