Physics-105-Lecture-Notes-04-16-2019

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April 17, 2019

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Abstract

A single PDF with all lectures in a single document can be downloaded at `https://www.dropbox.com/sh/8sqzvxghvbjifco/AAC9LoSRnsRQDp7pYedgWpQMa?dl=0`. The password is 'analytic.mech.dsp'. This file was automatically generated using a script, so there might be some errors. If there are, you can contact me at `mailto:ctdunc@berkeley.edu`.

0.1 Something New

Considedr N masses connected by some medium with tension τ, whihcm eans we need y_n $n = 1 \to N$, withh yeach y_k given as the displacement above equilibrium Lagrangian given as

$$\mathcal{L} = \frac{1}{2}\left(\sum_{k=1}^{N} m\dot{y}_k^2 - \sum_{k=0}^{n} \frac{\tau}{d}(y_k - y_{k+1})^2\right) : y_0 = 0, y_{N+1} = 0$$

we need to assume tht $m_{ik} = m\delta_{ik}$, and we have that

$$V_{ik} = \frac{\tau}{d}\begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots \\ 0 & -1 & 2 & -1 & \dots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & -1 & 2 \end{bmatrix}$$

So, if we want to find the eigenmodes, we just take $||\hat{V} - \omega^2\hat{m}|| = 0$, which gives us

$$\det\left|\begin{bmatrix} 2 - \frac{\omega^2}{\omega_0^2} & -1 & \dots & 0 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 2 - \frac{\omega^2}{\omega_0^2} \end{bmatrix}\right|$$

If we wanted to solve this for two masses, we have

$$\det\begin{vmatrix} x & -1 \\ -1 & x \end{vmatrix}$$

in 3 dimensions, we should have

$$\det\begin{vmatrix} x & -1 & 0 \\ -1 & x & -1 \\ 0 & -1 & x \end{vmatrix}$$

which gives eigenfrequencies $\omega = \omega_0\sqrt{2}$ and $\omega = \omega_0\sqrt{2 \pm \sqrt{2}}$. Also always true that $\omega \not\to \infty$. You can just use a computer to do this, but we can use recursive relations. Lets call Λ_N the determinant of our giant matrix of said form. You can use a recurrence relation $\Lambda_N = x\Lambda_{N-1} - \Lambda_{N-2}$. If we write down lagranges equation, we get

$$m\ddot{y}_k = \frac{\partial\mathcal{L}}{\partial y_k} = -\frac{\tau}{d}(y_k - y_{k+1}) - \frac{\tau}{d}(y_k - y_{k-1})$$

if we guess teh form is of $y_k = e^{i(k\gamma+\delta)}$, we can reqrite the above as

$$\omega^2 e^{ik\gamma} = \omega_0^2 e^{ik\gamma}(1 - e^{i\gamma}) + \omega_0^2 e^{ik\gamma}(1 - e^{-i\gamma})$$

which can simplify do

$$\omega^2 = \omega_0^2(2 - 2\cos\gamma) = 4\omega_0^2\sin^2\frac{\gamma}{2}$$

of course γ not arbitrary, because we must have $y_0 = y_{N+1} = 0$, which means we must have

$$y_k = \cos(ik\gamma + \delta)\cos(\omega t + \varphi)$$

which givess $y_0 = \cos\delta\cos(\omega t + \varphi) \Rightarrow \delta = \frac{(2n+1)\pi}{2}$. and $\gamma = \frac{\pi n}{N+1}$, which just gives out standing waves in the end! We can think of γ/d as the wavenumber, which, if we think about $\frac{2\pi}{\lambda} = \frac{\gamma}{d}$, our wavenumber is not arbitrary, which means we need to have that equal $\frac{\pi n}{d(N+1)}$, where the denominator is L, the length between boundaries of the medium. Or, $\frac{L}{\lambda} = \frac{n}{2}$.

0.2 Traveling Wave

This is *not a general solution* however. Imagine the case where we perturb one mass in the center of hundreds of masses. γ is not fixed here, because the boundary conditions don't know about the pertubation until later. We can try to find a solution for some traveling wave $y_k = A_k e^{i(k\gamma - \omega t)}$, where there's some dispersion relation $\omega = 2\omega_0 \sin\frac{\gamma}{2}$. If we assume small γ, we have immediately that $\omega = \omega_0\gamma = \omega_0 d\vec{k}$ where $\vec{k}$ is the wavevector, with some speed of sound $c_S = \sqrt{\frac{\tau}{md}} \times d = \sqrt{\frac{\tau d}{m}}$. We can be more strict though. Let's consider the continuous limit of our system, letting $d \to 0, k \to \infty, m \to 0$, with linear mass density $\rho = \frac{m}{d}$. Now, we have some function $y(x,t)$, with

$$\ddot{y} = \omega_0^2 \frac{\partial y_{k+1/2}}{\partial x} d - \omega_0^2 \frac{\partial y_{k-1/2}}{\partial x} d = \omega_0^2 d^2 \frac{\partial^2 y}{\partial x^2}$$

which is just the wave equation, for constant density, linear mass density.

$$\frac{\partial^2 y}{\partial t^2} = \omega_0^2 d^2 \frac{\partial^2 y}{\partial x^2}$$

Going through the derivation again, we have the more general form of

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{\rho(x)} \frac{\partial}{\partial x}\left[\tau(x)\frac{\partial y}{\partial x}\right]$$

If we rewrite the wave equation as

$$\frac{\partial^2 y}{\partial t^2} = c_s^2 \frac{\partial^2 y}{\partial x^2}$$

we find that in this wave equation, $\frac{\omega}{c_s} L = \pi n$, which is just a large limit of the formula we had before.

0.3 Lagrangian Density

We can also do this using the lagrangian, by making an argument

$$\mathcal{L} = \frac{1}{2}\rho(x)\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\tau(x)\left(\frac{\partial y}{\partial x}\right)^2$$

The way we arrive at this conclusion is by varying the functional

$$\delta \int_{t_1}^{t_2} \int_{\mathcal{D}} \mathcal{L}(x, t, y, \partial_x y, \partial_t y) = 0$$

A WHOLE LOT OF ALGEBRA LATER, the correct answer will fall outof the thing. the final form, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t y)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x y)}\right) = \frac{\partial \mathcal{L}}{\partial y}$$

0.4 Hamiltonian Density

We have some H, we can introduct some continuous medium version of this, with $\mathcal{L}(x, t, y, \partial_t y, \partial_x y)$, with some generalized momenta $\vec{p} = \frac{\partial \mathcal{L}}{\partial(\partial_t y)}$, then introduce hamiltonain denstiy

$$\mathcal{H} = p\partial_t y = \mathcal{L} = \frac{1}{2}\rho(x)\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}\tau(x)\left(\frac{\partial y}{\partial x}\right)^2$$