POVRŠINSKI INTEGRALI (zadaci- I deo)

1. Izračunati površinski integral $I = \iint_S (6x + 4y + 3z)dS$ **ako je S deo ravni** $x + 2y + 3z = 6$ **, koja pripada prvom oktantu.**

Rešenje:

Važno je još jednom napomenuti da: **POVRŠINSKI INTEGRAL PRVE VRSTE NE ZAVISI OD ORIJENTACIJE KRIVE**

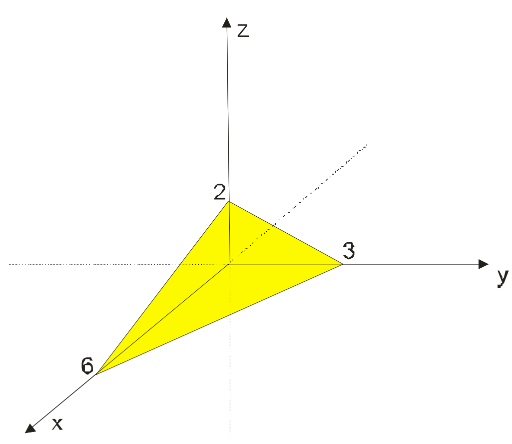
Koristićemo:

ii) Ako jednačina površi S ima oblik z=z(x,y) , gde je z=z(x,y) jednoznačna neprekidno diferencijabilna funkcija, onda je:

$$\iint_S f(x,y,z)ds = \iint_D f[x,y,z(x,y)]\sqrt{1+p^2+q^2}\,dxdy \qquad \text{i}$$

$$p = \frac{\partial z}{\partial x} \quad \text{i} \quad q = \frac{\partial z}{\partial y}$$

Najpre nacrtamo sliku i postavimo problem.



$$x + 2y + 3z = 6......./:6$$

$$\frac{x}{6} + \frac{y}{3} + \frac{z}{2} = 1$$

Segmentni oblik jednačine prave nam daje preseke sa osama (ovo nam sad baš i ne treba al nije rdjavo da pomenemo…)

Naš posao je da izrazimo z iz date jednačine i nadjemo prve parcijalne izvode, odnosno p i q.

$$x+2y+3z=6$$
$$3z=-x-2y+6....../:3$$
$$z=-\frac{x}{3}-\frac{2y}{3}+2$$

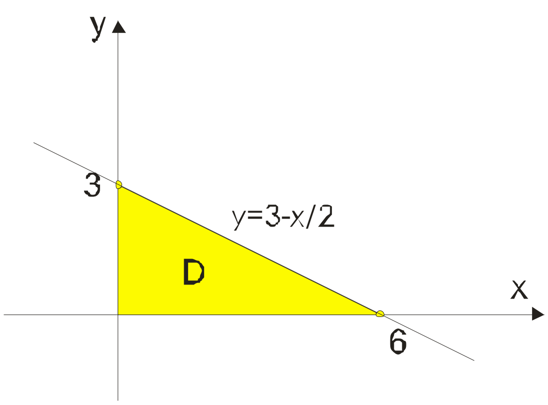
Odavde je:

$$p=\frac{\partial z}{\partial x}=-\frac{1}{3}$$
$$q=\frac{\partial z}{\partial y}=-\frac{2}{3}$$

Spustimo se u ravan z = 0 da odredimo granice.

$$\boxed{z=0}\rightarrow x+2y+3\cdot 0=6\rightarrow x+2y=6\rightarrow \boxed{y=3-\frac{x}{2}}$$

Pogledajmo sliku :



$$D:\begin{cases} 0\le x\le 6 \\ 0\le y\le 3-\frac{x}{2} \end{cases}$$

Sad možemo da predjemo na rešavanje integrala:

$$I=\iint_S (6x+4y+3z)dS=\iint_D (6x+4y+\cancel{3}\cdot\frac{6-x-2y}{\cancel{3}})\sqrt{1+p^2+q^2}\,dxdy$$
$$=\iint_D (6x+4y+6-x-2y)\sqrt{1+(-\frac{1}{3})^2+(-\frac{2}{3})^2}\,dxdy=$$
$$=\iint_D (5x+2y+6)\sqrt{\frac{14}{9}}\,dxdy=$$
$$=\frac{\sqrt{14}}{3}\int_0^6 dx\int_0^{3-\frac{x}{2}}(5x+2y+6)dy \quad \text{(sredimo sve - podsetite se dvojnih integrala)}$$
$$=\boxed{54\sqrt{14}}$$

