Stoksova formula

Dat je krivoliniski integral $\int\limits_c P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$

gdje je c kontura u prostoru.
Stoksova formula glasi:

$$\int\limits_c P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \iint\limits_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

površinski integral prve vrste

$$\int\limits_c P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \iint\limits_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

površinski integral druge vrste

gdje je S površina u prostoru ograničena konturom c a $\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ jedinični vektor normale na površinu S.

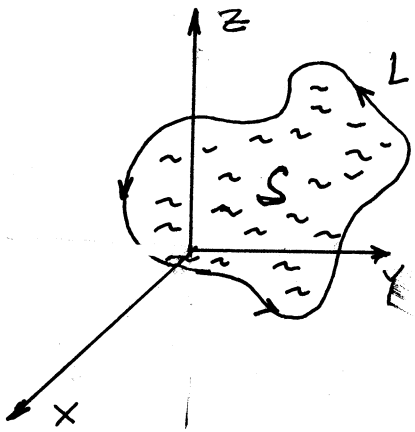
$$\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma$$

Vidimo da Stoksova formula povezuje krivoliniski integral druge vrste sa površinskim integralom prve i druge vrste.
Ranije smo spomenuli Greenovu formulu koja povezuje krivoliniski integral druge vrste sa dvostrukim integralom.
Formula Gauss-Ostrogradski povezuje površinski integral druge vrste sa trostrukim integralom.

(#) Integral $I=\int\limits_L (y^2+z^2)\,dx+(x^2+z^2)\,dy+(x^2+y^2)\,dz$

uzet po nekoj zatvorenoj konturi L, pretvoriti pomoću formule Stoksa u površinski integral, nad površinom koju zatvara spomenuta kontura.

R.:



$$\int\limits_L P\,dx+Q\,dy+R\,dz=\iint\limits_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$R(x,y,z)=x^2+y^2$

$P(x,y,z)=y^2+z^2$

$Q(x,y,z)=x^2+z^2$

$\frac{\partial R}{\partial y}=2y \qquad \frac{\partial Q}{\partial z}=2z$

$\frac{\partial R}{\partial x}=2x \qquad \frac{\partial P}{\partial z}=2z$

$\frac{\partial Q}{\partial x}=2x \qquad \frac{\partial P}{\partial y}=2y$

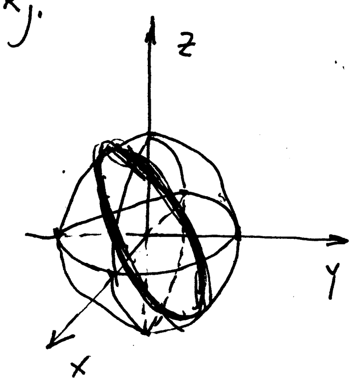
$$I=\iint\limits_S (2y-2z)\,dy\,dz-(2x-2z)\,dx\,dz+(2x-2y)\,dx\,dy=$$

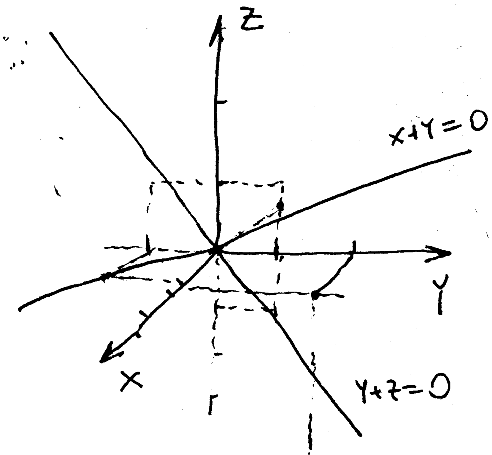
$$=2\iint\limits_S (y-z)\,dy\,dz+(z-x)\,dx\,dz+(x-y)\,dx\,dy$$

Izračunati krivoliniski integral $\int y\,dx + z\,dy + x\,dz$

ako je c krug dobijen presjekom c sfere $x^2+y^2+z^2=a^2$ i ravni $x+y+z=0$.

Rj.





$$\int_c y\,dx + z\,dy + x\,dz \overset{\text{Stoksova formula}}{=\!=} \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

$$\frac{\partial Q}{\partial x}=0 \qquad \frac{\partial P}{\partial y}=1 \qquad \begin{matrix} P=y \\ Q=z \\ R=x \end{matrix}$$

S je površina ograničena krugom

$$\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\cos\alpha - \left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)\cos\beta + \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\cos\gamma$$

$$\frac{\partial R}{\partial y}=0 \qquad \frac{\partial Q}{\partial z}=1 \qquad \frac{\partial R}{\partial x}=1 \qquad \frac{\partial P}{\partial z}=0$$

$$\int_c y\,dx + z\,dy + x\,dz = \iint_S (-\cos\alpha - \cos\beta - \cos\gamma)\,dS$$

gdje je $\vec{n}_0 = (\cos\alpha, \cos\beta, \cos\gamma)$ vektor (jedinični) normale na površinu S

