Physics 105: Analytic Mechanics **Spring 2019**

Lecture 3: Calculus of Variations

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Homework 1 is due Friday, Feb 8, and it consists of problems 6.1, 6.2, 6.11, 6.17, 6.22, 6.24 from Taylor.

General OH: Campbell 355, Tuesday 11-12.

3.1 The Idea of Lagrangian Mechanics

There are two major problems with using Newtonian mechanics for everything. One is it is not very general, dealing only with the motion of particles, and the other is it usually deals well only with orthogonal Cartesian coordinates. The calculus of variations is the mathematical foundation for dealing with mechanics in generalized coordinates $q_i (i = 1, \ldots, N)$ or in Hamiltonian mechanics generalized momenta p_i. These q_is define a *configuration space*, and when they are coupled with the p_is, it creates *phase space*. Instead of a path parameterized by x, y, z, we have a path in the configuration space.

We know from Newtonian mechanics that

$$\vec{F} = m\ddot{q} \tag{3.1}$$

We need two initial or boundary conditions to completely specify q here. For example, let $\ddot{q} = g$, considering the case of free fall in a gravitational field:

$$\begin{aligned} \vec{F} &= mg && (3.2)\\ \ddot{q} &= g && (3.3)\\ \dot{q} &= gt + v_0 \text{ where } v_0 = \dot{q}(t=0) && (3.4)\\ q &= \frac{1}{2}gt^2 + v_0 t + q_0 \text{ where } q_0 = q(t=0) && (3.5) \end{aligned}$$

Here, we need the initial conditions; what the particle is doing, in terms of its velocity and position, at $t = 0$. Lagrangian mechanics deals with boundary conditions instead, starting from $q(t_1)$ and $q(t_2)$. Lagrangian mechanics starts with these two data points, and asks "what path in q space does the system follow from $q(t_1)$ to $q(t_2)$?".

3.2 Calculus of Variations

Consider the example of a plane flying between SFO and EWR. Airlines want to save money, so other than paying their workers very poorly, they can minimize fuel. The path that minimizes fuel may not be the shortest path in terms of distance, for example, if there are strong head winds over the main path. It may be around those on a path that is longer. We express this idea with a *functional*,

$$F(\vec{x}) = \int_{t_1}^{t_2} f(\vec{x}, \dot{\vec{x}}, t)dt \tag{3.6}$$

where we want to minimize F, which in this example might be fuel usage.

The usual canonical example of a problem that can be solved quickly with the calculus of variations is the *brachistochrone problem*. Consider two points A and B in a plane, with gravity directed downward. What trajectory would a particle take from A to B to minimize its travel time? This is not a straight line between them, because of the force of gravity.

We can write the travel time as an integral over the path,

$$t = \int_{(x_A, y_A)}^{(x_B, y_B)} \frac{ds}{v} \tag{3.7}$$

which is the same sort of functional expression, where we want to minimize a quantity that is expressed as an integral. (This problem will be solved later after we have built the machinery of variational calculus.)

3.3 The Machinery of Variational Calculus

Again, the generalized statement of a functional is

$$F(\vec{x}) = \int_{t_1}^{t_2} f(\vec{x}, \dot{\vec{x}}, t)dt \tag{3.8}$$

By definition of the optimal path, any perturbation away from that path will increase the value of F. Let this perturbation be $\eta(t)$ and add the requirement that $\eta(t_1) = \eta(t_2) = 0$.

$$x(t) = \bar{x}(t) + \eta(t) \approx \bar{x}(t) \tag{3.9}$$

Then, we can rewrite the functional expression. By definition, $F(\bar{\vec{x}}) \leq F(\vec{x})$, where

$$F(\vec{x}) = \int_{t_1}^{t_2} f(\bar{\vec{x}} + \eta, \dot{\bar{\vec{x}}} + \dot{\eta}, t)dt \tag{3.10}$$

We expand this as a Taylor series,

$$F(\vec{x}) = \int_{t_1}^{t_2} \left(f(\bar{x}, \dot{\bar{x}}, t) + \frac{\partial f}{\partial x} \cdot \eta + \frac{\partial f}{\partial \dot{x}} \cdot \dot{\eta} + \dots \right) dt \tag{3.11}$$

Therefore the difference between the two paths is

$$F(\vec{x}) - F(\vec{\bar{x}}) = \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x}\eta + \frac{\partial f}{\partial \dot{x}}\dot{\eta} \right) dt \tag{3.12}$$

We apply integration by parts to the second part of this,

$$\int_{t_1}^{t_2} \frac{\partial f}{\partial \dot{x}}\dot{\eta}dt = \left.\frac{\partial f}{\partial \dot{x}}\eta\right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)\eta dt \tag{3.13}$$

The first term drops out, because we stipulated initially that a deviation would not change the end points of a path, i.e. $\eta(t_1) = \eta(t_2) = 0$. Therefore overall we get

$$F(x) = F(\bar{x}) + \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} \right) \cdot \eta dt \tag{3.14}$$

If we consider the deviation to be small, it follows that the integral must be zero, i.e.

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = \frac{\partial f}{\partial x} \tag{3.15}$$

This is the *Euler-Lagrange* equation.

In general, this will have to be applied as many times as there are dimensions in the configuration space.