2. Rešiti integral $I = \iint_S (x^2 + y^2)dS$ **ako je S sfera** $x^2 + y^2 + z^2 = a^2$.

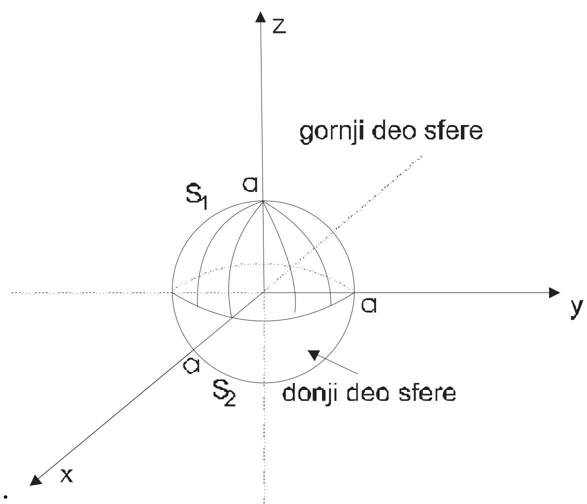
Rešenje:

Najpre moramo izraziti z iz date jednačine:

$$x^2 + y^2 + z^2 = a^2$$
$$z^2 = a^2 - x^2 - y^2$$
$$z = \pm\sqrt{a^2 - x^2 - y^2}$$

Ovde treba voditi računa da posebno moramo raditi za gornji deo sfere $z = +\sqrt{a^2 - x^2 - y^2}$ (iznad z = 0 ravni)

i posebno za $z = -\sqrt{a^2 - x^2 - y^2}$ (ispod z = 0 ravni).



Pogledajmo sliku:

$S_1 : z_1 = +\sqrt{a^2 - x^2 - y^2}$ i $S_2 : z_2 = -\sqrt{a^2 - x^2 - y^2}$

Za $S_1 : z_1 = +\sqrt{a^2 - x^2 - y^2}$ (a slično je i za $S_2 : z_2 = -\sqrt{a^2 - x^2 - y^2}$) imamo:

$$p = \frac{\partial z}{\partial x} = \frac{1}{2\sqrt{a^2 - x^2 - y^2}} \cdot \left(a^2 - x^2 - y^2\right)'_{po\ x} = \frac{1}{2\sqrt{a^2 - x^2 - y^2}} \cdot (-2x) = \boxed{\frac{-x}{\sqrt{a^2 - x^2 - y^2}}}$$

$$q = \frac{\partial z}{\partial y} = \frac{1}{2\sqrt{a^2 - x^2 - y^2}} \cdot \left(a^2 - x^2 - y^2\right)'_{po\ y} = \frac{1}{2\sqrt{a^2 - x^2 - y^2}} \cdot (-2y) = \boxed{\frac{-y}{\sqrt{a^2 - x^2 - y^2}}}$$

Za $S_2 : z_2 = -\sqrt{a^2 - x^2 - y^2}$ će samo biti plusevi….

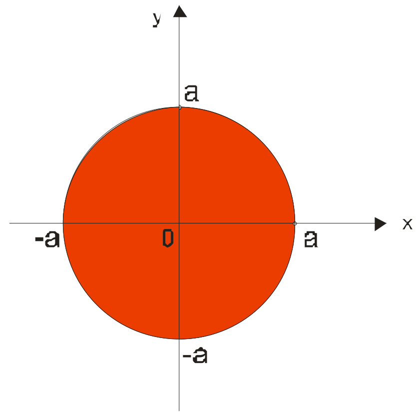
Izračunajmo "na stranu":

$$\sqrt{1 + p^2 + q^2} = \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}}\right)^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}} =$$

$$= \sqrt{\frac{a^2 \cancel{- x^2} \cancel{- y^2} \cancel{+ x^2} \cancel{+ y^2}}{a^2 - x^2 - y^2}} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}$$

