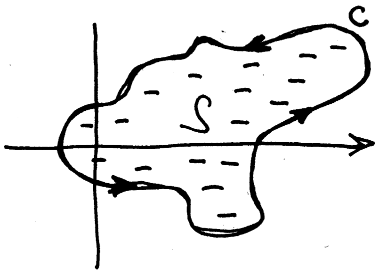
Greenova formula za ravan

Ako je c po djelovima glatka granica područja S, a f-je $P(x,y)$ i $Q(x,y)$ neprekidne zajedno sa svojim parcijalnim izvodima prvog reda u zatvorenom području $S+C$, onda vrijedi Greenova formula



$$\int_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

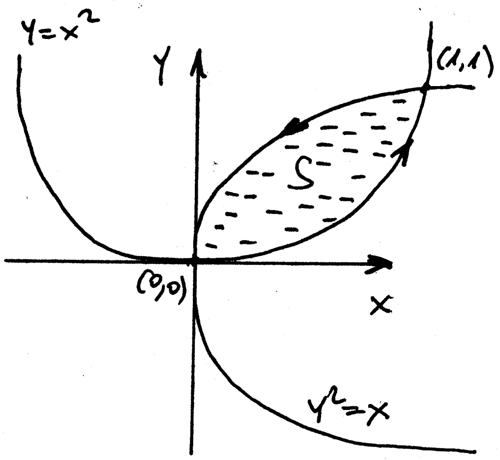
c - zatvorena kontura

S - oblast ograničena konturom

(#) Izračunati integral $\displaystyle\int_C (2xy - x^2)\,dx + (x + y^2)\,dx$

gdje je c kontura površine ograničene sa $y = x^2$ i $y^2 = x$.

Rj.



$P(x,y) = 2xy - x^2 \qquad \dfrac{\partial P}{\partial y} = 2x$

$Q(x,y) = x + y^2 \qquad \dfrac{\partial Q}{\partial x} = 1$

$$\int_C P\,dx - Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

formula Greena

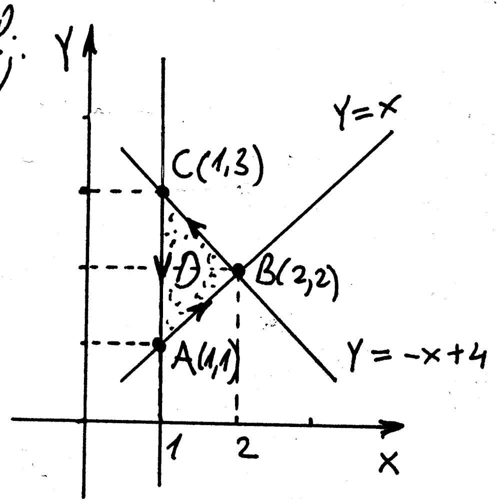
$$\int_C (2xy - x^2)\,dx + (x+y^2)\,dx = \iint_S (1-2x)\,dx\,dy = \int_0^1 \left[\int_{x^2}^{\sqrt{x}} (1-2x)\,dy\right] dx =$$

$$= \int_0^1 \left(y\Big|_{x^2}^{\sqrt{x}} - 2xy\Big|_{x^2}^{\sqrt{x}} \right) dx = \int_0^1 \left(\sqrt{x} - x^2 - 2x(\sqrt{x} - x^2)\right) dx =$$

$$= \int_0^1 \left(2x^3 - x^2 - 2x^{\frac{3}{2}} + x^{\frac{1}{2}}\right) = 2\cdot\frac{1}{4}x^4\Big|_0^1 - \frac{1}{2}x^3\Big|_0^1 - 2\cdot\frac{2}{5}x^{\frac{5}{2}}\Big|_0^1 + \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{1}{30}$$

(#) Izračunati $\int\limits_C 2(x^2+y^2)dx+(x+y)^2dy$ gdje je c kontura trougla ΔABC pozitivno orjentisana ($A(1,1)$, $B(2,2)$, $C(1,3)$).