$x+y+z=0$

$\vec{n}=(1,1,1)$ vektor normale na ravan $x+y+z=0$ (a time i na našu površinu S)

$|\vec{n}| = \sqrt{1+1+1} = \sqrt{3}$

$$\vec{n}_0 = \left(\underset{\cos\alpha}{\frac{1}{\sqrt{3}}}, \underset{\cos\beta}{\frac{1}{\sqrt{3}}}, \underset{\cos\gamma}{\frac{1}{\sqrt{3}}}\right)$$

$$\iint_S (-\cos\alpha-\cos\beta-\cos\gamma)\,dS = \iint_S \left(-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right) dS = -\frac{3}{\sqrt{3}}\iint_S dS$$

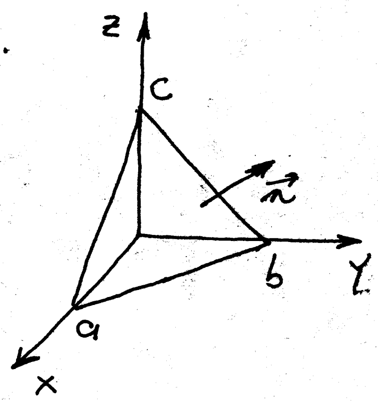
$\iint_S dS$ je površina oblasti S (S je krug poluprečnika a $P_{kruga} = a^2\pi$)

$$\int_c y\,dx + z\,dy + x\,dz = -\frac{3}{\sqrt{3}}a^2\pi = -\sqrt{3}\,a^2\pi$$

(#) Izračunati krivolinijski integral $-\int_C y^2 dx + z^2 dy + x^2 dz$

pri čemu je c kontura ΔABC gdje su tačke $A(a,0,0)$, $B(0,b,0)$ i $C(0,0,c)$, $a,b,c>0$.

Rj.



$$-\int_C y^2dx + z^2dy + x^2 dz \overset{\text{Stoksova formula}}{=\!=\!=} -\iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$P = y^2$, $Q = z^2$, $R = x^2$

$\frac{\partial Q}{\partial x} = 0$, $\frac{\partial P}{\partial y} = 2y$, $\frac{\partial R}{\partial y} = 0$, $\frac{\partial Q}{\partial z} = 2z$

$\frac{\partial R}{\partial x} = 2x$ $\quad \frac{\partial P}{\partial z} = 0$

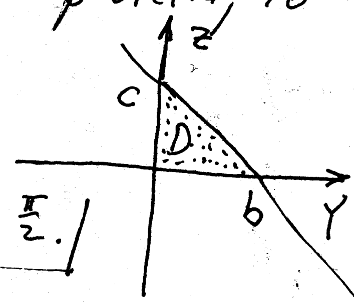
$$\begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = -2z\,dydz - 2x\,dzdx - 2y\,dxdy$$

$$-\int_C y^2dx + z^2dy + x^2dz = 2\iint_S (z\,dydz + x\,dzdx + y\,dxdy)$$

S oblast ograničena ΔABC

Izračunajmo $\iint_S z\,dydz$. Površinu S proicirajmo na yOz ravan:

Ugao koji zaklapa vektor normale $\vec{n}$ na površinu S je između 0 i $\frac{\pi}{2}$.



$\frac{y}{b} + \frac{z}{c} = 1$

$\frac{y}{b} + \frac{z}{c} = 1$

$cy + bz = bc$

$bz = bc - cy$

$z = c - \frac{c}{b}y = \frac{c}{b}(b-y)$

$D: \begin{cases} 0 \le y \le b \\ 0 \le z \le \frac{c}{b}(b-y) \end{cases}$

$\vec{n} = (\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$

$\cos\alpha > 0$

$$\iint_S z\,dydz = +\iint_D z\,dydz = \int_0^b \Big[\int_0^{\frac{c}{b}(b-y)} z\,dz\Big]dy = \int_0^b \frac{1}{2}z^2\Big|_0^{\frac{c}{b}(b-y)} dy = \frac{1}{2}\left(\frac{c}{b}\right)^2 \int_0^b (b-y)^2 dy$$

$$= \left| \begin{array}{ll} b-y=t & -dy=dt \\ y=0 \Rightarrow t=b & dy=-dt \\ y=b \Rightarrow t=0 & \end{array} \right| = \frac{1}{2}\frac{c^2}{b^2}\int_0^b t^2dt = \frac{1}{2}\cdot\frac{c^2}{b^2}\frac{t^3}{3}\Big|_0^b = \frac{1}{2}\cdot\frac{bc^2}{3}$$

Analogno izračunamo $\iint_S x\,dzdx = \frac{1}{2}\frac{a^2c}{3}$ i $\iint_S y\,dxdy = \frac{1}{2}\cdot\frac{ab^2}{3}$ $\Rightarrow I = \frac{ab^2 + bc^2 + ca^2}{3}$