Sad spuštamo problem u ravan z = 0:

$$z = 0 \wedge x^2 + y^2 + z^2 = a^2 \rightarrow x^2 + y^2 = a^2$$



Oblast D je unutrašnjost ovog kruga! (Ista je i za S_1 i za S_2)

$$I = \iint_S (x^2 + y^2) dS = \iint_{S_1} + \iint_{S_2} = \iint_D (x^2 + y^2)\sqrt{1 + p^2 + q^2} dxdy + \iint_D (x^2 + y^2)\sqrt{1 + p^2 + q^2} dxdy =$$

$$= 2\iint_D (x^2 + y^2)\sqrt{1 + p^2 + q^2} dxdy = 2\iint_D \frac{a(x^2 + y^2)}{\sqrt{a^2 - x^2 - y^2}} dxdy$$

Uvodimo smene:

$$\left.\begin{matrix} x = r\cos\varphi \\ y = r\sin\varphi \end{matrix}\right\} \rightarrow x^2 + y^2 = a^2 \rightarrow a^2 = r^2 \rightarrow r = a \rightarrow 0 \le r \le a \wedge 0 \le \varphi \le 2\pi \wedge |J| = r$$

$$I = 2\iint_D \frac{a(x^2 + y^2)}{\sqrt{a^2 - x^2 - y^2}} dxdy = 2\int_0^{2\pi} d\varphi \int_0^a \frac{ar^2}{\sqrt{a^2 - r^2}} rdr = 2\underbrace{\boxed{\int_0^{2\pi} d\varphi}}_{ovo\ je\ 2\pi} \int_0^a \frac{ar^2}{\sqrt{a^2 - r^2}} rdr = 4a\pi \int_0^a \frac{r^2}{\sqrt{a^2 - r^2}} rdr$$

Metodom smene ćemo rešiti ovaj integral bez granica (lakše malo)

$$\int \frac{r^2}{\sqrt{a^2 - r^2}} rdr = \left| \begin{matrix} a^2 - r^2 = t^2 \rightarrow r^2 = a^2 - t^2 \\ -2rdr = 2tdt \\ rdr = -tdt \end{matrix} \right| = \int \frac{a^2 - t^2}{t}(-t)\, dt = \int \left(t^2 - a^2\right) dt = \frac{t^3}{3} - a^2 t =$$

$$= \frac{\left(\sqrt{a^2 - r^2}\right)^3}{3} - a^2\sqrt{a^2 - r^2}$$

Sad mu ubacimo granice:

$$\left(\frac{\left(\sqrt{a^2 - r^2}\right)^3}{3} - a^2\sqrt{a^2 - r^2} \right)\Bigg|_0^a = (a^3 - \frac{a^3}{3}) - (0 - 0) = \frac{2a^3}{3}$$

Vratimo se u zadatak:

$$I = 4a\pi\int_0^a \frac{ar^2}{\sqrt{a^2-r^2}}rdr = 4a\pi\cdot\frac{2a^3}{3} = \boxed{\frac{8a^4\pi}{3}}$$ i evo konačnog rešenja!

3. **Rešiti integral** $I = \iint_S \frac{1}{x^2+y^2+z^2}dS$ **ako je S deo cilindra** $x^2+y^2=R^2$ **ograničenog ravnima x=0, y=0, z=0 i z = m.**

Rešenje:

Primećujemo da je cilindar $x^2+y^2=R^2$ uz z osu i da ne možemo odavde izraziti z. Onda ćemo izraziti ili x ili y

i raditi po njima sve isto kao i po z....

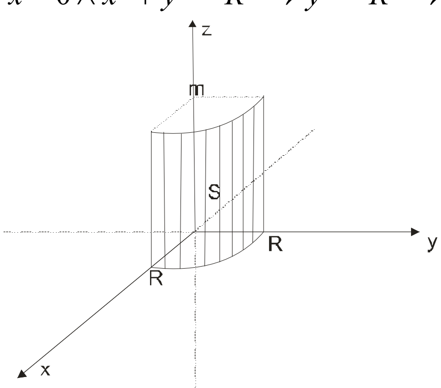
$$x^2+y^2=R^2$$
$$x^2=R^2-y^2$$
$x=\pm\sqrt{R^2-y^2}$ nama treba x > 0 pa je:
$$x=+\sqrt{R^2-y^2}$$
Odavde imamo:

