Formula Gauss-Ostrogradski

Ova formula daje vezu između površinskog integrala druge vrste i trostrukog integrala.

$$\iint P(x,y,z)\,dy\,dz + Q(x,y,z)\,dx\,dz + R(x,y,z)\,dx\,dy =$$

$$= \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

gdje je Ω oblast u prostoru ograničena datom površinom S (S je zatvorena površina).

(1.) Izračunati $\iint\limits_{S} xy\,dx\,dy + yz\,dy\,dz + zx\,dz\,dx$ gdje je S bilo koja zatvorena površ.

Rj.

$$\iint\limits_{S} yz\,dy\,dz + zx\,dx\,dz + xy\,dx\,dy = \iint\limits_{S} P\,dy\,dz + Q\,dx\,dz +$$

$$+ R\,dx\,dy \overset{\text{Formula Gauss-Ostr.}}{=\!=\!=} \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

gdje je Ω oblast u prostoru ograničena datom površinom S.

$$\frac{\partial P}{\partial x} = 0, \quad \frac{\partial Q}{\partial y} = 0, \quad \frac{\partial R}{\partial z} = 0$$

$$\iint\limits_{S} yz\,dy\,dz + zx\,dx\,dz + xy\,dx\,dy = \iiint\limits_{\Omega} 0\,dx\,dy\,dz =$$

$$\Omega : \begin{cases} a \le x \le b \\ c \le y \le d \\ e \le z \le f \end{cases} \qquad = \int_a^b dx \int_c^d dy \int_e^f 0\,dz = 0$$

(#) Površinski integral po zatvorenoj površini pretvoriti uz pomoć formule Ostrogradskog u trostruki integral po zapremini tijela, koje je ograničeno spomenutom površinom
$$\iint_S \sqrt{x^2+y^2+z^2}\left[\cos(\vec{n},x)+\cos(\vec{n},y)+\cos(\vec{n},z)\right]dS$$
gdje je $\vec{n}$ vanjska normala na površinu S.

Rj: $\cos(\vec{n},x)$ je kosinus ugla između normale i x-ose. $\cos(\vec{n},y)$ i $\cos(\vec{n},z)$ je kosinus ugla između normale na površinu S i y-ose i z-ose redom.

Uvedimo oznake $\cos(\vec{n},x)=\cos\alpha$, $\cos(\vec{n},y)=\cos\beta$ i $\cos(\vec{n},z)=\cos\gamma$.

Prema formuli Stoksa znamo da je
$$dy\,dz = dS\cos\alpha$$
$$dz\,dx = dS\cos\beta$$
$$dx\,dy = dS\cos\gamma$$

$$I=\iint_S \sqrt{x^2+y^2+z^2}\left(\cos(\vec{n},x)+\cos(\vec{n},y)+\cos(\vec{n},z)\right]dS =$$
$$=\iint_S \sqrt{x^2+y^2+z^2}\,(dy\,dz+dz\,dx+dx\,dy)$$

$$\iint_S P\,dy\,dz+Q\,dx\,dz+R\,dx\,dy=\iiint_\Omega\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz$$

$$\frac{\partial P}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2+z^2}} \qquad I=\iiint_\Omega \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$$

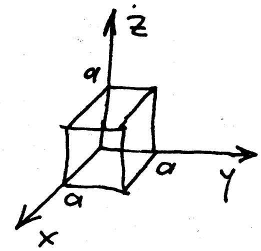
(#) Izvačunati $\iint\limits_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy$ gdje je

S - vanjska strana kocke $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$.

Rj.

$$\iint\limits_S P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy \overset{\text{formula Gauss-Ostr.}}{=} \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

$$\frac{\partial P}{\partial x} = 2x, \quad \frac{\partial Q}{\partial y} = 2y, \quad \frac{\partial R}{\partial z} = 2z$$



$$\Omega : \begin{cases} 0 \le x \le a \\ 0 \le y \le a \\ 0 \le z \le a \end{cases}$$

Prema tome:

$$\iint\limits_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy =$$

$$= \iiint\limits_{\Omega} (2x + 2y + 2z)\,dx\,dy\,dz = 2\int_0^a dx \int_0^a dy \int_0^a (x+y+z)\,dz =$$

$$= 2\int_0^a dx \int_0^a \left(xz\Big|_0^a + yz\Big|_0^a + \frac{1}{2}z^2\Big|_0^a \right) dy = 2\int_0^a dx \int_0^a \left(ax + ay + \frac{1}{2}a^2\right) dy =$$

