Physics-105-Lecture-Notes-04-09-2019

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April 9, 2019

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Abstract

A single PDF with all lectures in a single document can be downloaded at `https://www.dropbox.com/sh/8sqzvxghvbjifco/AAC9LoSRnsRQDp7pYedgWpQMa?dl=0`. The password is 'analytic.mech.dsp'. This file was automatically generated using a script, so there might be some errors. If there are, you can contact me at `mailto:ctdunc@berkeley.edu`.

1 Small Oscillations

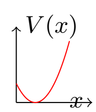
1.1 Stationary Points

Consider a lagrangian $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x)$, with generalized coordinates $(x, \dot{x})$. Recall ELE

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

a stationary point is a point x_0 such that $\dot{x}_0 = 0 \Rightarrow = x_0 \forall t$, which gives that $\frac{\partial V}{\partial x} = 0$. It's a point with no force acting on a system.

mass on spring A mass on a spring (simple harmonic oscillator) has potential $V(x) = \frac{1}{2}k(x - x_*)^2$, which looks as



at those stationary points, there are small oscillations that we can taylor expand to have that

$$m\delta\ddot{x} = -\frac{\mathrm{d}^2V}{\mathrm{d}x^2}_{x=x_i}\delta x$$
$$\delta x = Ae^{-i\omega t}$$
$$\omega^2 = \frac{1}{m}\frac{\mathrm{d}^2V}{\mathrm{d}x^2}_{x=x_i}$$

The most general expression for a lagrangian is

$$\mathcal{L}(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

with

$$T = \frac{1}{2}\sum T_{ik}\dot{q}_i\dot{q}_k$$

or, in other words

$$T = \frac{1}{2}\sum_i m_i|\dot{\vec{r}_i}|^2$$

which allows some reexpression of T as

$$T = \frac{1}{2}\sum_{i,\alpha\beta} m_i\frac{\partial \vec{r}_i}{\partial q_\alpha}\frac{\partial \vec{r}_i}{\partial q_\beta}\dot{q}_\alpha\dot{q}_\beta$$
$$= \frac{1}{2}\sum_{\alpha,\beta}\left[\sum_i m_i\frac{\partial \vec{r}_i}{\partial q_\alpha}\frac{\partial \vec{r}_i}{\partial q_\beta}\right]\dot{q}_\alpha\dot{q}_\beta$$

if we wanted to write this down as a symmetric tensor ($T_{ij} = T_{ji}$), then we should take some kinetic energy of the form

$$T = \frac{1}{2}(T_{11}\dot{q}_1^2 + T_{12}\dot{q}_1\dot{q}_2 + T_{21}\dot{q}_1\dot{q}_2 + T_{2@}\dot{q}_2^2)$$

and reexpress it as

$$T = \frac{1}{2}(T_{11}\dot{q}_1^2 + \frac{T_{12} + T_{21}}{2}\dot{q}_1\dot{q}_2 + \frac{T_{12} + T21}{2}\dot{q}_2\dot{q}_1 + T_{22}\dot{q}_2^2)$$

For a more abstrat system, for a stationary point q^0 where the superscript denotes stationary status, not exponentiation, we have that $\dot{q}_k = 0$ for ever k index, and that $\frac{\partial V}{\partial q_\alpha} = 0 \forall \alpha \in$ our range. For the case where T_{ik} depends on q, then our lagrangian is

$$\mathcal{L} = \frac{1}{2}\sum_{i,k} T_{ik}(q)\dot{q}_i\dot{q}_k - V(q)$$

which we can apply standard ops to to derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_k T_{\alpha k}\dot{q}_k\right] = \frac{1}{2}\sum_{i,k}\frac{\partial T_{ik}}{\partial q_\alpha}\dot{q}_i\dot{q}_k - \frac{\partial V}{\partial q_\alpha}$$

the final form comes out to be

$$\sum_k T_{\alpha k}\ddot{q}_k = -\frac{\partial V}{\partial q_\alpha} + \frac{1}{2}\sum_{i,k}\frac{\partial T_{ik}}{\partial q_\alpha}\dot{q}_i\dot{q}_k - \sum_{k,s}\frac{\partial T_{\alpha k}}{\partial q_s}\dot{q}_s\dot{q}_k$$

which are newtons equations for this lagrangian. Now, we are considering the behavior of a system around a stationary point in n generalized coordinates. We take the lagrangian and expand

$$\mathcal{L} = \frac{1}{2}\sum_{i,k} T_{ik}(q)\dot{q}_i\dot{q}_k - V(q_i)$$

if we expand up to quadratic terms, we are taking

$$\mathcal{L} = \frac{1}{2}\sum_{i,k} T_{ik}(q^{(0)} + \delta q)\delta\dot{q}_i\delta\dot{q}_k - V(q^{(0)} - \sum_i \frac{\partial V}{\partial q_i}_0 \delta q_i - \frac{1}{2}\sum_{i,j}\frac{\partial^2 V_{ij}}{\partial q_i \partial q_j}\delta q_i \delta q_j$$

Constant terms we set to zero, and we get that (setting the mass tensor $T_{ik}(q^{(0)} = m_{ik}$ and V_{ij} another tensor whos name i forget, we have

$$\mathcal{L} = \frac{1}{2}\sum_{i,k} m_{ik}\dot{q}_i\dot{q}_k - \frac{1}{2}\sum_{ik} V_{ik}q_i q_k$$

in a single equation, we just end up with $m\ddot{x} = -kx$ which is what we expect. This is apparently pretty easy in particular systems, so let's take a look at an example.