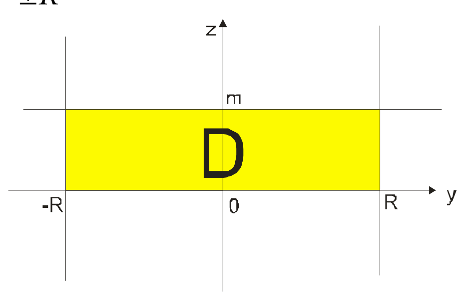
$$p=\frac{\partial x}{\partial y}=\frac{1}{2\sqrt{R^2-y^2}}\cdot\left(R^2-y^2\right)'_{po\ y}=\frac{1}{2\sqrt{R^2-y^2}}\cdot(-2y)=\frac{-y}{\sqrt{R^2-y^2}}\cdot$$
$$q=\frac{\partial x}{\partial z}=0$$

$$\sqrt{1+p^2+q^2}=\sqrt{1+\left(\frac{-y}{\sqrt{R^2-y^2}}\right)^2+0}=\sqrt{1+\frac{y^2}{R^2-y^2}}=\sqrt{\frac{R^2-\cancel{y^2}+\cancel{y^2}}{R^2-y^2}}=\frac{R}{\sqrt{R^2-y^2}}$$

Pošto smo odabrali da radimo po x , da bi odredili granice integrala, spuštamo se u ravan x = 0 .

$$x=0\wedge x^2+y^2=R^2\rightarrow y^2=R^2\rightarrow y=\pm R$$





$$D:\begin{cases}-R\le y\le R\\ 0\le z\le m\end{cases}$$

Sad da rešimo zadati integral:

$$I=\iint_S \frac{1}{x^2+y^2+z^2}dS=\iint_D \frac{1}{R^2\cancel{-y^2}\cancel{+y^2}+z^2}\sqrt{1+p^2+q^2}\,dxdy=$$

$$=\int_{-R}^{R}\frac{Rdy}{\sqrt{R^2-y^2}}\int_0^m\frac{dz}{R^2+z^2}=\text{na stranu sredimo:}$$

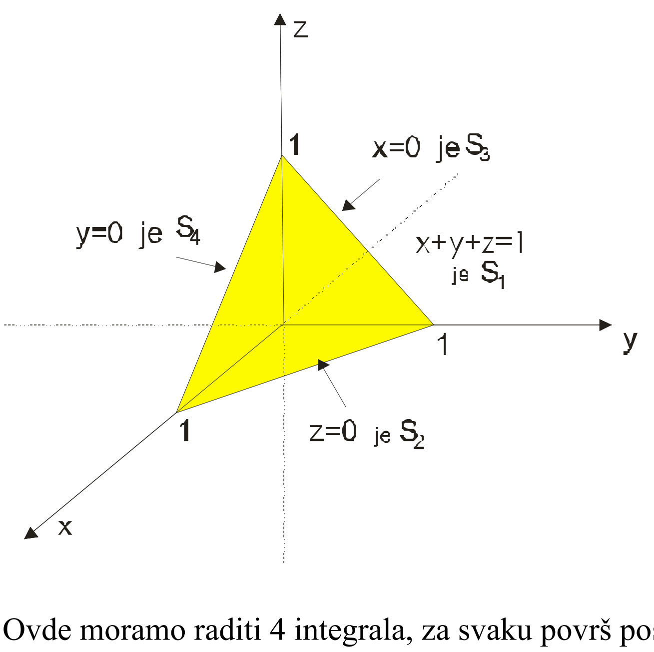
$$\int_0^m\frac{dz}{R^2+z^2}=\frac{1}{R}arctg\frac{z}{R}\Big|_0^m=\frac{1}{R}arctg\frac{m}{R}-\frac{1}{R}arctg\frac{0}{R}=\boxed{\frac{1}{R}arctg\frac{m}{R}}$$

$$I=\frac{1}{\cancel{R}}arctg\frac{m}{R}\int_{-R}^{R}\frac{\cancel{R}dy}{\sqrt{R^2-y^2}}=arctg\frac{m}{R}\cdot\int_{-R}^{R}\frac{dy}{\sqrt{R^2-y^2}}=....=\boxed{\frac{\pi}{2}arctg\frac{m}{R}}$$

4. Rešiti integral $I=\iint_S\frac{1}{(1+x+y)^2}dS$ **ako je S površ tetraedra ograničenog ravnima** $x+y+z=1, x=0, y=0, z=0$

Rešenje:

Nacrtajmo sliku i postavimo problem.