3.4 Applying Variational Calculus

Consider two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$. We can use the E-L equations to find that the shortest path between them is a line. The functional expression is

$$L = \int_{(x_1,y_1)}^{(x_2,y_2)} ds = \int_{x_1}^{x_2} \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{3.16}$$

$$L = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx \tag{3.17}$$

This is the same form as the general functional setup for a variational calculus problem, $f(y, y', x) = \sqrt{1 + y'^2}$, therefore we can apply the E-L equation:

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \tag{3.18}$$

where $\frac{\partial f}{\partial y}$ is trivially zero. We get

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0 \tag{3.19}$$

or

$$y' = c\sqrt{1+y'^2} \tag{3.20}$$
$$y' = \text{const} \tag{3.21}$$
$$\frac{dy}{dx} = m \tag{3.22}$$
$$y = mx + b \tag{3.23}$$

as we expected.

3.5 Brachistochrone Problem

We previously wrote the time equation for the brachistochrone problem. Let point A be the origin for convenience. We know that $m\ddot{x} = 0$ and $m\ddot{y} = mg$. From conservation of energy, we can find the velocity (as is required in the time equation),

$$\frac{1}{2}mv^2 = mgy \implies v = \sqrt{2gy} \tag{3.24}$$

and the differential element ds is the same as in the straight-line example. Therefore we get

$$t = \int_0^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}dx \tag{3.25}$$

This gives us the function under the integral that is required as input to the E-L equation,

$$f(y, y', x) = \frac{\sqrt{1+y'^2}}{\sqrt{y}} \tag{3.26}$$

to which we can then apply the equations and get the path. However, there is a shortcut using the Hamiltonian, which we will see in a few months. Take H as follows:

$$H = y'\frac{\partial f}{\partial y'} - f \tag{3.27}$$

Then, take an x derivative:

$$\frac{dH}{dx} = y'' \frac{\partial f}{\partial y'} + y' \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - y' \frac{\partial f}{\partial y} - y'' \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial x} \tag{3.28}$$

which can be simplified to

$$\frac{dH}{dx} = y' \left(\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y}\right) - \frac{\partial f}{\partial x} \tag{3.29}$$

which just gives us $\frac{dH}{dx} = -\frac{\partial f}{\partial x}$, because the first part is exactly the E-L equation and is therefore zero. This is an alternative statement of conservation of energy: when there is no time dependence, H is conserved.

Returning to the brachistochrone problem, we have

$$f = \sqrt{\frac{1 + y'^2}{y}} \tag{3.30}$$

and so the Hamiltonian is

$$\frac{-1}{\sqrt{y(1 + y'^2)}} = C \tag{3.31}$$

(as we just saw, the Hamiltonian is a constant)

Solving and setting $\sqrt{y} = \sqrt{C} \sin\theta$, we get

$$x = \frac{C}{2}(2\theta - \sin 2\theta) \tag{3.32}$$

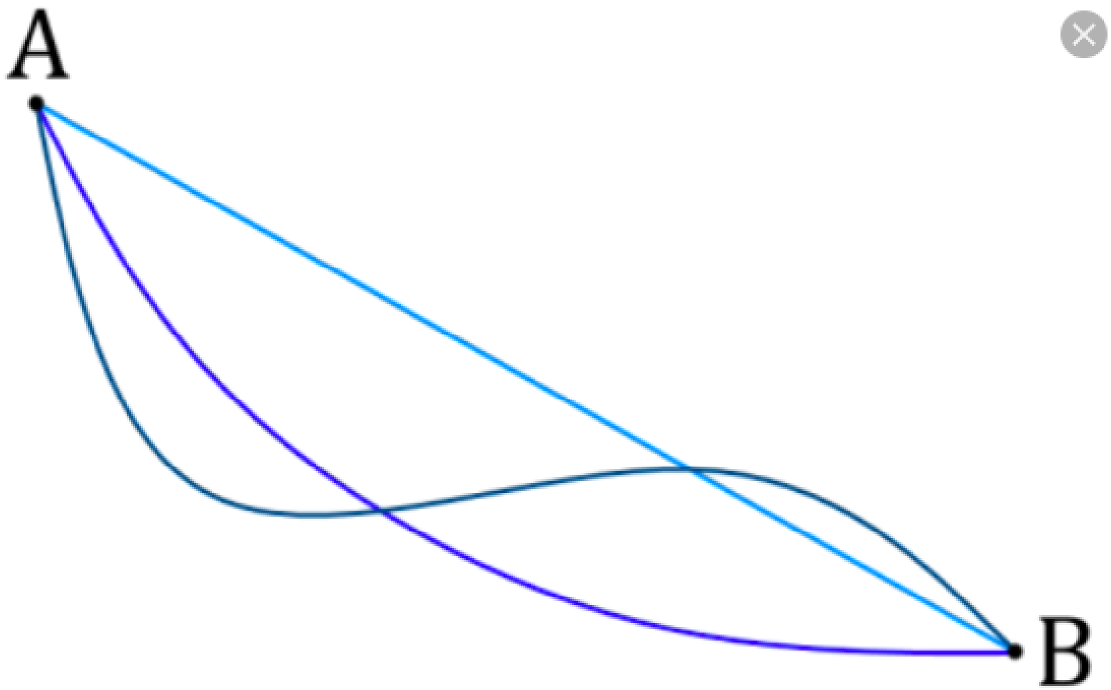
$$y = C \sin^2\theta = C(1 - \cos^2 2\theta) \tag{3.33}$$

Therefore, setting $\varphi = 2\theta$, we get

$$x = C(\varphi - \sin\varphi) \tag{3.34}$$

$$y = C(1 - \cos\varphi) \tag{3.35}$$

This is the equation for a cycloid.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