Rj.:



$$P(x,y)=2(x^2+y^2)=2x^2+2y^2$$

$$Q(x,y)=(x+y)^2=x^2+2xy+y^2$$

$$\int\limits_C P(x,y)\,dx+Q(x,y)\,dy=\iint\limits_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

formula Grina

$$y-y_1=\frac{x_2-x_1}{y_2-y_1}(x-x_1)$$

$$y-2=\frac{-1}{1}(x-2)$$

$$y-2=-x+2 \quad\Rightarrow\quad y=-x+4$$

$$\frac{\partial P}{\partial y}=4y \qquad \frac{\partial Q}{\partial x}=2x+2y \qquad \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=2x+2y-4y=2x-2y$$

$$D:\begin{cases}1\le x\le 2\\ x\le y\le 4-x\end{cases} \qquad \int\limits_C 2(x^2+y^2)dx+(x+y)^2dy=\iint\limits_D(2x-2y)\,dx\,dy=$$

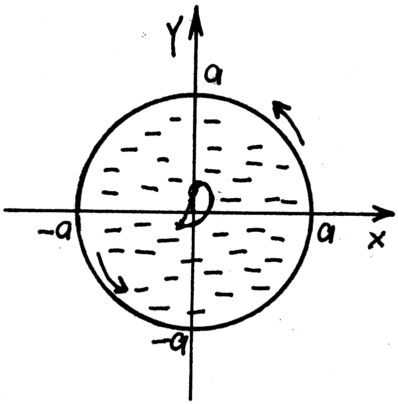
$$=\int_1^2\left[\int_x^{4-x}(2x-2y)\,dy\right]dx=\int_1^2\Big(\underbrace{2xy\Big|_x^{4-x}}_{4-x-x}-2\cdot\frac{1}{2}\underbrace{y^2\Big|_x^{4-x}}_{(4-x)^2-x^2}\Big)dx=$$

$$=\int_1^2\big(2x(4-2x)-(16-8x)\big)dx=\int_1^2(8x-4x^2-16+8x)\,dx=\int_1^2(-4x^2+16x-16)\,dx$$

$$=-4\cdot\frac{1}{3}x^3\Big|_1^2+16\cdot\frac{1}{2}x^2\Big|_1^2-16x\Big|_1^2=-\frac{4}{3}\cdot 7+8\cdot 3-16=8-\frac{28}{3}=-\frac{4}{3}$$

\# Izračunati $\int\limits_C xy^2\,dy - x^2y\,dx$ gdje je c krug $x^2+y^2=a^2$. Integraciju izvesti u pozitivnom smjeru.

Rj.



$$P(x,y) = -x^2y \qquad \frac{\partial P}{\partial y} = -x^2$$

$$Q(x,y) = xy^2 \qquad \frac{\partial Q}{\partial x} = y^2 \qquad D:\ x^2+y^2 \le a^2$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 + x^2 = x^2 + y^2$$

$$\int\limits_C P(x,y)\,dx + Q(x,y)\,dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Formula Greena

polarne koordinate $\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array} \Rightarrow D': \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ 0 \le r \le a \\ 0 \le \varphi \le 2\pi \end{cases} \quad dx\,dy = r\,dr\,d\varphi$

$$\int\limits_C xy^2\,dy - x^2y\,dx = \iint\limits_D (x^2+y^2)\,dx\,dy = \iint\limits_{D'} (r^2\cos^2\varphi + r^2\sin^2\varphi)\cdot r\,dr\,d\varphi =$$

$$= \int_0^{2\pi}\left[\int_0^a r^3\,dr\right] d\varphi = \int_0^{2\pi} \frac{1}{4} r^4 \Big|_0^a \,d\varphi = \frac{a^4}{4}\cdot \varphi\Big|_0^{2\pi} = \frac{\pi a^4}{2}$$

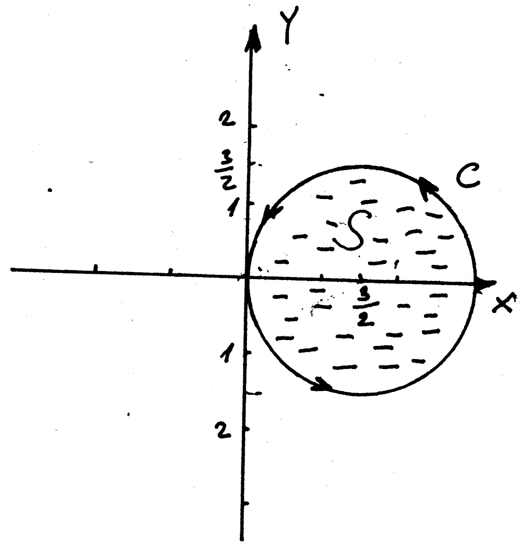
(#) Izračunati krivoliniski integral.

$$I = \int_C (xy + x + y)\,dx + (xy + x - y)\,dy \quad \text{ako je } C: x^2 + y^2 = 3x.$$

Rj.