Ovde moramo raditi 4 integrala, za svaku površ posebno. **Dakle** $I=\iint_{S_1}+\iint_{S_2}+\iint_{S_3}+\iint_{S_4}$

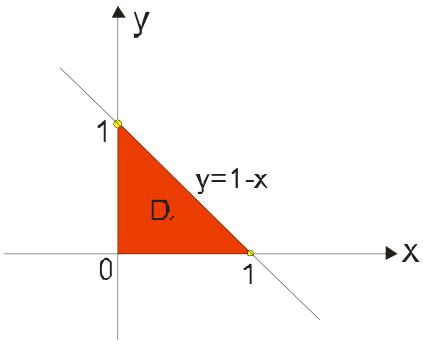
Za S_1 **(ravan** $x+y+z=1$**) imamo:**

$$x+y+z=1\to z=1-x-y\to\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=-1$$

$$\sqrt{1+p^2+q^2}=\sqrt{1+1+1}=\sqrt{3}$$

Ako se spustimo u ravan z = 0 imamo $x+y=1 \to y=1-x$

Nacrtajmo sliku i odredimo granice:



$$D_1:\begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{cases}$$

$$\iint_{S_1} \frac{1}{(1+x+y)^2} dS = \iint_{D_1} \frac{1}{(1+x+y)^2} \sqrt{3} dxdy = \sqrt{3}\int_0^1 dx \int_0^{1-x} \frac{1}{(1+x+y)^2} dy =$$

Na stranu :

$$\int \frac{1}{(1+x+y)^2} dy = \begin{vmatrix} 1+x+y=t \\ dy = dt \end{vmatrix} = \int t^{-2} dt = \frac{t^{-1}}{-1} = -\frac{1}{t} = -\frac{1}{1+x+y}, \textit{pa je}$$

$$-\frac{1}{1+x+y}\Bigg|_0^{1-x} = -\frac{1}{1+x+1-x} - \left(-\frac{1}{1+x+0}\right) = -\frac{1}{2} + \frac{1}{1+x}$$

vratimo se u integral

$$\sqrt{3}\int_0^1 dx \int_0^{1-x} \frac{1}{(1+x+y)^2} dy = \sqrt{3}\int_0^1 \left(-\frac{1}{2} + \frac{1}{1+x}\right) dx = \sqrt{3}\left(-\frac{1}{2}x + \ln|1+x|\right)\Bigg|_0^1 = \sqrt{3}(\ln 2 - \frac{1}{2})$$

Za S_2 (ravan $z=0$) imamo:

$$z=0 \to \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$$

$$\sqrt{1+p^2+q^2} = \sqrt{1} = 1$$

$$x+y=1 \to y=1-x$$

$$D_1:\begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{cases}$$

Dakle, oblast je ista kao za prethodni deo.... A i integral se slično rešava!

$$\iint_{S_2} \frac{1}{(1+x+y)^2} dS = \iint_{D_1} \frac{1}{(1+x+y)^2} dxdy = \int_0^1 dx \int_0^{1-x} \frac{1}{(1+x+y)^2} dy = \ln 2 - \frac{1}{2}$$

Za S_3 (ravan $x=0$) imamo:

$$x=0 \to \frac{\partial x}{\partial z}=\frac{\partial x}{\partial y}=0$$

$$\sqrt{1+p^2+q^2}=\sqrt{1}=1$$

$$y+z=1 \to z=1-y$$

$$D_3:\begin{cases} 0\le y\le 1 \\ 0\le z\le 1-y \end{cases}$$

$$\iint_{S_3}\frac{1}{(1+x+y)^2}dS=\iint_{D_3}\frac{1}{(1+0+y)^2}dydz=\int_0^1 dy\int_0^{1-y}\frac{1}{(1+y)^2}dy=1-\ln 2$$

Za S_4 (ravan $y=0$) imamo:

$$y=0 \to \frac{\partial y}{\partial z}=\frac{\partial y}{\partial x}=0$$

$$\sqrt{1+p^2+q^2}=\sqrt{1}=1$$

$$x+z=1 \to z=1-x$$

$$D_4:\begin{cases} 0\le x\le 1 \\ 0\le z\le 1-x \end{cases}$$

$$\iint_{S_4}\frac{1}{(1+x+y)^2}dS=\iint_{D_4}\frac{1}{(1+x+0)^2}dxdz=\int_0^1 dx\int_0^{1-x}\frac{1}{(1+x)^2}dz=1-\ln 2$$