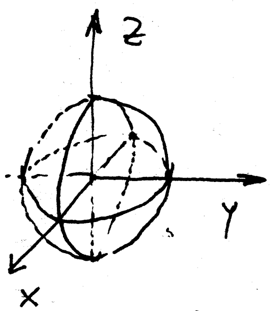
$$2a\int_0^a \left(xy\Big|_0^a + \frac{1}{2}y^2\Big|_0^a + \frac{1}{2}ay\Big|_0^a \right) dx = 2a\int_0^a \left(ax + \frac{1}{2}a^2 + \frac{1}{2}a^2\right) dx = 2a^2\int_0^a (x+a)\,dx =$$

$$= 2a^2\left(\frac{1}{2}a^2 + a^2\right) = 3a$$

(#) Izračunati $\iint\limits_S x^3 dy dz + y^3 dz dx + z^3 dx dy$ gdje je

S - vanjski dio sfere $x^2+y^2+z^2=R^2$.

Rj.



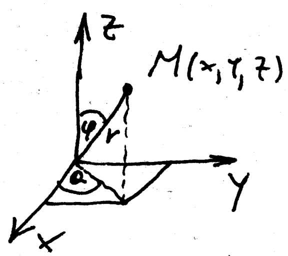
$$I = \iint\limits_S x^3 dy dz + y^3 dz dx + z^3 dx dy \overset{\text{formula Gauss-Ostrogradski}}{=}$$

$$= \iiint\limits_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx dy dz$$

$$P = x^3, \quad \frac{\partial P}{\partial x} = 3x^2, \quad Q = y^3, \quad \frac{\partial Q}{\partial y} = 3y^2, \quad R = z^3, \quad \frac{\partial R}{\partial z} = 3z^2$$

$$I = \iiint\limits_\Omega (3x^2+3y^2+3z^2) dx dy dz \qquad \Omega: x^2+y^2+z^2 \le R^2$$

Uvedimo sferne koordinate:



$$x = r \sin\varphi \cos\theta$$
$$y = r \sin\varphi \sin\theta$$
$$z = r \cos\varphi$$

$$\Omega': \begin{cases} 0 \le r \le R \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \\ dx dy dz = r^2 \sin\varphi \, d\varphi \, d\theta \, dr \end{cases}$$

$$x^2+y^2+z^2 = r^2$$

$$I = 3 \iiint\limits_{\Omega'} r^2 r^2 \sin\varphi \, d\varphi \, d\theta \, dr = 3 \int_0^R r^4 dr \int_0^{2\pi} d\theta \int_0^\pi \sin\varphi \, d\varphi =$$

$$= 3 \cdot \frac{1}{5} r^5 \Big|_0^R \cdot \theta \Big|_0^{2\pi} \cdot (-\cos\varphi) \Big|_0^\pi = \frac{3}{5} \cdot R^5 \cdot 2\pi \cdot 2 = \frac{12}{5} R^5 \pi$$

Zadaci za vježbu

1. Izračunati integral $\int x^3y^3\,dx+dy+z\,dz$ gdje je kontura L - krug $x^2+y^2=R^2$, $z=0$:

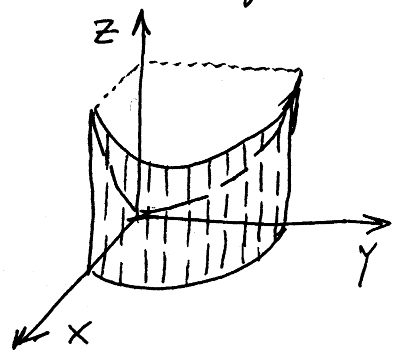
   a) direktno

   b) pomoću formule Stoksa, gdje ćete primjetiti da vadimo po površini polusfere $z=+\sqrt{R^2-x^2-y^2}$.

   Integral po krugu u ravni xOy izračunati u pozitivnom smjeru.

2. Izračunati $\iint\limits_S \sqrt{x^2+y^2+z^2}\,[\cos(\vec{n},x)+\cos(\vec{n},y)+\cos(\vec{n},z)]\,dS$ ($\vec{n}$ - vanjska normala na površinu S) gdje je S-sfera poluprečnika R s centrom u koordinatnom ishodištu.

3. Izračunati $\iint\limits_S y^2z\,dx\,dy+xz\,dy\,dz+x^2y\,dx\,dz$, gdje je S spoljna strana površine, koja se nalazi u prvom oktantu i sastavljena je od paraboloida $z=x^2+y^2$ rotiranog, cilindra $x^2+y^2=1$ i koordinatnih ravni (pogledati sliku).



Rješenja:

1. $-\frac{\pi R^6}{8}$ 2. 0 3. $\frac{\pi}{8}$

(Ova stranica je ostavljena prazna)