Abstract

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0.0.1 Deriving Higher Order Corrections

Now, lets say we have some initial state $|\psi_i(0)\rangle = |i\rangle$, and we want to transition it into some other state. We should have in the schroedinger picture $|\psi_i(t)\rangle = U(t,0)\,|i\rangle$ or, in the interaction picture

$$|\psi_i(t)\rangle = U_0(t,t_0)U_I(t,t_0)\,|i\rangle$$

Now, recall our $d_f(t) = e^{\frac{i}{\hbar}E_f t}\langle f|\psi_i(t)\rangle$, so we can write

$$\begin{aligned} d_f(t) &= e^{\frac{i}{\hbar}E_f t}\langle f|\psi_i(t)\rangle \\ &= e^{\frac{i}{\hbar}E_f t}\langle f|\,U_0(t,t_0)U_I(t,t_0))\,|i\rangle \\ &= \langle f|\,U_i(t,t_0)\,|i\rangle \end{aligned}$$

This gives us a formula for d_f.

$$d_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_{t_0}^{t}\langle f|\,V_I(t_1)\,|i\rangle\,dt_1$$

and, if we want to go to second order,

$$d_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_{t_0}^{t}\langle f|\,V_I(t_1)\,|i\rangle\,dt_1 - \frac{1}{\hbar^2}\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\,\langle f|\,V_I(t_1)V_I(t_2)\,|i\rangle$$

in the second term, we're going to insert a unit matrix, resolution of identity, so that we have

$$d_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_{t_0}^{t}\langle f|\,V_I(t_1)\,|i\rangle\,dt_1 - \frac{1}{\hbar^2}\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\sum_m\langle f|\,V_I(t_1)\,|m\rangle\langle m|\,V_I(t_2)\,|i\rangle$$

if we now insert the form for V_I we derived earlier, we should have, letting $V_{fi}(t) = \langle f|\,V\,|i\rangle$,

$$d_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_{t_0}^{t}dt_1 e^{i\omega_{fi}t_1}V_{fi} - \sum_m\frac{1}{\hbar^2}\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2 e^{i\omega_{fm}t_1}e^{i\omega_{mi}t_2}V_{fm}(t_1)V_{mi}(t_2)$$

This is really hinting at the path integral formulation of quantum mechancis, but we're gonna yeet that now.

0.1 Ex: Harmonic Perturbation

Consider our perturbation to be $V(t) = \hat{V}e^{-i\omega t+\varepsilon t}$ If we assume that $\langle f|\,\hat{V}\,|i\rangle = 0$, we can immediately compute that

$$d_f^{(2)} = -\frac{1}{\hbar^2}\sum_m\langle f|\,\hat{V}\,|m\rangle\langle m|\,\hat{V}\,|i\rangle\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2 e^{i(\omega_{fm}-\omega-i\varepsilon)t_1}e^{i(\omega_{mi}-\omega-i\varepsilon)t_2}$$

This is kind of a nasty integral, but we can get it by taking

$$d_f^{(2)} = \frac{1}{\hbar}e^{i(\omega_{fi}-2\omega)}\frac{e^{2\varepsilon t}}{\omega_{fi}-2\omega-2i\varepsilon}\sum_m\frac{\langle f|\,V\,|m\rangle\langle m|\,V\,|i\rangle}{\omega_{mi}-\omega-i\varepsilon}$$

Then, we want the transition rate, which we find by taking the time derivative of the modulus as $\varepsilon\to 0$, which gives

$$\lim_{\varepsilon\to 0}W_{fi} = \lim_{\varepsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}t}|d_f^{(2)}|^2 = \frac{2\pi}{\hbar^4}\left|\sum_m\frac{V_{fm}V_{mi}}{\omega_{mi}-\omega-i\varepsilon}\right|^2\delta(\omega_{fi}-2\omega)$$

where the δ appears since there is a lorenzian term in there. It's not super necessary to keep the ε in the denominator of the sum, but he's about to explain why we did.

$$W_{fi} = \frac{2\pi}{\hbar^4}\left|\sum_m\frac{V_{fm}V_{mi}}{\omega_{mi}-\omega}\right|^2\delta(\omega_{fi}-2\omega)$$

Basically, just watch out for resonances. Our interpretation of the intermediate terms should be in terms of paths,