$$x^2 + y^2 = 3x$$
$$x^2 - 3x + y^2 = 0$$
$$x^2 - 2\cdot x\cdot \frac{3}{2} + \frac{9}{4} - \frac{9}{4} + y^2 = 0$$
$$\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$$

C: Krug sa centrom u tački $\left(\frac{3}{2}, 0\right)$ poluprečnika $r = \frac{3}{2}$



I način: Greenova formula za ravan

$$\int_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

C - zatvorena kontura
S - oblast ograničena konturom

$P = xy + x + y$, $\frac{\partial P}{\partial y} = x + 1$, $Q = xy + x - y$, $\frac{\partial Q}{\partial x} = y + 1$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y + 1 - (x + 1) = y - x$$

Kako je C krug, oblast ograničena krugom je unutrašnjost kruga. Da bi smo lakše opisali unutrašnjost kruga uvedimo polarne koordinate

$$x = \frac{3}{2} + r\cos\varphi$$
$$y = r\sin\varphi$$
$$dx\,dy = r\,dr\,d\varphi$$

$$D: \begin{cases} 0 \le r \le \frac{3}{2} \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$I = \int_C (xy + x + y)\,dx + (xy + x - y)\,dy = \iint_S (y - x)\,dx\,dy = \iint_D \left(r\sin\varphi - \left(\frac{3}{2} + r\cos\varphi\right)\right)\cdot r\,dr\,d\varphi$$

$$= \int_0^{3/2}\left[\int_0^{2\pi}\left(r^2\sin\varphi - \frac{3}{2}r - r^2\cos\varphi\right)d\varphi\right]dr = \int_0^{3/2}\left(\underbrace{-r^2\cos\varphi\Big|_0^{2\pi}}_{=0} - \frac{3r}{2}\varphi\Big|_0^{2\pi} \underbrace{- r\sin\varphi\Big|_0^{2\pi}}_{=0}\right)dr$$

$$= \int_0^{3/2} -3\pi r\,dr = -3\pi\frac{r^2}{2}\Big|_0^{3/2} = -\frac{3}{2}\pi\cdot\frac{9}{4} = -\frac{27}{8}\pi$$

II način: Klasičan način

c kriva u ravni opisana jednačinom $y = \eta(x)$, $a \le x \le b$

$$\int_c P(x,y)\,dx + Q(x,y)\,dy = \int_a^b [P(x,\eta(x)) + Q(x,\eta(x))\cdot \eta'(x)]\,dx$$

Ako je c data kriva opisana parametarskim jednačinama $x = \mu(t)$, $y = \eta(t)$ gdje je $t_1 \le t \le t_2$ tada

$$\int_c P(x,y)\,dx + Q(x,y)\,dy = \int_{t_1}^{t_2} [P(\mu(t),\eta(t))\mu'(t) + Q(\mu(t),\eta(t))\eta'(t)]\,dt$$

U našem slučaju c je kružnica. Parametarizirajmo kružnicu

$$x = \frac{3}{2} + r\cos\varphi$$
$$y = r\sin\varphi$$

U našem slučaju $r = \frac{3}{2}$ a umjesto promjenjive φ stavimo promjenjivu t

$$\frac{\partial x}{\partial t} = -\frac{3}{2}\sin t \qquad x = \frac{3}{2} + \frac{3}{2}\cos t$$
$$\frac{\partial y}{\partial t} = \frac{3}{2}\cos t \qquad y = \frac{3}{2}\sin t$$

gdje $0 \le t \le 2\pi$

$$I = \int_c (xy + x + y)\,dx + (xy + x - y)\,dy = \int_0^{2\pi} \Big[\Big(\Big(\frac{3}{2} + \frac{3}{2}\cos t\Big)\Big(\frac{3}{2}\sin t\Big) + \Big(\frac{3}{2} + \frac{3}{2}\cos t\Big) +$$
$$+\Big(\frac{3}{2}\sin t\Big)\Big)\Big(-\frac{3}{2}\sin t\Big) + \Big(\Big(\frac{3}{2} + \frac{3}{2}\cos t\Big)\Big(\frac{3}{2}\sin t\Big) + \Big(\frac{3}{2} + \frac{3}{2}\cos t\Big) -$$
$$-\Big(\frac{3}{2}\sin t\Big)\Big)\frac{3}{2}\cos t\Big]\,dt = \ldots$$

na klasičan način ovo je komplikovano ali se može izračunati

$$I = -\frac{27}{8}\pi$$

(Ova stranica je ostavljena prazna)