1.1.1 Example: Coupled Pendulum

Consider two identitical masses connected by two identical ropes, ith generalized coordinates ϕ_1, ϕ_2, in a cartesian x, y system. So,

$$x_1 = e\sin\phi_1 \qquad y_1 = -e\cos\phi_1$$
$$x_2 = e\sin\phi_1 + e\sin(\phi_1 + \phi_2) \qquad y_2 = -e\cos\phi_1 - e\cos(\phi_1 + \phi_2)$$

With the conclusion that

$$T_1 = \frac{1}{2}ml^2\dot{\phi}_1^2$$

and

$$T_2 = \frac{1}{2}m[l^2\dot{\phi}_1^2 + l^2(\dot{\phi}_1 + \dot{\phi}_2)^2 + 2l^2\dot{\phi}_1(\dot{\phi}_1 + \dot{\phi}_2)\cos\phi_2]$$

The total kinetic energy then, is (after a lot of algebraic simplification

$$T = \frac{1}{2}ml^2\left[2\dot{\phi}_1^2 + (\dot{\phi}_1 + \dot{\phi}_2)^2 + 2\dot{\phi}_1(\dot{\phi}_1 + \dot{\phi}_2)\cos\phi_2\right]$$

Potential energy is given by

$$\begin{aligned} V &= -mgl\cos\phi_1 - mg(l\cos\phi_1 + l\cos(\phi_1 + \phi_2) \\ &= -mgl(2\cos\phi_1 + \cos(\phi_1 + \phi_2) \end{aligned}$$

So we want

$$\frac{\partial V}{\partial \phi_1} = 0 \Rightarrow \sin(\phi_1) + \sin(\phi_1 + \phi_2) = 0$$

If we want to standardize our kinetic energy, we should rewrite it as

$$\begin{aligned} T &= \frac{1}{2}ml^2[2\dot{\phi}_1^2 + \dot{\phi}_1^2 + 2\dot{\phi}_1\dot{\phi}_2 + \dot{\phi}_2^2 + 2\dot{\phi}_1^2\cos\phi_2 + 2\dot{\phi}_1\dot{\phi}_2\cos\phi_2] \\ &= \frac{1}{2}ml^2[(3 + 2\cos\phi_2)\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2(1 + \cos\phi_2)] \end{aligned}$$

which gives

$$\begin{aligned} T_{11} &= (3 + 2\cos\phi_2)ml^2 \\ T_{12} = T_{21} &= (1 + \cos\phi_2)ml^2 \\ T_{22} &= ml^2 \end{aligned}$$

Now, we have to expand the system, so that

$$V = -mgl\left[\left(1 - \frac{\phi_1^2}{2}\right)2 + \left(1 - \frac{(\phi_1+\phi_2)^2}{2}\right)\right]$$
$$V = \frac{1}{2}mgl[2\phi_1^2 + (\phi_1+\phi_2)^2] = \frac{1}{2}mgl[3\phi_1^2 + 2\phi_1\phi_2 + \phi_2^2]$$

when we ignore constants, which is allowable because of the lagrangian formalism. Kinetic energy about our expansion goes as

$$\frac{1}{2}ml^2\left[5\dot\phi_1^2 + 4\dot\phi_1\dot\phi_2 + \dot\phi_2^2\right]$$

Finally, this gives us the lagrangian

$$\mathcal{L} = \frac{1}{2}ml^2\left(5\dot\phi_1^2 + 4\dot\phi_1\dot\phi_2 + \dot\phi_2^2\right) - \frac{1}{2}mgl(3\phi_1^2\phi_1\phi_2 + \phi_2^2)$$

which means we can write down

$$m_{ik} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$
$$V_{ik} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\omega_0^2$$

where $\omega_0^2 = g/l$. This just gives us a solution of the form $q_k = A_k e^{-i\omega t}$, which we know how to solve.

$$-\omega^2 m_{\alpha k} + V_{\alpha k})A_k = 0$$

which is a statement about whether or not the solution has nontrivial solutions, i.e. it only does if

$$\det\left(\hat{V} - \omega^2\hat{m}\right) = 0$$

Now, let's try taking

$$\sum_{ik} A_i^{(s)}(V_{ik} - \omega_s^2 m_{ik})A_k^{(s)} = 0 \Rightarrow \omega_S^2 = \frac{V_{ik}A_i^{(s)}A_k^{(s)}}{m_{ik}A_i^{(s)}A_k^{(s)}}$$

We now want to solve

$$\det\left(\begin{bmatrix} 3\omega_0^2 - 5\omega^2 & -\omega_0^2 - 2\omega^2 \\ \omega_0^2 - 2\omega^2 & \omega_0^2 - \omega^2 \end{bmatrix}\right)$$

which gives a characterisitc equation

$$\omega^4 - 4\omega^2\omega_0^2 + 2\omega_0^2 = 0$$

which gives that

$$\omega_1^2 = \omega_0^2(2 + \sqrt{2})$$

and

$$\omega_2^2 = \omega_0^2(2 - \sqrt{2})$$

and then we find the eigenvectors of this matrix using usual linear algebra methods.