E sad ćemo sabrati sva 4 rešenja:

$$I=\iint_{S_1}+\iint_{S_2}+\iint_{S_3}+\iint_{S_4}=\sqrt{3}(\ln 2-\frac{1}{2})+\left(\ln 2-\frac{1}{2}\right)+\left(1-\ln 2\right)+\left(1-\ln 2\right)=\boxed{\left(\sqrt{3}-1\right)\ln 2+\frac{3-\sqrt{3}}{2}}$$

5. Rešiti integral $I=\iint_S z^2 dS$ **ako je S:** $\begin{cases} x=r\cos\varphi\sin\alpha \\ y=r\sin\varphi\sin\alpha \\ z=r\cos\alpha \\ 0\le\varphi\le 2\pi \wedge 0\le r\le a \wedge \alpha=const. \end{cases}$

Rešenje:

<u>Da se podsetimo:</u>

i) Ako je S deo po deo glatka dvostrana površ zadata jednačinama:

x=x(u,v)
y=y(u,v)
z=z(u,v)

gde (u,v) pripada D a funkcija f(x,y,z) je definisana i neprekidna na površi S , onda je:

$$\iint_S f(x,y,z)ds = \iint_D f[x(u,v),y(u,v),z(u,v)]\sqrt{EG-F^2}dudv$$

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$

$$F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$

Malo ćemo “**korigovati”** formulice i upotrebiti ih u ovoj situaciji!

Najpre da nadjemo parcijalne izvode koji nam trebaju :

$$x = r\cos\varphi\sin\alpha \rightarrow \frac{\partial x}{\partial r} = \cos\varphi\sin\alpha \wedge \frac{\partial x}{\partial\varphi} = -r\sin\varphi\sin\alpha$$

$$y = r\sin\varphi\sin\alpha \rightarrow \frac{\partial y}{\partial r} = \sin\varphi\sin\alpha \wedge \frac{\partial x}{\partial\varphi} = r\cos\varphi\sin\alpha$$

$$z = r\cos\alpha \rightarrow \frac{\partial z}{\partial r} = \cos\alpha \wedge \frac{\partial z}{\partial\varphi} = 0$$

Sad tražimo E,G i F

$$E = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = \left(\cos\varphi\sin\alpha\right)^2 + \left(\sin\varphi\sin\alpha\right)^2 + \left(\cos\alpha\right)^2 =$$

$$= \cos^2\varphi\sin^2\alpha + \sin^2\varphi\sin^2\alpha + \cos^2\alpha = \sin^2\alpha\left(\underbrace{\cos^2\varphi + \sin^2\varphi}_{ovo\ je\ 1}\right) + \cos^2\alpha = 1$$

$$G = \left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2 + \left(\frac{\partial z}{\partial\varphi}\right)^2 = \left(-r\sin\varphi\sin\alpha\right)^2 + \left(r\cos\varphi\sin\alpha\right)^2 + 0 =$$

$$= r^2\sin^2\varphi\sin^2\alpha + r^2\cos^2\varphi\sin^2\alpha = r^2\sin^2\alpha\left(\sin^2\varphi + \cos^2\varphi\right) = r^2\sin^2\alpha$$

$$F = \frac{\partial x}{\partial r}\frac{\partial x}{\partial\varphi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial\varphi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial\varphi} =$$

$$= \cos\varphi\sin\alpha\cdot(-r\sin\varphi\sin\alpha) + \sin\varphi\sin\alpha\cdot r\cos\varphi\sin\alpha + 0 = 0$$

U zadatku nam je već dato da je: $D:\begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le a \end{cases}$

Sad rešavamo po formuli:

$$I = \iint_S z^2 dS = \iint_D (r\cos\alpha)^2 \sqrt{EG - F^2}\, dr d\varphi = \int_0^{2\pi} d\varphi \int_0^a r^2 \cos^2\alpha \sqrt{r^2 \sin^2\alpha}\, dr = \int_0^{2\pi} d\varphi \int_0^a r^2 \cos^2\alpha \cdot r\left|\sin\alpha\right| dr =$$

= pazite, i α je konstanta , pa sve ide ispred integrala!

$$= \cos^2\alpha \left|\sin\alpha\right| \int_0^{2\pi} d\varphi \int_0^a r^3 dr = \cos^2\alpha \left|\sin\alpha\right| \cdot 2\pi \cdot \frac{a^4}{4} = \boxed{\cos^2\alpha \left|\sin\alpha\right| \cdot \frac{a^4\pi}{2}}$$